Asymptotic grid minors in infinite graphs

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Joint work with Sandra Albrechtsen

Halin's grid theorem

Theorem (Halin 1965)

Let G be a graph that contains infinitely many disjoint equivalent one-way infinite paths. Then the half-grid is a minor of G.

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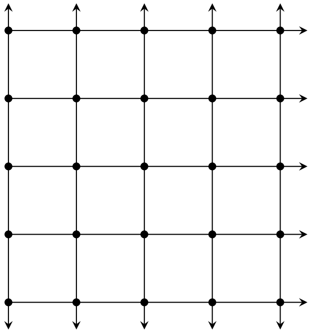
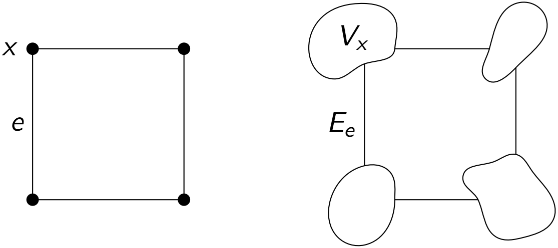


Minors



Quasi-isometries

Let G, H be graphs. A map $\varphi\colon V(G) \to V(H)$ is a quasi-isometry (and we call G and H quasi-isometric) if there exist $\gamma \geq 1$, $c \geq 0$ such that

1. $\frac{1}{\gamma} d_G(u,v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u,v) + c$ for all $u, v \in V(G)$ and
2. for all $x \in V(H)$ there exists $v \in V(G)$ with $d_H(x, \varphi(v)) \leq c$.

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Quasi-isometries play an important role in geometric group theory since Cayley graphs of the same finitely generated group but for distinct finite generating sets are quasi-isometric.

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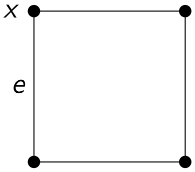
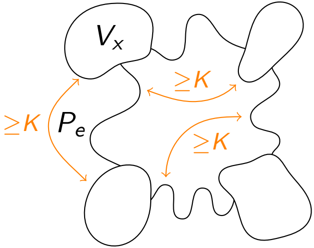
$\Rightarrow$ We look for minor-notions that appear in the coarse structure of the graphs.

For $K \in \mathbb{N}$, a graph H is a K-fat minor if:





Asymptotic minors

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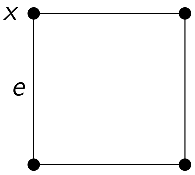

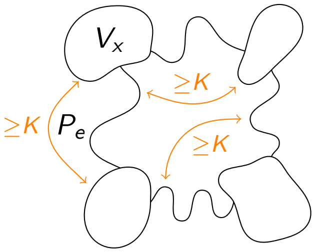

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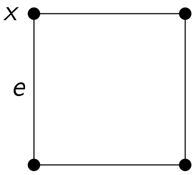

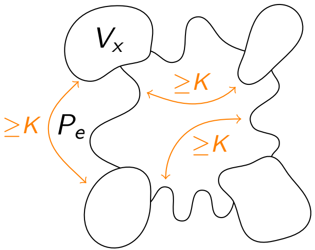

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Observation (Georgakopoulos & Papasoglu)

Asymptotic minors are preserved by quasi-isometries.

Ends

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An end is thick if it contains infinitely many pairwise disjoint rays.

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Problem (Georgakopoulos & Papasoglu 2025)

Every one-ended graph that contains the disjoint union of countably many rays as an asymptotic minor also contains the half-grid as an asymptotic minor.

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Remark

There exists a counterexample to the problem that is not locally finite.

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Theorem (Georgakopoulos & H. 2024)

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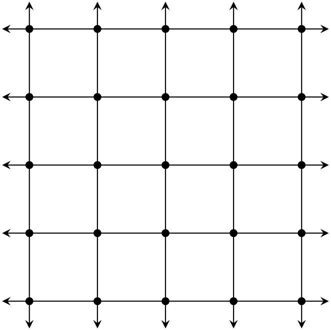

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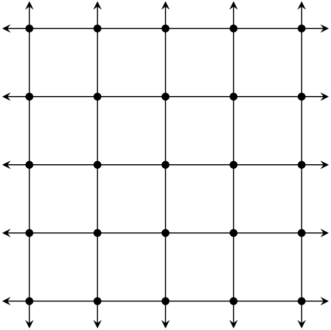

Problem (Georgakopoulos & Papasoglu 2025)

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A condition on the cycle space

The cycle space of a graph G is generated by cycles of bounded length if there is some $n \in \mathbb{N}$ such that for each cycle C there exist finitely many cycles $C_1, \ldots, C_k$ of length at most n such that the edges of C are exactly those that lie in an odd number of C_i.

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Remark

Let G be a locally finite Cayley graph of a finitely presented group. Then its cycle space is generated by cycles of bounded length.

Full-grids as asymptotic minors

Theorem

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. Then G has a thick end if and only if G contains the full-grid as an asymptotic minor.

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Corollary

Let Γ be a finitely presented group. Then Γ is virtually free if and only if none of its locally finite Cayley graphs contains the full-grid as an asymptotic minor.

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Observation (Georgakopoulos)

Every locally finite Cayley graph of the lamplighter group contains the countably infinite clique as an asymptotic minor.

Diverging minors

A graph G contains a graph H as a diverging minor if G contains a model $(\mathcal{V}, \mathcal{E})$ of H with the following property:
for every two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of vertices and/or edges of H such that $d_H(x_n, y_n) \to \infty$, we have $d_G(X_n, Y_n) \to \infty$ where $X_n := V_{x_n}$ if $x_n \in V(H)$ and $X_n := V(P_{x_n})$ if $x_n \in E(H)$ and analogously $Y_n := V_{y_n}$ or $Y_n := V(P_{y_n})$.

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Remark

All presented theorems stay true if we exchange *asymptotic minor* by *diverging minor* in the conclusions.