Calculus – 11. Series

(turn in: January 16, 2004)

1. Prove that for $a, b > 1$ and $x, y \in \mathbb{R}$
   (a) $(a^x)^y = a^{xy}$
   (b) $a^x b^x = (ab)^x$
   (c) $\left(\frac{1}{a}\right)^x = a^{-x}$

2. Prove that for all $z, w \in \mathbb{C}$
   (a) $\cosh^2 z - \sinh^2 z = 1$
   (b) $\cos(z + w) = \cos z \cos w - \sin z \sin w$
   (c) $\cos z = \cosh(\mathrm{i}z)$
   (d) $\tanh(z) = -\mathrm{i} \tan(\mathrm{i}z)$

   Prove that for $y \in \mathbb{R}$
   (e) $\operatorname{arcosh}(y) = \log(y + \sqrt{y^2 - 1}), \quad y \geq 1$
   (f) $\operatorname{arcoth}(y) = \frac{1}{2} \log \frac{y+1}{y-1}, \quad |\,y\,| > 1$

3. Prove that for all $z \in \mathbb{C}$

$$\mathrm{e}^{\bar{z}} = \overline{\mathrm{e}^z}$$

   Hint. Use Proposition 2.31.

4. Compute the following limits

$$\text{(a)} \quad \lim_{x \to 0+0} x \log x$$

$$\text{(b)} \quad \lim_{x \to 0} \frac{\tan x}{x}$$

$$\text{(c)} \quad \lim_{z \to 0} \frac{\mathrm{e}^z - 1}{z}$$

   Hint. For (a) substitute $x = 1/\mathrm{e}^z$; for (c) use Proposition 17 with $n = 2$.