Alternative Set Theories

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Naive Set Theory

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For every φ, the set $\{x \mid \varphi(x)\}$ exists.

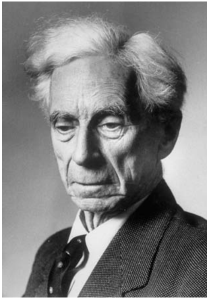
Russell's Paradox

Russell's Paradox

Let $K := \{x \mid x \notin x\}$

Then $K \in K \leftrightarrow K \notin K$

$\lightning$



ZFC

The commonly accepted axiomatization of set theory is ZFC. All results in mainstream mathematics can be formalized in it.

Philosophy of ZFC

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- Everything is a set.
- Sets are constructed out of other sets, bottom up.
- **Comprehension** can only select a subset out of an existing set (avoid paradoxes).
- Certain definable collections $\{x \mid \varphi(x)\}$ are "too large" to be sets.

Logic of ZFC

Logic of ZFC

- Classical predicate logic.
- One-sorted.
- One binary non-logical relation symbol $\in$.
- In this language, ZFC is an infinite (but recursive) collection of axioms.

Other set theories

All these factors are liable to change! Several alternative set theories have been proposed, for a variety of reasons:

- Philosophical (more intuitive conception)
- The need to have proper classes as formal objects (e.g., "class forcing")
- Capturing a fragment of mathematics (e.g., predicative fragment, intuitionistic fragment etc.)
- Application to other fields (e.g., computer science)
- Simply out of curiosity...

Alternative systems

The alternative systems we intend to study are the following:

1. NGB (von Neumann-Gödel-Bernays)
2. MK (Morse-Kelley)
3. NF (New Foundations)
4. KP (Kripke-Platek)
5. IZF/CZF (Intuitionistic and constructive set theory)
6. $ZF^- + AFA$ (Antifoundation)
7. Modal or other set theory?

von Neumann-Gödel-Bernays (NBG)

NGB (von Neumann-Gödel-Bernays)

- We have **sets** and **classes**; some classes are sets, others are not.
- It can be formalized either in a **two-sorted language** or using a predicate $M(X)$ stating "X is a set".
- You still need **set existence** axioms, along with **class existence** axioms.
- NGB can be **finitely axiomatized**.

Axiomatization of NGB

Axiomatization of NGB

- Set axioms:
  - Pairing
  - Infinity
  - Union
  - Power set
  - Replacement
- Class axioms:
  - Extensionality
  - Foundation
  - **Class comprehension schema** for φ which quantify only over sets:

$$C := \{x \mid \varphi(x)\} \text{ is a class}$$

- Global Choice.

Class comprehension vs. finite axiomatization

As stated above, class comprehension is a schema. However, it can be replaced by finitely many instances thereof (roughly speaking: one axiom for each application of a logical connective/quantifier).

Morse-Kelley (MK)

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Morse-Kelley set theory is a variant of NGB with the class comprehension schema allowing **arbitrary** formulas (also those that quantify over classes).

MK is **not** finitely axiomatizable.

Consistency strength

- NGB is a **conservative extension** of ZFC: for any theorem φ involving **only sets**, if NGB $\vdash \varphi$ then ZFC $\vdash \varphi$. In particular, if ZFC is consistent then NGB is consistent.
- MK $\vdash$ Con(ZFC), and therefore the consistency of MK does not follow from the consistency of ZFC.
- If κ is inaccessible, then $(V_\kappa, \mathrm{Def}(V_\kappa)) \models$ NGB while $(V_\kappa, \mathcal{P}(V_\kappa)) \models$ MK.
- The consistency strength of MK is strictly between ZFC and ZFC + Inaccessible.

Kripke-Platek (KP)

Kripke-Platek set theory (KP)

Captures a small part of mathematics — stronger than 2nd order arithmetic but noticeably weaker than ZF.

Idea: get rid of "impredicative" axioms of ZFC: in particular Power Set, (full) Separation and (full) Replacement.

Instead, have Separation and Replacement for Δ_0-formulas only.

Applications of KP

KP has applications in many standard areas of set theory as well as recursion theory and constructibility.

One example: KP is sufficient to develop the theory of **Gödel's Constructible Universe** L.

L is not only the **minimal** model of ZFC, but also the minimal model of KP (this is because "$V = L$" is absolute for KP).

Quine's New Foundations (NF)

Quine's New Foundations.

The idea is: avoid Russell's paradox by a **syntactical limitation** on φ in the comprehension scheme $\{x \mid \varphi(x)\}$.

NF has its roots in **type theory**, as it was first developed in *Principia Mathematica*.

Stratified sentences

A sentence ϕ in the language of set theory (only $=$ and $\in$ symbols) is **stratified** if it is possible to assign a non-negative integer to each variable x occurring in ϕ, called the **type of** x, in such a way that:

1. Each variable has the same type whenever it appears,
2. In each occurrence of "$x = y$" in ϕ, the types of x and y are the same, and
3. In each occurrence of "$x \in y$" in ϕ, the type of y is one higher than the type of x.

Example: $x = x$ is stratified. $x \in y$ is stratified. $x \notin x$ is not stratified.

Axiomatization of NF

Axiomatization of NF

- Extensionality,
- Stratified comprehension scheme:

$$\{x \mid \varphi(x)\} \text{ exists}$$

for every **stratified** formula φ.

Finite axiomatization of NF

Alternatively, we may replace the stratified comprehension scheme by **finitely many instances** thereof, each having intuitive motivation:

- The empty set exists: $\{x \mid \bot\}$
- The singleton set exists: $\{x \mid x = y\}$
- The union of a set a exists: $\{x \mid \exists y \in a\,(x \in y)\}$
- ...

as well as other "non-ZFC-ish" axioms, e.g.:

- The universe exists: $\{x \mid \top\}$
- The compelement of A exists: $\{x \mid x \notin A\}$
- ...

The full stratified comprehension scheme is a **consequence** of these instances.

Properties of NF

NF is weird!

- $V := \{x \mid \top\}$, the universe of all sets, exists, and $V \in V$. Therefore:
$$\cdots \in V \in V \in V.$$
- NF $\vdash \neg$AC.
- Therefore: NF $\vdash$ Infinity.
- The consistency of NF was an **open problem** since 1937 till (about) 2010.

NFU

Ronald Jensen considered a weakening of NF called NFU (New Foundations with Urelements), weakening Extensionality to

$$\forall x \forall y\, (x \neq \varnothing \wedge y \neq \varnothing \wedge \forall z\, (z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y)$$

NFU was known to be consistent for a long time, NFU $\nvdash \neg$AC and NFU $\nvdash$ Infinity. So NFU$+$ Infinity $+$AC is consistent.

Non-well-founded set theory

Aczel's non-well-founded set theory

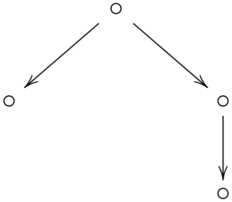
We already saw that $V \in V$ holds in NF.

Peter Aczel considered the question: "how non-well-founded can set theory be"? In other words, is it consistent that **any kind of** non-well-founded set that you can think of, exists?

This leads to the **Anti-Foundation Axiom** AFA.

Pictures of sets

Idea: **sets** can be pictured using **graphs**, with "$x \to y$" representing "$y \in x$", e.g.:



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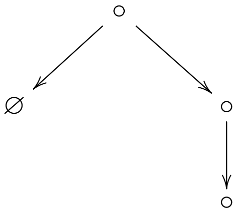

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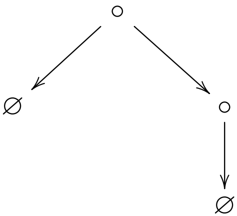

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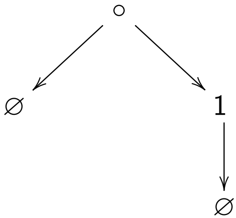

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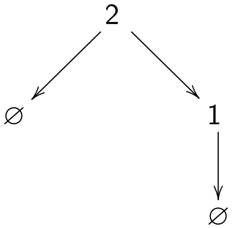

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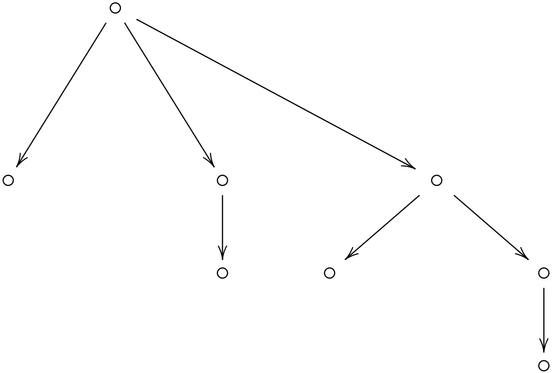

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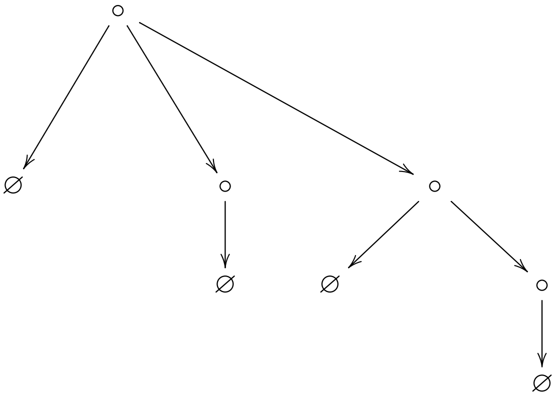

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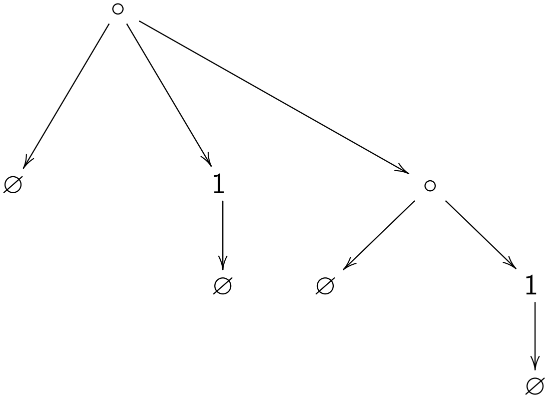

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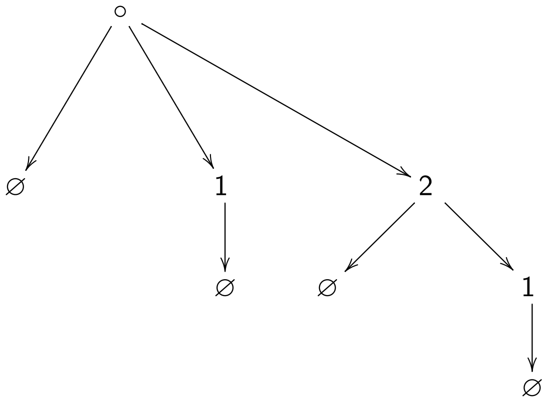

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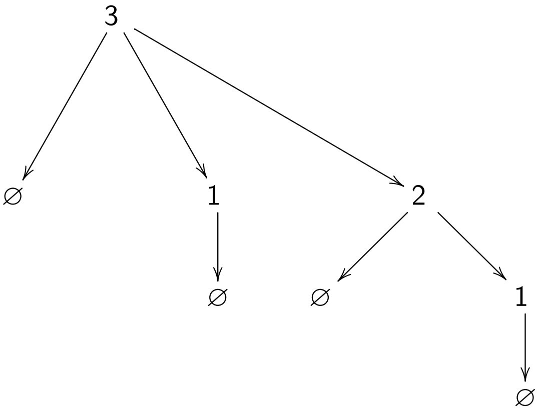

Pictures of sets

Every graph **without infinite paths or cycles** corresponds to a unique set (Mostowski Collapse).

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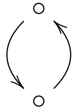

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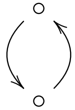

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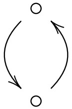

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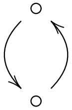

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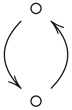

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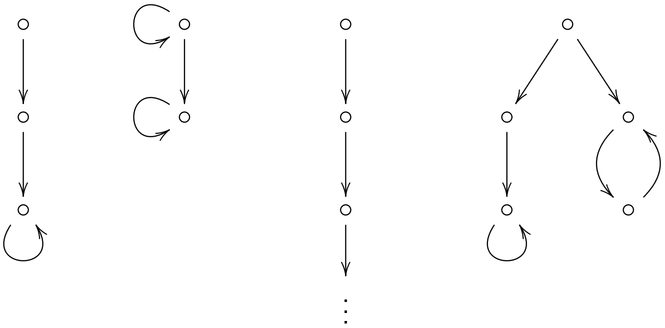
More pictures of Ω

More pictures of the set Ω:



AFA

The anti-foundation axiom AFA is (roughly speaking):

> *Every connected pointed graph represents a unique set.*

AFA is consistent with ZF − Foundation.

Intuitionistic/Constructive Set Theory

IZF (the simplest variant)

Suppose we take all the ZFC axioms exactly as they are, but change the logic from **classical logic** to **intuitionistic logic**? I.e., ϕ is a theorem of the system iff ZFC $\vdash \phi$ in intuitionistic predicate logic.

Axiom of Choice

Let ϕ be some formula. Consider:

$$A := \{n \in \{0,1\} \mid n = 0 \lor (n = 1 \land \phi)\}$$

$$B := \{n \in \{0,1\} \mid n = 1 \lor (n = 0 \land \phi)\}$$

Since A and B are non-empty, by AC, there is a choice function $f : \{A, B\} \to \{0,1\}$.

Since **equality of natural numbers is intuitionistically decidable**, $f(A) = f(B)$ or $f(A) \neq f(B)$.

If $f(A) = f(B) = 0$ then $0 \in B$, hence ϕ. Similarly if $f(A) = f(B) = 1$.

Suppose $f(A) \neq f(B)$. Towards contradiction, suppose ϕ. Then, **by extentionality**, $A = B$, hence $f(A) = f(B)$: contradiction! Hence $\neg\phi$.

(Recall that $(\phi \to \bot) \to \neg\phi$ is valid, only $(\neg\phi \to \bot) \to \phi$ is invalid).

Axiom of Choice

Thus AC (formulated as above) implies $\phi \vee \neg\phi$ for any formula ϕ. We have obtained the **Law of Excluded Middle**, thus the ZFC axioms with intuitionistic logic is just normal ZFC.

Foundation

You may think AC is inherently non-constructive, and thus shouldn't even be taken into account, etc.

But consider **Foundation**: $\forall X\, (\exists y \in X \;\rightarrow\; \exists y \in X\, \forall z \in X\, (z \notin y))$.

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A is non-empty so there is a $y \in A$ which is $\in$-minimal. By definition of A, either $y = 1$ or $y = 0 \wedge \phi$. But the former case implies that $0 \notin A$, hence $\neg\phi$.

Hence, in either case, $\phi \vee \neg\phi$.

Again we have proved the Law of Excluded Middle (from a seemingly harmless statement).

Set Induction

Set Induction is the following axiom schema for all ϕ:

$$\forall x\, [(\forall y \in x\, \phi(y)) \to \phi(x)] \quad \to \quad \forall x\, \phi(x)$$

This principle does **not** imply Excluded Middle.

Likewise, there are variants of AC which do not imply Excluded Middle and are compatible with an intuitionistic system.

IZF

IZF is the theory consisting of the ZF axioms without Choice, but with Foundation replaced by Set Induction, and Replacement by a stronger principle called Collection.

Other theories, most notably **CZF**, implements other changes as well (mostly based on conceptual/philosophical justifications).

Other Logics

Other logics

Set Theories based on other logics?

Modal Set Theory would be some form of ZF or ZFC done in some form of **modal predicate logic**. There is no universal consensus on what should count as modal predicate logic — let alone for modal ZF or ZFC. Therefore this is a highly experimental subject.

Robert Passmann is currently writing his Master Thesis on modal set theory, so I hope he can tell us something about it!

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Paraconsistent set theory?

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