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A–posteriori error control of a state constrained elliptic control problem

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Abstract: We develop an adaptive finite element concept for elliptic optimal control problems with constraints on the state. In order to generate goal–oriented meshes, we extend the Dual Weighted Residual (DWR) concept proposed by Becker and Rannacher [1] for PDE-constrained optimization to the state constrained case. Using the optimality system of the underlying PDE-constrained optimal control problem we obtain a representation for the error in the objectives. Based on this representation we define local error indicators, whose performance we investigate by means of a numerical example.

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1 Introduction

In this paper we develop an a posteriori error estimator for PDE-constrained optimization problems in the presence of state-constraints. For its construction we extend the DWR concept of [1, 2] to elliptic optimal control problems with state constraints, where the refinement goal consists in the construction of finite element meshes which allow to resolve well the value of the cost functional. The main analytical result of this work consists in proving an error representation for the values of the cost functional of the form

$$J(y,u) - J(y_h,u_h) = \frac{1}{2}\left(\rho^y(p - i_h p) + \rho^p(y - i_h y) + \langle \mu + \mu_h, y_h - y\rangle\right),$$

where ρ^p, ρ^y denote the dual and primal residual of the underlying PDE, i_h denotes an appropriate interpolation operator, and μ, μ_h denote the multipliers associated to the state constraints of the continuous and discrete problem, respectively, compare (4.1). To anticipate discussion let us point out two basic facts of our approach;

i.) Under common assumptions no residual ρ^u associated to the optimality conditions (2.6),(3.5) appears in our approach. This is due to the fact that we do not discretize controls explicitly, see [9] for details on this concept. This result remains valid in the case of control constraints, see Remark 4.2.

ii.) Differences of multipliers do not occur in our concept. This is of particular importance for multipliers associated to state constraints, since these may be represented by measures. As a consequence there is no need to construct a computable approximation to μ which carries more information than μ_h. In fact we use $\mu \equiv \mu_h$ in our numerical approach.

Let us briefly comment on adaptive approaches in PDE-constrained optimization. For problems with neither constraints on controls nor on states an excellent overview of the DWR approach is contained in [1]. Problems also dealing with constraints on the control are discussed in [10]. A posteriori analysis of an adaptive algorithm for elliptic control problems with constraints on the control is presented in [8]. An extension of the DWR concept to PDE-constrained optimization problems in the presence of control constraints is proposed in [12]. To the best of the authors knowledge the present work presents the first contribution to adaptive approaches for PDE-constrained optimization problems in the presence of state constraints.

The rest of this work is organized as follows. In §2 we present the mathematical setting, §3 sketches the finite element discretization of problem (2.3), derives the corresponding optimality system, and states some properties of the discrete approximations to y, p, u, μ^a and μ^b. In §4 we specify the local error indicators and test their efficiency by means of a numerical example in §5.

2 Mathematical setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the differential operator

$$Ay := -\sum_{i,j=1}^{d} \partial_{x_j}\big(a_{ij} y_{x_i}\big) + \sum_{i=1}^{d} b_i y_{x_i} + cy,$$

along with its formal adjoint operator

$$A^* y = -\sum_{i=1}^{d} \partial_{x_i}\big(\sum_{j=1}^{d} a_{ij} y_{x_j} + b_i y\big) + cy$$

where for simplicity the coefficients a_{ij}, b_i and c are assumed to be smooth functions on $\bar{\Omega}$. We associate with A the bilinear form

$$a(y, z) := \int_{\Omega} \big(\sum_{i,j=1}^{d} a_{ij}(x) y_{x_i} z_{x_j} + \sum_{i=1}^{d} b_i(x) y_{x_i} z + c(x) yz\big) dx, \qquad y, z \in H^1(\Omega)$$

and subsequently assume that there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } x \in \Omega.$$

Furthermore we suppose that the form a is coercive on $H^1(\Omega)$, i.e. there exists $c_1 > 0$ such that

$$a(v, v) \geq c_1 \|v\|^2_{H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega). \tag{2.1}$$

From the above assumptions it follows that for a given $f \in (H^1(\Omega))'$ the elliptic boundary value problem

$$\begin{aligned} Ay &= f \quad \text{in } \Omega \\ \sum_{i,j=1}^d a_{ij} y_{x_i} \nu_j &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.2}$$

has a unique weak solution $y \in H^1(\Omega)$ which we denote by $y = \mathcal{G}(f)$. Here, ν is the unit outward normal to $\partial\Omega$. Furthermore, if $f \in L^2(\Omega)$, then the solution y belongs to $H^2(\Omega)$ and satisfies

$$\|y\|_{H^2(\Omega)} \leq C\|f\|,$$

where we have used $\|\cdot\|$ to denote the $L^2(\Omega)$–norm.

We are interested in goal–oriented adaptive solution strategies for the following control problem

$$\begin{gathered} \min_{u \in U := L^2(\Omega)} J(y,u) = \frac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - u_0\|^2_U \\ \text{subject to } y = \mathcal{G}(u) \text{ and } a(x) \leq y(x) \leq b(x) \text{ in } \Omega. \end{gathered} \tag{2.3}$$

Here, we suppose that $\alpha > 0$, $u_0, y_0 \in H^1(\Omega)$, and $a, b \in W^{2,\infty}(\Omega)$ are given. Under the assumption

$$\bar{a} := \max_{x \in \bar{\Omega}} a(x) < \min_{x \in \bar{\Omega}} b(x) =: \underline{b} \tag{2.4}$$

our problem satisfies the *Slater condition* or interior point condition, i.e.

$$\exists\, \tilde{u} \in U : \qquad a < \mathcal{G}(\tilde{u}) < b \text{ in } \bar{\Omega},$$

since the function $\tilde{u} := \frac{c}{2} \cdot (\bar{a} + \underline{b}) \in U$ together with the constant function $\tilde{y} := \frac{1}{2}(\bar{a} + \underline{b})$ then solves the elliptic PDE (2.2) and satisfies $a < \mathcal{G}(\tilde{u}) = \tilde{y} = \frac{1}{2}(\bar{a} + \underline{b}) < b$ in $\bar{\Omega}$.

Since the state constraints form a convex set it is not difficult to establish the existence of a unique solution $u \in U$ to problem (2.3). In order to characterize this solution we introduce the space $\mathcal{M}(\bar{\Omega})$ of Radon measures which is defined as the dual space of $C^0(\bar{\Omega})$ and endowed with the norm

$$\|\mu\|_{\mathcal{M}(\bar{\Omega})} = \sup_{f \in C^0(\bar{\Omega}), |f| \leq 1} \int_{\bar{\Omega}} f d\mu.$$

Using [4, Theorem 5.2] we have

Theorem 2.1. *Let $u \in U$ denote the unique solution to (2.3). Then there exist unique $\mu^a, \mu^b \in \mathcal{M}(\bar{\Omega})$ and a unique function $p \in W^{1,s}(\Omega)$ for all $1 \leq s < \frac{d}{d-1}$ such that with $y = \mathcal{G}(u)$ there holds*

$$\int_\Omega pAv = \int_\Omega (y - y_0)v + \int_{\bar{\Omega}} v d(\mu^b - \mu^a) \quad \forall v \in H^2(\Omega) \text{ with } \sum_{i,j=1}^d a_{ij} v_{x_i} \nu_j = 0 \text{ on } \partial\Omega \tag{2.5}$$

$$p + \alpha(u - u_0) = 0 \tag{2.6}$$

$$\mu^a \geq 0, \qquad y(x) \geq a(x) \text{ in } \Omega \quad \text{and} \quad \int_{\bar{\Omega}} (y - a) d\mu^a = 0 \tag{2.7}$$

$$\mu^b \geq 0, \qquad y(x) \leq b(x) \text{ in } \Omega \quad \text{and} \quad \int_{\bar{\Omega}} (b - y) d\mu^b = 0. \tag{2.8}$$

3 Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$ and vertices $x_1, \ldots, x_m$. Furthermore one defines $h_{\min} := \min_{T\in\mathcal{T}_h} \operatorname{diam}(T)$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved. In addition, we assume that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. Let us define the space of linear finite elements,

$$X_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}$$

with Lagrange basis $\{v_i \in X_h : i = 1, \ldots, m\}$ and appropriate modification for boundary elements. In what follows it is convenient to introduce a discrete approximation of the operator $\mathcal{G}$. For a given function $v \in L^2(\Omega)$ we denote by $z_h = \mathcal{G}_h(v) \in X_h$ the solution of the discrete Neumann problem

$$a(z_h, v_h) \;=\; \int_\Omega v v_h \quad \text{for all } v_h \in X_h. \tag{3.1}$$

Problem (2.3) is now approximated by the following sequence of control problems depending on the mesh parameter h:

$$\begin{aligned} &\min_{u\in U} J_h(y_h, u) := \frac{1}{2}\|y_h - y_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - u_{0,h}\|^2_U \\ &\text{subject to } \; y_h = \mathcal{G}_h(u) \text{ and } a(x_j) \le y_h(x_j) \le b(x_j) \text{ for } j = 1, \ldots, m. \end{aligned} \tag{3.2}$$

Here, $u_{0,h}$ denotes an approximation to u_0 which is assumed to satisfy

$$\|u_0 - u_{0,h}\| \le Ch. \tag{3.3}$$

Problem (3.2) represents a convex infinite-dimensional optimization problem of similar structure as problem (2.3), but with only finitely many equality and inequality constraints for the state, which define a convex set of admissible functions. Again we can apply [4, Theorem 5.2] which yields

Lemma 3.1. *Problem (3.2) has a unique solution $u_h \in U$. There exist unique $\mu^a_1, \ldots, \mu^a_m, \mu^b_1, \ldots, \mu^b_m \in \mathbb{R}$ and a unique function $p_h \in X_h$ such that with $y_h = \mathcal{G}_h(u_h)$, $\mu^a_h = \sum_{j=1}^m \mu^a_j \delta_{x_j}$ and $\mu^b_h = \sum_{j=1}^m \mu^b_j \delta_{x_j}$ we have*

$$a(v_h, p_h) = \int_\Omega (y_h - y_0)v_h + \int_{\bar{\Omega}} v_h d(\mu^b_h - \mu^a_h) \qquad \forall v_h \in X_h, \tag{3.4}$$

$$p_h + \alpha(u_h - u_{0,h}) = 0, \tag{3.5}$$

$$\mu^a_j \ge 0, \quad y_h(x_j) \ge a(x_j), \quad j = 1, \ldots, m \;\; \textit{and} \;\; \int_{\bar{\Omega}} (y_h - I_h a) d\mu^a_h = 0, \tag{3.6}$$

$$\mu^b_j \ge 0, \quad y_h(x_j) \le b(x_j), \quad j = 1, \ldots, m \;\; \textit{and} \;\; \int_{\bar{\Omega}} (I_h b - y_h) d\mu^b_h = 0. \tag{3.7}$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

Remark 3.2. Problem (3.2) is still an infinite–dimensional optimization problem, but with finitely many state constraints. By (3.5) it follows that $u_h \in X_h$, i.e. the optimal discrete solution is discretized implicitly through the optimality condition of the discrete problem. Hence in (3.2) U may be replaced by X_h to obtain the same discrete solution u_h, which results in a finite–dimensional discrete optimization problem instead.

The finite element analysis of problems (2.3),(3.2) is carried out in [6]. For the convenience of the reader we in the following summarize the main results. From [6, Theorem 2.3, Theorem 3.6, Corollary 3.7] we have (compare also [7])

Theorem 3.3. Let y_h, u_h, p_h, μ_h^a and μ_h^b denote the solutions to (3.4)–(3.7) and let y, u, p, μ^a and μ^b denote the solutions to (2.5)–(2.8). Then

$$\|\mu_h^{a,b}\|_{\mathcal{M}(\bar\Omega)}, \|u_h\|_{W^{1,s}(\Omega)} \le C \text{ for all } 0 < h \le 1, \text{ i.e. } \mu_h^{a,b} \rightharpoonup \mu^{a,b} \text{ weak-* in } \mathcal{M}(\bar\Omega) \tag{3.8}$$

for a subsequence $h \to 0$, and for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|u - u_h\| + \|y - y_h\|_{H^1(\Omega)} \le C_\epsilon h^{2-\frac{d}{2}-\epsilon}. \tag{3.9}$$

Furthermore, for $K \subset\subset \bar\Omega$ with $K \cap \operatorname{supp}(\mu^{a,b}) = \emptyset$ there holds

$$\mu_h^{a,b}(K) \le C_\epsilon h^{2-\frac{d}{2}-\epsilon}. \tag{3.10}$$

4 Local error indicators

From here onwards let us assume $a = I_h a$, $b = I_h b$ and $u_0 = u_{0,h}$, which is fulfilled for functions in X_h for instance. Let us further abbreviate

$$\mu := \mu^b - \mu^a, \quad \mu_h := \mu_h^b - \mu_h^a,$$

and let us use the notation

$$\langle \mu, v \rangle := \int_{\bar\Omega} v d\mu \quad \text{for all} \quad v \in C^0(\bar\Omega) \quad \text{and} \quad \mu \in \mathcal{M}(\bar\Omega).$$

Following [1] we introduce the dual, control and primal residual functionals determined by the discrete solution y_h, u_h, p_h, μ_h^a and μ_h^b of (3.4)-(3.7) by

$$\begin{aligned}
\rho^p(\cdot) &:= J_y(y_h, u_h)(\cdot) - a(\cdot, p_h) + \langle \mu_h, \cdot \rangle, \\
\rho^u(\cdot) &:= J_u(y_h, u_h)(\cdot) + (\cdot, p_h) \quad \text{and} \\
\rho^y(\cdot) &:= -a(y_h, \cdot) + (u_h, \cdot),
\end{aligned}$$

where we have used $(\cdot,\cdot)$ to denote the $L^2(\Omega)$ scalar product. In addition we introduce the error stemming from the complementarity conditions (2.7), (2.8), (3.6) and (3.7), respectively by

$$e^\mu(y) := \langle \mu + \mu_h, y_h - y \rangle.$$

It follows from (3.5) that $\rho^u(\cdot) \equiv 0$. This is due to the fact that we do not discretize the control, so that the discrete structure of the solution u_h of problem (2.3) is induced by the optimality condition (3.5).
We are now in the position to prove the analogue to [11, Theorem 1] for the state constrained case.

Theorem 4.1 (Compare [11, Theorem 1] and [1])**.** There holds the error representation

$$J(y, u) - J(y_h, u_h) = \frac{1}{2}\rho^p(y - i_h y) + \frac{1}{2}\rho^y(p - i_h p) + \frac{1}{2}e^\mu(y) \tag{4.1}$$

with arbitrary quasi–interpolants $i_h y$ and $i_h p \in X_h$.

Proof. It follows from (2.6) and (3.5) that

$$u_h - u = \frac{1}{\alpha}(p - p_h) \tag{4.2}$$

holds. This yields

$$\begin{aligned}
& 2(J(y_h,u_h) - J(y,u)) \\
= \; & (y_h - y_0, y_h - i_h y) + (y_h - y_0, i_h y - y) + (y - y_0, y_h - y) \\
& + \alpha(u_h - u_0, \tfrac{1}{\alpha}(p - p_h)) - \alpha(u - u_0, \tfrac{1}{\alpha}(p_h - p)) \\
= \; & J_y(y_h,u_h)(y_h - i_h y) + (y_h - y_0, i_h y - y) + J_y(y,u)(y_h - y) \\
& -(u_h, p_h - p) + 2(u_0, p_h - p) - (u, p_h - p).
\end{aligned}$$

Since by (4.2), (3.1)

$$(u_0, p_h - p) = -(u_h, p - i_h p) - a(y_h, i_h p) + a(y, p_h)$$

holds, we obtain

$$\begin{aligned}
& 2(J(y_h,u_h) - J(y,u)) \\
= \; & a(y_h - i_h y, p_h) - \langle \mu_h, y_h - i_h y\rangle + (y_h - y_0, i_h y - y) \\
& + a(y_h - y, p) - \langle \mu, y_h - y\rangle \\
& -(u_h, p_h - p) - 2(u_h, p - i_h p) - 2a(y_h, i_h p) + 2a(y, p_h) - (u, p_h - p) \\
= \; & a(y_h - i_h y, p_h) - \langle \mu_h, y - i_h y\rangle + (y_h - y_0, i_h y - y) + a(y_h - y, p) \\
& -(u_h, p_h - p) - 2(u_h, p - i_h p) - 2a(y_h, i_h p) + 2a(y, p_h) - (u, p_h - p) \\
& + \langle \mu_h^a, a - y\rangle - \langle \mu^a, a - y_h\rangle + \langle \mu_h^b, y - b\rangle - \langle \mu^b, y_h - b\rangle \\
= \; & [a(y_h, p_h) - (u_h, p_h)] + a(y, p_h) - a(i_h y, p_h) - \langle \mu_h, y - i_h y\rangle \\
& -J_y(y_h,u_h)(y - i_h y) + [(u,p) - a(y,p)] + [(u_h, i_h p) - a(y_h, i_h p)] + [a(y, p_h) - (u, p_h)] \\
& + a(y_h, p) - a(y_h, i_h p) - (u_h, p - i_h p) - e^{\mu}(y),
\end{aligned}$$

where we have used

$$\begin{aligned}
& \langle \mu^a, a - y_h\rangle - \langle \mu_h^a, a - y\rangle = \langle \mu^a, y - y_h\rangle + \langle \mu_h^a, y - y_h\rangle, \text{ and} \\
& \langle \mu^b, y_h - b\rangle - \langle \mu_h^b, y - b\rangle = \langle \mu^b, y_h - y\rangle + \langle \mu_h^b, y_h - y\rangle.
\end{aligned}$$

Since the terms within the squared brackets vanish, we finally obtain

$$\begin{aligned}
& 2(J(y_h,u_h) - J(y,u)) \\
= \; & -J_y(y_h,u_h)(y - i_h y) + a(y - i_h y, p_h) - \langle \mu_h, y - i_h y\rangle \\
& + a(y_h, p - i_h p) - (u_h, p - i_h p) - e^{\mu}(y) \\
= \; & -\rho^p(y - i_h y) - \rho^y(p - i_h p) - e^{\mu}(y). \qquad \blacksquare
\end{aligned}$$

Remark 4.2. If we introduce control constraints of the form $c \le u \le d$ almost everywhere in Ω with sufficiently smooth bounds c, d satisfying $I_h c = c$ and $I_h d = d$ we obtain the error representation

$$J(y,u) - J(y_h,u_h) = \frac{1}{2}\left(\rho^p(y - i_h y) + \rho^y(p - i_h p) + e^{\mu}(y) + (\lambda + \lambda_h, u_h - u)\right),$$

Here, with $\lambda^{c,d}, \lambda_h^{c,d} \in L^2(\Omega)$ denoting the multipliers associated to the control constraints, we set $\lambda := \lambda^d - \lambda^c$ and $\lambda_h^d - \lambda_h^c$. We emphasize that neither differences of the multipliers μ, μ_h

nor differences of the multipliers λ, λ_h appear in this error representation. We exploit this fact in the definition of the error estimators, since it now is meaningful to replace the continuous multipliers μ, λ by their discrete counterparts μ_h, λ_h. This idea is different from the one used in [12] to construct an a posteriori error estimator for control constrained optimization problems, and takes care of the fact that a better approximation to $\mu \in \mathcal{M}(\bar\Omega)$ than μ_h can hardly be constructed using only the values $\mu_1^{a,b}, \ldots, \mu_m^{a,b}$ of Lemma 3.1.

The goal now consists in deriving an a posteriori error representation of the form

$$|J(y,u) - J(y_h,u_h)| \approx \frac{1}{2}|\sum_{T\in\mathcal{T}_h} \rho_T^p((y-i_hy)_{|T}) + \rho_T^y((p-i_hp)_{|T}) + e_T^\mu(y_{|T})|,$$

and in an final step to replace continuous quantities by computable analogues. To begin with let us first consider $\rho^y(p-i_hp)$. It follows from the definition of the bilinear form a that

$$\rho^y(p-i_hp) = -a(y_h, p-i_hp) + (u_h, p-i_hp) =$$
$$= \sum_{T\in\mathcal{T}_h}\int_T \Big(\sum_{i,j=1}^d -a_{ij}(x)(y_h)_{x_i}(p-i_hp)_{x_j} - \sum_{i=1}^d b_i(x)(y_h)_{x_i}(p-i_hp) - c(x)y_h(p-i_hp) + u_h(p-i_hp)\Big)dx,$$

so that we may define

$$\rho_T^y((p-i_hp)_{|T}) :=$$
$$:= \int_T \Big(\sum_{i,j=1}^d -a_{ij}(x)(y_h)_{x_i}(p-i_hp)_{x_j} - \sum_{i=1}^d b_i(x)(y_h)_{x_i}(p-i_hp) - c(x)y_h(p-i_hp) + u_h(p-i_hp)\Big)dx.$$

For $\rho^p(y-i_hy)$ the situation is more involved, since it contains the term $\langle \mu_h, y-i_hy\rangle$. We interpret this contribution as a quadrature rule of an integral of a certain function. To begin with we set for $i = 1, \ldots, m$

$$n_i := \operatorname{card}\big(\{T \in \mathcal{T}_h : x_i \in \bar T\}\big) \in \mathbb{N}$$

and introduce the Lagrange-interpolants $N_h > 0$ and $I_h\mu_h \in X_h$ by

$$N_h := \sum_{i=1}^m n_i v_i \qquad \text{and} \qquad I_h\mu_h := \sum_{i=1}^m \mu_i v_i.$$

Denoting by $x_j^T (j = 1, \ldots, d+1)$ the finite element nodes of a simplex T and by μ_j^T the corresponding coefficients of μ_h we have

$$\langle \mu_h, y-i_hy\rangle = \sum_{i=1}^m \mu_i (y-i_hy)(x_i) = \sum_{T\in\mathcal{T}_h} \frac{|T|}{d+1}\sum_{j=1}^{d+1} \frac{d+1}{|T|}\frac{(y-i_hy)(x_j^T)\mu_j^T}{N_h(x_j^T)},$$

so that $\langle \mu_h, y-i_hy\rangle$ may be considered as the application of the quadrature rule

$$\int_T g(x)\,\mathrm{d}x \approx \frac{|T|}{d+1}\sum_{j=1}^{d+1} g(x_j^T) \tag{4.3}$$

to the expression

$$\sum_{T\in\mathcal{T}_h}\int_T \frac{d+1}{|T|}\frac{(y-i_hy)I_h\mu_h}{N_h}(x)\,\mathrm{d}x.$$

We use the quadrature rule (4.3) since the quadrature weights μ_j^T $(j = 1, \dots, d+1)$ are only given in the vertices of a simplex T. The previous considerations motivate to define the local adjoint residual by

$$\rho_T^p((y - i_h y)_{|T}) :=$$
$$:= \int\limits_T \Big(\sum_{i,j=1}^{d} -a_{ij}(x)(y - i_h y)_{x_i}(p_h)_{x_j} - \sum_{i=1}^{d} b_i(x)(y - i_h y)_{x_i}(p_h) - c(x)(y - i_h y)p_h\Big)dx +$$
$$+ \int\limits_T (y_h - y_0)(y - i_h y)dx + \sum_{j=1}^{d+1} \frac{(y - i_h y)(x_j^T)(\mu_j^{b,T} - \mu_j^{a,T})}{N_h(x_j^T)}.$$

Let us finally consider $e^\mu(y)$. Remark 4.2 motivates to approximate this term according to

$$e^\mu(y) = \langle \mu + \mu_h, y_h - y\rangle \approx 2\langle \mu_h, y_h - y\rangle =$$
$$= 2\sum_{i=1}^{m} \mu_i (y_h - y)(x_i) = \sum_{T\in\mathcal{T}_h} \sum_{j=1}^{d+1} \frac{2\mu_j^T}{N_h(x_j^T)}(y_h - y)(x_j^T),$$

where $\mu_i := \mu_i^b - \mu_i^a$ $(i = 1, \dots, m)$, and $\mu_j^T := \mu_j^{b,T} - \mu_j^{a,T}$ $(j = 1, \dots, d+1)$ denote the discrete multipliers in the element-wise renumbering. We now set

$$e_T^\mu(y_{|T}) := \sum_{j=1}^{d+1} \frac{2\mu_j^T}{N_h(x_j^T)}(y_h - y)(x_j^T).$$

In order to obtain computable local indicators, we approximate $y - i_h y$ and $p - i_h p$ on every triangle T by $(i_{2h}^{(2)} y_h - y_h)_{|T}$ and $(i_{2h}^{(2)} p_h - p_h)_{|T}$ as suggested in [11, Remark 1]. Here, $i_{2h}^{(2)} y_h$ denotes a quadratic Lagrange interpolation of y_h on a coarser mesh using function values of y_h at element vertices (similarly for p_h). For approximating $(y_h - y)(x_j^T)$ we compute $(y_h - i^{(2)} y_h)(x_j^T)$. The quadratic interpolation operator $i^{(2)}$ differs from $i_{2h}^{(2)}$ in interpolating the function values of y_h in the midpoints of element edges. Its use is caused by the fact that our approximation to $e^\mu(y)$ relies on function evaluations in the finite element nodes x_i $(i = 1, \dots, m)$. If the interpolants $i_{2h}^{(2)} y_h$ and $i^{(2)} y_h$ violate the state constraints we use $\max(a, \min(b, i_{2h}^{(2)} y_h))$ and $\max(a, \min(b, i^{(2)} y_h))$, respectively instead.
Our error estimator finally takes the form

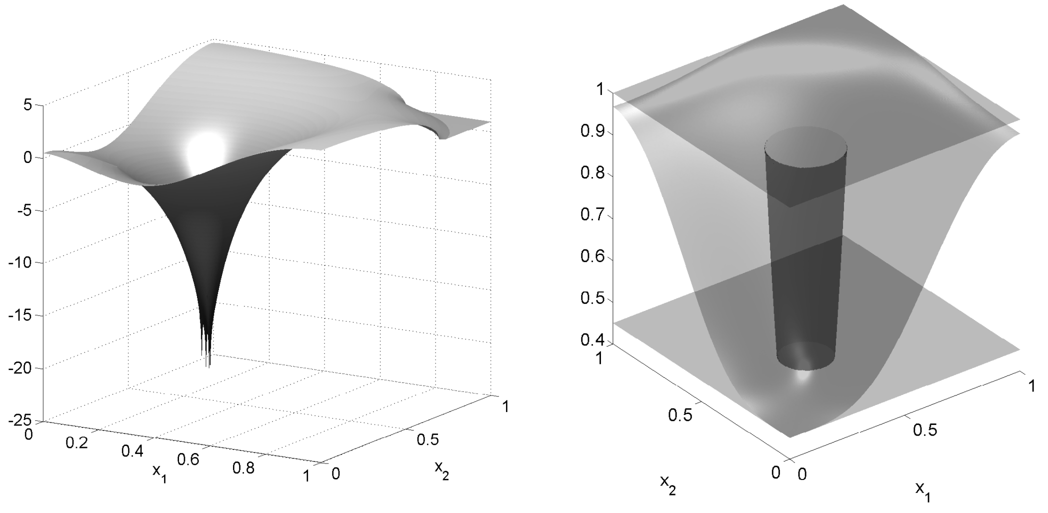
$$\eta := \left| \frac{1}{2} \sum_{T\in\mathcal{T}_h} \rho_T^p((i_{2h}^{(2)} y_h - y_h)_{|T}) + \rho_T^y((i_{2h}^{(2)} p_h - p_h)_{|T}) + e_T^\mu((i^{(2)} y_h)_{|T}) \right|. \tag{4.4}$$

In the following numerical example we investigate the efficiency of the estimator η in terms of

$$I_{\text{eff}} := \frac{|J(y,u) - J(y_h, u_h)|}{\eta}.$$

5 Numerical example

We set $d = 2$ and consider the domain $\Omega := (0,1)^2$ with the elliptic differential operator A defined by $a_{ij} = \delta_{ij}$, $b_i = 0$, $(i, j = 1, 2)$, and $c = 1$. The regularization parameter in the cost



(a) u (b) $y,\ a,\ b$

Figure 1: Solution on a uniform mesh with 577^2 nodes

functional J is set to $\alpha = 1$. The desired control and state functions u_0 and y_0 as well as the bounds a and b for the state are given by

$$
\begin{aligned}
u_0(x) &= 60, & y_0(x) &= 0.5, \\
a(x) &= 0.45 \quad \text{and} & b(x) &= \min\Big(1, \max\Big(0.5,\ 50\,\big|x - (0.3, 0.3)^T\big|^2\Big)\Big)
\end{aligned}
$$

for every $x \in \bar{\Omega}$. The corresponding optimal control problem reads

$$
\min_{u \in L^2(\Omega)} J(y,u) := \frac{1}{2}\,\|y - y_0\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|u - u_0\|^2_{L^2(\Omega)}
$$

$$
\text{subject to} \quad \begin{aligned} -\Delta y + y &= u \quad \text{in} \quad \Omega \\ \partial_\nu y &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \qquad \text{and} \qquad a(x) \le y(x) \le b(x) \quad \forall x \in \bar{\Omega}.
$$

In order to avoid specialties introduced by test problems admitting exact solutions we consider a fully generic test case by taking the numerical solution (y, u) obtained on an equidistant grid containing 557^2 nodes as substitute for the exact solution, see Fig.1. The reference functional value $J^* := J(y, u)$ takes the value $J^* = 1759.04733$. The support of the corresponding multiplier μ is depicted in Fig. 2. We start the numerical run on a uniform triangulation containing 484 nodes. On a mesh with 113569 nodes obtained by congruent refinement we obtain $|J^* - J(y_h, u_h)| \approx 0.00726$. Local refinement using the so called tolerance reduction strategy (see [2]) together with the estimator η leads to meshes where this value of the error already is reached with less then a quarter of unknowns. Specifically, for $m = 23216$ we already obtain $|J^* - J(y_h, u_h)| \approx 0.00516$. The development of the error in the objective is presented in Fig. 3. The efficiency of our estimator is documented in Tab. 2, and Tab. 1 contains the efficiency of global refinement. We observe that the estimator η slightly under- and overestimates, respectively, the real error, but always has the same magnitude as the true error. Figure 4 shows two meshes obtained by the tolerance reduction strategy. These meshes clearly indicate that the largest errors in the numerical approximation have their origin in the sub-square $[0.3, 0.5]^2$. In this area the discrete multipliers take their largest values.
All solutions of the discrete optimization problems are computed under Matlab by a Moreau-Yosida-based active set strategy described in [3].

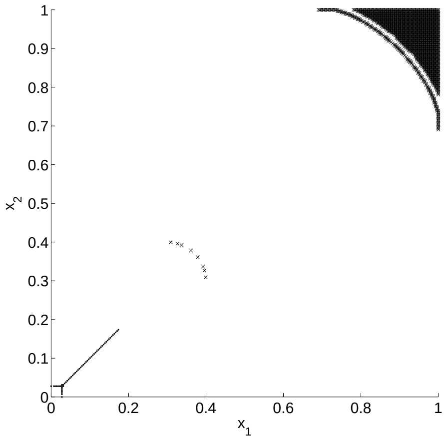


Figure 2: • $\text{supp}(\mu_h^a)$, × $\text{supp}(\mu_h^b)$

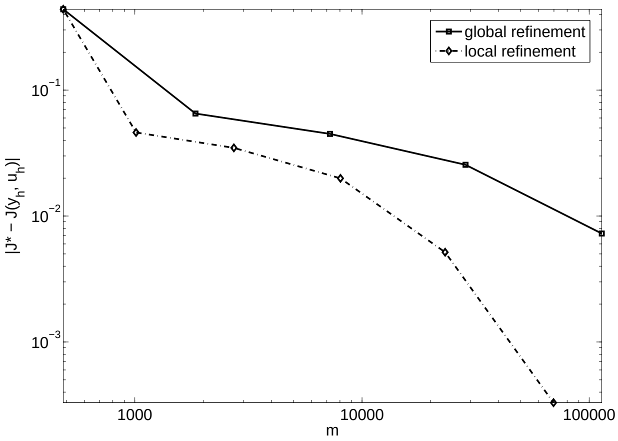
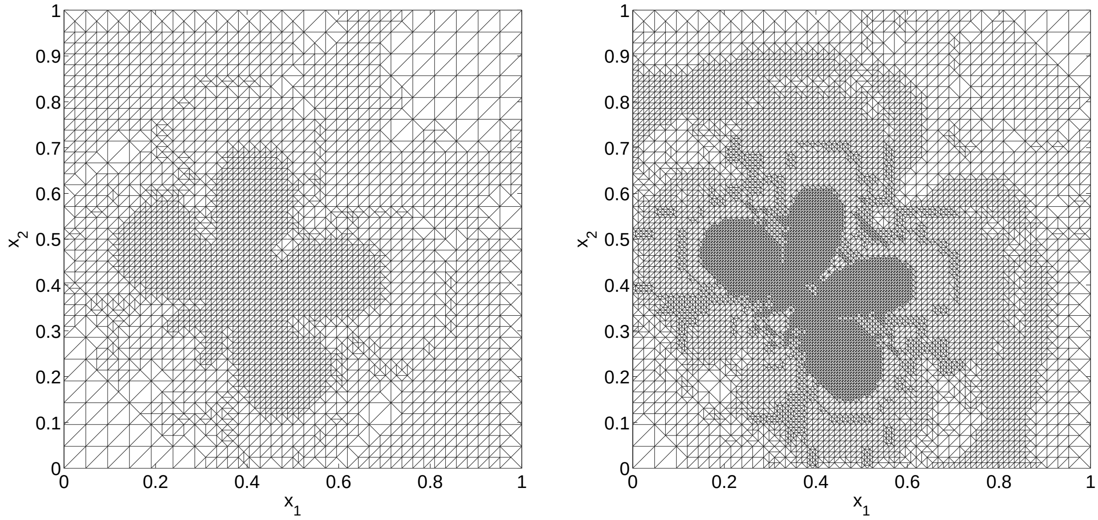


Figure 3: Error in the cost functional J



(a) $m = 2730$

(b) $m = 8038$

Figure 4: Locally refined meshes from η

i	$m = i^2$	$h = \frac{\sqrt{2}}{i-1}$	$h_{\min} = \frac{1}{i-1}$	$\lvert J^* - J(y_h, u_h)\rvert$	I_{eff}
22	484	0.0673	0.0476	0.43855	2.0
43	1849	0.0337	0.0238	0.06514	0.9
85	7225	0.0168	0.0119	0.04492	2.5
169	28561	0.0084	0.0060	0.02553	5.6
337	113569	0.0042	0.0030	0.00726	6.4

Table 1: Mesh data, error and efficiency index for global refinement

m	h	$h_{\min}$	$\lvert J^* - J(y_h, u_h)\rvert$	I_{eff}
484	0.0673	0.0476	0.43855	2.0
1013	0.0673	0.0238	0.04606	0.5
2730	0.0673	0.0119	0.03477	1.2
8038	0.0673	0.0060	0.01992	1.9
23216	0.0673	0.0030	0.00516	1.4
69645	0.0673	0.0015	0.00033	0.3

Table 2: Mesh data, error and efficiency index for local refinement

6 Conclusions

We extend the DWR concept developed by Becker and Rannacher in [1, 2] to optimal control of elliptic partial differential equations in the presence of state constraints. Our approach among other things avoids the discretization of controls and the construction of computable expressions for differences of multipliers. This is of particular importance if the multipliers happen to appear as measures. Furthermore, our estimator only contains contributions of PDE residuals, and of errors stemming from complementarity integrals. We present a generic numerical test which delivers an efficiency close to 1 for our estimator.

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