Alternative Set Theories – Report

Nuno Maia, Uva-ID: 11840072

The presentation adressed the work developed by Graham Priest regarding (1) the philosophical motivations for a Paraconsistent Set Theory and (2) the techniques to build paraconsistent models using Priest's *Logic of Paradox* (LP).

1

Intuitively, the axioms of naïve set theory

(Comprehension:) $\exists y \forall x (x \in y \leftrightarrow \phi(x))$ (where '$\phi(x)$' denotes a condition on x)
(Extensionality:) $\forall z((z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$

formalize what it seems to be our intuitive understanding of the notion of 'set', being in accordance with Cantor's early intuitons on the subject:

> By an "aggregate", we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought' [1]

Further, the axioms seem to guarantee (1) *existence:* there is a set y of all and only those x such that $\phi(x)$; (2) *objectivity:* y is an object (in particular, it may be a subject of predication); (3) *identity criteria* for individuating different sets. The difficulty with this analysis is that it quickly gives rise to set-theoretical antionomies and, in particular, Russell's Paradox: by letting $\phi(x) := x \notin x$ and by introducing the following instance of (**Comprehension**) – $\forall x(x \in R \leftrightarrow x \notin x)$ – we quickly have $R \in R \leftrightarrow R \notin R$, from where it follows $R \in R \wedge R \notin R$.

According to the paraconsistent proposal, the naïve axioms can only be retained by accepting contradictions in our underlying logic – urging a different notion of logical consequence such that it does not validate *ex contradictione quodlibet* ($A, \neg A \models B$), in order to avoid explosion in triviality. Historically, since the paraconsistent approach never gain much avowal, two different responses have been proposed:

- Deny the Law of Excluded Middle (LEM): it is suggested that the paradox depends on an instance of the LEM (namely, that either R is a member or is not a member of itself) and, therefore, it could be blocked by denying this principle . However:
  - Russell's paradox can be recreated without LEM: given the way R is constructed we have $R \in R \leftrightarrow R \notin R$ and so, $R \in R \rightarrow R \notin R$; we also know $R \in R \rightarrow R \in R$, from where it follows $R \in R \rightarrow R \in R \wedge R \notin R$. By Non-Contradiction we have $\neg(R \in R \wedge R \notin R)$ and, by Modus Tollens, $R \notin R$. Also, since $R \notin R \rightarrow R \in R$ and $R \notin R \rightarrow R \notin R$ (by similar reasoning) we have $R \in R$. Hence, $R \in R \wedge R \notin R$.
  - There are other set-theoretical paradoxes that do not make essential use of LEM; e.g. Mirimanoff's[2]; Cantor's paradox[3].
- Deny Bivalence: Weak Bochvar-Kleene semantics is an extension of two-valued classical truth-tables, introducing a third truth-value normally interpreted as 'indeterminate'/'paradoxical'. Here, when the atoms of a compound formula take only classical values, the valuation of the entire formula is also defined classicaly; but, when an atom of a compound formula takes the value 'paradoxical' the entire formula is rendered 'paradoxical' as well. In our case, the formula $R \in R \wedge R \notin R$ would not produce an explicit contradiction since '$R \in R$' is allowed to take an indeterminate value, making the all formula indeterminate rather than false. However,

[1] Cantor, G., 1895, *Beiträge zur Begründung der transfiniten Mengenlehre*, "mathematische Annalen", 46; 481-512, tr, in *Contributions to the Founding of the Theory of Transfinite Numbers*, New York: Dover 1955.

[2] Call a sequence of $x_0, x_1, x_2, ...$ such that $... \in x_n \in ... \in x_2 \in x_1 \in x_0$ a 'regress from x_0'. Call a set x well-founded if there is no infinite regress from x. Now, for $W = \{w \mid w \text{ is well-founded }\}$ we have that (1) since all member of W are well-founded, W is also well-founded which implies $W \in W$; (2) then, there is an infinite regress from W of the form $... \in W \in ... \in W \in W \in W$ which implies $W \notin W$

[3] By Cantor's Theorem, $P(V)$ is bigger than V – but V is supposed to be the most inclusive of all sets; contradiction!

- it is unclear how this third-value is exactly to be understood and its introduction seems rather *ad hoc*;
- Though we might still want to keep the tautological character of, for example, $L \vee \neg L$ (even though L may be neither true nor false), this semantics does not allow to express tautologies. To circunvent this difficult an assertion operator (A) was introduced to force bivalence: the compound formula $A\alpha$ (read as 'it is true that α') is true when α is true and false otherwise (that is, when α takes the value 'false' or 'paradoxical'). As Church noticed[4], by introducing the A operator, Russell's Paradox can be restored: let $\phi(x) := \neg A(x \in x)$ and consider the following instance of (**Comprehension**) – $\forall x(x \in R^* \leftrightarrow \neg A(x \in x))$ – again, we quickly have $R^* \in R^* \leftrightarrow \neg(R^* \in R^*)$, from where it follows $R^* \in R^* \wedge \neg(R^* \in R^*)$. From the way R^* is constructed, the latter formula is equivalent to $\neg A(R^* \in R^*) \wedge \neg\neg A(R^* \in R^*)$, that is, $\neg A(R^* \in R^*) \wedge A(R^* \in R^*)$ – in Bochvar-Kleene this formula will always be false.

Since these strategies do yield unnwanted results (in particular, Russell's Paradox can be rebuilt even denying LEM or Bivalence), ZF proposes to apply restrictions directly with respect to the nature of sets (*i.e.* existence is denied to $V, R, W, ...$). According to ZF, the domain of the totality of objects in the theory (namely V) is not an object of the theory on pain of allowing inconsistent sets such as R. Existence is denied to V since:

> Every set M possesses at least one subset M_0 that is not an element of M. [5]

for suppose that this was not the case. Then, consider the set $M_=0$ of all elements of M which are not in themselves. Since $M_0 \subset M$ it follows, by assumption, $M_0 \in M$. If $M_0 \in M_0$ then, from the way M_0 is constructed, we have $M_0 \notin M_0$. So assume $M_0 \notin M_0$. Again, from the way M_0 is constructed, we have $M_0 \in M_0$. Contradiction! It follows

> not all objects x of the domain D can be elements of one and the same set; that is, the domain D is not itself a set, and that disposes of the Russell antinomy so far we are concerned. [6]

Following Priest[7], the problem with this proposal is that it goes against 'Cantor's Domain Principle (CDP)':

> **(CDP:)** Each variable pressuposes the existence of its domain of variation [8]

I.e. for a sentence containg a variable to have a determinate meaning, the range of the quantifiers governing the variable must be a determinate totality. However, in ZF variables range over *all* sets; according to (CDP), this entails that ZF pressuposes the existence of a domain of variation that corresponds to the extension of the term 'set'. However, this set is just V whose existence is denied in ZF. This means that the intelligibility of ZF presuposes the existence of a set whose existence must be denied on pain of contradiction:

> Now, what is a set? It cannot be the extension of "( ) is set", since this extension would be a universal set [...] but there is none. So in standard set theory there is no set/extension corresponding to our usage of "( ) is a set". [...] Standard set theory is using a fundamental notion that cannot be explained by this theory! Or uses a fundamental notion that is incoherent given that very theory! [9]

---

[4]Church, A., 1939, *Review of Bochvar (1939)*, "Journal of Symbolic Logic", 4; 98-9.

[5]Zermelo, E., 1908, *Untersuchungen über die Grundlagen der Mengenlehre I*, "Mathematische Annalen", 65: 261-81, tr. *Investigations on the Foundations of Set Theory I*, in van Heijenoort, J. 1967, ed. *From Frege to Gödel. A source Book in Mathematical Logic*, Harvard: Harvard University Press, pp.199 -215.

[6]Ibid. idem.

[7]Priest, G. 2006, *In Contradiction: A Study of the Transconsistent*, Dordrecht: Martinus Nijhoff, Oxford: Oxford University Press.

[8]Cantor's early formulation states that:

> In order for there to be a variable quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the study would collapse.

quoted *in* Hallet, M., 1984, *Cantorian Set Theory and Limitation of Size*, Oxford: Clarendon Press.

[9]Bremer, B., *An Introduction to Paraconsistent Logics*, Frankfurt a.M.: Peter Lang.

Priest also notes that both set-theoretical and semantical paradoxes behave according to the same schema (already discovered by Russel[10]): Given a property ϕ and a function δ, consider the following conditions

(1) $\Omega = \{y \mid \phi(y)\}$ exists.
(2) if X is a subset of Ω: (a) $\delta(X) \notin X$ and (b) $\delta(X) \in \Omega$

given these conditions, we have a contradiction, when we let $X = \Omega$ and we have $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$. As an example:

Russell's Paradox: Let $\Omega = \{x \mid \phi(x)\}$ where $\phi(x) :=$'$x \notin x$'. Further , δ is the identity function. Now, for $X \subseteq \Omega$, it follows:

1) $\delta(X) = X \notin X$ [for suppose otherwise – then, $X \in X$ which implies, by the the fact that every element of X does not belong to itself, that $X \notin X$ (Contradiction!)];

2) $\delta(X) = X \in \Omega$ [from (1), because we have that $X \notin X$ and that Ω is the set of elements that do not belong to themselves].

Then, when $X = \Omega$ it follows $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ – Contradiction!

Tarski's Liar Sentence:
Let $\Omega := \{\langle x\rangle \mid \phi(x)\}$ where $\phi(x) :=$ '$T(\langle x\rangle)$' (that is, the formula expressed by x is true). Further, assume $\delta : A \mapsto \alpha$ where $A \subseteq \Omega$ and $\alpha = \langle \alpha \notin A\rangle$, that is, α means 'This sentence is not in A'. Now, for $X \subseteq \Omega$, and for $\delta(X) = x = \langle x \notin X\rangle$ it follows:

1) $\delta(X) \notin X$ [for suppose otherwise – then

1*) $\delta(X) \in X$
2*) $\langle x \notin X\rangle \in X$ **(By def. of $\delta(x)$)**
3*) $\langle x \notin X\rangle \in \Omega$ **(Since $X \subseteq \Omega$)**
4*) $T(\langle x \notin X\rangle)$ **(By the construction of Ω)**
5*) $x \notin X$ **(By the fact that $\langle x \notin X\rangle$ is true, from 4*)**
6*) $\langle x \notin X\rangle \notin X$ **(By the definition of x, that is, $x = \langle x \notin X\rangle$)**
7*) $\delta(X) \notin X$ **(By the definition of $\delta(X)$)**
8*) $\perp$ **(Since (7*) contradicts the assumption in (1*))**];

2) By (1) we know $\delta(X) \notin X$, from where it follows $\langle x \notin X\rangle \notin X$ (by the definition of $\delta(X)$). Further, since $\langle x \notin X\rangle$ means that 'This sentence is not in X' and this is true (because it is indeed the case that it is not in X, that is, $\langle x \notin X\rangle \notin X$), we have that $\langle x \notin X\rangle \in \Omega$. Then, $\delta(X) \in \Omega$.

Then, when $X = \Omega$ it follows $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ – Contradiction!

The possibility of providing a common structure for both set-theoretic and semantical paradox sugests, it is argued, that they also have a common solution – being this solution the acceptance of true contradictions (motivating the adoption of a paraconsistent set-theory). Again, Priest:

> If two paradoxes are of different kinds, it is reasonable to expect them to have different kinds of solution; on the other hand, if two paradoxes are of the same kind, then it is reasonable to expect them to have the same kind of solution. [...] The only satisfactory *uniform* approach to all these paradoxes is the dialetheic one, which takes paradoxical contradictions to be exactly what they appear to be. [11]

2

Let LP be a system for first-order logic with the usual connectives $\neg$, $\wedge$, $\forall$ ($\vee$ and $\exists$ are defined in the usual way). We define an interpretation $\mathfrak{M}$ as an ordered pair $\langle D, I\rangle$ where D is a non-empty domain of quantification and I is a function which maps each individual constant, c, into D and each n-ary predicate, P, into a pair $\langle I^+(P), I^-(P)\rangle$ (where $I^+(P)$ is interpreted as the extension of P (the set of objects satisfying P)). Also, we make the restriction that $I^+(P) \cup I^-(P) = D^n$, though it is not necessary that $I^+(P) \cap I^-(P) = \emptyset$. A formula α in the language of $\mathfrak{M}$ is assigned a truth-value $v(\alpha)$, from the set $\{\{1\,\{0\}, \{1, 0\}\}$, by the following recursive clauses:

[10] *Vide* Russell, B. 1903, *The Principles of Mathematics*, Cambridge: Cambridge University Press.
[11] *Op. Cit.* Priest, 2006.

$1 \in v(Pc_1...c_n) \Leftrightarrow \langle I(c_1)...I(c_n)\rangle \in I^+(P)$
$0 \in v(Pc_1...c_n) \Leftrightarrow \langle I(c_1)...I(c_n)\rangle \in I^-(P)$

$1 \in v(\neg\alpha) \Leftrightarrow 0 \in v(\alpha)$
$0 \in v(\neg\alpha) \Leftrightarrow 1 \in v(\alpha)$

$1 \in v(\alpha \wedge \beta) \Leftrightarrow 1 \in v(\alpha)$ and $1 \in v(\beta)$
$0 \in v(\alpha \wedge \beta) \Leftrightarrow 0 \in v(\alpha)$ or $0 \in v(\beta)$

$1 \in v(\forall\alpha) \Leftrightarrow$ for all $d \in D, 1 \in v(\alpha(x/d))$[12]
$0 \in v(\forall\alpha) \Leftrightarrow$ for some $d \in D, 0 \in v(\alpha(x/d))$

We also define semantical consequence as:
$\mathfrak{M}$ is a LP-model for α ($\mathfrak{M} \models \alpha$) iff $1 \in v(\alpha)$
α is an LP-semantical-consequence of σ ($\sigma \models \alpha$) iff every model of σ is a model of α [13]

Collapsed Model: Let $\mathfrak{M} = \langle D, I\rangle$ be an arbitrary interpretation. Let π be an equivalence relation defined on D and for arbitrary $d \in D$ let $[d]$ be the equivalence class of d under π. Then, we call $\mathfrak{M}_\pi = \langle D_\pi, I_\pi\rangle$ is the collapsed model of $\mathfrak{M}$ under π and we define it as follows:

- $D_\pi = \{[d]; d \in D\}$;
- for every constant c, $I_\pi(c) = I[c]$;
- for $a_1, ..., a_n \in D_\pi$:
  - $\langle a_1, ..., a_n\rangle \in I^+_\pi(P)$ iff $\exists x_1 \in a_1, ..., x_n \in a_n, \langle x_1, ..., x_n\rangle \in I^+(P)$;
  - $\langle a_1, ..., a_n\rangle \in I^-_\pi(P)$ iff $\exists x_1 \in a_1, ..., x_n \in a_n, \langle x_1, ..., x_n\rangle \in I^-(P)$

Collapsing Lemma: For every formula α, in the language of $\mathfrak{M}$, $v_\pi(\alpha) \supseteq v(\alpha)$.
Proof: By recursion on the complexity of α.[14]

- **(Atomic Case):** $1 \in v(Pc_1...c_n) \Longrightarrow \langle I(c_1), ..., I(c_n)\rangle \in I^+(P) \Longrightarrow \exists x_1 \in I[(c_1)], ..., x_n \in I[(c_n)]$, $\langle x_1, ..., x_n\rangle \in I^+(P) \Longrightarrow \langle I[(c_1)], ..., I[(c_n)]\rangle \in I^+_\pi(P) \Longrightarrow \langle I_\pi(c_1), ..., I_\pi(c_n)\rangle \in I^+_\pi(P) \Longrightarrow 1 \in v_\pi(Pc_1...c_n)$.
- **($\neg$):** $1 \in v(\neg\alpha) \Longrightarrow 0 \in v(\alpha) \Longrightarrow 0 \in v_\pi(\alpha) \Longrightarrow 1 \in v_\pi(\neg\alpha)$.
- **($\wedge$):** $1 \in v(\alpha \wedge \beta) \Longrightarrow 1 \in v(\alpha)$ and $1 \in v(\beta) \Longrightarrow 1 \in v_\pi(\alpha)$ and $1 \in v_\pi(\beta) \Longrightarrow 1 \in v_\pi(\alpha \wedge \beta)$.
- **($\forall$):** $1 \in v(\forall x\alpha) \Longrightarrow 1 \in v(\alpha(x/d))$ for all $d \in D \Longrightarrow 1 \in v_\pi(\alpha(x/d))$ for all $[d] \in D_\pi \Longrightarrow 1 \in v_\pi(\forall x\alpha)$.

Toy-Model for ZF and the axioms of naïve set theory:
Using the Collapsing Lemma, we can build a model for ZF and for **(Comprehension).** Take $\mathfrak{M} = \langle D, I\rangle$ to be a model for ZF and let α be some ordinal in D. Now, define a to be an object in $\mathfrak{M}$ such that $a = V_\alpha$.
We can build a collapsed model $\mathfrak{M}_\pi$ by defining a partition π under D such that π:

- Puts every member of rank $\leqslant \alpha$ (in $\mathfrak{M}$) in its own singleton;
- Puts all objects of rank $>\alpha$ **(including a)** into a single set.

We observe that since $\mathfrak{M}$ is a model for ZF, the collapsed model $\mathfrak{M}_\pi$ is also a model for ZF, by the Collapsing Lemma. Now we have that, for arbitrary $b \in D$:

- **(1.1)** If (in $\mathfrak{M}$) b is of rank less than α, then $b \in a$ is true in $\mathfrak{M}$; by the Collapsing Lemma, $[b] \in [a]$ is true in $\mathfrak{M}_\pi$;

[12] Where '$\alpha(x/d)$' denotes α with all free occurences of 'x' replaced by 'd'
[13] We observe that in LP $A, \neg A \not\models B$ – consider an interpretation where $\{1, 0\} \in v(A)$ but $0 \in v(B)$ and $1 \notin v(B)$.
[14] We will only prove the truth cases. For details, *Vide* Priest, G., 1991, *Minimally Inconsistent LP*, "Studia Logica", 50, pp. 321-31.

- **(1.2)** If (in $\mathfrak{M}$) b is of rank greater or equal than α, there is some element c that is also of rank greater or equal than α (*e.g.* $c = \{b\}$) such that $b \in c$. Since c is of rank greater than α, we have that $I_\pi(c) = [a]$. Hence, in $\mathfrak{M}_\pi$, it is true that $[b] \in [a]$;

- **(2)** There are elements that do not have rank less than α such that b is not a member of them (e.g. $\{c\}$, where c is distinct from b and has rank greater than α). Since these elements have been identified with $[a]$ in $\mathfrak{M}_\pi$, we have that $[b] \in [a]$ is false.

From (1.1) and (1.2) we have that $\mathfrak{M}_\pi \models [b] \in [a]$ and from (2) we have that $\mathfrak{M}_\pi \not\models [b] \in [a]$. Then, it follows that $v_\pi([a] \in [b]) = \{1, 0\}$ and that $\mathfrak{M}_\pi \models [b] \in [a] \leftrightarrow \phi(x/[b])$ from where we get $\mathfrak{M}_\pi \models \forall x(x \in [a] \leftrightarrow \phi(x))$ and, finally, $\mathfrak{M}_\pi \models \exists y \forall x(x \in y \leftrightarrow \phi(x))$. So $\mathfrak{M}_\pi$ satisfies the axiom of Comprehension and since it is a collapsed model built from a model for ZF, by the Collapsing Lemma, it also satisfies all the theorems that $\mathfrak{M}$ satisfies, including Extensionality. Hence, $\mathfrak{M}_\pi$ is a paraconsistent model for ZF and for the naïve axioms of set theory.