# Calculus – 8. Series, Solutions

1. Depending on $a > 0$ decide whether the series $\sum a_n$ converges or diverges, where

$$\text{(a) } a_n = \frac{a^n}{1+a^n}, \quad \text{(b) } a_n = \frac{n^a}{n!}.$$

*Solution.* (a) Suppose first $a > 1$. Since $a_n = \dfrac{1}{1+1/a^n}$ and $1/a^n \underset{n\to\infty}{\longrightarrow} 0$ we find $a_n \underset{n\to\infty}{\longrightarrow} 1$; and the necessary condition for convergence of the series $\sum a_n$ (Corollary 19) is not satisfied. Hence $\sum a_n$ diverges. If $0 < a < 1$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{a^n}{1+a^n}} = \frac{a}{\sqrt[n]{1+a^n}} < a < 1.$$

Hence $\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} \le a < 1$ and the root test applies; the series converges.

(b) Consider

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^a n!}{(n+1)! n^a} = \left(1 + \frac{1}{n}\right)^a \frac{1}{n+1}.$$

We show that for for arbitrary *real* exponents $r$ and $s$, $r < s$ implies $a^r < a^s$ if $a > 1$ (cf. Homework 4.1 (b) for rational exponents). For $t \in \mathbb{R}$ set $M_t := \{a^q \mid q \in \mathbb{Q},\ q < t\}$. By definition $a^r = \sup M_r$ and $a^s = \sup M_s$. Since $r < s$, $M_r \subseteq M_s$ and Homework 1.4 (b) gives $a^r < a^s$. In particular $n \ge 1$ implies

$$\frac{a_{n+1}}{a_n} \le 2^a \frac{1}{n+1}$$

Since $2^a$ is a constant,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

The ratio test (Corollary 26) indicates convergence.

2. Let $\ell_2$ be the set of all sequences $(a_n)$ such that the series $\sum a_n^2$ converges. Suppose $(a_n)$ and $(b_n)$ are in $\ell_2$ with $\sum a_n^2 = A$ and $\sum b_n^2 = B$. Prove that

   (a) $\sum a_n b_n$ converges and $\sum a_n b_n \le \sqrt{AB}$;
   (b) $(a_n + b_n)$ is in $\ell_2$.

*Hint.* The first part of (a) is easy. For the second part of (a) consider the partial sums and use the Cauchy–Schwarz inequality. For (b) use (a).

*Proof.* (a) *First proof.* The arithmetic-geometric mean inequality gives

$$|\, a_k b_k \,| = \sqrt{a_k^2 b_k^2} \le (a_k^2 + b_k^2)/2.$$

The comparison test (Proposition 20 (a)) now indicates that $\sum a_k b_k$ converges. Moreover,

$$\sum_{k=1}^n |\, a_k b_k \,| \le \frac{1}{2}\left(\sum_{k=1}^n a_k^2 + \sum_{k=1}^n b_k^2\right) \le \frac{1}{2}\left(\sum_{k=1}^\infty a_k^2 + \sum_{k=1}^\infty b_k^2\right) = \frac{1}{2}(A+B).$$

But this estimate is not sufficient to prove the second part of (a). However, by the Cauchy–Schwarz inequality we have

$$\begin{aligned}\sum_{k=1}^{n} |\, a_k b_k \,| &\le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\\ &\le \left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2} = \sqrt{AB}.\end{aligned}$$

Hence $\sqrt{AB}$ is an upper bound for the partial sum of $\sum a_k b_k$. Noting that $\overline{\lim}\, s_n = \lim s_n$ since $s_n$ converges, Proposition 12 (a) yields

$$\sum_{k=1}^{\infty} |\, a_k b_k \,| \le \sqrt{AB}.$$

This proves (a).

Remark: We have shown that the Cauchy–Schwarz inequality holds for series as well. The proofs of Hölder's inequality and Minkowski's inequality for series are along the same lines.

(b) Since

$$(a_n + b_n)^2 = a_n^2 + 2a_n b_n + b_n^2$$

The convergence of $\sum a_n^2$, $\sum b_n^2$, and $\sum a_n b_n$ yields the convergence of $\sum (a_n + b_n)^2$.

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3. Compute the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n\, z^n$ where

(a) $a_n = n^s, \quad s \in \mathbb{Q}.$

(b) $a_n = q^{n^2} \quad q \in \mathbb{R}.$

(c) $a_n = \begin{cases} a^n, & \text{if } n \text{ is odd,} \\ b^n, & \text{if } n \text{ is even,} \end{cases} \qquad a, b \in \mathbb{R}.$

*Solution.* (a)

$$\alpha := \overline{\lim_{n\to\infty}}\, \sqrt[n]{n^s} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^s. \tag{1}$$

Since we have not yet established limit laws for powers we will add this here.

> If $(x_n)$ is a sequence of positive real numbers and $\lim x_n = x > 0$ then $\lim x_n^s = x^s$ for all rational numbers $s \in \mathbb{Q}$.

The proof for real numbers $s$ is a little bit more elaborate. We will do this later after the redefinition of the power function.

For integers $s \in \mathbb{Z}$, this follows from the product and quotient rules (Proposition 3(b) and (c)). Suppose $s = 1/k$ with some positive integer $k \in \mathbb{N}$. Then Lemma 1.16 (b) shows with $k$, $\sqrt[k]{x_n}$, and $\sqrt[k]{x}$ in place of $n$, $x$, and $y$

$$k x_n^{\frac{k-1}{k}} \left(\sqrt[k]{x} - \sqrt[k]{x_n}\right) \leq x - x_n \leq k x^{\frac{k-1}{k}} \left(\sqrt[k]{x} - \sqrt[k]{x_n}\right)$$

Since $(x_n)$ is bounded the sandwich theorem gives

$$\lim_{n\to\infty} (\sqrt[k]{x} - \sqrt[k]{x_n}) = 0,$$

which proves the claim for $s = 1/k$. For $s = -1/k$, $k \in \mathbb{N}$, use Proposition 3 (d). Finally, for arbitrary $s = p/q$ use Proposition 3 (c). Consequently, $\lim_{n\to\infty} x_n^s = x^s$ is shown for all rational $s \in \mathbb{Q}$.
Now we can proceed in (1) using $\sqrt[n]{n} \underset{n\to\infty}{\longrightarrow} 1$,

$$\alpha = \left(\lim_{n\to\infty} \sqrt[n]{n}\right)^s = 1.$$

Hence, $R = 1/\alpha = 1$ is the radius of convergence.

(b)

$$\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|\, q^{n^2} \,|} = \lim_{n\to\infty} |\, q \,|^{\frac{n^2}{n}} = \lim_{n\to\infty} |\, q \,|^n .$$

*Case 1.* $|\, q \,| < 1$. This gives $\alpha = 0$ and therefore $R = +\infty$.
*Case 2.* $|\, q \,| > 1$. This gives $\alpha = +\infty$ and therefore $R = 0$.
*Case 3.* $|\, q \,| = 1$. This gives $\alpha = 1$ and $R = 1$.

(c) Let $M = \max\{|\, a \,|, |\, b \,|\}$. Then

$$\alpha = \overline{\lim_{n\to\infty}} \sqrt[n]{|\, a_n \,|} = M,$$

since $\sqrt[2n]{|\, b^{2n} \,|} \underset{n\to\infty}{\longrightarrow} |\, b \,|$ and $\sqrt[2n+1]{|\, a^{2n+1} \,|} \underset{n\to\infty}{\longrightarrow} |\, a \,|$. Hence

$$R = \frac{1}{M} = \min\{1/|\, a \,|, 1/|\, b \,|\}.$$

4. Compute the sum of the series.

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{(n-1)n(n+1)} + \cdots$$

*Hint.* Find numbers $A$, $B$, and $C$ such that

$$\frac{1}{(n-1)n(n+1)} = \frac{A}{n-1} + \frac{B}{n} + \frac{C}{n+1} \quad \text{for all integers } n,\ n \neq -1, 0, 1. \qquad (2)$$

Compute the partial sums explicitely.

*Solution. First method.* Multiplying (2) by $(n-1)n(n+1)$ gives

$$1 = A(n^2+n) + B(n^2-1) + C(n^2-n) = n^2(A+B+C) + n(A-C) - B.$$

Comparing the coefficients of the polynomials on the lhs (which is constant 1) and on the rhs (which is a quadratic polynomial) we find a system of linear equations in $A$, $B$, and $C$

$$\begin{aligned} n^2: &\qquad 0 = A+B+C,\\ n^1: &\qquad 0 = A-C,\\ n^0: &\qquad 1 = -B. \end{aligned}$$

The solution is $A = C = \frac{1}{2}$, $B = -1$.

*Second method.* Multiplying (2) by $n-1$ we find

$$\frac{1}{n(n+1)} = A + B\frac{n-1}{n} + C\frac{n-1}{n+1}.$$

Taking the limit $n \to 1$ gives

$$\frac{1}{1\cdot 2} = A.$$

Similarly, multiplication of (2) by $n$ and inserting $n \to 0$ gives $B = -1$. Finally, multiplication of (2) by $n+1$ and inserting $n \to -1$ gives $C = 1/2$.

Using this and index shifts we obtain

$$\begin{aligned} s_n &= \sum_{k=2}^{n} \frac{1}{(k-1)k(k+1)} = \frac{1}{2}\sum_{k=2}^{n}\frac{1}{k-1} - \sum_{k=2}^{n}\frac{1}{k} + \frac{1}{2}\sum_{k=2}^{n}\frac{1}{k+1}\\ &= \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k} - \sum_{k=2}^{n}\frac{1}{k} + \frac{1}{2}\sum_{k=3}^{n+1}\frac{1}{k}\\ &= \sum_{k=3}^{n-1}\frac{1}{k}\left(\frac{1}{2} - 1 + \frac{1}{2}\right) + \frac{1}{2}\left(1+\frac{1}{2}\right) - \left(\frac{1}{2}+\frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n+1}\right)\\ &= \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}. \end{aligned}$$

Taking the limit $n \to \infty$, the sum of the series is $1/4$.

5. (a) Prove directly (without using the Cauchy criterion) that $\sum 1/n = +\infty$.
(b) Prove that $\sum a_n$ diverges if $\lim\limits_{n\to\infty} n\, a_n = 1$.

*Proof.* (a) We give an estimate for the $2^n$th partial sum.

$$\begin{aligned} s_{2^n} = \sum_{k=1}^{2^n}\frac{1}{k} &> 1 + \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n}+\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right)\\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \qquad \left(\; n \text{ times } \frac{1}{2}\right)\\ &> \frac{n}{2} \end{aligned}$$

Given $E > 0$ choose $n_0 > 2E$ then $n \geq 2^{n_0}$ implies

$$s_n \geq s_{2^{n_0}} > \frac{n_0}{2} > E.$$

This shows $\sum 1/k = +\infty$.

(b) If $\lim na_n = 1$, for all but finitely many $n$ we have $na_n > \frac{1}{2}$. This implies $a_n > \dfrac{1}{2n}$. The comparison test (Proposition 19 (b) with $d_n = 1/n$ and $C = \frac{1}{2}$) shows that $\sum a_n$ diverges. ■