7 Baire property and Lebesgue measure

7.1 Lebesgue measure

Here we will focus on the actual real number line $\mathbb{R}$, and assume some familiarity with standard measure theory. The standard Lebesgue measure on $\mathbb{R}$ will be denoted by μ. Also, we will assume here that $\mathbf{\Gamma}$ is, additionally, closed under finite unions, intersections and complements, and contains the F_σ sets (the $\mathbf{\Delta}^1_n$ sets, projective sets etc. satisfy this property). Notice that $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, so any pointclass $\mathbf{\Gamma}$ generates a pointclass on $\mathbb{R} \setminus \mathbb{Q}$, and we can also extend it to $\mathbb{R}$ by stipulating for $A \subseteq \mathbb{R}$ that $A \in \mathbf{\Gamma}$ iff $A \setminus \mathbb{Q} \in \mathbf{\Gamma}$.

The result of this section was first proved by Mycielski-Świerczkowski [MŚ64], but we present a proof due to Harrington. We will define the *covering game* G_μ, for which we first need to fix some setup. Note that it is sufficient to prove that every subset of $[0,1]$ in $\mathbf{\Gamma}$ is measurable.

Definition 7.1.

1. Fix an enumeration $\{I_n \mid n \in \mathbb{N}\}$ of all possible finite unions of open intervals in $[0,1]$ with rational endpoints (e.g., I_n is of the form $(q_0, q_1) \cup \cdots \cup (q_k, q_{k+1})$ for some k, $q_i \in \mathbb{Q}$, etc.) This is possible since $\mathbb{Q}$ is countable.
2. For $x \in 2^\mathbb{N}$, let $a : 2^\mathbb{N} \longrightarrow [0,1]$ be the function given by

$$a(x) := \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}$$

It is not hard to see that a, as a function from the Baire space to $[0,1]$, is continuous and that its range is all of $[0,1]$ (but a is not injective, e.g., both $\langle 1,0,0,0,\dots\rangle$ and $\langle 0,1,1,1,\dots\rangle$ map into $\frac{1}{2}$ —think of x as the binary expansion of $a(x)$).

For every $\epsilon > 0$, we define a game $G_\mu(A,\epsilon)$.

Definition 7.2. Let A be a subset of $[0,1]$ and $\epsilon > 0$. The game $G_\mu(A,\epsilon)$ is defined as follows:

- At each turn, Player I picks 0 or 1, and Player II picks natural numbers.

I :	x_0		x_1		x_2		$\dots$	
II :		y_0		y_1		y_2		

- At every move n, Player II must make sure that

$$\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}},$$

(otherwise she loses).

- Player I wins iff $a(x) \in A \setminus \bigcup_{n=0}^{\infty} I_{y_n}$.

So the idea here is that Player I attempts to play a real number in $A \subseteq [0,1]$, essentially by using the infinite binary expansion of that real, while Player II attempts to "cover" that real with a countable union of the I_n's, but of an increasingly smaller measure.

Showing that this game can be formulated as a game within the same pointclass is a bit more involved.

Lemma 7.3. *Given A in $\mathbf{\Gamma}$ and ϵ, there exists a set $A^{\mu,\epsilon} \subseteq \mathbb{N}^{\mathbb{N}}$ which is in $\mathbf{\Gamma}$ and such that $G_\mu(A,\epsilon) = G(A^{\mu,\epsilon})$.*

Proof. First of all, clearly the functions $f(z)(n) := z(2n)$ and $g(z)(n) := z(2n+1)$ are continuous. Then, if z is the result of the game in the standard sense, Player I should win $G_{\mu,\epsilon}(A)$ iff

1. $\forall n(f(z)(n) \in \{0,1\})$,
2. $a(f(z)) \in A$, and
3. $a(f(z)) \notin \bigcup_{n=1}^{\infty} I_{g(z)(n)}$,
   or if
4. $\exists n$ such that $\mu(I_{g(z)(n)}) \geq \frac{\epsilon}{2^{2(n+1)}}$

So define the following sets:

1. $C_1 := \{z \mid \forall n(z(n) \in \{0,1\})\}$,
2. $C_2 := \{(a,y) \in \mathbb{R} \times \mathbb{N}^{\mathbb{N}} \mid \forall n(a \notin I_{y(n)})\}$,
3. $C_3 := \{z \mid \exists n(\mu[I_{z(n)}] \geq \frac{\epsilon}{2^{2(n+1)}})\}$.

And let

$$A^{\mu,\epsilon} := \big(f^{-1}[C_1] \;\cap\; (a \circ f)^{-1}[A] \;\cap\; ((a \circ f) \times g)^{-1}[C_2]\big) \;\cup\; g^{-1}[C_3]$$

Clearly $G_\mu(A,\epsilon) = G(A^{\mu,\epsilon})$. Using reasoning as in Exercise 6.4 (1)–(5), it is easy to see that C_1 is closed and C_3 is open. Concerning C_2, let's write the complement of C_2 in the following form

$$(\mathbb{R} \times \mathbb{N}^{\mathbb{N}}) \setminus C_2 = \bigcup_{n=0}^{\infty} \{(a,y) \mid a \in I_{y(n)}\}$$

$$= \bigcup_{n=0}^{\infty} \{(a,y) \mid \exists m\, (a \in I_m \;\wedge\; y(n) = m)\}$$

$$= \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} \{(a,y) \mid a \in I_m \;\wedge\; y(n) = m\}$$

$$= \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} I_m \times \{y \mid y(n) = m\}$$

Since the I_m are finite unions of open intervals, they are all open; also the $\{y \mid y(n) = m\}$ are open by Exercise 6.4 (3). So the product of these two sets is open in the product topology of $\mathbb{R} \times \mathbb{N}^{\mathbb{N}}$, and hence the countable union is open. Therefore the complement of C_2 is open, hence C_2 is closed.

Now all the functions involved are continuous and it is easy to see that $A^{\mu,\epsilon} \in \mathbf{\Gamma}$ (recall that we assumed $\mathbf{\Gamma}$ to be closed under finite unions and intersections). □

Now let us return to the main result.

Theorem 7.4. *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ and ϵ be given. Then*

1. *If Player I has a winning strategy in $G_\mu(A, \epsilon)$ then there is a measurable set Z with $\mu(Z) > 0$ such that $Z \subseteq A$ (i.e., the inner measure of A is > 0).*
2. *If Player II has a winning strategy in $G_\mu(A, \epsilon)$ then there is an open O such that $A \subseteq O$ and $\mu(O) < \epsilon$ (i.e., the* outer measure *of A is $< \epsilon$).*

Proof.

1. Let σ be a winning strategy for I. It is clear that the mapping $y \mapsto \sigma * y$ is continuous. But then, also the mapping $y \mapsto a(f(\sigma * y))$ is continuous (where f is defined as in Lemma 7.3). Let $Z := \{a(f(\sigma * y)) \mid y \in \mathbb{N}^{\mathbb{N}}\}$. This is an *analytic* set (continuous image of a closed set) which is well-known to be measurable (by a classical result of Suslin from 1917). As we assumed σ to be winning, $Z \subseteq A$. But if $\mu(Z) = 0$ then (again by standard measure-theory) there exists a cover of Z by a sequence of sets $\{I_{y_n} \mid n \in \mathbb{N}\}$ satisfying $\forall n\, (\mu(I_{y_n}) < \frac{\epsilon}{2^{2(n+1)}})$. Then if II plays the sequence $y = \langle y_0, y_1, \dots \rangle$, we will get $a(f(\sigma * y)) \in Z \subseteq \bigcup_{n=0}^{\infty} I_{y_n}$, contradicting that σ is winning for I.

2. Now suppose II has a winning strategy τ. For each $s \in 2^*$ of length n, define $I_s := I_{(s*\rho)(2n-1)}$, i.e., I_s is the $I_{y_{n-1}}$ where y_{n-1} is the last move of the game in which I played s and II used τ. As τ is winning for II, for every $a \in A$ and every $x \in 2^{\mathbb{N}}$ such that $a(x) = a$, there must be some n such that $a \in I_{x \restriction n}$. In other words, $a \in \bigcup\{I_s \mid s \lhd x\}$ where x is such that $a(x) = a$. Therefore, in particular,

$$A \subseteq \bigcup_{s \in 2^{\mathbb{N}}} I_s = \bigcup_{n=1}^{\infty} \bigcup_{s \in \{0,1\}^n} I_s.$$

Now notice that, since τ was winning, for every s of length $n \geq 1$, $\mu(I_s) < \epsilon / 2^{2n}$. Therefore

$$\mu\left(\bigcup_{s \in \{0,1\}^n} I_s\right) < \frac{\epsilon}{2^{2n}} \cdot 2^n = \frac{\epsilon}{2^n}.$$

It follows that

$$\mu\left(\bigcup_{s \in 2^{\mathbb{N}}} I_s\right) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

Therefore, indeed, A is contained in an open set of measure $< \epsilon$. □

Now it only remains to use the above dichotomy to show that it implies that every set is measurable.

Corollary 7.5. *Let $X \subseteq [0,1]$ be any set in $\boldsymbol{\Gamma}$ and assume $\mathsf{Det}(\boldsymbol{\Gamma})$. Then X is measurable.*

Proof. Let δ be the outer measure of X. Then there is a G_δ set B such that $X \subseteq B$ and $\mu(B) = \delta$. Now consider the games $G_\mu(B \setminus X, \epsilon)$, for all ϵ. Notice that our closeure properties on $\boldsymbol{\Gamma}$ guarantee that $X \setminus B \in \boldsymbol{\Gamma}$. By Theorem 7.4 each such game is determined. But if, for at least one $\epsilon > 0$, I would have a winning strategy, then there would exist a measurable set $Z \subseteq B \setminus X$ of positive measure, implying that X is contained in a set $B \setminus Z$ of measure strictly less than δ, thus contradicting that the outer measure of X was δ. Therefore, by determinacy, II must have a winning strategy in every game $G_\mu(B \setminus X, \epsilon)$ for every $\epsilon > 0$. But that implies that, for every ϵ, $B \setminus X$ can be covered by an open set of measure $< \epsilon$, therefore $B \setminus X$ itself has measure 0. Since X is equal to a G_δ set modulo a measure-zero set, X itself must be measurable. □

7.2 Baire property

Next, we consider a well-known topological property, frequently seen as a counterpart to Lebesgue-measurability. The game used in this context is the Banach-Mazur game from Definition 5.1.

Recall the following topological definitions:

Definition 7.6. Let $X \subseteq \mathbb{N}^{\mathbb{N}}$. We say that

1. X is *nowhere dense* if every basic open $O(t)$ contains a basic open $O(s) \subseteq O(t)$ such that $O(s) \cap X = \varnothing$,
2. X is *meager* if it is the union of countably many nowhere dense sets.
3. X has the *Baire property* if it is equal to an open set modulo meager, i.e., if there is an open set O such that $(X \setminus O) \cup (O \setminus X)$ is meager.

Just as with the perfect set property, it is possible to show (using the Axiom of Choice) that there are sets without the Baire property. We will prove that it follows from determinacy (for boldface pointclasses $\boldsymbol{\Gamma}$).

So, let $G^{**}(A)$ be the Banach-Mazur game from Definition 5.1; recall that we already proved that the coding involved in the game is continuous. Originally, this theorem is due to Banach and Mazur; it can be found in [Oxt57].

Theorem 7.7. *Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set and $G^{**}(A)$ the Banach-Mazur game.*

1. *If Player II has a winning strategy in $G^{**}(A)$ then A is meager.*
2. *If Player I has a winning strategy in $G^{**}(A)$ then $O(s) \setminus A$ is meager for some basic open $O(s)$.*

Proof.

1. This part of the proof is similar to the proof with the $*$-game in the previous section. Let τ be a winning strategy of Player II. For a position $p := \langle s_0, t_0, \dots, s_n, t_n \rangle$ write $p^* := s_0{}^\frown t_0{}^\frown \dots {}^\frown s_n{}^\frown t_n$. For any position p and $x \in \mathbb{N}^{\mathbb{N}}$ we say that
   - p is *compatible with* x if $p^* \lhd x$.