Grid minors in transitive graphs

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Based on joint works with Agelos Georgakopoulos
and with Sandra Albrechtsen

Outline

1. motivation
2. full-grid minors
3. coarse geometry
4. main result
5. sketch of the proof
6. final remarks and recent development

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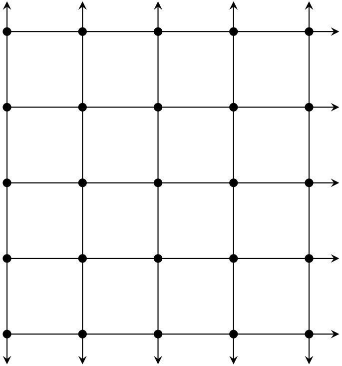

A symmetric version?

Question

Can we obtain for symmetric graphs a symmetric grid as a minor?

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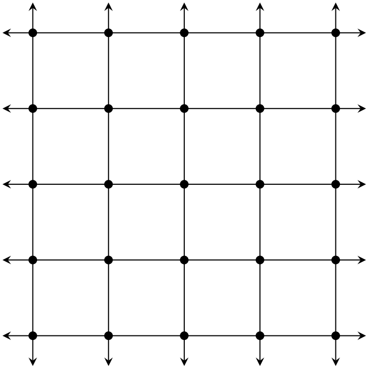

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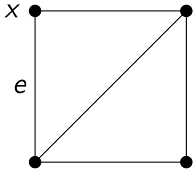
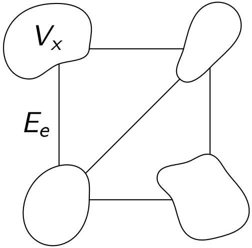
Minors





Two one-way infinite paths (=rays) in a graph are equivalent if there are infinitely many pairwise disjoint paths between them.
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An end is thick if it contains infinitely many pairwise disjoint rays.

A characterisation for full-grid minors

A graph is quasi-transitive if there are only finitely many orbits under its automorphism group.

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Theorem (Georgakopoulos & H. 2024)

Let G be a locally finite, quasi-transitive graphs. Then G has a thick end if and only if it contains the full-grid as a minor.

General strategy of the proof:

1. for one-ended planar graphs: direct construction

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1. for one-ended planar graphs: direct construction
2. for general planar graphs: use canonical tree-decompositions to identify a one-ended, quasi-transitive subgraph
3. for general graphs: apply a result by Esperet, Giocanti & Legrand-Duchesne that characterises quasi-transitive, locally finite graphs without some countable minor and finds a planar, quasi-transitive minor

Which ends contain full-grid minors?

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For accessible, quasi-transitive, locally finite graphs, every end inhabits the end of some full-grid minor.

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For inaccessible, quasi-transitive, locally finite graphs, we cannot prescribe any end (yet) that inhabits the end of some full-grid minor.

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Let G, H be graphs. A map $\varphi\colon V(G) \to V(H)$ is a quasi-isometry (and we call G and H quasi-isometric) if there exist $\gamma \geq 1$, $c \geq 0$ such that

1. $\frac{1}{\gamma} d_G(u,v) - c \leq d_H(\varphi(u), \varphi(v)) \leq \gamma d_G(u,v) + c$ for all $u, v \in V(G)$ and
2. for all $x \in V(H)$ there exists $v \in V(G)$ with $d_H(x, \varphi(v)) \leq c$.

A characterisation for full-grid minors

Theorem (Krön & Möller 2008)

Let G be a locally finite, quasi- transitive graph. Then G contains a thick end if and only if it is not quasi-isometric to a tree.

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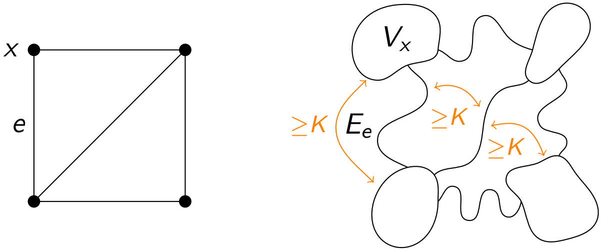
Theorem (Georgakopoulos & H. 2024)

Let G be a locally finite, quasi-transitive graphs. Then G is not quasi-isometric to a tree if and only if it contains the full-grid as a minor.

For $K \in \mathbb{N}$, a graph H is a K-fat minor if:



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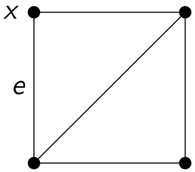

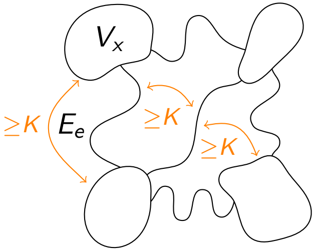

A graph G contains a graph H as an asymptotic minor if G contains H as a K-fat minor for every $K \in \mathbb{N}$.

A graph G contains a graph H as a diverging minor if G contains a model $(\mathcal{V}, \mathcal{E})$ of H with the following property:
for every two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of vertices and/or edges of H such that $d_H(x_n, y_n) \to \infty$, we have $d_G(X_n, Y_n) \to \infty$ where $X_n := V_{x_n}$ if $x_n \in V(H)$ and $X_n := V(E_{x_n})$ if $x_n \in E(H)$ and analogously $Y_n := V_{y_n}$ or $Y_n := V(E_{y_n})$.

Coarse minor questions

Question (Georgakopoulos & Papasoglu 2023^+)

Does every locally finite, quasi-transitive graph that is not quasi-isometric to a tree contain the full-grid as an asymptotic minor?

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The edge space $\mathcal{E}(G)$ of a graph G is the vector space over $\mathbb{F}_2$ of all functions $E(G) \to \mathbb{F}_2$: its elements correspond to the subsets of $E(G)$ and vector addition corresponds to symmetric difference.

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The cycle space $\mathcal{C}(G)$ of a graph G is the subspace of $\mathcal{E}(G)$ generated by all cycles in G (i.e. by their edge sets).

Cycle space

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The cycle space $\mathcal{C}(G)$ of a graph G is the subspace of $\mathcal{E}(G)$ generated by all cycles in G (i.e. by their edge sets).

The cycle space $\mathcal{C}(G)$ of a graph G is generated by cycles of bounded length if there is some $n \in \mathbb{N}$ such that the cycles in G of length at most n generate $\mathcal{C}(G)$.

Main theorem

Theorem (Albrechtsen & H. 2024^+)

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G is not quasi-isometric to a tree, then G contains the full-grid as an asymptotic minor and as a diverging minor.

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Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G is not quasi-isometric to a tree, then G contains the full-grid as an asymptotic minor and as a diverging minor.

Corollary

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. Tfae:

- *G is not quasi-isometric to a tree.*
- *G contains the half-grid as a minor.*
- *G contains the full-grid as a minor.*
- *G contains the full-grid as an asymptotic minor.*
- *G contains the full-grid as a diverging minor.*

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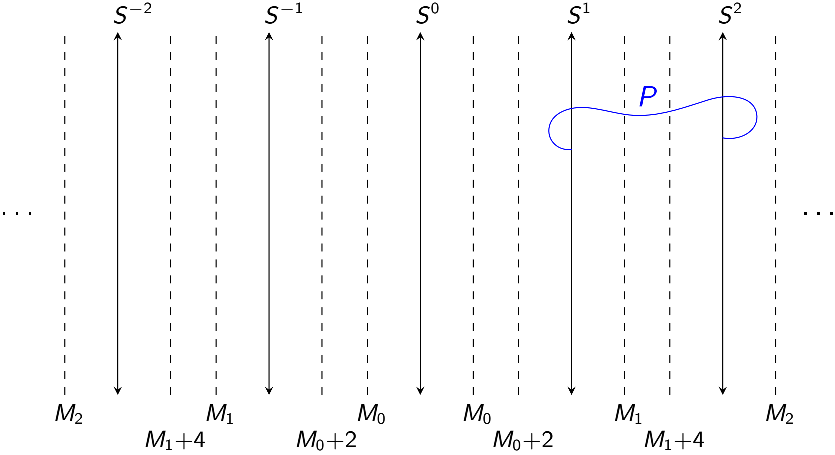
Corollary

Let Γ be a finitely presented group. Then Γ is not virtually free if and only if none of its locally finite Cayley graphs contain the full-grid as an asymptotic minor.

Escaping subdivisions of the full-grid



- $S^i \subseteq G[S^0, M_i] - B_G(S^0, M_{i-1} + 2i)$ for all $i \geq 1$ and
- $P \subseteq G[B_G(S^0, M_i)] - B_G(S^0, M_{i-2} + i)$

A model $((V_i)_{i \in \mathbb{N}}, (E_{ij})_{i \neq j \in \mathbb{N}})$ of $K_{\aleph_0}$ in a graph G is ultra fat if

- $d_G(V_i, V_j) \geq \min\{i, j\}$ for all $i \neq j \in \mathbb{N}$,
- $d_G(V_i, E_{kl}) \geq \min\{i, k, l\}$ for all $i, k, l \in \mathbb{N}$ with $i \notin \{k, l\}$, and
- $d_G(E_{ij}, E_{kl}) \geq \min\{i, j, k, l\}$ for all $i, j, k, l \in \mathbb{N}$ with $\{i, j\} \neq \{k, l\}$.

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Thus: a model $((V_i)_{i\in\mathbb{N}}, (E_{ij})_{i\neq j\in\mathbb{N}})$ of $K_{\aleph_0}$ is ultra fat if and only if, for every $n \in \mathbb{N}$,

$$((V_i)_{i\geq n}, (E_{ij})_{i\neq j\geq n})$$

is n-fat.

Theorem (Albrechtsen & H. 2024^+)

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length. If G is not quasi-isometric to a tree, then G contains either an ultra fat $K_{\aleph_0}$-minor or the full-grid as an escaping subdivision.

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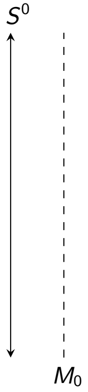

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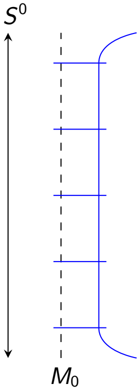

Lemma

Let G be a graph whose cycle space is generated by cycles of length at most $\kappa \in \mathbb{N}$, and let Y be a connected subgraph of G. Then for every component C of $G - Y$ that attaches to Y, the graph $C \cap G[B_G(N_G(Y), \lfloor \frac{\kappa-2}{2} \rfloor)]$ is connected.

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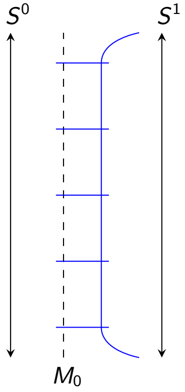

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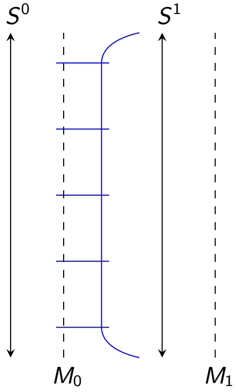

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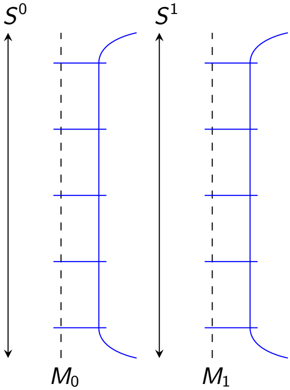

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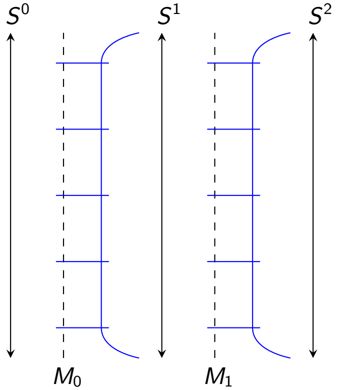

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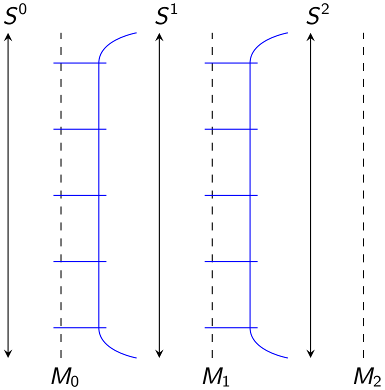

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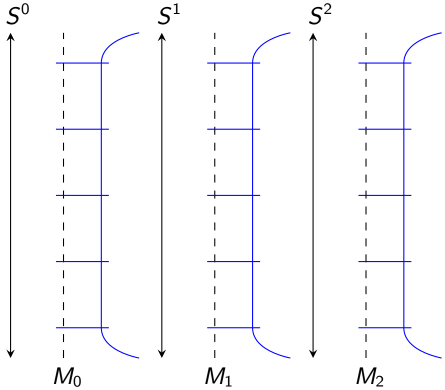

LEMMA

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Dropping the symmetry condition

Theorem (Albrechtsen & H. 2024^+)

Let G be a graph of finite maximum degree whose cycle space is generated by cycles of bounded length. If G has a thick end, then G contains the half-grid as an asymptotic minor and as a diverging minor.

Proposition

Let G be a locally finite, quasi-transitive graph whose cycle space is generated by cycles of bounded length and that is not quasi-isometric to a tree. Then there exist equivalent rays R_1, R_2, R_3 in G such that $R_1 \cap R_2 = R_1 \cap R_3 = R_2 \cap R_3 = \{v\}$ for some $v \in V(G)$ and such that $R_1 \cup R_2 \cup R_3$ is quasi-geodesic in G.

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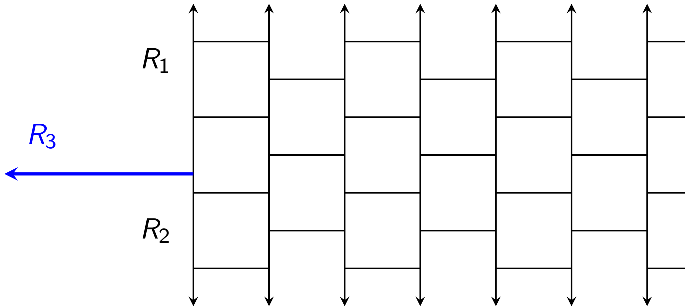

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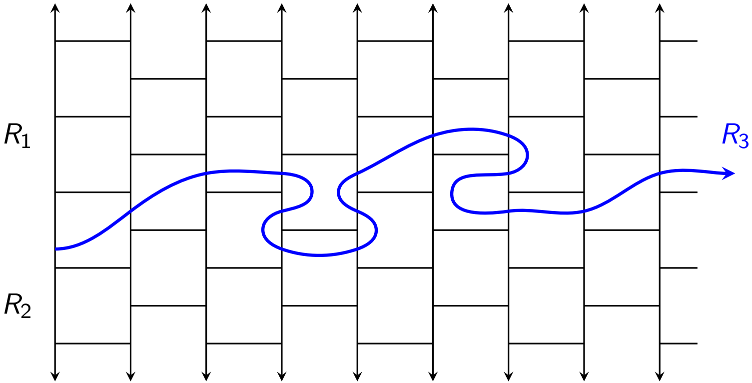

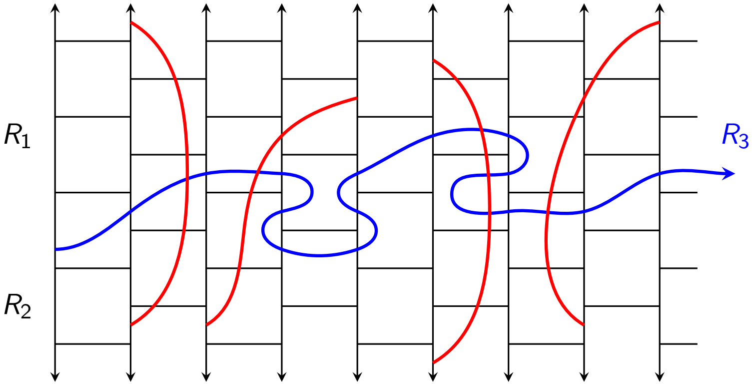
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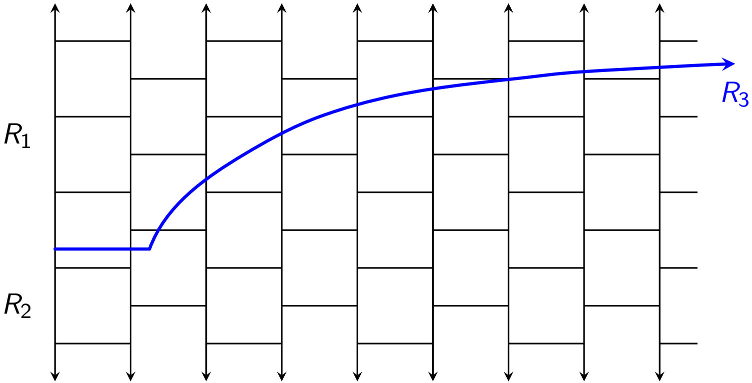
R_2
R_3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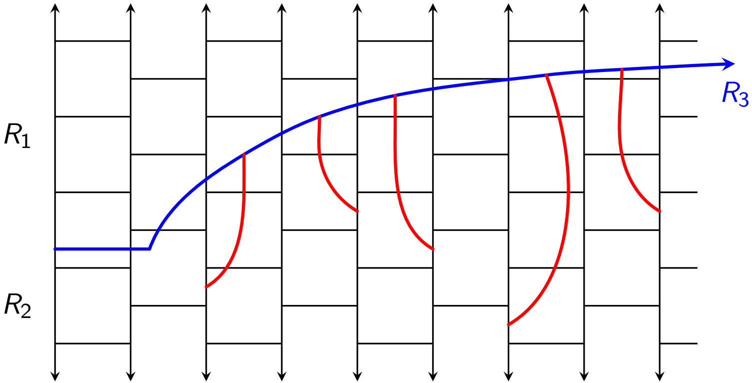

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Does every locally finite, quasi-transitive graph that is not quasi-isometric to a tree contain the full-grid as an asymptotic / diverging minor?

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Does every locally finite, quasi-transitive graph that is not quasi-isometric to a tree contain either an ultra fat $K_{\aleph_0}$-minor or an escaping subdivision of the full grid?

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Observation (Georgakopoulos)

Every locally finite Cayley graph of the lamplighter group has an ultra fat $K_{\aleph_0}$-minor.

Theorem (Albrechtsen & H.)

Every one-ended, quasi-transitive, locally finite graph contains the half-grid as an asymptotic minor.

Recent development

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Every one-ended, quasi-transitive, locally finite graph contains the half-grid as an asymptotic minor.

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Every one-ended, locally finite graph that contains the disjoint union of countably many rays as asymptotic minor also contains the half-grid as an asymptotic minor.

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These theorems solve problems by Georgakopoulos and Papasoglu.