# Calculus – 2. Series, Solutions

1. (a) Let

$$M = \left\{ \frac{1}{2^n} + \frac{1}{m} \,\middle|\, m, n \in \mathbb{N} \right\}.$$

Compute $\max M$, $\min M$, $\sup M$, and $\inf M$ if they exist.

A one-line induction proof shows that $2^n \geq 2$ for all positive integers $n$ ($2^1 \geq 2$ and $2^{n+1} = 2 \cdot 2^n \underset{\text{Ind.hyp.}}{\geq} 2 \cdot 2 \geq 2$). Hence $m \geq 1$ and $2^n \geq 2$ for all $m, n \in \mathbb{N}$. Proposition 9 (e) then implies

$$\frac{1}{2^n} + \frac{1}{m} \leq \frac{1}{2} + 1 = \frac{3}{2}.$$

Hence, $3/2$ is an upper bound for $M$. Since $3/2 \in M$ we have $\max M = 3/2$. By the remark after Definition 4, $\max E = \sup E$ if the maximum exists. Hence $\sup M = 3/2$.

Since $m$ and $2^n$ are positive for all positive integers $m, n$, Proposition 9 (e) gives $1/m$ and $1/2^n$ are also positive. Hence

$$0 < \frac{1}{2^n} + \frac{1}{m},$$

and 0 is a lower bound of $M$. We will show that $0 = \inf M$. For, let $\varepsilon > 0$. Our aim is to show that $\varepsilon$ is not a lower bound, and we are done. First note that $2^n > n$ for $n \geq 3$ (proof by induction on $n$: $2^3 = 8 > 3$ and $2^{n+1} = 2 \cdot 2^n \underset{\text{Ind.hyp.}}{>} 2n > n + 1$). The Archimedean property of the real numbers furnishes positive integers $m$ and $n \geq 3$ with

$$m\varepsilon > 2 \quad \text{and} \quad n\varepsilon > 2.$$

The first inequality implies $1/m < \varepsilon/2$ while the second inequality yields $2^n\varepsilon > n\varepsilon > 2$ and therefore, $1/2^n < \varepsilon/2$. Summing up both inequalities we arrive at

$$\frac{1}{m} + \frac{1}{2^n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\varepsilon$ is not a lower bound for $M$, so that $\inf M = 0$. Since $0 \notin M$, $\min M$ does not exist.

(b) Let $A \subset \mathbb{R}_+$ be a set with $\inf A > 0$. Prove that

$$\sup A^{-1} = \frac{1}{\inf A},$$

where $A^{-1} = \{1/a \mid a \in A\}$.

*Proof.* Put $t = \inf A > 0$. Then $0 < t \leq a$ for all $a \in A$. Proposition 9 (e) gives $0 < 1/a \leq 1/t$; hence $1/t$ is an upper bound for $A^{-1}$. Suppose $s$ is positive with $s < 1/t$. We will show that $s$ is not an upper bound for $A^{-1}$.

First $0 < s < 1/t$ implies $t < 1/s$. Since $t$ is the greatest lower bound of $A$, $1/s$

is not a lower bound for $A$. Hence, there is some $a \in A$ with $a < 1/s$. Since all $a$ are positive, this implies $1/a > s$. Hence $s$ is not an upper bound for $A^{-1}$. This completes the proof. ∎

2. Solve for $x \in \mathbb{R}$

$$|\,2x - 4\,| < |\,x - 1\,|. \tag{1}$$

*Solution. Case 1:* $2x - 4 \geq 0$. This implies $x \geq 2$, in particular, $x \geq 1$. Therefore, $|\,2x - 4\,| = 2x - 4$ and $|\,x - 1\,| = x - 1$. Inequality (1) reads as

$$2x - 4 < x - 1 \iff x < 3.$$

Our partial solution in Case 1 is the interval $[2, 3)$.
*Case 2:* $2x - 4 < 4$ *and* $x - 1 \geq 0$. This implies $1 \leq x < 2$, $|\,2x - 4\,| = -2x + 4$, and $|\,x - 1\,| = x - 1$. Inequality (1) now reads as

$$-2x + 4 < x - 1 \iff \frac{5}{3} < x.$$

Our partial solution in Case 2 is $(\frac{5}{3}, 2)$.
*Case 3:* $x - 1 < 0$. This implies $x < 1$ and $2x - 4 < 0$. Therefore, $|\,2x - 4\,| = -2x + 4$, and $|\,x - 1\,| = -x + 1$. Inequality (1) reads as

$$-2x + 4 < -x + 1 \iff 3 < x.$$

There is no solution in Case 3. The inequality (1) is fulfilled if and only if $x$ belongs to the open interval $(5/3, 2)$.

3. Solve for $x \in \mathbb{R}$

$$\frac{1}{x - 1} + \frac{1}{x + 1} \geq 1. \tag{2}$$

*Solution. Case 1:* $x > 1$. This implies $x - 1 > 0$ and $x + 1 > 0$ so that multiplication of (2) by $(x - 1)(x + 1)$ does not change the relation sign. We can make the following equivalent transformations.

$$x + 1 + x - 1 \geq (x - 1)(x + 1) = x^2 - 1 \iff 0 \geq x^2 - 2x - 1$$
$$0 \geq (x - 1)^2 - 2 \iff \sqrt{2} \geq |\,x - 1\,|$$
$$1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}.$$

Taking care of our assumption $x > 1$ we find $1 < x \leq 1 + \sqrt{2}$.
*Case 2:* $x < -1$. This implies $x - 1 < 0$ and $x + 1 < 0$ so that multiplication of (2) by $(x - 1)(x + 1)$ does not change the sign. As in Case 1 we get

$$1 - \sqrt{2} \leq x \leq 1 + \sqrt{2}.$$

Taking care of $x < -1$, there is no solution in this case.
*Case 3:* $-1 < x < 1$. This implies $x - 1 < 0$ and $x + 1 > 0$ so that multiplication of (2) changes the relation sign. We obtain

$$\sqrt{2} \leq |\, x - 1 \,|$$

which is equivalent to

$$x \geq 1 + \sqrt{2} \quad \text{or} \quad x \leq 1 - \sqrt{2}.$$

Taking care of our assumption $-1 < x < 1$ we find $-1 < x < 1 - \sqrt{2}$.
The inequality (2) is fulfilled, if and only if $x$ belongs to one of the intervals $(-1, 1 - \sqrt{2}]$ or $(1, 1 + \sqrt{2}]$.

4. Define a relation $\prec$ on $\mathbb{R}^2 = \{(x, y) \mid x,\, y \in \mathbb{R}\}$ by

$$(x, y) \prec (x', y') \quad \text{if} \quad (x < x' \quad \text{or} \quad (x = x' \quad \text{and} \quad y < y')).$$

Prove that $(\mathbb{R}^2, \prec)$ is an ordered set. Is $(\mathbb{R}^2, \prec)$ order complete?

*Proof.* First we show that any two elements $p$ and $q$ of $\mathbb{R}^2$ are comparable, i. e. $\mathbb{R}^2$ has the property in Definition 1 (i). Let $p = (x, y)$ and $q = (x', y')$, and suppose first $p \not\prec q$. That is, $x > x'$ or both $x = x'$ and $y \geq y'$. In the first case, $q \prec p$. If $x = x'$ and $y = y'$ then $p = q$, and finally if $x = x'$ and $y > y'$ again $q \prec p$. We have seen that one and only one of the relations $p \prec q$, $p = q$, and $q \prec p$ is true.
We will show transitivity. Suppose further $r = (x'', y'')$, $p \prec q$, and $q \prec r$. Then, $x < x'$ and $x' \leq x''$ implies $x < x''$ so that $p \prec r$ in this case. If $x = x'$ and $y < y'$, then $x' < x''$ implies $x < x''$ which means $p \prec r$. We are left with the case $x = x'$, $y < y'$, $x' = x''$, and $y' < y''$. We conclude $x = x''$ and $y < y''$ which also means $p \prec r$. This completes the proof; $(\mathbb{R}^2, \prec)$ is an ordered set. ■

$(\mathbb{R}^2, \prec)$ is not order complete.

*Proof.* We will construct a subset $E$ of $\mathbb{R}^2$ which is bounded above; however $E$ has not a least upper bound.
(a) The subset $E := \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$ is bounded above by $(0, 0)$ since $x < 0$ for all $(x, y) \in E$ and therefore $(x, y) \prec (0, 0)$.

Suppose to the contrary that $p = (a, b) = \sup E$. Since $p$ is the least upper bound and $(0, 0)$ is an upper bound by (a), $p \preccurlyeq (0, 0)$. Hence $a < 0$ or both $a = 0$ and $b \leq 0$. The first case is impossible since otherwise $(a, b) \prec (a/2, b) \in E$, and $p$ is not an upper bound of $E$. Hence $p = (0, b)$.
Putting $q = (0, b - 1)$, $q$ is also an upper bound of $E$ (since again $x < 0$ for all $(x, y) \in E$). Moreover, $q = (0, b - 1) \prec (0, b) = p$ since $0 = 0$ and $b - 1 < b$. This shows that $p$ is not the least upper bound; a contradiction! Hence, $(\mathbb{R}^2, \prec)$ is not order complete. ■

5. Let $n$ be a positive integer and $x_1, \ldots, x_n$ real numbers. Prove that

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |\, x_k \,|, \tag{3}$$

$$\left| \prod_{k=1}^{n} x_k \right| = \prod_{k=1}^{n} |\, x_k \,|. \tag{4}$$

*Proof.* We prove the statements using induction on $n$. The case $n = 1$ is obvious for both since we have $|\, x_1 \,| = |\, x_1 \,|$. (a) Suppose (3) is true for some fixed positive integer $n$ and all $x_1, \ldots, x_n \in \mathbb{R}$. We will prove that the statement is true for arbitrary $n+1$ real numbers $x_1, \ldots, x_n, x_{n+1}$. Using the triangle inequality (Lemma 14 (d)) and the induction hypothesis, we compute

$$\left| \sum_{k=1}^{n+1} x_k \right| = \left| \sum_{k=1}^{n} x_k + x_{n+1} \right| \underset{\text{L. 14}}{\leq} \left| \sum_{k=1}^{n} x_k \right| + |\, x_{n+1} \,| \underset{\text{i. h.}}{\leq} \sum_{k=1}^{n} |\, x_k \,| + |\, x_{n+1} \,| = \sum_{k=1}^{n+1} |\, x_k \,|\,.$$

This proves the induction assertion.

(b) Suppose (4) is true for some fixed positive integer $n$ and all $x_1, \ldots, x_n \in \mathbb{R}$. We will prove that the statement is true for arbitrary $n+1$ real numbers $x_1, \ldots, x_n, x_{n+1}$. Using Lemma 14 (c) and the induction hypotesis, we compute

$$\left| \prod_{k=1}^{n+1} x_k \right| = \left| \prod_{k=1}^{n} x_k \cdot x_{n+1} \right| \underset{\text{L. 14}}{=} \left| \prod_{k=1}^{n} x_k \right| \cdot |\, x_{n+1} \,| \underset{\text{ind. hyp.}}{=} \prod_{k=1}^{n} |\, x_k \,| \cdot |\, x_{n+1} \,| = \prod_{k=1}^{n+1} |\, x_k \,|\,.$$

This proves the induction assertion. ∎