The Theories of Classes

Report of Alternative Set Theories

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1 Theories of Classes

When we work in set theories – especially when we work in ZFC – the statements of our theory do only refer to sets and every time we speak of classes we are actually using only abbreviations for first-order formulas. For this reason one may be interested in **class theories**, namely in alternative theories in which one can talk both about sets and classes in the "official language".

In particular we will introduce two important class theories:

- **NBG** is a class theory which has been developed in the first half of the 20_{th} century in a series of papers by von Neumann, Bernays and Gödel. Some interesting features of NBG are the fact that it is finitely axiomatizable and that it can be proved to be a conservative extension of ZFC.

- **MK** is a class theory whose comprehension principle for classes is stronger than that of NBG. More precisely, NBG can be proved to be a subtheory of MK. It is also interesting to notice that MK is strong enough to prove the consistency of NBG (and thus also of ZFC). MK was firstly presented in an appendix to a book of topology by Kelley in 1955 an then developed by Mostowski, Quine and Morse.

2 The Axioms of NBG

In what follows I will provide an axiomatization of NBG which closely refers to the original axiomatization proposed by Gödel in his 1940 monograph "The consistency of the Axiom of Choice and of the generalized continuum-hypothesis with the axioms of set theory".

The language we use to axiomatize NBG is first-order-logic with equality and three additional predicates:

$$Cls(X) = \text{“}X \text{ is a class”}$$

$$M(X) = \text{“}X \text{ is a set”}$$

$$X \in Y = \text{“}X \text{ belongs to } Y\text{”}$$

We then define:

$$Pr(X) = \neg M(X) = \text{“}X \text{ is a proper class ”}$$

Moreover, we adopt the convention that capital letter variables ($X, Y, Z, ...$) refer to classes and small letter variables ($x, y, z, ...$) refer to sets. Notice that for us this is just a convention and that to be fully precise we should always make explicit if something is a set or a proper class. For instance, every time we write something like $\forall x(Px)$, we should rather write $\forall X(M(X) \rightarrow P(X))$. Differently, if one adopts a two-sorted language, then one can drop the uses of the predicates for sets and classes and use different variables to distinguish among them.

Let us now present the **axioms of NBG**. We distinguish them in four groups A,B,C,D.

Group A

1. $\forall x(Cls(x))$ [Every set is a class]
2. $\forall X \forall Y(X \in Y \rightarrow M(X))$ [The elements of classes are sets]
3. $\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \rightarrow X = Y)$ [Extensionality]
4. $\forall x \forall y \exists x \forall u(u \in z \leftrightarrow u = x \vee u = y)$ [Pairing]

In the axioms of the next group we assume the standard definition of ordered pairs and ordered n-tuples.

Group B

1. $\exists X \forall u \forall v(\langle u, v\rangle \in X \leftrightarrow u \in v)$ [Axiom of $\in -$reduction]
2. $\forall X \forall Y \exists Z \forall u(u \in Z \leftrightarrow u \in X \vee u \in Y)$ [Axiom of intersection]
3. $\forall X \forall Z \forall u(u \in Z \leftrightarrow u \notin X)$ [Axiom of complement]
4. $\forall X \exists Z \forall u(u \in Z \leftrightarrow \exists v(\langle u, v\rangle \in X))$ [Axiom of domain]
5. $\forall X \exists Z \forall u \forall v(\langle u, v\rangle \in Z \leftrightarrow u \in X)$
6. $\forall X \exists Z \forall u \forall v \forall w(\langle u, v, w\rangle \in Z \leftrightarrow \langle v, w, u\rangle \in X)$
7. $\forall X \exists Z \forall u \forall v \forall w(\langle u, v, w\rangle \in Z \leftrightarrow \langle u, w, v\rangle \in X)$

Let us now define the following properties:

$$Em(X) := \forall y \neg(y \in X)$$

$$Fn(X) := \forall u \forall v \forall z(\langle v, u\rangle \in X \wedge \langle v, w\rangle \rightarrow u = w)$$

We then have the following new group of axioms:

Group C

1. $\exists x(\neg Em(x) \wedge \forall y(y \in x \rightarrow \exists z(z \in x \wedge y \subset z))$ [Axiom of infinity]
2. $\forall x \exists y \forall u \forall v(u \in v \wedge v \in x \rightarrow u \in x)$ [Axiom of union]
3. $\forall x \exists y \forall u(u \subseteq x \rightarrow u \in y)$ [Axiom of power set]
4. $\forall x \forall F(Fn(F) \rightarrow \exists y \forall u(u \in y \leftrightarrow \exists v(v \in x \wedge \langle u, v\rangle \in F))$ [Axiom of replacement]

Group D: Axiom of Regularity

1. $\forall X(\neg Em(X) \rightarrow \exists u(u \in X \wedge Em(u \cap X))$

Group E: Axiom of Choice

1. $\exists F(Fn(F) \wedge \forall x(\neg Em(x) \rightarrow \exists y(y \in x \wedge \langle x, y\rangle \in F))$

Differently from ZFC, one can immediately notice that the list of axioms of NBG is finite. Indeed, the reason why ZFC is not finitely axiomatizable is that both replacement and separation are axioms schemas and not axioms.

3 The Class Existence Theorem

Interestingly, in NBG one can prove the class comprehension scheme for a limited class of formulas, i.e. for formulas in which only set variables are quantified. Most specifically, this class comprehension principle is provably equivalent to the axioms of Group B and it often replaces them in the very axiomatization of NBG (cp. Jech). However, we decided to list the axioms of group B to make clear the finitely axiomatizability of NBG. We now define:

Definition 3.1. We say that a formula $\varphi(x_1, ..., x_n, Y_1, ..., Y_m)$ whose variables occur among $x_1, ..., x_n, Y_1, ..., Y_m$ is a *predicative formula* if only set variables are quantified in φ.

We can now state and prove the Class-Existence Theorem:

Theorem 3.1 (Class Existence Theorem)**.** Let $\varphi(x_1, ..., x_n, Y_1, ..., Y_m)$ be a predicative formula, then:

$$\vdash \exists Z \forall X_1, ..., \forall X_n(\langle x_1 ... x_n\rangle \in Z \leftrightarrow \varphi(x_1, ..., x_n, Y_1, ..., Y_m))$$

Proof. We prove this theorem by induction on the number k of connectives and quantifiers in φ. First notice the following:

- We can consider formulas in which no subformula of the form $Y_i \in W$ occurs, since every instance of this kind can be replaced by $\exists x(x = Y_i \wedge x \in W)$
- Moreover, we can also assume that φ does not contain subformulas of the form $X \in X$

We limit ourselves to outline the general strategy of the proof. The *base case* for $k = o$ has three cases: when $\varphi = x_i \in x_j$, when $\varphi = x_j \in x_i$ and when $\varphi = x_i \in Y_l$. To prove the theorem it is then sufficient to apply intelligently the axioms of group B. The *induction step* splits in three cases too: when $\varphi = \neg\psi$, when $\varphi = \psi \to \sigma$ and when $\varphi = \forall x \psi$. The proofs for these cases then follow by applying the appropriate axioms of group B. □

4 NBG is a conservative extension of ZFC

First, we define:

Definition 4.1. We say that a theory T_2 is a *conservative extension* of a theory T_1 if the language of T_2 extends the language of T_1 and, for every statement in the language of T_1 we have:

$$T_1 \vdash \varphi \Leftrightarrow T_2 \vdash \varphi$$

It has been proved that NBG is a conservative extension of ZFC. Therefore we have, for every statement φ about sets, that:

$$\text{ZFC} \vdash \varphi \Leftrightarrow \text{NBG} \vdash \varphi$$

This result is interesting for what concerns the issue of the relative consistency of these theories. Suppose that either ZFC or NBG is inconsistent, this means it derives $\perp$ and thus that also the other derives $\perp$. So from the former theorem we then have:

$$\text{ZFC is consistent} \Leftrightarrow \text{NBG is consistent}$$

5 MK and its Axioms

The class theory MK is axiomatized in the same language of NBG, of which it is an extension. Indeed, the axioms of MK are the very same axioms of NBG, but we replace the class axioms of the group B with a new axioms schema:

$$(\star) \;\; \exists Y \forall x (x \in Y \leftrightarrow \varphi(x))$$

where φ is any formula of MK/NBG and Y is not free in it. We thus have as an axiom of MK the full axiom of class comprehension, for the formula $\varphi(x)$ does not have to be a predicative formula as in the case of NBG. It is also worth remarking that we need to extend in a similar way the axiom of replacement, by allowing a function F to quantify also over classes and not over sets only. Finally, notice that since we replaced the axiom of replacement and the axioms of group B of NBG with axiom schemes, then MK is clearly not finitely axiomatizable.

It is easy then to show that one can prove all the axioms of the group B of NBG from the new axiom $(\star)$. Therefore, we have that for every formula φ of the language of NBG/MK:

$$\text{NBG} \vdash \varphi \Rightarrow \text{MK} \vdash \varphi$$

However, one can show that the converse does not hold. Indeed, as was first shown by Mostowski, one can prove in MK that NBG is consistent, which by Gödel's second incompleteness theorem cannot be proved in NBG. So we have:

$$\text{MK} \vdash \varphi \not\Rightarrow \text{NBG} \vdash \varphi$$

Thus, from the two previous results, it then follows that MK is a proper extension of NBG.