Numerical analysis and algorithms in control and state constrained optimization with pdes

Klaus Deckelnick[1], **Andreas Günther**[2], and **Michael Hinze**[2*]

[1] Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

[2] Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.

We consider an elliptic optimal control problem with control and pointwise state constraints. The cost functional is approximated by a sequence of functionals which are obtained by discretizing the state equation with the help of linear finite elements and enforcing the state constraints in the nodes of the triangulation. The control variable is not discretized. A general error bound for control and state is obtained which forms the starting point for optimal error estimates in both in two and three space dimensions. For the numerical implementation of the discrete concept fix-point iterations or generalized Newton methods are proposed.

1 Optimization problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider an uniformly elliptic, coercive differential operator $Ay := -\sum_{i,j=1}^d \partial_{x_j}\left(a_{ij}y_{x_i}\right) + \sum_{i=1}^d b_i y_{x_i} + cy$, with associated bilinear form a defined on $H^1(\Omega)$. We are interested in finite element analysis of the following control problem

$$\min_{u\in U_{ad}} J(u) = \frac{1}{2}\int_\Omega |y-y_0|^2 + \frac{\alpha}{2}\|u-u_0\|_U^2 \text{ subject to } y = \mathcal{G}(Bu) \text{ and } y(x) \le b(x) \text{ in } \Omega. \tag{1}$$

Here, $\mathcal{G}$ denotes the solution operator associated to A, and $U_{ad} \subseteq U$ denotes the set of admissible controls which is assumed to be a closed and convex subset of the Hilbert space U. Furthermore, we suppose that $\alpha > 0$ and that $y_0 \in H^1(\Omega)$, $u_0 \in U$, $b \in W^{2,\infty}(\Omega)$ are given, and that $B : U \to (H^1(\Omega))^*$ is linear and bounded. Clearly, (1) admits a unique solution $u \in U_{ad}$. We suppose a so called *Slater condition*. i.e. there exists some $\tilde{u} \in U_{ad}$ such that $\mathcal{G}(B\tilde{u}) < b$ in $\bar{\Omega}$.

Finite element analysis for elliptic control problems in the presence of control and state constraints is presented by Casas in [1] who proves convergence of finite element approximations for finitely many state constraints. Casas and Mateos extend these results in [2] to a less regular setting for the states and prove convergence of finite element approximations to semi-linear distributed and boundary control problems. In [9] Meyer considers a fully discrete strategy to approximate an elliptic control problem with pointwise state and control constraints. His results are similar to those presented by the authors in [3,4]. Constraints on the gradient of the state are considered by the authors in [5]. The discretization concept introduced in the next section is developed in [6], where also tailored fixed point iterations and generalized Newton methods are proposed for the numerical solution of the corresponding discrete problem (2), compare also [7].

2 Finite element discretization

Let $\mathcal{T}_h$ be a quasi–uniform triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \text{diam}(T)$ and vertices $x_1, \ldots, x_m$. Furthermore, let $X_h \subset H^1(\Omega)$ denote some associated finite element space consisting of continuous, piecewise polynomial functions. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved. Problem (1) is now approximated by the following sequence of control problems depending on the mesh parameter h:

$$\min_{u\in U_{ad}} J_h(u) := \frac{1}{2}\int_\Omega |y_h-y_0|^2 + \frac{\alpha}{2}\|u-u_{0,h}\|_U^2 \text{ subject to } y_h = \mathcal{G}_h(Bu) \text{ and } y_h(x_j) \le b(x_j) \text{ for } j = 1, \ldots, m. \tag{2}$$

Here, $\mathcal{G}_h$ denotes the finite element solution operator associated to $\mathcal{G}$, and $u_{0,h}$ denotes an approximation to u_0 which is assumed to satisfy $\|u_0 - u_{0,h}\| \le Ch$. Clearly, problem (2) admits a unique solution $u_h \in U_{ad}$. We note that the set U_{ad} is not discretized so that u_h in general does not represent a finite element function. This is different to the common approaches considered in the literature to approximate (1). There holds

* Corresponding author, email: michael.hinze@uni-hamburg.de

Theorem 2.1 *Let u, u_h denote the unique solutions to* (1), (2) *with associated states y, y_h. Then*

$$\alpha\|u-u_h\|_U^2+\|y-y_h\|^2 \le \langle B(u_h-u),(p-p^h)\rangle_{(H^1)^*,H^1}+\int_\Omega (y-y_h)(y-y^h)-$$
$$-\alpha\,(u_0-u_{0,h},u_h-u)_U+\int_{\bar\Omega}(y-y^h)d\mu+\int_{\bar\Omega}(I_hb-b)d\mu+\int_{\bar\Omega}(y^h-y)d\mu_h+\int_{\bar\Omega}(b-I_hb)d\mu_h. \quad (3)$$

Here, $y_h=\mathcal{G}_h(Bu_h), y^h=\mathcal{G}_h(Bu)$, I_h denotes the Lagrange interpolation operator, and $p^h\in X_h$ denotes the unique solution of $a(w_h,p^h)=\int_\Omega(y-y_0)w_h+\int_{\bar\Omega}w_h d\mu$ for all $w_h\in X_h$. Furthermore, p, p_h denote adjoint and discrete adjoint states associated to the state and discrete state equation, respectively, and μ, μ_h denote multipliers associated to the constraints on the state and its finite element discretization, respectively. A proof of this theorem can be deduced from the proof of [4, Theorem 3.2]. Estimate (3) is optimal, since it allows to estimate the error $\|u-u_h\|$ in the controls using the weakest possible norms of $p-p^h$ and $y-y^h$, respectively. Using standard finite element analysis together with the fact that μ_h is uniformly bounded w.r.t. h in the space of regular Borel measures ([4, Lemma 2.4]) it is possible to deduce from (3) for continuous, linear finite elements

$$\|u-u_h\|_U,\ \|y-y_h\|_{H^1}\le\begin{cases} Ch^{1-\frac{d}{4}},\\ Ch^{\frac{3}{2}-\frac{d}{2s}}\sqrt{|\log h|},\ \text{if } Bu\in W^{1,s}(\Omega)\text{ for } s\in\left[1,\frac{d}{d-1}\right).\end{cases} \quad (4)$$

2.1 Numerical example

The following test problem is taken - in a slightly modified form - from [8, Example 6.2]. Let $\Omega:=B_1(0)\subset\mathbb{R}^2$, $\alpha>0$, $U_{ad}\equiv U:=L^2(\Omega)$, $B:L^2(\Omega)\to H^1(\Omega)^*$ denote the injection,

$$y_0(x):=4+\frac{1}{\pi}-\frac{1}{4\pi}|x|^2+\frac{1}{2\pi}\log|x|,\quad u_0(x):=4+\frac{1}{4\alpha\pi}|x|^2-\frac{1}{2\alpha\pi}\log|x|$$

and $b(x):=|x|^2+4$. We consider problem (2) for this setting, where we use linear, continuous finite elements for the approximation of y_h. By checking the optimality conditions of first order one verifies that $u\equiv 4$ is the unique solution of (1) with corresponding state $y\equiv 4$, see [3]. The experimental order of convergence for a sequence of conform, uniform refinements of the unit disc up to $RL=5$ refinement levels is reported in Table 1 for the error functionals $E(h):=\|u-u_h\|,\|y-y_h\|$. It confirms the analytical findings of (4) for the controls u, u_h. The order of convergence for y, y_h is better than expected which may be explained by the fact that $y\in X_h$.

The experimental order of convergence for an error functional $E(h)>0$ is defined by

$$\text{EOC} := \frac{\log E(h_1)-\log E(h_2)}{\log h_1-\log h_2}.$$

Its value is equal to κ if $E(h)\le Ch^\kappa$, with some constant $C>0$.

RL	$\|u-u_h\|$	$\|y-y_h\|$
1	0.788985	0.536461
2	0.759556	1.147861
3	0.919917	1.389378
4	0.966078	1.518381
5	0.986686	1.598421

Table 1 Experimental order of convergence

References

[1] Casas, E.: *Error Estimates for the Numerical Approximation of Semilinear Elliptic Control Problems with Finitely Many State Constraints*, ESAIM, Control Optim. Calc. Var. **8**, 345–374 (2002).
[2] Casas, E., Mateos, M.: *Uniform convergence of the FEM. Applications to state constrained control problems*. Comp. Appl. Math. **21** (2002).
[3] Deckelnick, K., Hinze, M.: *Convergence of a finite element approximation to a state constrained elliptic control problem*, MATH-NM-01-2006, Institut für Numerische Mathematik, TU Dresden (2006).
[4] Deckelnick, K., Hinze, M.: *A finite element approximation to elliptic control problems in the presence of control and state constraints*, Hamburger Beiträge zur Angewandten Mathematik, Preprint HBAM2007-01 (2007).
[5] Deckelnick, K., Günther, A., Hinze, M.: *Finite element approximation of elliptic control problems with constraints on the gradient*, Priority Program 1253, German Research Foundation, Preprint-Number SPP1253-08-02 (2007).
[6] Hinze, M.: *A variational discretization concept in control constrained optimization: the linear-quadratic case*, J. Computational Optimization and Applications **30**, 45-63 (2005).
[7] Hinze, M., Pinnau, R., Ulbrich, M., Ulbrich, S.: *Modelling and Optimization with Partial Differential Equations*, Lecture Notes of the autumn school with same title, Department Mathematik, Universität Hamburg (2005).
[8] Meyer, C., Prüfert, U., Tröltzsch, F.: *On two numerical methods for state-constrained elliptic control problems*, Technical Report 5-2005, Institut für Mathematik, TU Berlin (2005).
[9] Meyer, C.: *Error estimates for the finite element approximation of an elliptic control problem with pointwise constraints on the state and the control*, WIAS Preprint 1159 (2006).