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Algebraic Topology (Master)
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Exercise sheet 1

Question 1.1

Let A and B be two abelian groups. We denote by $\mathrm{Hom}(A, B)$ the set of group homomorphisms from A to B.

a) Show that $\mathrm{Hom}(A, B)$ is an abelian group.

b) Construct an explicit isomorphism $\varphi\colon \mathrm{Hom}(\mathbb{Z}, A) \cong A$ for all abelian groups A.

c) Let $n > 1$ be a natural number. Describe $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A)$ as a subgroup of A. What is $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ or $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$?

Question 1.2

Let $\mathbb{D}^n$ be the chain complex whose only non-trivial entries are in degrees n and $n-1$ with $\mathbb{D}^n_n = \mathbb{D}^n_{n-1} = \mathbb{Z}$. Its only non-trivial boundary operator is the identity.

Similarly, let $\mathbb{S}^n$ be the chain complex whose only non-trivial entry is in degree n with $\mathbb{S}^n_n = \mathbb{Z}$.

a) Assume that (C_*, d) is an arbitrary chain complex. Describe the abelian group of chain maps from $\mathbb{D}^n$ to C_* and from $\mathbb{S}^n$ to C_* in terms of subobjects of C_n.

b) What is the homology of $\mathbb{D}^n$ and $\mathbb{S}^m$?

c) Let $f_*\colon C_* \to C'_*$ be a chain map and assume that f_n is a monomorphism for all n. Do we then know that $H_n(f_*)$ is also a monomorphism?

Question 1.3

a) What are the homology groups of the chain complex

$$C_* = (\ldots \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z} \to \ldots)?$$

b) Is there a chain homotopy from the identity of C_* to the zero map, i.e. can there be maps $s_n\colon C_n \to C_{n+1}$ with $d \circ s + s \circ d = \mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$?

Question 1.4

Given a map $f : A_* \to B_*$ between chain complexes we define a new chain complex, the *mapping cone* by $C(f)_n = A_{n-1} \oplus B_n$ and the differential sends (a, b) to $(-d_A(a), d_B(b) - f(a))$.

a) Check this defines a chain complex.

b) Show that given any chain complex D_* it is equivalent to specify a chain map $C(f)_* \to D_*$ and to give a map $g : B_* \to D_*$ and a chain homotopy $g \circ f \simeq 0$.

c) What is the homology of $C(\mathbf{1}_A)$?

These questions will be discussed in the class on 12/4/23. You may hand in your solutions (in pairs) the day before.

Questions with an asterisk are more challenging.