Forcing and the structure of the real line: the Bogotá lectures

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Abstract

These are the lecture notes of a minicourse on the interplay between forcing theory and combinatorial properties of the real line, held at Universidad de los Andes (Bogotá) from January 29 till February 4, 2009 (first five lectures). The sixth lecture was done at Universidad Nacional (Bogotá) on February 9, and repeated at Instituto Venezolano de Investigaciones Científicas (Caracas) on February 19. Basic knowledge of forcing theory and iterated forcing, as expounded in [Ku1, Chapters VII and VIII] and [Je, Sections 14 to 16], were a prerequisite for the course.

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Lecture 1: Cohen Forcing

1.1. Set theory of the reals. *Set theory of the reals* investigates (families of) subsets of the real line from the point of view of combinatorial set theory. There is a close connection with *descriptive set theory* (the study of subsets of the reals from the point of view of complexity). One of the main topics of set theory of the reals are cardinal invariants of the continuum, see below.

For aspects of measure and category in set theory of the reals, see the exhaustive monograph [BJ].

1.2. Cardinal invariants of the continuum. *Cardinal invariants of the continuum* are cardinal numbers which describe the combinatorial structure of the real line. They are usually defined in terms of ideals on the reals, or some structure very closely associated with the reals like $\mathcal{P}(\omega)/\text{fin}$, or ... Typically, they assume values between $\aleph_1$, the first uncountable cardinal, and $\mathfrak{c} = |2^\omega| = |\omega^\omega| = |\mathbb{R}|$, the cardinality of the continuum. So they are uninteresting under the continuum hypothesis $\mathfrak{c} = \aleph_1$. Also, most of the cardinal invariants are equal to $\mathfrak{c}$ under Martin's axiom MA. However, in other models of set theory, they may assume different values, and thus they provide a means for characterizing the structure of the real line in various models. Axioms which are expressed in terms of equalities or inequalities between cardinal invariants play an important role in applications of set theory to other areas of mathematics like general topology or group theory ...

For a treatment of the main cardinal invariants of the continuum, see the survey article [Bl]. (This survey focuses on ZFC-provable inequalities between the cardinals, and only has a short appendix about consistency results.)

1.3. The eventual dominance ordering. For $f, g \in \omega^\omega$, say $f \leq^* g$ (g *eventually dominates* f) if $f(n) \leq g(n)$ holds for all but finitely many n. $\leq^*$ is a preorder (i.e., it is reflexive and transitive), but it is not antisymmetric. We can easily turn it into an antisymmetric ordering by identifying functions which are equal on a tail, and introducing a partial order on the resulting quotient structure: note that $=^*$ is an equivalence relation, let $[f] = \{g \in \omega^\omega : g =^* f\}$ denote the equivalence class of f, and let $[f] \leq [g]$ if $f \leq^* g$; then $(\{[f] : f \in \omega^\omega\}, \leq)$ is a p.o. However, $(\omega^\omega, \leq^*)$ is simpler to work with and we shall usually stick with the latter structure.

A word on notation. Let $\forall^\infty n$ stand for *for all but finitely many* n ($\exists k \forall n \geq k$); we often say *for almost all* n instead. Similarly, $\exists^\infty n$ means *there are infinitely many* n ($\forall k \exists n \geq k$).

A family of functions $\mathcal{F} \subseteq \omega^\omega$ is called *bounded* if it is eventually dominated by a single function, i.e., there is $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in \mathcal{F}$. Otherwise $\mathcal{F}$ is *unbounded*. $\mathcal{F}$ is *dominating* (or *cofinal*) if every g is eventually dominated by a member of $\mathcal{F}$. Clearly, every dominating family is unbounded.

The *(un)bounding number* $\mathfrak{b}$ is the smallest size of an unbounded family, and the *dominating number* $\mathfrak{d}$ is the least cardinality of a dominating family. $\mathfrak{b}$ and $\mathfrak{d}$ are two of the most important cardinal invariants of the continuum.

Proposition 1.1. *$\aleph_1 \leq \mathfrak{b} \leq cf(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$. Furthermore, $\mathfrak{b}$ is a regular cardinal.*

Proof. $\aleph_1 \leq \mathfrak{b}$ is a standard *diagonal argument*: given f_i, $i \in \omega$, the function f defined by $f(n) = \max\{f_i(n) : i \leq n\}$ eventually dominates the f_i.

$cf(\mathfrak{b}) = \mathfrak{b}$: If $\kappa < \mathfrak{b}$ and $\mathcal{F}_\alpha$, $\alpha < \kappa$, are bounded, say by f_α, then $\bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$ is bounded by any function which eventually dominates all the f_α.

$\mathfrak{b} \leq cf(\mathfrak{d})$: Suppose $\kappa < \mathfrak{b}$ and $\mathcal{F}_\alpha$, $\alpha < \kappa$, are not dominating. Then there are functions f_α witnessing this (i.e., f_α is not eventually dominated by any member of $\mathcal{F}_\alpha$). Let f be a function which eventually dominates all the f_α. Then f is not bounded by any member of $\bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$. So $\bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$ is not dominating, and $\mathfrak{b} \leq cf(\mathfrak{d})$ follows.

The rest is trivial. $\square$

1.4. The almost inclusion ordering. For $A, B \in \mathcal{P}(\omega)$, say $A \subseteq^* B$ (A is *almost contained in* B, A is *almost included in* B) if $A \setminus B$ is finite. Like $\leq^*$, $\subseteq^*$ is a preorder which can be turned into a partial order by going over to equivalence classes of subsets of ω (which we usually won't do). The quotient structure $(\mathcal{P}(\omega)/\text{fin}, \leq)$ where $[A] \leq [B]$ if $A \subseteq^* B$ is one of the important realizations of the real line (see Lecture 6).

Recall $[\omega]^\omega$ is the collection of all infinite subsets of ω. For $A, B \in [\omega]^\omega$, say that A *splits* B if both $A \cap B$ and $B \setminus A$ are infinite. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called a *splitting family* if every infinite subset of ω is split by a member of $\mathcal{A}$. $\mathcal{A}$ is *unsplit* (or *unreaped*) if no single set splits all members of $\mathcal{A}$. This is equivalent to saying that for all $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that either $A \subseteq^* B$ or $A \cap B$ is finite. As for *unbounding* and *dominating*, the concepts of *splitting* and *unsplit* are dual to each other.

The *splitting number* $\mathfrak{s}$ is the least size of a splitting family, and the *(un)reaping number* $\mathfrak{r}$ is the smallest cardinality of an unreaped family.

By $\omega^{\uparrow\omega}$ we denote the collection of all *strictly increasing functions* $f \in \omega^\omega$ (i.e., $f(n) < f(n+1)$ for all n) such that $f(0) > 0$. Notice that these two conditions imply that $f(n) > n$ for all n.

Theorem 1.2. *1. $\aleph_1 \leq \mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{c}$.*

2. In fact, there are functions $\Phi : [\omega]^\omega \to \omega^{\uparrow\omega}$ and $\Psi : \omega^{\uparrow\omega} \to [\omega]^\omega$ such that whenever $\Phi(A) \leq^* f$ then $\Psi(f)$ splits A.

Proof. (1) $\aleph_1 \leq \mathfrak{s}$ is a diagonal argument, and $\mathfrak{r} \leq \mathfrak{c}$ is obvious.

To see $\mathfrak{s} \leq \mathfrak{d}$, let $\mathcal{F} \subseteq \omega^{\uparrow\omega}$ be a dominating family, and use (2) to argue that $\{\Psi(f) : f \in \mathcal{F}\}$ is a splitting family.

Similarly, if $\mathcal{A}$ is an unreaped family, $\{\Phi(A) : A \in \mathcal{A}\}$ must be unbounded by (2), and $\mathfrak{b} \leq \mathfrak{r}$ follows.

(2) For $A \in [\omega]^\omega$ and $n \in \omega$, let

$$\Phi(A)(n) = \min(A \setminus n) + 1,$$

that is, $\Phi(A)(n)$ is the least element of A which is $\geq n$. In particular, $[n, \Phi(A)(n)) \cap A \neq \emptyset$ for all n. For $f \in \omega^{\uparrow\omega}$, define recursively $f^k(0)$ by $f^0(0) = 0$, $f^1(0) = f(0)$, and $f^{k+1}(0) = f(f^k(0))$. Then let

$$\Psi(f) = \bigcup_k [f^{2k}(0), f^{2k+1}(0)).$$

Assume $\Phi(A) \leq^* f$. Then, for almost all k, $\Phi(A)(f^k(0)) \leq f^{k+1}(0)$, and thus A has nontrivial intersection with almost all intervals $[f^k(0), f^{k+1}(0))$. Hence $A \cap \Psi(f)$ and $A \setminus \Psi(f)$ are both infinite, and $\Psi(f)$ splits A as required. $\square$

Note that the functions Φ and Ψ in the above proof are even continuous. Proofs on the order-relationship of cardinal invariants are often done by exhibiting such functions.

The cardinals we have defined so far may be displayed in the following diagram (part of *Van Douwen's diagram*).

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{c} \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathfrak{r} \arrow[dr, dash] & & \mathfrak{d} \arrow[dl, dash] \arrow[dr, dash] & \\
 & \mathfrak{b} \arrow[dr, dash] & & \mathfrak{s} \arrow[dl, dash] \\
 & & \aleph_1 &
\end{tikzcd}$$

1.5. Cohen forcing as forcing with finite partial functions. Let I be an infinite index set. For any set J, let $\mathsf{Fn}(I, J)$ denote the collection of finite partial functions from I to J. It is ordered by reverse inclusion, i.e., $s \leq t$ iff $s \supseteq t$. $\mathbb{C}_I = \mathsf{Fn}(I, 2)$ is called *Cohen forcing*. It is a ccc forcing notion. In case $I = \omega$, we write $\mathbb{C}$ instead of $\mathbb{C}_\omega$. $\mathbb{C}_I$ generically adds a new function $c_I \in 2^I$ given by $c_I = \bigcup\{s \in \mathbb{C}_I : s \in G\}$ where G denotes the $\mathbb{C}_I$-generic filter over the ground model V. c_I is called a *Cohen function* and, in case $I = \omega$, $c = c_\omega$ is a *Cohen real*. It is well-known (and easy to see) that G can be reconstructed from c_I, that is:

Observation 1.3. $V[c_I] = V[G]$.

Proof. For the purposes of this proof, let $c_G = \bigcup\{s \in \mathbb{C}_I : s \in G\}$ whenever G is a filter. c_G is a (not necessarily total) function from I to 2. For such a function d, let $H_d = \{s \in \mathbb{C}_I : s \subseteq d\}$. H_d is a filter. Obviously $G \subseteq H_{c_G}$ and $d = c_{H_d}$. To see $H_{c_G} \subseteq G$, let $s \in H_{c_G}$. Thus $s \subseteq c_G$ and there must be $t_i \in G$, $i < n$, such that $s \subseteq \bigcup_i t_i$. Since G is a filter, $\bigcup_i t_i \in G$ and $s \in G$ follow.

Since $V[H_{c_G}] \subseteq V[c_G] \subseteq V[G]$, we obtain that the models are equal. $\square$

A word on forcing notation. We will be sloppy with our forcing notation. E.g., we will in general not distinguish the forcing extension $V[G]$ and the Boolean-valued model $V^{\mathbb{C}_I}$. Also we usually think of forcing as adding a new real or a new function rather than a generic filter. Since $V[c_I] = V[G]$, this is justified.

Lemma 1.4. *Any countable forcing notion is forcing equivalent to $\mathbb{C}$. In particular, $\mathsf{Fn}(\omega,\omega)$ is equivalent to $\mathbb{C}$.*

See [Ku1, VII Exercise (C4)].

Lemma 1.5. *Let $\kappa < |I|$ and let $\dot f$ be a $\mathbb{C}_I$-name for a function with domain κ and values in V. Then there is $J \subseteq I$ of size at most κ such that $\dot f$ is (equivalent to) a $\mathbb{C}_J$-name.*

Proof. For each $\alpha < \kappa$, let A_α be a maximal antichain in $\mathbb{C}_I$ consisting of conditions which decide the value of $\dot f(\alpha)$. By the ccc, the set $J = \bigcup\{\mathrm{dom}(s) : s \in \bigcup_{\alpha<\kappa} A_\alpha\}$ has size at most κ, and it is clear that $\dot f$ can be construed as a $\mathbb{C}_J$-name. $\square$

Lemma 1.6. (product lemma) *Let J_0 and J_1 be disjoint such that $I = J_0 \cup J_1$.*

1. *If c_I is $\mathbb{C}_I$-generic over V, $c_I{\restriction}J_i$ is $\mathbb{C}_{J_i}$-generic over both V and $V[c_I{\restriction}J_{1-i}]$.*
2. *If c_{J_0} is $\mathbb{C}_{J_0}$-generic over V and c_{J_1} is $\mathbb{C}_{J_1}$-generic over $V[c_{J_0}]$, then $c_{J_0} \cup c_{J_1}$ is $\mathbb{C}_I$-generic over V.*

This is a consequence of the product lemma for forcing, [Ku1, VIII Theorem 1.4]. We include a proof for the sake of completeness.

Proof. 1. Genericity over V is trivial. To see genericity of $c_I{\restriction}J_1$ over $V[c_I{\restriction}J_0]$, let $D \in V[c_I{\restriction}J_0]$ be dense in $\mathbb{C}_{J_1}$. Let $\dot D$ be a $\mathbb{C}_{J_0}$-name for D, and put $E = \{s \in \mathbb{C}_I : s{\restriction}J_0 \Vdash s{\restriction}J_1 \in \dot D\}$. E belongs to V and it is easy to see that it is dense. Since c_I is generic, $s \subseteq c_I$ for some $s \in E$. This means that $s{\restriction}J_1 \in D$ in $V[c_I{\restriction}J_0]$. Hence $c_I{\restriction}J_1$ is indeed generic over $V[c_I{\restriction}J_0]$.

2. Let $D \in V$ be dense in $\mathbb{C}_I$. We claim that, in $V[c_{J_0}]$, $E = \{s \in \mathbb{C}_{J_1} : \exists t \in D$ such that $s = t{\restriction}J_1$ and $t{\restriction}J_0 \subseteq c_{J_0}\}$ is dense. To see this let $s \in \mathbb{C}_{J_1}$. Note that $F = \{u \in \mathbb{C}_{J_0} : u \cup v \in D$ for some $v = v_u \in \mathbb{C}_{J_1}$ with $v \le s\}$ is dense in V. By genericity, $u \subseteq c_{J_0}$ for some $u \in F$. Then $t = u \cup v_u$ witnesses that $v_u \in E$, as required.

By genericity over $V[c_{J_0}]$, $s \subseteq c_{J_1}$ for some $s \in E$. Thus there is $t \in D$ such that $s = t{\restriction}J_1$ and $t{\restriction}J_0 \subseteq c_{J_0}$. This implies $t \subseteq c_{J_0} \cup c_{J_1}$, and genericity of $c_{J_0} \cup c_{J_1}$ is proved. $\square$

1.6. The effect of Cohen forcing on $(\omega^\omega, \le^*)$ and $([\omega]^\omega, \subseteq^*)$. Any investigation of the effect of adding a real on a structure S like $(\omega^\omega, \le^*)$ comes in two parts: the *easier part* is always to show that the generic real (or some real defined in terms of the generic real) has a certain property with respect to S. The *harder part* is to prove that reals with certain other properties with respect to S do not get adjoined by the forcing; this is the contents of so-called *preservation theorems.*

1.6.1. Properties of the generic Cohen real.

Lemma 1.7. *A Cohen real $c \in \omega^\omega$ is unbounded over the ground model reals. I.e., for any $f \in \omega^\omega \cap V$, $c \not\leq^* f$.*

Proof. As hinted at by the formulation of the lemma, we think of $\mathbb{C}$ as forcing with $\omega^{<\omega}$. This is justified by Lemma 1.4. For $f \in \omega^\omega \cap V$ and $n \in \omega$, let $D_{f,n} = \{t \in \omega^{<\omega} : \exists m \geq n \;\; f(m) < t(m)\}$. Once we establish all $D_{f,n} \in V$ are dense, a straightforward genericity argument yields the lemma.

To see $D_{f,n}$ is dense, take any $s \in \omega^{<\omega}$, choose $m \geq \max\{n, |s|\}$, and extend s to $t \in \omega^{<\omega}$ such that $|t| = m + 1$ and $t(m) = f(m) + 1$. Then $t \in D_{f,n}$. $\square$

Lemma 1.8. *A Cohen real $C = \{n : c(n) = 1\} = c^{-1}(1) \in [\omega]^\omega$ splits all ground model infinite subsets of ω.*

Proof. This is similar, and details are left as an exercise.

Think of $\mathbb{C}$ as forcing with $2^{<\omega}$. For $A \in [\omega]^\omega \cap V$ and $n \in \omega$, let $E_{A,n} = \{t \in 2^\omega : \exists m_0, m_1 \in A \setminus n \;\; (t(m_0) = 0 \text{ and } t(m_1) = 1)\}$, and show that the $E_{A,n}$ are dense. $\square$

1.6.2. Preservation results.

Main Lemma 1.9. *Let $\dot{f}$ be a $\mathbb{C}$-name for a function in ω^ω. There is $g = g_{\dot{f}} \in \omega^\omega$ such that for all $h \in \omega^\omega$, if $h \not\leq^* g$, then $\Vdash_{\mathbb{C}} h \not\leq^* \dot{f}$.*

Proof. We use the countability of Cohen forcing. Let $\mathbb{C} = \{s_i : i \in \omega\}$. For each $i, n \in \omega$, let $g_i(n)$ be any k such that some extension of s_i forces $\dot{f}(n) = k$. Let g eventually dominate all g_i. We show that g works.

Assume $h \in \omega^\omega$ is such that $h \not\leq^* g$. Also let $s \in \mathbb{C}$ and $n \in \omega$. It clearly suffices to find $t \leq s$ and $m \geq n$ such that t forces $\dot{f}(m) < h(m)$.

To this end, let i be such that $s = s_i$, and choose $m \geq n$ such that $g_i(m) \leq g(m) < h(m)$. Next choose $t \leq s_i$ forcing $\dot{f}(m) = g_i(m)$. Then t forces $\dot{f}(m) < h(m)$, as well, as required. $\square$

Corollary 1.10. *Assume $\mathcal{F}$ is an unbounded family in V. Then $V^{\mathbb{C}_I} \models$ "$\mathcal{F}$ is unbounded" for any index set $I \in V$. In particular, Cohen forcing does not add dominating reals.*

Proof. Let f be a real in $V^{\mathbb{C}_I}$. By 1.5, there is a countable $J \subseteq I$ belonging to the ground model V such that f is in $V^{\mathbb{C}_J}$. Let $\dot{f}$ be its $\mathbb{C}_J$-name. Since $\mathbb{C}_J$ is forcing equivalent to $\mathbb{C}$, we may apply 1.9 to obtain $g_{\dot{f}}$. As $\mathcal{F}$ is unbounded, there is $h \in \mathcal{F}$ such that $h \not\leq^* g_{\dot{f}}$. $h \not\leq f$ follows, and thus f does not dominate $\mathcal{F}$. Hence $\mathcal{F}$ remains unbounded. $\square$

Main Lemma 1.11. *Let $\dot{A}$ be a $\mathbb{C}$-name for an infinite subset of ω. There are $B_i = B_{\dot{A},i} \in [\omega]^\omega$ such that whenever $C \in [\omega]^\omega$ splits all B_i then $\Vdash_{\mathbb{C}}$ "C splits $\dot{A}$".*

Proof. Exercise! (This is like the proof of Main Lemma 1.9.) □

We say a family $\mathcal{A} \subseteq [\omega]^\omega$ is *ω-splitting* if for every sequence $B_i \in [\omega]^\omega$, $i \in \omega$, there is $A \in \mathcal{A}$ which simultaneously splits all B_i. Clearly every ω-splitting family is a splitting family.

Corollary 1.12. *Assume $\mathcal{A}$ is an ω-splitting family in V. Then $V^{\mathbb{C}_I} \models$ "$\mathcal{A}$ is ω-splitting" for any index set $I \in V$. In particular, Cohen forcing does not add unreaped (unsplit) reals.*

Proof. Exercise! (This is like the proof of Corollary 1.10.) □

1.6.3. We now obtain the main result on the effect of $\mathbb{C}_\kappa$ on the cardinal invariants $\mathfrak{b}, \mathfrak{d}, \mathfrak{s}$, and $\mathfrak{r}$.

Theorem 1.13. *Let $\kappa \geq \aleph_1$. In $V^{\mathbb{C}_\kappa}$, $\mathfrak{b} = \mathfrak{s} = \aleph_1$ and $\mathfrak{d}, \mathfrak{r} \geq \kappa$. In particular, if $\kappa^\omega = \kappa$ in V, $\mathfrak{d} = \mathfrak{r} = \mathfrak{c} = \kappa$ will hold in the generic extension.*

Proof. Let $c_\alpha \in \omega^\omega$, $\alpha < \kappa$, denote the Cohen reals added by $\mathbb{C}_\kappa$. By this, we mean that we identify $\mathbb{C}_\kappa = \mathsf{Fn}(\kappa, 2)$ with $\mathbb{C}_{\kappa\times\omega} = \mathsf{Fn}(\kappa \times \omega, 2)$ and define $c_\alpha(n) = c_{\kappa\times\omega}(\alpha, n)$ where $c_{\kappa\times\omega}$ is the generic Cohen function. A standard genericity argument shows that the c_α are pairwise distinct new reals.

We argue that the c_α, $\alpha < \omega_1$, form an unbounded family in $V^{\mathbb{C}_{\kappa\times\omega}}$: by 1.5, every new real $f \in \omega^\omega$ belongs to $V^{\mathbb{C}_{I\times\omega}}$ for some countable $I \subseteq \kappa$. If $\alpha \in \omega_1 \setminus I$, c_α is Cohen over $V^{\mathbb{C}_{I\times\omega}}$ by the product lemma (1.6) and thus unbounded over $V^{\mathbb{C}_{I\times\omega}}$ by 1.7. Hence f does not dominate c_α, and $\mathfrak{b} = \aleph_1$ follows.

To see $\mathfrak{d} \geq \kappa$, let $\mathcal{F} \subseteq \omega^\omega$ be of size less than κ. By 1.5, there is $I \subseteq \kappa$ of size less than κ such that $\mathcal{F} \subseteq V^{\mathbb{C}_{I\times\omega}}$. If $\alpha \in \kappa \setminus I$, c_α is unbounded over $\mathcal{F}$ by the argument in the previous paragraph. Hence, $\mathcal{F}$ is not dominating, and $\mathfrak{d} \geq \kappa$ follows.

$\mathfrak{s} = \aleph_1$ and $\mathfrak{r} \geq \kappa$ are proved similarly.

To see the last part of the theorem, note that if $\kappa^\omega = \kappa$ in V, there are only κ many canonical $\mathbb{C}_\kappa$-names for reals. Hence $\mathbb{C}_\kappa$ forces $\mathfrak{c} = \kappa$, and the rest follows by what we proved already. □

If CH holds in the ground model, there is an alternative argument for $\mathfrak{b} = \mathfrak{s} = \aleph_1$: put $\mathcal{F} = \omega^\omega \cap V$ and notice that $\mathcal{F}$ is unbounded in $V^{\mathbb{C}_\kappa}$ by 1.10. Similarly for $\mathfrak{s}$.

Lecture 2: random forcing

2.1. σ-ideals and their cardinal invariants. Let $\mathcal{I}$ be a *σ-ideal* on the real numbers. This means $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ (or $\mathcal{I} \subseteq \mathcal{P}(\omega^\omega)$) is a family of subsets of 2^ω which is closed under subsets, under countable unions, and which contains all singletons, but does not contain 2^ω. We define four cardinal invariants related to $\mathcal{I}$:

- the *additivity* $\mathsf{add}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} \notin \mathcal{I}\}$,
- the *covering number* $\mathsf{cov}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{J} = 2^\omega\}$,
- the *uniformity* $\mathsf{non}(\mathcal{I}) = \min\{|X| : X \subseteq 2^\omega \text{ and } X \notin \mathcal{I}\}$,
- the *cofinality* $\mathsf{cof}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I}$ and for all $I \in \mathcal{I}$ there is $J \in \mathcal{J}$ with $I \subseteq J\}$.

A set $\mathcal{J}$ which satisfies the clause in the definition of the cofinality is called a *basis* of the ideal $\mathcal{I}$. It is easy to see that $\aleph_1 \leq \mathsf{add}(\mathcal{I}) \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$ and $\mathsf{add}(\mathcal{I}) \leq \mathsf{non}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. See [BJ] for more on these cardinals.

2.2. Cichoń's diagram. Equip 2 and ω with the discrete topology, and consider 2^ω and ω^ω with the product topology. This means that basic clopen sets are of the form $[s] = \{f \in \omega^\omega : s \subseteq f\}$ where $s \in \omega^{<\omega}$. The metric d given by $d(f,g) = 2^{-\min\{n : f(n) \neq g(n)\}}$ is compatible with this topology, and both 2^ω and ω^ω become *Polish spaces* (i.e., separable complete metric spaces) with this topology. The former is called *Cantor space*, the latter, *Baire space*. The Cantor space is compact (by Tychonoff's theorem). See [Ke] for more on descriptive set theory and Polish spaces.

Recall a subset X of a topological space is *nowhere dense* if its closure has empty interior. X is *meager* if it is a union of countably many nowhere dense sets. By the *Baire category theorem*, no non-empty open set in a Polish space is meager. In particular, the collection $\mathcal{M}$ of all meager subsets of 2^ω or ω^ω is a σ-ideal (the *meager ideal*).

Equip 2 with the measure which gives both $\{0\}$ and $\{1\}$ measure $\frac{1}{2}$, and ω, with the measure which gives all $\{n\}$ measure $2^{-(n+1)}$. Consider 2^ω and ω^ω with the product measures. Both are *probability measure spaces*, and the collection $\mathcal{N}$ of all measure zero subsets of 2^ω or ω^ω is a σ-ideal (the *null ideal*).

The ideals $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal* in the sense that there is a comeager null set X (and $2^\omega \setminus X$ then is a meager set of measure one): namely, for each n find an open dense set X_n of measure $\leq 2^{-n}$ and let $X = \bigcap_n X_n$.

Note that 2^ω with addition modulo 2 is a *Polish group*. Both ideals $\mathcal{M}$ and $\mathcal{N}$ are *translation-invariant* with respect to this group; i.e., for any $x \in 2^\omega$, $x+Y$ is meager iff Y is meager, and similarly for "null".

Proposition 2.1. $\mathsf{cov}(\mathcal{N}) \le \mathsf{non}(\mathcal{M})$ *and* $\mathsf{cov}(\mathcal{M}) \le \mathsf{non}(\mathcal{N})$. *In fact, if* $X, Y \subseteq 2^\omega$ *are non-meager and comeager, respectively, then* $X + Y = \{x + y : x \in X \text{ and } y \in Y\} = 2^\omega$. *Similarly for "non-meager" replaced by "non-null" and "comeager" replaced by "of measure one".*

Proof. The first part follows from the second part, the translation-invariance, and the orthogonality of the ideals $\mathcal{M}$ and $\mathcal{N}$.

To see the second part, let $z \in 2^\omega$. By translation-invariance, $z + X$ is non-meager. Hence $z + X \cap Y \neq \emptyset$. Find $x \in X$ and $y \in Y$ such that $z + x = y$. Then $z = x + y$. $\square$

Closed sets in either 2^ω or ω^ω are represented by trees: $T \subseteq \omega^{<\omega}$ is a *tree* if T is closed under initial segments. For a tree T, the set of its branches $[T] = \{f \in \omega^\omega : f{\restriction}n \in T \text{ for all } n\}$ is closed. Conversely, if $C \subseteq \omega^\omega$ is closed, there is a tree $T = T_C$ such that $C = [T]$. It is easy to see that C is compact iff T_C is *finitely branching* iff there is a function $f \in \omega^\omega$ dominating all members of C everywhere. C is nowhere dense iff for each $s \in T_C$, there is $t \supseteq s$ which does not belong to T_C. In particular, every compact set is nowhere dense in ω^ω, and every bounded family is meager. Thus we obtain:

Proposition 2.2. $\mathfrak{b} \le \mathsf{non}(\mathcal{M})$ *and* $\mathsf{cov}(\mathcal{M}) \le \mathfrak{d}$.

The two following theorems are more difficult to prove, and we shall only state them.

Theorem 2.3. (Miller and Truss) $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ *and* $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$.

See [BJ, 2.2.9 and 2.2.11] or [Bl, Theorem 5.6].

Theorem 2.4. (The Bartoszyński-Raisonnier-Stern Theorem) $\mathsf{add}(\mathcal{N}) \le \mathsf{add}(\mathcal{M})$ *and* $\mathsf{cof}(\mathcal{M}) \le \mathsf{cof}(\mathcal{N})$.

See [BJ, Theorem 2.3.7].

Hence the order-relationship between the cardinals defined so far can be summarized in the following diagram, called *Cichoń's diagram.*

$$\begin{tikzcd}
 & \mathsf{cov}(\mathcal{N}) \arrow[r, dash] & \mathsf{non}(\mathcal{M}) \arrow[r, dash] & \mathsf{cof}(\mathcal{M}) \arrow[r, dash] & \mathsf{cof}(\mathcal{N}) \arrow[r, dash] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[r, dash] \arrow[u, dash] & \mathfrak{d} \arrow[u, dash] & & \\
\aleph_1 \arrow[r, dash] & \mathsf{add}(\mathcal{N}) \arrow[r, dash] \arrow[uu, dash] & \mathsf{add}(\mathcal{M}) \arrow[r, dash] \arrow[u, dash] & \mathsf{cov}(\mathcal{M}) \arrow[r, dash] \arrow[u, dash] & \mathsf{non}(\mathcal{N}) \arrow[uu, dash] &
\end{tikzcd}$$

2.3. Random forcing versus Cohen forcing. Let I be an infinite index set. Equip 2^I with the product topology and the product measure (see 2.2). The family of *Baire sets* $\mathcal{B}_I \subseteq \mathcal{P}(2^I)$ is the σ-algebra generated by the clopen sets $[s] = \{f \in 2^I : s \subseteq f\}$ where $s \in \mathsf{Fn}(I, 2)$. In case $I = \omega$ (or, more generally, for countable I), $\mathcal{B}_I$ coincides with the *Borel sets* (the σ-algebra generated by the open sets). Every Baire set is of the form $X \times 2^{I \setminus J}$ where $I \subseteq J$ is countable and $X \subseteq 2^I$ is Baire (equivalently, Borel). All Baire sets are measurable and have the *property of Baire* (i.e., they are equivalent to an open set modulo a meager set). Let $\mathcal{M}_I$ and $\mathcal{N}_I$ denote the meager and null ideals on 2^I, respectively. Both are *Baire ideals* in the sense that they have a basis consisting of Baire sets.

Let $\mathbb{C}_I = \mathcal{B}_I/\mathcal{M}_I$, the *Cohen algebra*, and $\mathbb{B}_I = \mathcal{B}_I/\mathcal{N}_I$, the *random algebra*. Both forcing notions are ordered by inclusion: $[X] \leq [Y]$ if $X \setminus Y$ is meager (null, respectively). Both are ccc: for $\mathbb{C}_I$, this follows from the fact that 2^I is ccc as a topological space, i.e., any collection of pairwise disjoint open sets is at most countable; for $\mathbb{B}_I$, this follows from basic properties of the measure: any family of positive measurable sets whose pairwise intersections are null is at most countable.

A word on Cohen forcing. Cohen forcing has already been defined in Lecture 1, albeit with a rather different definition. To see that the two definitions are equivalent, notice that the function sending $s \in \mathsf{Fn}(I, 2)$ to $[[s]] \in \mathcal{B}_I/\mathcal{M}_I$ (where $[[s]]$ denotes the equivalence class of the clopen set $[s]$) is a dense embedding: any $X \in \mathcal{B}_I$ is equivalent to an open set modulo a meager set, and thus $[[s]] \leq [X]$ for some basic clopen set $[s]$. Hence $\mathsf{Fn}(I, 2)$ and $\mathcal{B}_I/\mathcal{M}_I$ are forcing equivalent.

All the results in this subsection (2.3) hold for both $\mathbb{B}_I$ and $\mathbb{C}_I$, with the obvious adjustments. We usually state them for $\mathbb{B}_I$ only.

In case $I = \omega$, write $\mathbb{B}$ instead of $\mathbb{B}_\omega$. $\mathbb{B}_I$ generically adds a new function $r_I \in 2^I$ given by $r_I = \bigcup\{s : [[s]] \in G\}$ where G is the $\mathbb{B}_I$-generic filter over V. r_I is a *random function* and $r = r_\omega$ is a *random real*.

To be able to discuss basis properties of random and Cohen forcing, we need the concept of *Baire codes*. If $M \subseteq N$ are models of (a large enough finite fragment of) ZFC and $s \in \mathsf{Fn}(I, 2)$, we may consider the clopen set $[s]$ in both M and N, $[s]^M$ and $[s]^N$. If N contains elements of 2^I which do not belong to M, these sets will be *distinct*, $[s]^M \subsetneq [s]^N$. However, they are given by the *same* s, and we may therefore construe them as being the same set as interpreted in different models. We say *s codes* the set $[s]$. Incidentally, we always have $[s]^M = [s]^N \cap M$.

More generally, for each Baire set X, there is a *Baire code* c which describes how X is built up recursively. We may think of X as the *interpretation* of c in a given model M (which contains c), $X = X_c^M$. Whenever $c \in M$ and $M \subseteq N$, we can interpret c in N and obtain X_c^N. Furthermore, if $c \in M \subseteq N$, we have $X_c^M \subseteq X_c^N$ and $X_c^M = X_c^N \cap M$. We shall forget about c and, instead of saying *the Baire code $c \in M$ codes the set X*, we shall simply say *X is a Baire set coded in M*.

For a detailed treatment of the construction of the Baire codes see [Ku2, Section 1]. See also [Je, Section 25].

Lemma 2.5. *Let r_I be random over M. For every Baire set X coded in M, $r_I \in X$ iff $[X] \in G$.*

Note we are rather sloppy in the formulation of this lemma: more formally, it should be "$r_I \in X^{M[G]}$ *iff* $[X^M] \in G$" because we interpret the code of X in two distinct models.

Proof. We make induction on the complexity of X. (I.e., this is an induction on the *Baire hierarchy.*)

- First assume $X = [s]$ is clopen (where $s \in \mathsf{Fn}(I, 2)$). Then, by definition of r_I, $r_I \in [s]$ iff $s \subseteq r_I$ iff $[[s]] \in G$ (the left-to-right implication in the last equivalence is an easy consequence of the fact that G is a filter).
- Next assume the lemma holds for Y and let $X = 2^\omega \setminus Y$. Then $r_I \in X$ iff $r_I \notin Y$ iff $[Y] \notin G$ iff $[X] \in G$ (where the latter holds by genericity).
- Finally assume the lemma holds for X_n and let $X = \bigcup_n X_n$. Then $r_I \in X$ iff $\exists n : r_I \in X_n$ iff $\exists n : [X_n] \in G$ iff $[X] \in G$ (the right-to-left implication in the last equivalence follows again from genericity).

This completes the induction and the proof of the lemma. $\square$

Lemma 2.5 allows us to define G in terms of r_I via $G = \{[X] : X \in V$ and $r_I \in X\}$, and we see that $V[r_I] = V[G]$ holds (cf. Observation 1.3).

Proposition 2.6. (Solovay's characterization of genericity) *A function $f \in 2^I$ is random generic over M iff f avoids all Baire null sets coded in M.*

Proof. First assume $f = r_I$ is $\mathbb{B}_I$-generic over M. If X is a Baire null set coded in M, then $[2^\omega \setminus X] \in G$, and thus $r_I \notin X$ by the previous lemma.

Now assume f avoids all Baire null sets coded in M. Let $G = \{[X] : X$ is a Baire set coded in M and $f \in X\}$. Clearly, $G \subseteq \mathbb{B}_I$ is a filter. Let $[X_n]$, $n \in \omega$, be a maximal antichain in $\mathbb{B}_I$ belonging to M. Then $\bigcup_n X_n$ has measure one and is coded in M, and $f \in \bigcup_n X_n$ follows. Thus $f \in X_n$ for some n and $[X_n] \in G$. Hence G is generic. By the previous lemma, it is immediate that the generic function r_I defined from G agrees with f. Hence f is generic. $\square$

Corollary 2.7. (existence of many generics over countable models) *Let M be a countable model of (a large enough finite fragment of) ZFC. Then the set $\mathsf{Ra}(M)$ of reals random over M is a Borel set of measure one, and the set $\mathsf{Co}(M)$ of Cohen reals over M is a comeager Borel set.*

Proof. Indeed, $\mathsf{Ra}(M) = 2^\omega \setminus \bigcup\{X : X \in \mathcal{N}$ is coded in $M\}$. The union on the right-hand side is a countable union of Borel null sets, and the corollary follows. The proof for $\mathsf{Co}(M)$ is similar. $\square$

Corollary 2.8. (downward absoluteness of genericity) *Let $M \subseteq N$ be models of ZFC. Assume r is random over N. Then r is random over M as well. Similarly for "Cohen" instead of "random".*

Proof. This is immediate by Solovay's characterization 2.6. □

Lemma 2.9. *Let $\kappa < |I|$ and let $\dot{f}$ be a $\mathbb{B}_I$-name for a function with domain κ and values in V. Then there is $J \subseteq I$ of size at most κ such that $\dot{f}$ is (equivalent to) a $\mathbb{B}_J$-name.*

Proof. This is similar to Lemma 1.5.

For each $\alpha < \kappa$, let A_α be a maximal antichain in $\mathbb{B}_I$ deciding the value of $\dot{f}(\alpha)$. For each $[X] \in A_\alpha$, we may find $J_X \subseteq I$ countable and a Borel set $Y \subseteq 2^{J_X}$ such that $X = Y \times 2^{I \setminus J_X}$. By the ccc, the set $J = \bigcup\{J_X : [X] \in \bigcup_{\alpha<\kappa} A_\alpha\}$ has size at most κ, and it is clear that $\dot{f}$ can be construed as a $\mathbb{B}_J$-name. □

We may as well forget about equivalence classes and think of $\mathbb{B}_I$ as forcing with $\mathcal{B}_I \setminus \mathcal{N}_I$, ordered by inclusion modulo null.

Say that $g : 2^I \to 2^\omega$ is a *Baire function* if the inverse image of every basic clopen set (equivalently, every Borel set) is a Baire set.

Lemma 2.10. *Let $\dot{f}$ be a $\mathbb{B}_I$-name for a real in 2^ω. Then there is a Baire function $g = g_{\dot{f}} : 2^I \to 2^\omega$ such that $\dot{f} = g(\dot{r}_I)$ is forced.*

Proof. Define $g^{-1}([s]) = [\![\dot{f}\restriction|s| = s]\!]$ for $s \in 2^{<\omega}$. Here, $[\![\varphi]\!]$ denotes the *Boolean value* of the statement φ, i.e., the maximal condition forcing φ. See [Ku1, VII Section 7] (or [Je, Section 14]) for a discussion of Boolean values. We note that, since we construe $\mathbb{B}_I$ as $\mathcal{B}_I \setminus \mathcal{N}_I$, $[\![\varphi]\!]$ is not unique, but any two Baire sets representing it are equivalent modulo null. In particular, we can choose the $[\![\dot{f}\restriction|s| = s]\!]$ such that they are disjoint for incompatible s, $[\![\dot{f}\restriction|t| = t]\!] \subseteq [\![\dot{f}\restriction|s| = s]\!]$ for $s \subseteq t$, and the union over $[\![\dot{f}\restriction|s| = s]\!]$ of the same length is 2^I. This entails that g is indeed a function from 2^I to 2^ω. By definition, it is obviously Baire.

Let G be a $\mathbb{B}_I$-generic filter, and let r_I be the corresponding random function. Then we have for all s:

$$\begin{aligned} f\restriction|s| = s \quad &\iff \quad [\![\dot{f}\restriction|s| = s]\!] \in G \iff g^{-1}([s]) \in G \\ &\iff \quad r_I \in g^{-1}([s]) \iff g(r_I) \in [s] \iff g(r_I)\restriction|s| = s \end{aligned}$$

The third equivalence comes from Lemma 2.5. So indeed $f = g(r_I)$. □

Lemma 2.11. *Let $\dot{X}$ be a $\mathbb{B}_I$-name for a Baire set in 2^J. Then there is a Baire set in the plane $Y \subseteq 2^I \times 2^J$ such that $\dot{X} = Y_{\dot{r}_I}$ is forced.*

Proof. Consider first the case where $\dot{X}$ is a Borel set in 2^ω. Assume $\dot{X}$ is forced to be Σ^0_α. Let $U \subseteq 2^\omega \times 2^\omega$ be a universal Σ^0_α set ([Je, Lemma 11.2], [Ke, Theorem 22.3]). Then there is a name $\dot{f}$ for a real such that $\dot{X} = U_{\dot{f}}$ is forced. Now use the previous lemma to get g and put $Y = \{(x, y) : (g(x), y) \in U\}$. Since U is a Borel set and g is a Baire function, it follows that Y is a Baire set. Also, it is easy to see that $\dot{X} = U_{\dot{f}} = U_{g(\dot{r}_I)} = Y_{\dot{r}_I}$ is forced.

If $\dot{X}$ is a Baire set in 2^J, there are $J_0 \subseteq J$ countable and a Borel set $\dot{X}_0$ in 2^{J_0} such that $\dot{X} = \dot{X}_0 \times 2^{J\setminus J_0}$. Choose $Y_0 \subseteq 2^I \times 2^{J_0}$ for $\dot{X}_0$ as in the previous paragraph, and let $Y = Y_0 \times 2^{J\setminus J_0}$. □

For the next lemma, we recall the Fubini and Kuratowski-Ulam Theorems (see [Ox, Sections 14 and 15] or [Ke, 8.K and 17.A]).

Theorem 2.12. *Let J_0 and J_1 be disjoint such that $I = J_0 \cup J_1$. Assume $Y \subseteq 2^I = 2^{J_0} \times 2^{J_1}$ is a Baire set.*

1. (Fubini) *The set $\{x : Y_x$ is null$\} \subseteq 2^{J_0}$ is measurable. Furthermore, Y is null iff $\{x : Y_x$ is null$\}$ has measure one.*
2. (Kuratowski-Ulam) *The set $\{x : Y_x$ is meager$\} \subseteq 2^{J_0}$ has the property of Baire. Furthermore, Y is meager iff $\{x : Y_x$ is meager$\}$ is comeager.*

Lemma 2.13. (product lemma) *Let J_0 and J_1 be disjoint such that $I = J_0 \cup J_1$.*

1. *If r_I is $\mathbb{B}_I$-generic over V, $r_I \restriction J_i$ is $\mathbb{B}_{J_i}$-generic over both V and $V[r_I \restriction J_{1-i}]$.*
2. *If r_{J_0} is $\mathbb{B}_{J_0}$-generic over V and r_{J_1} is $\mathbb{B}_{J_1}$-generic over $V[r_{J_0}]$, then $r_{J_0} \cup r_{J_1}$ is $\mathbb{B}_I$-generic over V.*

Proof. 1. We use Proposition 2.6. Genericity of $r_I \restriction J_1$ over V is easy and left to the reader.

To see genericity of $r_I \restriction J_1$ over $V[r_I \restriction J_0]$, let $X \subseteq 2^{J_1}$ be a Baire null set coded in $V[r_I \restriction J_0]$. Let $\dot{X}$ be a $\mathbb{B}_{J_0}$-name for X. Without loss of generality, we may assume that the trivial condition forces that $\dot{X}$ is a null set. By 2.11, there is a Baire set $Y \subseteq 2^I = 2^{J_0} \times 2^{J_1}$ such that $\mathbb{B}_{J_0}$ forces $\dot{X} = Y_{\dot{r}_{J_0}} = Y_{\dot{r}_I \restriction J_0}$. By Fubini, Y must be a null set.

For suppose Y was not null. Then $Z = \{x : Y_x \text{ is not null}\} \subseteq 2^{J_0}$ would be a measurable non-null set coded in V, by 2.12. Without loss of generality, we may assume Z is a Baire non-null set. Z would force that $\dot{X}$ is not null for if $r_{J_0} \in Z$ is random over V, $Y_{r_{J_0}} = X$ is not null. A contradiction.

By 2.6 we obtain that $r_I \notin Y$. Therefore $r_I \restriction J_1 \notin Y_{r_I \restriction J_0} = X$, as required.

2. This is also proved by a combination of 2.6 and 2.12. Let $Y \subseteq 2^I = 2^{J_0} \times 2^{J_1}$ be a Baire null set coded in V. By Fubini, $Z = \{x : Y_x \text{ is null}\} \subseteq 2^{J_0}$ is a measure one set coded in V. Hence $r_{J_0} \in Z$, i.e., $Y_{r_{J_0}}$ is Baire null set coded in $V[r_{J_0}]$. Thus $r_{J_1} \notin Y_{r_{J_0}}$. Therefore $r_{J_0} \cup r_{J_1} \notin Y$, as required. ☐

The analogous lemma for Cohen forcing can be proved using the Kuratowski-Ulam Theorem (2.12). However, a simpler proof is provided by 1.6.

2.4. The effect of random (and Cohen) forcing on Cichoń's diagram.

We start with investigating the effect of $\mathbb{B}$ on $(\omega^\omega, \leq^*)$.

Lemma 2.14. *Random forcing is ω^ω-bounding. That is, for every $\mathbb{B}_I$-name for a real $\dot{f} \in \omega^\omega$, there is $g = g_{\dot{f}}$ such that $\Vdash_{\mathbb{B}_I} \dot{f} \leq^* g$.*

Proof. Let $X \in \mathbb{B}_I$. By the σ-additivity of measure, we can find, for each n, a number $g(n)$ such that the measure of $X_n = [\![\dot{f}(n) \leq g(n)]\!] \cap X$ is at least $(1 - 2^{-(n+2)})\mu(X)$. This means the measure of $Y = \bigcap_n X_n$ is at least $\frac{1}{2}\mu(X)$ and Y forces that $\dot{f}(n) \leq g(n)$ for all n.

By the ccc we can therefore find a maximal antichain Y_i, $i \in \omega$, and functions g_i such that Y_i forces $\dot{f}(n) \leq g_i(n)$ for all n. Any g eventually dominating all g_i is as required. $\square$

Lemma 2.15. *Assume $A \subseteq 2^\omega$ has outer measure one in the ground model V. Then A has outer measure one in $V^{\mathbb{B}_I}$ as well.*

Note that we are not talking about a Borel set, but about an arbitrary set A. That is, we do not mean that we interpret a Borel code for A in two different models, but rather that we consider the *same* set A in V and $V^{\mathbb{B}_I}$. A typical application would be $A = 2^\omega \cap V$: by 2.15, it has outer measure one in the extension. On the other hand, it is easy to see it must have inner measure zero (this necessarily happens in any extension which adds reals). Therefore A is a non-measurable set in $V^{\mathbb{B}_I}$.

Proof. Let $\dot{X}$ be a $\mathbb{B}_I$-name for a set of measure < 1. We need to show that $A \not\subseteq \dot{X}$ is forced.

By 2.11, there is $Y \subseteq 2^I \times 2^\omega$ such that $\dot{X} = Y_{\dot{r}_I}$ is forced. Since the trivial condition forces that $\mu(\dot{X}) < 1$, $\{x : \mu(Y_x) < 1\}$ must be a measure one set, and by Fubini (2.12), it follows that $\mu(Y \cap (Z \times 2^\omega)) < \mu(Z)$ for any $Z \in \mathcal{B}_I \setminus \mathcal{N}_I$. (See the proof of 2.13 for the details of a similar argument.)

Fix $Z \in \mathcal{B}_I \setminus \mathcal{N}_I$. By Fubini again, $\{y : \mu(Y^y \cap Z) = \mu(Z)\}$ is a measurable set which is not of measure one. Since A has outer measure one, there is $y \in A$ such that $\mu(Y^y \cap Z) < \mu(Z)$. The condition $Z \setminus Y^y$ forces that $y \in A \setminus \dot{X}$, for if $r_I \in Z \setminus Y^y$ is random over V, $y \notin Y_{r_I} = X$. Since Z was arbitrary, the trivial condition forces that $A \setminus \dot{X}$ is non-empty. $\square$

An analogous argument shows:

Lemma 2.16. *Assume $A \subseteq 2^\omega$ is not meager in any non-empty open set, in the ground model V. Then A is not meager in any non-empty open set, in $V^{\mathbb{C}_I}$ as well.*

Proof. Exercise! $\square$

Theorem 2.17. *Let $\kappa \geq \aleph_1$. In $V^{\mathbb{B}_\kappa}$, $\mathsf{non}(\mathcal{N}) = \aleph_1$, $\mathsf{cov}(\mathcal{N}) \geq \kappa$, $\mathfrak{b} = \mathfrak{b}^V$, and $\mathfrak{d} = \mathfrak{d}^V$. In particular, if $\kappa^\omega = \kappa$, $\mathsf{cov}(\mathcal{N}) = \mathfrak{c} = \kappa$ will hold in the generic extension. Furthermore, if CH holds in the ground model, $\mathfrak{d} = \aleph_1$.*

Proof. $\mathfrak{b} = \mathfrak{b}^V$ and $\mathfrak{d} = \mathfrak{d}^V$ follow from Lemma 2.14.

First let $\mathcal{F}$ be an unbounded family in V. We claim that $\mathcal{F}$ is still unbounded in $V^{\mathbb{B}_\kappa}$. Indeed, if $f \in \omega^\omega$ in $V^{\mathbb{B}_\kappa}$, by 2.14 there is $g_f \in \omega^\omega$ in V with $f \leq^* g_f$. Since $\mathcal{F}$ is unbounded, there is $h \in \mathcal{F}$ such that $h \not\leq^* g_f$. Hence $h \not\leq^* f$, and $\mathcal{F}$ is indeed unbounded in $V^{\mathbb{B}_\kappa}$. Thus $\mathfrak{b} \leq \mathfrak{b}^V$.

Next let $\mathcal{F} \subseteq \omega^\omega$ be a family of functions in $V^{\mathbb{B}_\kappa}$ of size less than $\mathfrak{b}$. By 2.14, for each $f \in \mathcal{F}$ we find $g_f \in \omega^\omega$ in V with $f \leq^* g_f$. Then $\mathcal{G} = \{g_f : f \in \mathcal{F}\}$ is a bounded family in V, and therefore $\mathcal{F}$ is bounded in $V^{\mathbb{B}_\kappa}$. Hence $\mathfrak{b}^V \geq \mathfrak{b}$ follows.

The argument for $\mathfrak{d} = \mathfrak{d}^V$ is similar and left to the reader.

Let $r_\alpha \in 2^\omega$, $\alpha < \kappa$, denote the random reals added by $\mathbb{B}_\kappa$. That is, we identify $\mathbb{B}_\kappa$ with $\mathbb{B}_{\kappa\times\omega}$ and define $r_\alpha(n) = r_{\kappa\times\omega}(\alpha, n)$.

We argue that the r_α, $\alpha < \omega_1$, form a non-null set in $V^{\mathbb{B}_\kappa}$. Let X be a Borel null set in $V^{\mathbb{B}_\kappa}$. Since X is coded by a real, by 2.9 there is a countable $I \subseteq \kappa$ such that X is coded in $V^{\mathbb{B}_I}$. By 2.13, if $\alpha \in \omega_1 \setminus I$, then r_α is random over $V^{\mathbb{B}_I}$, and $r_\alpha \notin X$ by 2.6. $\mathsf{non}(\mathcal{N}) = \aleph_1$ follows.

To see $\mathsf{cov}(\mathcal{N}) \geq \kappa$, let $\mathcal{J}$ be a family of Borel null sets of size less than κ. By 2.9, there is $I \subseteq \kappa$ of size less than κ such that all members of $\mathcal{J}$ are coded in $V^{\mathbb{B}_I}$. As in the previous paragraph, we see that if $\alpha \in \kappa \setminus I$, then r_α is random over $V^{\mathbb{B}_I}$ and therefore does not belong to $\bigcup \mathcal{J}$. Hence, $\mathcal{J}$ is not covering, and $\mathsf{cov}(\mathcal{N}) \geq \kappa$ follows.

To see the last part of the theorem, note that if $\kappa^\omega = \kappa$ in V, there are only κ many canonical $\mathbb{B}_\kappa$-names for reals. Hence $\mathbb{B}_\kappa$ forces $\mathfrak{c} = \kappa$, and the rest follows by what we proved already. $\square$

If CH holds in the ground model, $\mathsf{non}(\mathcal{N}) = \aleph_1$ can alternatively be proved as follows: let $A = 2^\omega \cap V$, and note that A has outer measure one in $V^{\mathbb{B}_\kappa}$ by 2.15.

An analogous argument shows:

Theorem 2.18. *Let $\kappa \geq \aleph_1$. In $V^{\mathbb{C}_\kappa}$, $\mathsf{non}(\mathcal{M}) = \aleph_1$ and $\mathsf{cov}(\mathcal{M}) \geq \kappa$. In particular, if $\kappa^\omega = \kappa$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \kappa$ will hold in the generic extension.*

Proof. Exercise! $\square$

Lecture 3: Hechler forcing

3.1. General introduction to iterated forcing. How do we add many reals? Say we want to add κ reals f_α, $\alpha < \kappa$, such that f_α is generic not only over V, but also over $V[\{f_\beta : \beta < \alpha\}]$ ($\star$). One option are

- *Products.*
  We used them for Cohen or random forcing. We have the product lemma (1.6 and 2.13) which says that "generics are even generic over an initial segment of the extension" (property ($\star$) above).

Unfortunately, for most forcing notions adjoining real numbers, there is no such product lemma, and we need another approach:

- *Iterations.*
  They are explicitly described by recursion in such a way that f_α is added by a forcing notion in $V[\{f_\beta : \beta < \alpha\}]$ and will thus be generic over $V[\{f_\beta : \beta < \alpha\}]$.

There are two main techniques for iterations: *finite support iteration* discussed here, and *countable support iteration* discussed in the next lecture.

3.2. Finite support iteration of ccc forcing. We briefly review – without any proofs – the basics of *iterated forcing theory.*

Let $\mathbb{P}$ be a forcing notion, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing notion. We put

$$\mathbb{P} \star \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in \mathbb{P} \text{ and } \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\},$$

the *two-step iteration* of $\mathbb{P}$ and $\dot{\mathbb{Q}}$. The order is given by $(p', \dot{q}') \leq (p, \dot{q})$ if $p' \leq p$ and $p' \Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}$. It is well-known that forcing with $\mathbb{P} \star \dot{\mathbb{Q}}$ is the same as forcing first with $\mathbb{P}$ and then, over $V^{\mathbb{P}}$, with $\mathbb{Q}$.

Lemma 3.1. *Assume $\mathbb{P}$ is ccc and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is ccc. Then $\mathbb{P} \star \dot{\mathbb{Q}}$ is ccc. More generally, this holds with "ccc" replaced by "κ–cc".*

Let δ be an ordinal. Recursively define $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ to be a *δ-stage iteration with limit* $\mathbb{P}_\delta$ if all $\mathbb{P}_\alpha$, $\alpha \leq \delta$, are forcing notions consisting of functions with domain α and

1. (basic stage) $\mathbb{P}_0 = \{\emptyset\}$ is trivial,
2. (successor stage) $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a p.o., $\mathbb{P}_{\alpha+1} = \{p : p{\restriction}\alpha \in \mathbb{P}_\alpha$ and $\Vdash_\alpha p(\alpha) \in \dot{\mathbb{Q}}(\alpha)\} \cong \mathbb{P}_\alpha \star \dot{\mathbb{Q}}_\alpha$, and for $p, q \in \mathbb{P}_{\alpha+1}$, we put $q \leq_{\alpha+1} p$ if $q{\restriction}\alpha \leq_\alpha p{\restriction}\alpha$ and $q{\restriction}\alpha \Vdash_\alpha q(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha)$,
3. (limit stage) for limit $\beta \leq \delta$, $\mathbb{P}_\beta$ is either the *direct limit* or the *inverse limit* of the $\mathbb{P}_\alpha$, $\alpha < \beta$; i.e., either $\mathbb{P}_\beta = \{p : \exists \alpha < \beta$ ($\mathrm{supp}(p) \subseteq \alpha$ and $p{\restriction}\alpha \in \mathbb{P}_\alpha$)$\}$ or $\mathbb{P}_\beta = \{p : \forall \alpha < \beta\ \ p{\restriction}\alpha \in \mathbb{P}_\alpha\}$; for $p, q \in \mathbb{P}_\beta$, we put $q \leq_\beta p$ if $q{\restriction}\alpha \leq_\alpha p{\restriction}\alpha$ for all $\alpha < \beta$.

Here, $\mathrm{supp}(p) = \{\alpha : p(\alpha) \neq 1_{\dot{\mathbb{Q}}_\alpha}\}$ denotes the *support* of p. It is well-known that forcing with an iteration $\mathbb{P}_\delta$ is the same as first forcing with an initial segment $\mathbb{P}_\alpha$, $\alpha < \delta$, and then, over $V^{\mathbb{P}_\alpha}$, with the remainder forcing, $\mathbb{P}_\delta / \mathbb{P}_\alpha$.

An iteration is called *finite support iteration* (*fsi* for short) if all elements of $\mathbb{P}_\delta$ have finite support. This is equivalent to stipulating that direct limits are taken at all limit ordinals. An iteration is a *countable support iteration* (*csi* for short) if inverse limits are taken at all ordinals of countable cofinality and direct limits are taken at all limit ordinals of uncountable cofinality. This implies that all elements of $\mathbb{P}_\delta$ have at most countable support.

Lemma 3.2. *Fsi of ccc forcing notions are ccc. I.e., if $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is an fsi such that $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is ccc", then $\mathbb{P}_\delta$ is ccc.*

Lemma 3.3. *Let δ be a limit ordinal. Assume $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is an fsi of ccc forcing notions. Let $\kappa < cf(\delta)$ and let $\dot{f}$ be a $\mathbb{P}_\delta$-name for a function with domain κ and values in V. Then there is $\alpha < \delta$ such that $\dot{f}$ is (equivalent to) a $\mathbb{P}_\alpha$-name.*

Proof. Exercise! (This is like Lemmata 1.5 and 2.9.) □

3.3. Hechler forcing. Let $\mathbb{D} = \{(s, f) : s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subseteq f\}$, ordered by $(t, g) \leq (s, f)$ if $t \supseteq s$, g dominates f everywhere, and $t(i) \geq f(i)$ for all $i \in |t| \setminus |s|$. $\mathbb{D}$ is called *Hechler forcing*. It generically adds a new real $d = \bigcup\{s : (s, f) \in G \text{ for some } f \in \omega^\omega\}$, where G denotes the $\mathbb{D}$-generic filter over V. d is called a *Hechler real*. We again have $V[d] = V[G]$ (cf. Observation 1.3). A straightforward genericity argument shows:

Observation 3.4. *A Hechler real d real is* dominating*, that is, it eventually dominates all reals of V.*

Lemma 3.5. *Hechler forcing adds a Cohen real.*

Proof. Let $d \in \omega^\omega$ be a Hechler real over V. Define $c \in 2^\omega$ by $c(n) = d(n) \mod 2$. We claim that c is Cohen over V.

To see this, let $D \subseteq \mathbb{C}$ be dense and $(s, f) \in \mathbb{D}$. We need to find $(t, g) \leq (s, f)$ and $u \in D$ such that $(t, g) \Vdash_{\mathbb{D}} u \subseteq \dot{c}$. Let $v = s \mod 2$. Find $u \supseteq v$ with $u \in D$. Next choose $t \supseteq s$ with $t \geq f$ everywhere such that $u = t \mod 2$. Finally let g be such that $t \subseteq g$ and $g(n) = f(n)$ for $n \geq |t|$. Then $(t, g) \leq (s, f)$ and $(t, g) \Vdash_{\mathbb{D}} u \subseteq \dot{c}$. □

A subset P of a forcing notion $\mathbb{P}$ is called *centered* if any finitely many elements of P have a common extension, i.e., for all finite $F \subseteq P$ there is $q \in \mathbb{P}$ such that $q \leq p$ for all $p \in F$. $\mathbb{P}$ is *σ-centered* if there are centered sets P_n such that $\mathbb{P} = \bigcup_n P_n$.

Observation 3.6. *$\mathbb{D}$ is a σ-centered forcing notion.*

Proof. For $s \in \omega^{<\omega}$, let $D_s = \{(s, g) : g \in \omega^\omega \text{ and } s \subseteq g\}$. Clearly all D_s are centered and $\mathbb{D} = \bigcup_s D_s$. □

Lemma 3.7. *σ-centered forcing does not add random reals.*

Proof. Let $\mathbb{P}$ be σ-centered, $\mathbb{P} = \bigcup_i P_i$ where the P_i are centered. Also let $\dot{f}$ be a $\mathbb{P}$-name for a function in 2^ω.

Fix i. We claim that for each n there is $s \in 2^n$ such that no $p \in P_i$ forces $\dot{f}\restriction n \neq s$. For if there was no such s, we could find a condition $p_s \in P_i$ forcing $\dot{f}\restriction n \neq s$ for each $s \in 2^n$. A common extension of the p_s would then force $\dot{f}\restriction n \notin 2^n$, a contradiction.

By König's Lemma, there is a function $g_i \in 2^\omega$ such that for all n, no $p \in P_i$ forces $\dot{f}\restriction n \neq g_i\restriction n$.

Unfix i. We claim that whenever $h : \omega \to 2^{<\omega}$, $h(n) \in 2^n$, is such that for all i, there are infinitely many n with $g_i\restriction n = h(n)$, then $\mathbb{P}$ forces that there are infinitely many n with $\dot{f}\restriction n = h(n)$. To see this, fix m and a condition p. Next fix i such that $p \in P_i$ and $n \geq m$ such that $g_i\restriction n = h(n)$. Since p does not force $\dot{f}\restriction n \neq g_i\restriction n$, there is $q \leq p$ forcing $\dot{f}\restriction n = h(n)$, as required.

Put $A_h = \{f : \exists^\infty n \;\; f\restriction n = h(n)\}$ and note that this is a G_δ measure zero set. By the previous paragraph, any new real is contained in such a set from the ground model. Thus, by 2.6, no new real is random. $\square$

3.4. The effect of Hechler forcing on cardinal invariants of the continuum. As mentioned earlier (beginning of Subsection 1.6), any investigation of the effect of a forcing on the combinatorial structure of the reals boils down to studying *properties of the generic* and to proving *preservation theorems*. For *products*, one preservation result typically is sufficient, but for *iterations*, preservation splits into two – usually quite distinct – proofs:

- *preservation by the single-step forcing* which heavily uses the combinatorial properties of the forcing notion involved;
- *preservation in the limit step*, a general argument showing that if all $\mathbb{P}_\alpha$, $\alpha < \delta$, have a certain property, then so does $\mathbb{P}_\delta$; this does not depend at all on the forcing notion we are iterating.

3.4.1. Preservation in the single step.
For the purposes of the next proof, we use a slightly different representation of Hechler forcing[1]. Say $\mathbb{D}$ consists of all pairs (s, φ) such that $s \in \omega^{<\omega}$ and $\varphi : \omega^{<\omega} \to \omega$. The order is given by $(t, \psi) \leq (s, \varphi)$ if $t \supseteq s$, ψ dominates φ everywhere, and $t(i) \geq \varphi(t\restriction i)$ for all $i \in |t| \setminus |s|$.

Main Lemma 3.8. *Assume $\dot{A}$ is a $\mathbb{D}$-name for an infinite subset of ω. There are sets A_i, $i \in \omega$, such that whenever $B \in [\omega]^\omega$ splits all A_i, then $\Vdash_{\mathbb{D}}$ "B splits $\dot{A}$".*

[1]Denoting classical Hechler forcing by $\mathbb{D}$, and the representation here by $\mathbb{D}'$, it can be shown that $\mathbb{D}$ adds a $\mathbb{D}'$-generic, and that the two-step iteration of $\mathbb{D}'$, $\mathbb{D}' \star \dot{\mathbb{D}}'$, adds a $\mathbb{D}$-generic. Therefore the fsi of the two forcing notions have the same combinatorial properties. I do not know, though, whether $\mathbb{D}$ and $\mathbb{D}'$ are forcing equivalent.

Proof. This is a *rank argument*. Such rank arguments are quite common for establishing combinatorial properties of forcing notions adding dominating reals. In this sense, the present proof is a paradigm for many other proofs.

For $s \in \omega^{<\omega}$ and $n \in \omega$, say that *s favors that k is the n-th element of $\dot{A}$* if there is no condition with first coordinate s which forces that "k is not the n-th element of $\dot{A}$". Define the *rank* $\mathrm{rk}_n(s)$ by recursion on the ordinals, as follows:

- $\mathrm{rk}_n(s) = 0$ if for some k, s favors that k is the n-th element of $\dot{A}$;
- for $\alpha > 0$: $\mathrm{rk}_n(s) = \alpha$ if there is no $\beta < \alpha$ such that $\mathrm{rk}_n(s) = \beta$ and there are infinitely many ℓ such that $\mathrm{rk}_n(s\hat{\ }\ell) < \alpha$.

Clearly, $\mathrm{rk}_n(s)$ must either be a countable ordinal or undefined (in which case we write $\mathrm{rk}_n(s) = \infty$). We show the latter never happens.

Claim 3.9. $\mathrm{rk}_n(s) < \omega_1$ *for all s and n.*

Proof. Assume $\mathrm{rk}_n(s) = \infty$. Notice that for any t, if $\mathrm{rk}_n(t) = \infty$, then $\mathrm{rk}_n(t\hat{\ }\ell) = \infty$ for almost all ℓ. This allows us to recursively construct a function $\varphi : \omega^{<\omega} \to \omega$ such that whenever $t \supseteq s$ and $t(i) \geq \varphi(t{\restriction}i)$ for all $i \in |t| \setminus |s|$, then $\mathrm{rk}_n(t) = \infty$.

Consider the condition (s, φ). Find $(t, \psi) \leq (s, \varphi)$ and k such that (t, ψ) forces that k is the n-th element of $\dot{A}$. Then clearly $\mathrm{rk}_n(t) = 0$. However, by the preceding paragraph, $\mathrm{rk}_n(t) = \infty$, a contradiction. □

We continue with the proof of the main lemma. If s is such that $\mathrm{rk}_n(s) = 0$ for infinitely many n, find $k_n \geq n$ such that s favors that k_n is the n-th element of $\dot{A}$, and let A_s be the collection of the k_n. If s and n are such that $\mathrm{rk}_n(s) = 1$, there are infinitely many ℓ such that $\mathrm{rk}_n(s\hat{\ }\ell) = 0$, and for each such ℓ we may find k_ℓ such that $s\hat{\ }\ell$ favors that k_ℓ is the n-th element of $\dot{A}$. It is easy to see that for each k, the set $\{\ell : k_\ell = k\}$ must be finite for otherwise k witnesses that $\mathrm{rk}_n(s) = 0$. In particular, the collection $A_{s,n}$ of such k_ℓ must be infinite.

We claim that if B splits all A_s and all $A_{s,n}$, then it is forced to split $\dot{A}$.

Let (s, φ) be condition and let $m \in \omega$. We need to find $(t, \psi) \leq (s, \varphi)$ and $m_0, m_1 \geq m$ such that $m_0 \in B$, $m_1 \notin B$, and (t, ψ) forces both m_0 and m_1 belong to $\dot{A}$. Since the construction of m_0 and m_1 is analogous, it suffices to produce the former.

First assume there are infinitely many n such that $\mathrm{rk}_n(s) = 0$. Since $B \cap A_s$ is infinite, we find $m_0 \geq m$ in this intersection. By definition of A_s there is some n such that s favors that $k_n = m_0$ is the n-th element of $\dot{A}$, and thus there is $(t, \psi) \leq (s, \varphi)$ such that $(t, \psi) \Vdash_{\mathbb{D}} m_0 \in \dot{A}$.

Next assume $\mathrm{rk}_n(s) > 0$ for all but finitely many n. Choose $n \geq m$ such that $\mathrm{rk}_n(s) > 0$. Extend s to t such that $t(i) \geq \varphi(t{\restriction}i)$ for all $i \in |t| \setminus |s|$ and $\mathrm{rk}_n(t) = 1$. That this can be done is proved by induction on $\mathrm{rk}_n(s)$.

If $\mathrm{rk}_n(s) = 1$, put $t = s$. If $\mathrm{rk}_n(s) > 1$, we can find, by definition of rk_n, an ℓ such that $\ell \geq \varphi(s)$ and $1 \leq \mathrm{rk}_n(s\hat{\ }\ell) < \mathrm{rk}_n(s)$. By induction hypothesis, $s\hat{\ }\ell$ can be extended to the required t.

Since $B \cap A_{t,n}$ is infinite, we find $\ell \geq \varphi(t)$ and $k_\ell = m_0 \geq m$ such that $m_0 \in B \cap A_{t,n}$. Thus $t\hat{}\ell$ favors that m_0 is the n-th element of $\dot{A}$. Hence we can find a condition $(u, \psi) \leq (s, \varphi)$ such that $t\hat{}\ell \subseteq u$ and $(u, \psi) \Vdash_{\mathbb{D}} m_0 \in \dot{A}$. $\square$

3.4.2. Preservation in the iteration.

Proposition 3.10. *Let δ be a limit ordinal. Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be an fsi of ccc forcing notions. Assume that for all $\alpha < \delta$ we have*

($\star_\alpha$) *for any $\mathbb{P}_\alpha$-name $\dot{A}$ for an infinite subset of ω, there are $A_i \in [\omega]^\omega$, $i \in \omega$, such that whenever B splits all A_i, then $\Vdash_\alpha$ "B splits $\dot{A}$".*

Then ($\star_\delta$) holds as well.

Proof. By Lemma 3.3, no new reals arise in limit stages of uncountable cofinality so that the proposition vacuously holds if $cf(\delta) > \omega$. Hence assume $cf(\delta) = \omega$. To simplify notation suppose $\delta = \omega$.

Let $\dot{A}$ be a $\mathbb{P}_\omega$-name for an infinite subset of ω. Fix $n \in \omega$. Work in $V^{\mathbb{P}_n}$ for the moment. There is a decreasing sequence of conditions $p_k = p_{n,k}$ in the remainder forcing $\mathbb{P}_\omega / \mathbb{P}_n$ such that p_k decides the k-th element of $\dot{A}$. Say $p_k \Vdash_{[n,\omega)}$ "ℓ_k is the k-th element of $\dot{A}$". Let $A_n = \{\ell_k : k \in \omega\}$.

Work in the ground model V. We have a $\mathbb{P}_n$-name $\dot{A}_n$ for A_n. By ($\star_n$), we can find $A_{n,i}$, $i \in \omega$, such that whenever B splits all $A_{n,i}$, then $\Vdash_n$ "B splits $\dot{A}_n$". Unfix n. We claim that if B splits all $A_{n,i}$, $n, i \in \omega$, then $\Vdash_\omega$ "B splits $\dot{A}$".

To see this, let $p \in \mathbb{P}_\omega$ and $m \in \omega$. We need to find $q \leq p$ and $m_0, m_1 \geq m$ such that $m_0 \in B$, $m_1 \notin B$, and $q \Vdash_\omega m_0, m_1 \in \dot{A}$. As in the proof of Main Lemma 3.8, it suffices to find m_0.

Fix n such that $p \in \mathbb{P}_n$ and work in $V^{\mathbb{P}_n}$. We know that B splits A_n. Thus there is $m_0 \geq m$ such that $m_0 \in B \cap A_n$. There is k such that $m_0 = \ell_k$ and $p_k \Vdash_{[n,\omega)} m_0 \in \dot{A}$.

In V, we have $\mathbb{P}_n$-names $\dot{m}_0$ for m_0 and $\dot{p}_k$ for p_k. By strengthening $p \in \mathbb{P}_n$, if necessary, we may assume p decides $\dot{m}_0$ to be m_0 and $\dot{p}_k$ to be p_k, a partial function with domain $[n, \omega)$, so that $q = p\hat{}p_k$ is a condition. Then $q \Vdash_\omega m_0 \in \dot{A}$. $\square$

By a similar argument we show that an iteration of σ-centered forcing does not add random reals. The exact statement is as follows:

Proposition 3.11. *Let δ be a limit ordinal. Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be an fsi of ccc forcing notions. Assume that for all $\alpha < \delta$ we have*

($\star\star_\alpha$) *for any $\mathbb{P}_\alpha$-name $\dot{f}$ for a function in 2^ω, there are $g_i \in 2^\omega$, $i \in \omega$, such that: whenever $h : \omega \to 2^{<\omega}$, $h(n) \in 2^n$, is such that for all i, there are infinitely many n with $g_i \restriction n = h(n)$, then $\Vdash_\alpha$ "there are infinitely many n with $\dot{f} \restriction n = h(n)$".*

Then ($\star\star_\delta$) holds as well.

Corollary 3.12. *Iterations of σ-centered forcing do not add random reals.*

Proof. Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be an fsi of σ-centered forcing. By induction on α, show that $(\star\star_\alpha)$ holds. The case $\alpha = 0$ is trivial. Suppose $(\star\star_\beta)$ holds and let $\alpha = \beta + 1$. Since $\mathbb{P}_\alpha = \mathbb{P}_\beta \star \dot{\mathbb{Q}}_\beta$ and $\mathbb{Q}_\beta$ is forced to be σ-centered, the induction hypothesis together with the argument in the proof of Lemma 3.7 yield $(\star\star_\alpha)$. If α is a limit ordinal and $(\star\star_\beta)$ holds for $\beta < \alpha$, $(\star\star_\alpha)$ follows by Proposition 3.11.

The argument in the proof of Lemma 3.7 now shows that $(\star\star_\delta)$ implies that $\mathbb{P}_\delta$ does not add random reals. □

3.4.3. The effect of $\mathbb{D}$ on cardinal invariants.

Theorem 3.13. *Let $\kappa \geq \aleph_1$ be a regular cardinal. In $V^{\mathbb{D}_\kappa}$, $\mathfrak{s} = \mathsf{cov}(\mathcal{N}) = \aleph_1$, $\mathsf{add}(\mathcal{M}) = \mathfrak{b} = \mathfrak{d} = \mathsf{cof}(\mathcal{M}) = \kappa$, and $\mathfrak{r}, \mathsf{non}(\mathcal{N}) \geq \kappa$.*

Proof. By 3.5 we know that $\mathbb{D}$ adds a Cohen real. Let c_α, $\alpha < \omega_1$, denote the Cohen reals added by the first $\aleph_1$ many Hechler reals. By 1.8 and 3.3, we know that the c_α form an ω-splitting family $\mathcal{A}$ in $V^{\mathbb{D}_{\aleph_1}}$. Now show by induction on $\aleph_1 \leq \alpha \leq \kappa$ that $(\star_\alpha)$ holds with respect to $V^{\mathbb{D}_{\aleph_1}}$.

The case $\alpha = \aleph_1$ is trivial. Suppose $(\star_\beta)$ holds and let $\alpha = \beta + 1$. Since $\mathbb{D}_\alpha = \mathbb{D}_\beta \star \dot{\mathbb{D}}$, the induction hypothesis together with Main Lemma 3.8 yield $(\star_\alpha)$. If δ is a limit ordinal and $(\star_\alpha)$ holds for $\alpha < \delta$, $(\star_\delta)$ follows by Proposition 3.10.

Therefore $(\star_\kappa)$ holds with respect to $V^{\mathbb{D}_{\aleph_1}}$, and thus $\mathcal{A}$ remains ω-splitting in $V^{\mathbb{D}_\kappa}$. Hence $\mathfrak{s} = \aleph_1$.

$\mathsf{cov}(\mathcal{N}) = \aleph_1$ is similar. More explicitly, one shows that if $c : \omega \to 2^{<\omega}$, $c(n) \in 2^n$, is Cohen over a model M, then $A_c = \{f : \exists^\infty n \;\; f{\restriction}n = c(n)\}$ is a G_δ measure zero set containing all reals from M (exercise!). Therefore the A_{c_α}, $\alpha < \omega_1$, form a covering family for the null ideal in $V^{\mathbb{D}_{\aleph_1}}$, and this family is still covering in $V^{\mathbb{D}_\kappa}$ by a combination of 3.11 and the proof of 3.7.

To see $\mathfrak{b} \geq \kappa$, let $\mathcal{F} \subseteq \omega^\omega$ be of size less than κ in the generic extension. By 3.3, there is $\alpha < \kappa$ such that $\mathcal{F}$ is contained in $V^{\mathbb{D}_\alpha}$. Since the Hechler real d_α added in the α-th step of the iteration dominates $V^{\mathbb{D}_\alpha}$ (3.4), $\mathcal{F}$ is bounded in $V^{\mathbb{D}_\kappa}$. Hence $\mathfrak{b} \geq \kappa$.

Using once again that a Hechler real is dominating, one also sees that the sequence d_α, $\alpha < \kappa$, of Hechler generics is a dominating family. Hence $\mathfrak{d} \leq \kappa$ follows, and $\mathfrak{b} = \mathfrak{d} = \kappa$ must hold.

Using the Miller-Truss theorem (2.3) and the Cohen reals added by $\mathbb{D}$ (3.5), one in fact sees that $\mathsf{add}(\mathcal{M}) = \mathsf{cof}(\mathcal{M}) = \kappa$.

More explicitly, let $\mathcal{J}$ be a family of Borel meager sets of size less than κ. By 3.3, there is $\alpha < \kappa$ such that all members of $\mathcal{J}$ are coded in $V^{\mathbb{D}_\alpha}$. Let c_α be the Cohen real added in the α-th step of the iteration. Then, by 2.6, $c_\alpha \notin \bigcup \mathcal{J}$ so that $\mathcal{J}$ is not covering. $\mathsf{cov}(\mathcal{M}) \geq \kappa$ follows, and 2.3 yields $\mathsf{add}(\mathcal{M}) = \kappa$.

A similar argument proves $\mathsf{non}(\mathcal{M}) \leq \kappa$, and $\mathsf{cof}(\mathcal{M}) = \kappa$ follows.

Since $\mathfrak{b} \leq \mathfrak{r}$ (1.2) and $\mathsf{add}(\mathcal{M}) \leq \mathsf{cov}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$ (2.1), $\mathfrak{r}, \mathsf{non}(\mathcal{N}) \geq \kappa$ follow easily. □

If CH holds in the ground model, $\mathfrak{s} = \aleph_1$ can more easily be proved by showing that $\mathcal{A} = [\omega]^\omega \cap V$ is ω-splitting in $V^{\mathbb{D}_\kappa}$, using again 3.8 and 3.10. Similarly for $\mathsf{cov}(\mathcal{N}) = \aleph_1$.

3.5. Limitations of the method of fsi of ccc forcing. An alternative argument for showing that Cohen reals are added in the iteration in 3.13 goes by observing that they naturally arise in limit stages of countable cofinality of fsi.

Lemma 3.14. *Let δ be an ordinal of countable cofinality and let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be an fsi such that all $\dot{\mathbb{Q}}_\alpha$ are forced to be nontrivial. Then $\mathbb{P}_\delta$ adds a real which is Cohen over all $V^{\mathbb{P}_\alpha}$, $\alpha < \delta$.*

Proof. To simplify notation suppose $\delta = \omega$. For each n, let $\dot{q}_n$ be a $\mathbb{P}_n$-name for a condition in $\dot{\mathbb{Q}}_n$ such that $\Vdash_n \dot{q}_n \neq 1_{\dot{\mathbb{Q}}_n}$. In the extension, define $c \in 2^\omega$ by $c(n) = 1$ iff the interpretation of $\dot{q}_n$ belongs to the generic filter. We claim that c is Cohen over all $V^{\mathbb{P}_m}$.

To see this fix m. Without loss of generality, $m = 0$. Let D be a dense subset of $\mathbb{C}$ in V, and let $p \in \mathbb{P}_\omega$. We need to find $r \leq p$ and $t \in D$ such that r forces $t \subseteq \dot{c}$. Fix k such that $p \in \mathbb{P}_k$. Strengthening p if necessary, we may assume that for all $n < k$, either $\Vdash_n p(n) \leq \dot{q}_n$ or $\Vdash_n$ "$p(n)$ and $\dot{q}_n$ are incompatible". This defines $s \in 2^k$ given by $s(n) = 1$ iff the first alternative holds. Clearly p forces $s \subseteq \dot{c}$. Find $t \in D$ with $s \subseteq t$. Let $\ell = |t|$ and extend p to $r \in \mathbb{P}_\ell$ such that $r \restriction k = p$ and for $k \leq n < \ell$, $r(n) = \dot{q}_n$ if $t(n) = 1$, and $r(n)$ is forced to be incompatible with $\dot{q}_n$ if $t(n) = 0$. Then q forces $t \subseteq \dot{c}$, as required. $\square$

More generally, if δ is a limit ordinal and $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is an fsi with nontrivial $\dot{\mathbb{Q}}_\alpha$, then $\mathbb{P}_\delta$ adds a $\mathbb{C}_\delta$-generic over V.

Lecture 4: proper forcing

4.1. Introduction to properness. There are many combinatorial problems which cannot be solved by fsi of ccc forcing. Reasons are that

- there may be no ccc forcing with the right combinatorial properties,
- it may be necessary that no Cohen reals – or no larger Cohen functions – are adjoined to solve the problem at hand (by 3.14 we know that Cohen reals are added in limit stages of fsi, no matter what the individual forcing notions are).

Therefore we look for a larger class of forcing notions which still preserve cardinals – or at least $\aleph_1$ – so that they may be used for blowing up the continuum *and* we also look for a method for iterating them. This leads to proper forcing *and* csi of proper forcing. We note in this context that fsi will not work for non-ccc forcing.

Lemma 4.1. *Assume* $(\mathbb{P}_n, \dot{\mathbb{Q}}_n : n \in \omega)$ *is an fsi such that* $\Vdash_n$ "$\dot{\mathbb{Q}}_n$ *is not ccc" for infinitely many* n. *Then* $\mathbb{P}_\omega$ *collapses* $\aleph_1$.

Proof. For simplicity, assume all $\dot{\mathbb{Q}}_n$ are forced not to be ccc, and let $\{\dot{q}_n^\alpha : \alpha < \omega_1\}$ be a $\mathbb{P}_n$-name for an antichain of $\dot{\mathbb{Q}}_n$. In the extension, define $f : \omega \to \omega_1$ by $f(n) = \alpha$ if the interpretation of $\dot{q}_n^\alpha$ belongs to the generic filter, and $f(n) = 0$ otherwise. It is easy to see that f is onto. (In fact, an argument exactly like the one in the proof of 3.14 shows that f is generic for $\mathsf{Fn}(\omega, \omega_1)$, the Levy collapse of ω_1 to ω.) $\square$

4.2. Proper forcing. Let χ be a cardinal. $H(\chi)$ denotes the collection of sets which are *hereditarily of size less than* χ. That is, $x \in H(\chi)$ iff the transitive closure of x has cardinality less than χ. The $H(\chi)$ are transitive sets, and for regular $\chi > \aleph_0$, $H(\chi)$ is a model of ZFC minus the power set axiom. If χ is large with respect to a p.o. $\mathbb{P}$ (typically, $\chi > 2^{|\mathbb{P}|}$ is enough), then any statement relevant for forcing purposes which is true in the universe (in V) is already true in $H(\chi)$, and vice-versa. Thus we may as well discuss forcing-theoretic properties of $\mathbb{P}$ within $H(\chi)$.

Now let $N \prec H(\chi)$ be countable, i.e., N is a countable *elementary submodel* of $H(\chi)$. Assume $\mathbb{P} \in N$. We say that $q \in \mathbb{P}$ is $(N, \mathbb{P})$-*generic* if for all dense $D \subseteq \mathbb{P}$ with $D \in N$, $D \cap N$ is *predense* below q. (Recall that the latter means that any condition stronger than q is compatible with some condition from $D \cap N$.)

Lemma 4.2. *The following are equivalent:*

1. *q is $(N, \mathbb{P})$-generic.*
2. *For all dense $D \subseteq \mathbb{P}$ with $D \in N$, $q \Vdash D \cap N \cap \dot{G} \neq \emptyset$.*

3. *For all maximal antichains $A \subseteq \mathbb{P}$ with $A \in N$, $q \Vdash A \cap N \cap \dot{G} \neq \emptyset$.*

4. $q \Vdash N[\dot{G}] \cap \mathrm{On} = N \cap \mathrm{On}$.

5. $q \Vdash N[\dot{G}] \cap V = N$.

Here, $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter, and On denotes as usual the ordinals.

Proof. The equivalence of (1) and (2) is immediate.

It is also easy to see that it doesn't matter whether we quantify over dense sets or over maximal antichains (or over predense sets or over open dense sets, for that matter). Hence (2) and (3) are equivalent.

To see that (3) implies (5), let $\dot{x} \in N$ be a $\mathbb{P}$-name for an element of V (that is, $\dot{x}$ belongs to N *as a $\mathbb{P}$-name*). Let $A \in N$ be a maximal antichain of conditions deciding the value of $\dot{x}$. Say $p \Vdash \dot{x} = x_p$ for $p \in A$. If $p \in A \cap N$, then, since $N \models$ "p decides the value of $\dot{x}$", we must have $x_p \in N$. By (3), $q \Vdash A \cap N \cap \dot{G} \neq \emptyset$. Hence $q \Vdash \dot{x} \in \{x_p : p \in A \cap N\}$. Therefore q forces that $\dot{x}$ belongs to N, as required (that is, q forces *the interpretation of* $\dot{x}$ is in N).

(5) implies (4) is trivial.

To see that (4) implies (3), let $A \in N$ be a maximal antichain. In N, let $f : A \to \mathrm{On}$ be one-to-one. We may think of f as a $\mathbb{P}$-name $\dot{\alpha}$ for an ordinal, namely, $p \Vdash \dot{\alpha} = f(p)$ for $p \in A$. Since $f \in N$, $\dot{\alpha} \in N$ is immediate. By (4), $q \Vdash \dot{\alpha} \in N$. Since $p \in N$ iff $f(p) \in N$, $q \Vdash A \cap N \cap \dot{G} \neq \emptyset$. $\square$

A forcing notion $\mathbb{P}$ is called *proper* if for all large enough regular cardinals χ, all countable $N \prec H(\chi)$ with $\mathbb{P} \in N$, and all conditions $p \in \mathbb{P} \cap N$, there is $q \leq p$ which is $(N, \mathbb{P})$-generic.

Properness has a number of equivalent definitions, one in terms of stationary sets, one in terms of games, and one in terms of distributive laws for cBa's. However, the model-theoretic definition given above is by far the most useful one, and we shall not discuss the other definitions in detail. Let us prove, however, that properness implies that stationary subsets of ω_1 are preserved.

Theorem 4.3. *Let $\mathbb{P}$ be proper. Assume $S \subseteq \omega_1$ is stationary. Then $\mathbb{P}$ forces that S is stationary.*

Proof. The proof is a paradigmatic *properness argument*, and we go through the details.

Let $\dot{C}$ be a $\mathbb{P}$-name for a club subset of ω_1. We need to show that $\mathbb{P}$ forces that $\dot{C} \cap S \neq \emptyset$. Let $p \in \mathbb{P}$. We need to find $q \leq p$ and an ordinal γ such that $q \Vdash \gamma \in \dot{C} \cap S$.

Choose χ large enough and $N \prec H(\chi)$ countable such that $\mathbb{P}, p, S, \dot{C} \in N$ and $N \cap \omega_1 \in S$. To see that this is possible, first note that $N \cap \omega_1$ must indeed be an ordinal: let $\alpha \in N \cap \omega_1$. Since $H(\chi)$ thinks α is countable, so does N. Therefore N contains a bijection $f : \omega \to \alpha$. Since $\omega \subseteq N$, $\alpha \subseteq N$ follows, and $N \cap \omega_1$ is an ordinal. Next note that the collection $\{N \cap \omega_1 : N \prec H(\chi)\}$ contains a club subset of ω_1. This is so because the union of a countable increasing chain of countable elementary submodels is again a countable elementary submodel. Thus we may indeed choose N with $N \cap \omega_1 \in S$.

Let q be $(N, \mathbb{P})$-generic. Since $N \cap \omega_1 \in S$, the following claim completes the proof of the theorem.

Claim 4.4. $q \Vdash N \cap \omega_1 \in \dot{C}$.

Proof. For fixed $\alpha \in \omega_1 \cap N$, let $D_\alpha = \{r \in \mathbb{P} : r \Vdash \beta \in \dot{C} \text{ for some } \beta \geq \alpha\}$. Clearly $D_\alpha \in N$. Let $r \in D_\alpha \cap N$. Then we have

$$H(\chi) \models \exists \beta \geq \alpha \ \ r \Vdash \beta \in \dot{C}.$$

By elementarity, we see

$$N \models \exists \beta \geq \alpha \ \ r \Vdash \beta \in \dot{C}.$$

Hence r forces $\beta \in \dot{C}$ for some β with $\alpha \leq \beta < N \cap \omega_1$. Since $D_\alpha \cap N$ is predense below q,

$$q \Vdash \text{“}\exists \beta \text{ with } \alpha \leq \beta < N \cap \omega_1 \text{ and } \beta \in \dot{C}.\text{”}$$

Unfixing α, we see that q forces this for all $\alpha \in \omega_1 \cap N$. Since $\dot{C}$ is forced to be club, q forces $\omega_1 \cap N \in \dot{C}$, as required. □

□

Corollary 4.5. *Let $\mathbb{P}$ be proper. Then $\mathbb{P}$ preserves $\aleph_1$.*

Proof. Let S be any subset of ω_1 such that $\omega_1 \setminus S$ is unbounded. If $\mathbb{P}$ forces that $|\omega_1^V| = \aleph_0$, then $\mathbb{P}$ forces that there is an increasing ω-sequence in $\omega_1 \setminus S$ converging to ω_1. This sequence trivially is club, so that S cannot be stationary in the generic extension. Hence many stationary sets are destroyed. □

An alternative argument for 4.5 goes by proving first that a proper p.o. forces that any countable set of ordinals is covered by a countable set of ordinals from the ground model (see, e.g, [Go, Fact 3.13] or [Sh, III Lemma 1.16]).

Proper forcing notions form a much larger class than ccc forcing notions.

Proposition 4.6. *1. Any ccc forcing is proper.*

2. Any σ-closed forcing is proper.

Proof. (1) Let $\mathbb{P}$ be ccc. Let χ and $N \prec H(\chi)$ be as required, and notice that *any* condition $p \in \mathbb{P}$ is $(N, \mathbb{P})$-generic. To see this, let $A \in N$ be a maximal antichain in $\mathbb{P}$. By ccc, A is countable (in $H(\chi)$), and by elementarity, this is true in N as well. Hence there is a bijection $f : \omega \to A$ in N. Since $\omega \subseteq N$, $A \subseteq N$ follows. But clearly, p forces that $A \cap \dot{G}$ is nonempty.

(2) Let $\mathbb{P}$ be σ-closed. Again let χ and $N \prec H(\chi)$ be as required, and let $p \in \mathbb{P} \cap N$. Let $\{D_i : i \in \omega\}$ enumerate all dense subsets of $\mathbb{P}$ which belong to N. Recursively construct conditions $p_i \in N$ such that $p_0 = p$ and $p_{i+1} \leq p_i$ belongs to $D_i \cap N$. This is clearly possible. Since $\mathbb{P}$ is σ-closed, the sequence of the p_i has a lower bound q. Then $q \Vdash p_i \in \dot{G}$ for all i. Hence q is $(N, \mathbb{P})$-generic. □

We shall see further examples for proper forcing notions in the next two lectures.

For a more comprehensive discussion of proper forcing and countable support iteration of proper forcing, see [Sh, Chapter III], [Go] or [Ab].

4.3. Preservation of properness in countable support iterations.

Lemma 4.7. *Let $\mathbb{P} \star \dot{\mathbb{Q}}$ be a two-step iteration. Let $N \prec H(\chi)$ be countable with $\mathbb{P} \star \dot{\mathbb{Q}} \in N$. Then $(p, \dot{q})$ is $(N, \mathbb{P} \star \dot{\mathbb{Q}})$-generic iff p is $(N, \mathbb{P})$-generic and p forces that $\dot{q}$ is $(N[\dot{G}], \dot{\mathbb{Q}})$-generic. In particular, if $\mathbb{P}$ is proper and forces that $\dot{\mathbb{Q}}$ is proper, then $\mathbb{P} \star \dot{\mathbb{Q}}$ is proper.*

Proof. Clearly,

$$(p, \dot{q}) \Vdash N[\dot{G}][\dot{H}] \cap \mathrm{On} = N \cap \mathrm{On}$$

is equivalent to

$$p \Vdash \text{“}N[\dot{G}] \cap \mathrm{On} = N \cap \mathrm{On} \text{ and } \dot{q} \Vdash N[\dot{G}][\dot{H}] \cap \mathrm{On} = N[\dot{G}] \cap \mathrm{On}\text{”}.$$

Hence the first part follows from Lemma 4.2.

To see that properness is preserved in two-step iterations, let χ and $N \prec H(\chi)$ be as required, and let $(p, \dot{q}) \in N$. Since $\mathbb{P}$ is proper, there is $p' \leq p$ such that p' is $(N, \mathbb{P})$-generic. Since $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is proper and that $N[\dot{G}] \prec H(\chi)[\dot{G}] = H(\chi)^{V[\dot{G}]}$, there is a $\mathbb{P}$-name $\dot{q}'$ such that p forces that $\dot{q}' \leq \dot{q}$ is $(N[\dot{G}], \dot{\mathbb{Q}})$-generic. $(p', \dot{q}') \leq (p, \dot{q})$ follows and, by the first part, $(p', \dot{q}')$ is $\mathbb{P} \star \dot{\mathbb{Q}}$-generic. ∎

Main Lemma 4.8. *Let $\bar{\mathbb{P}} = (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be a csi such that $\Vdash_\alpha$ “$\dot{\mathbb{Q}}_\alpha$ is proper”. Let $N \prec H(\chi)$ be countable with $\bar{\mathbb{P}} \in N$. Then, for all $\beta \in N \cap (\delta + 1)$, all $\alpha \in N \cap (\beta + 1)$, and all $\mathbb{P}_\alpha$-names $\dot{p} \in N$ for a condition in $\mathbb{P}_\beta$, if*

- (i) *$q \in \mathbb{P}_\alpha$,*
- (ii) *q is $(N, \mathbb{P}_\alpha)$-generic,*
- (iii) *$q \Vdash_\alpha \dot{p} \restriction \alpha \in \dot{G}_\alpha \cap N$,*

then there is q' such that

- (i') *$q' \in \mathbb{P}_\beta$, $q' \restriction \alpha = q$,*
- (ii') *q' is $(N, \mathbb{P}_\beta)$-generic,*
- (iii') *$q' \Vdash_\beta \dot{p} \in \dot{G}_\beta \cap N$.*

Proof. We make induction on $\beta \geq \alpha$.

Basic step. The case $\beta = \alpha$ is trivial.

Successor step. Let $\beta = \gamma + 1$. By induction hypothesis, we know the lemma holds for the pair (α, γ), and we may thus assume without loss of generality that $\alpha = \gamma$. Since $\dot{\mathbb{Q}}_\gamma$ is forced to be proper, there is a $\mathbb{P}_\gamma$-name $\dot{q}$ such that q forces

"$\dot{q} \in \dot{\mathbb{Q}}_\gamma$, $\dot{q} \leq \dot{p}(\gamma)$ and $\dot{q}$ is $(N[\dot{G}_\gamma], \dot{\mathbb{Q}}_\gamma)$-generic". By Lemma 4.7, we know that $q' = (q, \dot{q})$ is $(N, \mathbb{P}_\beta)$-generic. Since q' forces that $\dot{p}(\gamma)$ belongs to $\dot{G}_{\dot{\mathbb{Q}}_\gamma} \cap N[\dot{G}_\gamma]$, it also forces that $\dot{p}$ belongs to $\dot{G}_\beta \cap N$.

Limit step. Let $\beta \in N$ be a limit ordinal. Put $\gamma = \sup(N \cap \beta)$. Clearly, γ is a limit ordinal of countable cofinality. Let $\alpha_n \in N$ be such that $\alpha_0 = \alpha$ and $\sup_n \alpha_n = \gamma$. Also let D_n, $n \in \omega$, list the dense sets of $\mathbb{P}_\beta$ belonging to N.

First: construct $\dot{p}_n \in N$ such that $\dot{p}_0 = \dot{p}$, all $\dot{p}_n$ are $\mathbb{P}_{\alpha_n}$-names for conditions in $\mathbb{P}_\beta$, and

- $\Vdash_{\alpha_{n+1}} \dot{p}_{n+1} \leq \dot{p}_n$,
- $\Vdash_{\alpha_{n+1}} \dot{p}_{n+1} \in D_n$,
- $\Vdash_{\alpha_{n+1}}$ "if $\dot{p}_n{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$ then $\dot{p}_{n+1}{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$".

Let us argue why this is possible. Work in N. It suffices to prove that given $r \in \mathbb{P}_{\alpha_{n+1}} \cap N$, we can find $s \in \mathbb{P}_{\alpha_{n+1}} \cap N$, $s \leq r$, and $p_{n+1} \in N$ such that

- $s \Vdash_{\alpha_{n+1}} p_{n+1} \leq \dot{p}_n$,
- $p_{n+1} \in D_n$,
- $s \Vdash_{\alpha_{n+1}}$ "if $\dot{p}_n{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$ then $p_{n+1}{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$".

For then we can produce the required $\mathbb{P}_{\alpha_{n+1}}$-name in N. By strengthening r, we may assume that it decides $\dot{p}_n$, say $r \Vdash \dot{p}_n = p_n$ for some $p_n \in N$. We may also assume r decides whether $p_n{\restriction}\alpha_{n+1}$ belongs to $\dot{G}_{\alpha_{n+1}}$.

Assume first r forces that $p_n{\restriction}\alpha_{n+1} \notin \dot{G}_{\alpha_{n+1}}$. Since p_n and D_n both belong to N and $N \models$ "D_n is dense in $\mathbb{P}_\beta$", we may find $p_{n+1} \leq p_n$ in N belonging to D_n, and $s = r$ works.

Next assume r forces $p_n{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$. Then $r \leq p_n{\restriction}\alpha_{n+1}$ must hold. So r and p_n have a common lower bound $r' \in \mathbb{P}_\beta$ in N. Let $p_{n+1} \leq r'$ be an element of D_n in N. Then $s := p_{n+1}{\restriction}\alpha_{n+1}$ forces that $p_{n+1}{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$, and $s \leq r$.

Next: construct q_n such that

- $q_0 = q$, $q_n \in \mathbb{P}_{\alpha_n}$, $q_{n+1}{\restriction}\alpha_n = q_n$,
- q_n is $(N, \mathbb{P}_{\alpha_n})$-generic,
- $q_n \Vdash_{\alpha_n} \dot{p}_n{\restriction}\alpha_n \in \dot{G}_{\alpha_n} \cap N$.

q_{n+1} is obtained from q_n by applying the induction hypothesis to α_n, α_{n+1}, and $\dot{p}_n{\restriction}\alpha_{n+1}$. Thus we obtain that q_{n+1} forces that $\dot{p}_n{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}} \cap N$. By applying the above property of the $\dot{p}_n$, we also obtain that q_{n+1} forces that $\dot{p}_{n+1}{\restriction}\alpha_{n+1} \in \dot{G}_{\alpha_{n+1}} \cap N$, as required. This completes the recursive construction.

Let $q' = \bigcup_n q_n$. This is a condition in $\mathbb{P}_\beta$ (even in $\mathbb{P}_\gamma$) and we have $q' \leq q_n$ for all n. We claim that q' forces that $\dot{p}_n \in \dot{G}_\beta \cap N$. To see this, first note that since q_n is generic and $q' \leq q_n$ by construction, q' forces that $\dot{p}_n \in N$. Next notice that q' forces that $\dot{p}_n{\restriction}\alpha_n \in \dot{G}_{\alpha_n} \cap N$. Since the $\dot{p}_n$ are forced to be decreasing, this means that q' forces that $\dot{p}_n \restriction \alpha_m \in \dot{G}_{\alpha_m} \cap N$ for all $m \geq n$. Let $q'' \leq q'$ be an arbitrary condition deciding $\dot{p}_n$, say $q'' \Vdash_\beta \dot{p}_n = p_n$ for some $p_n \in N$. Since $q'' \Vdash_\beta p_n \restriction \alpha_m \in \dot{G}_{\alpha_m}$ for all m, we obtain $q'' \restriction \alpha_m \leq p_n \restriction \alpha_m$ for all m. Next, $p_n \in N$ implies that $\mathrm{supp}(p_n) \subseteq N$ so that $\mathrm{supp}(p_n) \subseteq \gamma$. This means we may construe p_n as a condition in $\mathbb{P}_\gamma$, and $q'' \leq p_n$ now follows. Thus $q'' \Vdash_\beta p_n \in \dot{G}_\beta$. Since q'' was arbitrary below q', $q' \Vdash_\beta \dot{p}_n \in \dot{G}_\beta$ as well.

In particular, q' forces $\dot{p} \in \dot{G}_\beta \cap N$ so that (iii') holds, and it also forces that $\dot{p}_{n+1} \in D_n \cap \dot{G}_\beta \cap N$ so that (ii') holds. □

Theorem 4.9. *Csi of proper forcing notions are proper. I.e., if $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is a csi such that $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is proper", then $\mathbb{P}_\delta$ is proper.*

Proof. This is a consequence of Main Lemma 4.8 with $\alpha = 0$ and $\beta = \delta$. □

Theorem 4.10. *Assume CH. Let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2)$ be a csi of proper forcing such that $\mathbb{P}_\alpha$ forces $\dot{\mathbb{Q}}_\alpha$ has size at most $\aleph_1$, for all α. Then $\mathbb{P}_\alpha$, $\alpha < \omega_2$, has a dense subset of size $\aleph_1$ and CH still holds in $V^{\mathbb{P}_\alpha}$. Furthermore, $\mathbb{P}_{\omega_2}$ is $\aleph_2$-cc and thus preserves all cardinals.*

This is similar to 4.9 and 4.8. See [Ab, Theorem 2.10] or [Sh, III Theorem 4.1].

Next comes the analog of 3.3 (see also 1.5 and 2.9).

Lemma 4.11. *Let δ be a limit ordinal of uncountable cofinality. Assume $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ is a csi of proper forcing notions. Let f be a new real in $V^{\mathbb{P}_\delta}$. Then there is $\alpha < \delta$ such that f already belongs to $V^{\mathbb{P}_\alpha}$.*

Proof. Let $\dot{f}$ be a $\mathbb{P}_\delta$-name for f. By 4.9, we know that $\mathbb{P}_\delta$ is proper. Let χ be large enough, let $p \in \mathbb{P}_\delta$, and let $N \prec H(\chi)$ be such that $\dot{f}, p$ in N. Put $\alpha = \sup(N \cap \delta)$. By properness, there is $q \leq p$ which is $(N, \mathbb{P}_\delta)$-generic. We claim that q forces that $\dot{f}$ belongs to the extension via $\mathbb{P}_\alpha$.

Indeed, if for each n we let A_n be the maximal antichain of conditions deciding $\dot{f}(n)$, then $A_n \in N$, and q forces that the generic meets $A_n \cap N$. However all conditions in $A_n \cap N$ have support contained in α. Therefore, if we know the $\mathbb{P}_\alpha$-generic filter, we also know which condition of $A_n \cap N$ is in the filter, and therefore we know the value of $\dot{f}(n)$. Hence the interpretation of $\dot{f}$ is in the $\mathbb{P}_\alpha$-generic extension. □

4.4. Limitations of the method of csi of proper forcing.

An argument exactly as in Lemma 3.14 shows that csi add Cohen subsets of ω_1 in limit steps of cofinality ω_1.

Lemma 4.12. *Let δ be an ordinal of cofinality ω_1 and let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ be a csi such that all $\dot{\mathbb{Q}}_\alpha$ are forced to be nontrivial. Then $\mathbb{P}_\delta$ adds a subset of ω_1 which is $\mathsf{Fn}(\omega_1, 2, \omega_1)$-generic over V (and even over all $V^{\mathbb{P}_\alpha}$, $\alpha < \delta$).*

Proof. Exercise! □

It is well-known that $\mathsf{Fn}(\omega_1, 2, \omega_1)$ forces CH (and even $\diamondsuit$), see [Ku1, VII Theorem 8.3]. As a consequence we obtain

Corollary 4.13. *Let* $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 + \omega_1)$ *be a csi of proper forcing. Then* $\mathbb{P}_{\omega_2+\omega_1}$ *forces* CH. *In particular, if all* $\dot{\mathbb{Q}}_\alpha$ *add reals, then* $\aleph_2$ *is collapsed to* $\aleph_1$.

Proof. By 4.11, any new real added by $\mathbb{P}_{\omega_2+\omega_1}$ is already added by $\mathbb{P}_\alpha$ for some $\alpha < \omega_2 + \omega_1$. The argument of the proof of 4.12 in fact shows that the $\mathsf{Fn}(\omega_1, 2, \omega_1)$-generic added by $\mathbb{P}_{\omega_2+\omega_1}$ codes all reals adjoined at an earlier stage. Since it has size ω_1, it can code only ω_1 many reals, and $\mathfrak{c} = \aleph_1$ follows.

If all $\dot{\mathbb{Q}}_\alpha$ do add reals, then $\aleph_2^V$ many reals get added. Hence $\aleph_2^V$ is collpased to $\aleph_1$. $\square$

Lecture 5: Sacks forcing

5.1. Sacks forcing. A tree $T \subseteq 2^{<\omega}$ is called *perfect* if for all $t \in T$ there is $u \supseteq t$ such that both $u\hat{\ }0$ and $u\hat{\ }1$ belong to T. T is perfect iff the closed set $[T]$ consisting of the branches of T (see Lecture 2) is *perfect*; that is, $[T]$ has no isolated points. *Sacks forcing* $\mathbb{S}$ consists of all perfect trees ordered by inclusion; i.e., $S \leq T$ iff $S \subseteq T$. It generically adds a new real s with $\{s\} = \bigcap\{[T] : T \in G\}$ where G denotes the $\mathbb{S}$-generic filter over V. To see that this intersection is indeed a singleton, first note that it must be non-empty because G is a filter and the sets $[T]$ are compact. Then notice that by genericity it cannot contain more than one element. s is called a *Sacks real*. As usual, $V[s] = V[G]$.

An alternative description of s is obtained as follows. The *stem* of a (perfect) tree T, $\mathrm{stem}(T)$, is the unique $u \in T$ such that $u\hat{\ }0, u\hat{\ }1 \in T$ and u is comparable with all members of T. Then $s = \bigcup\{\mathrm{stem}(T) : T \in G\}$.

It is easy to see that every perfect T can be split into continuum many perfect trees T_α, $\alpha < \mathfrak{c}$, which have pairwise finite intersection (this is like the construction of an almost disjoint family of size continuum). In particular, for $\alpha \neq \beta$, $[T_\alpha] \cap [T_\beta] = \emptyset$, and T_α and T_β are incompatible. Thus $\mathbb{S}$ is not $\mathfrak{c}$-cc, but of course it is $\mathfrak{c}^+$-cc (because $|\mathbb{S}| = \mathfrak{c}$). Hence, under CH, it is $\aleph_2$-cc and preserves cardinals $\geq \aleph_2$. Therefore, the main issue is the preservation of $\aleph_1$. This is obtained by a so-called *fusion argument*. This kind of argument is very common for forcing notions consisting of trees, and in this sense, the proof of 5.1 below is paradigmatic.

For $u \in T$, define $T_u = \{t \in T : t \subseteq u \text{ or } u \subseteq t\}$, the *subtree of T determined by u*. By recursion on n, define the *n-th splitting level* $\mathrm{split}_n(T)$ of T, as follows. $\mathrm{split}_0(T) = \{\mathrm{stem}(T)\}$, and $\mathrm{split}_{n+1}(T) = \{\mathrm{stem}(T_{u\hat{\ }i}) : u \in \mathrm{split}_n(T)$ and $i \in 2\}$. Notice that $|\mathrm{split}_n(T)| = 2^n$. The *fusion orderings* $\leq_n$ are given by $S \leq_n T$ if $S \leq T$ and $\mathrm{split}_n(S) = \mathrm{split}_n(T)$. Clearly, $S \leq_{n+1} T$ implies $S \leq_n T$ which in turn implies $S \leq T$. A sequence of conditions T_n, $n \in \omega$, is a *fusion sequence* if $T_{n+1} \leq_n T_n$ for all n. For such a fusion sequence, define the *fusion* $S := \bigcap_n T_n$. It is easy to see that the fusion is again a perfect tree. In fact, it can also be described as $\{u : \exists n \exists t \ (t \in \mathrm{split}_n(T_n) \text{ and } u \subseteq t)\}$.

Main Lemma 5.1. (fusion for Sacks forcing) *Let $N \prec H(\chi)$ be countable, and let $T \in \mathbb{S} \cap N$. Let A_n, $n \in \omega$, list (some of) the maximal antichains of $\mathbb{S}$ which belong to N. Then there are $S \leq T$ and finite sets $B_n \subseteq A_n$ of size at most 2^n with $B_n \in N$ which are predense below S. In particular, if the A_n list all maximal antichains of N, S is $(N, \mathbb{S})$-generic.*

Proof. In N, we recursively construct conditions T_n and finite sets B_n such that

- $T_0 \leq T$,
- $T_{n+1} \leq_n T_n$,

- B_n is predense below T_n,
- $|B_n| \leq 2^n$.

We emphasize that while the list of all A_n in general does not belong to N, any finite initial segment of this list does. Therefore any finite segment of this construction can be done in N (in particular, the T_n and B_n belong to N), while the construction as a whole is *not* an element of N.

For the basic step note that, by elementarity, $N \models$ "A_0 is a maximal antichain". Hence there is $U_0 \in A_0 \cap N$ compatible with T. Let $T_0 \in N$ be a common extension and put $B_0 = \{U_0\}$.

Suppose T_n and B_n have been constructed. For $u \in \mathrm{split}_n(T_n)$ and $i \in 2$, consider $(T_n)_{u^\frown i} \in N$. There is $U_{u^\frown i} \in A_{n+1} \cap N$ compatible with $(T_n)_{u^\frown i}$. Let $(T_{n+1})_{u^\frown i} \in N$ be a common extension of these two conditions, and put $T_{n+1} = \bigcup\{(T_{n+1})_{u^\frown i} : u \in \mathrm{split}_n(T_n) \text{ and } i \in 2\}$. Then T_{n+1} belongs again to N, $T_{n+1} \leq_n T_n$ holds, and $B_{n+1} = \{U_{u^\frown i} : u \in \mathrm{split}_n(T_n) \text{ and } i \in 2\}$ is a finite subset of A_{n+1} of size at most 2^{n+1} which belongs to N and which is predense below T_{n+1}. This completes the recursive construction.

Now, let S be the fusion of the sequence of T_n's. Since $S \leq_n T_n$ for all n, all B_n are predense below S, as required. Since $B_n \in N$ is finite, $B_n \subseteq N$ is immediate, and therefore S is $(N, \mathbb{S})$-generic. $\square$

Corollary 5.2. *Sacks forcing $\mathbb{S}$ is proper and thus preserves $\aleph_1$ and, under CH, all cardinals.*

Proof. Properness is immediate from Main Lemma 5.1. The preservation of $\aleph_1$, then, follows from Corollary 4.5. $\square$

We say that a forcing notion $\mathbb{P}$ has the *Sacks property* if for every condition $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{f} \in \omega^\omega$ there are a condition $q \leq p$ and a function $F : \omega \to [\omega]^{<\omega}$ with $|F(n)| \leq 2^n$ such that q forces that $\dot{f}(n) \in F(n)$ for all n. Recall (Lecture 2) that $\mathbb{P}$ is *ω^ω-bounding* if for all $p \in \mathbb{P}$ and all $\mathbb{P}$-names $\dot{f} \in \omega^\omega$ there are $q \leq p$ and $g \in \omega^\omega$ such that q forces that $\dot{f}(n) \leq g(n)$ for all n. The following is easy to see.

Observation 5.3. *If $\mathbb{P}$ has the Sacks property, then $\mathbb{P}$ is ω^ω-bounding.*

The converse need not be true. In fact, random forcing is ω^ω-bounding (see Lemma 2.14), but does not have the Sacks property (the latter follows from properties of random forcing discussed in Lecture 2 and the consequences of the Sacks property exhibited below).

Corollary 5.4. *Sacks forcing $\mathbb{S}$ has the Sacks property.*

Proof. Let $T \in \mathbb{S}$ and let $\dot{f}$ be an $\mathbb{S}$-name for a real in ω^ω. Let $N \prec H(\chi)$ be countable such that $T, \dot{f} \in N$. Let A_n be a maximal antichain of conditions deciding $\dot{f}(n)$. Clearly $A_n \in N$, and we may apply Main Lemma 5.1 to obtain $S \leq T$ and B_n of size at most 2^n such that B_n is predense below S. Let $F(n) = \{k : \exists U \in B_n \ \ U \Vdash \dot{f}(n) = k\}$. Then $F(n)$ has size at most 2^n, and S forces that $\dot{f}(n)$ belongs to $F(n)$ for all n. $\square$

5.2. The countable support product of Sacks forcing. How do we add many Sacks reals? As we saw earlier (Lemma 4.1 at the beginning of Lecture 4), a finite support product or a finite support iteration of Sacks forcing will collapse cardinals. By the discussion of properness (Theorem 4.9 in Lecture 4), we know we can iterate Sacks forcing with countable support (see below).

However, there is an alternative: the *countable support product* (*csp*, for short) of Sacks forcing. Let I be an index set. The forcing $\mathbb{S}_I$ consists of all functions $f : I \to \mathbb{S}$ such that $\mathrm{supp}(f) = \{i \in I : f(i) \neq 1_{\mathbb{S}}\}$ is at most countable. $\mathbb{S}_I$ is ordered coordinatewise by $g \leq f$ if $g(i) \leq f(i)$ for all i (this implies in particular that $\mathrm{supp}(g) \supseteq \mathrm{supp}(f)$). $\mathbb{S}_I$ adds $|I|$ many Sacks reals. Namely, each s_i added in coordinate $i \in I$ is Sacks over the ground model.

Properties of $\mathbb{S}_I$ are very similar to those of $\mathbb{S}$, and we refrain from giving detailed proofs (basically, the arguments are just combinatorially more complicated versions of the arguments for $\mathbb{S}$). In particular:

Lemma 5.5. *$\mathbb{S}_I$ is a proper forcing notion with the Sacks property. Thus it preserves $\aleph_1$ and, under CH, all cardinals. Also it is ω^ω-bounding.*

However, unlike for Cohen and random forcing (Lemmata 1.6 and 2.13), we do *not* have a product lemma for the csp of Sacks forcing. In particular, if $i \neq j$, s_j is *not* $\mathbb{S}$-generic over $V[s_i]$.

We saw earlier (Lemma 2.14 and Theorem 2.17) that the ω^ω-bounding property is closely related to (preservation of) the cardinal invariant $\mathfrak{d}$. In a similar vein, the Sacks property is closely related to (preservation of) $\mathsf{cof}(\mathcal{N})$. Say $F : \omega \to [\omega]^{<\omega}$ is a *slalom* if $|F(n)| \leq 2^n$ for all n.

Theorem 5.6. (Bartoszyński's characterization of additivity and cofinality of the null ideal)

1. $\mathsf{add}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ *and for all slaloms F there is $f \in \mathcal{F}$ such that $f(n) \notin F(n)$ for infinitely many n*$\}$.
2. $\mathsf{cof}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F}$ *is a family of slaloms and for all $f \in \omega^\omega$ there is $F \in \mathcal{F}$ such that $f(n) \in F(n)$ for (almost) all n*$\}$.

See [BJ, Theorem 2.3.9].

Theorem 5.7. *Assume CH holds in the ground model V. Let κ be arbitrary, and let $\mathbb{S}_\kappa$ be the csp of $\mathbb{S}$. In $V^{\mathbb{S}_\kappa}$, $\mathsf{cof}(\mathcal{N}) = \aleph_1$. If additionally $\kappa^\omega = \kappa$, $\mathfrak{c} = \kappa$ holds in the generic extension.*

Proof. By 5.5, $\mathbb{S}_\kappa$ has the Sacks property. Thus, for every $f \in \omega^\omega \cap V^{\mathbb{S}_\kappa}$ there is a slalom $F \in V$ such that $f(n) \in F(n)$ for all n. By CH and 5.6, $\mathsf{cof}(\mathcal{N}) = \aleph_1$ follows.

Since $\mathbb{S}_\kappa$ adds κ many Sacks reals, $\mathfrak{c} \geq \kappa$ is obvious. A standard argument shows that if $\kappa^\omega = \kappa$, then there are only κ many canonical $\mathbb{S}_\kappa$-names for reals. Thus $\mathfrak{c} \leq \kappa$ as well. $\square$

5.3. The countable support iteration of Sacks forcing. We have the following general preservation theorem for the Sacks property.

Theorem 5.8. *Csi of proper forcing notions with the Sacks property have the Sacks property. I.e., if* $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta)$ *is a csi such that* $\Vdash_\alpha$ *"*$\dot{\mathbb{Q}}_\alpha$ *is proper and has the Sacks property", then* $\mathbb{P}_\delta$ *has the Sacks property.*

This is similar to 4.9. See [Go, Section 6, Application 5].

Theorem 5.9. *Assume* CH *holds in the ground model* V*. Let* $\mathbb{S}_{\omega_2}$ *be the csi of* $\mathbb{S}$*. In* $V^{\mathbb{S}_{\omega_2}}$*,* $\mathsf{cof}(\mathcal{N}) = \aleph_1$ *and* $\mathfrak{c} = \aleph_2$*.*

Proof. This is like the proof of Theorem 5.7. However, instead of 5.5, we use 5.8.

More explicitly, by 5.2 and 4.9, $\mathbb{S}_{\omega_2}$ is proper and preserves $\aleph_1$, and by CH and 4.10, it preserves $\aleph_2$. Moreover, by 5.4 and 5.8, $\mathbb{S}_{\omega_2}$ has the Sacks property, and $\mathsf{cof}(\mathcal{N}) = \aleph_1$ follows from 5.6 and CH. Again, it is obvious that $\mathbb{S}_{\omega_2}$ adds exactly $\aleph_2$ many reals so that $\mathfrak{c} = \aleph_2$ follows. $\square$

In fact, one also has $\mathfrak{r} = \aleph_1$ in the iterated Sacks model. For this, one uses the *P-point preservation theorem* [BJ, Theorem 6.2.6].

Lecture 6: Mathias forcing

6.1. Mathias forcing. *Mathias forcing* $\mathbb{M}$ consists of all pairs (s, A) where s is a finite subset of ω, A is an infinite subsets of ω, and $\max(s) < \min(A)$. The order is given by $(t, B) \leq (s, A)$ if $t \supseteq s$, $B \subseteq A$, and $t \setminus s \subseteq A$. It generically adds a new real m with $m = \bigcup\{s : (s, A) \in G \text{ for some } A\}$ where G denotes the $\mathbb{M}$-generic filter over V. m is called a *Mathias real*. As usual, $V[m] = V[G]$. m is an unsplit real (this is a straightforward genericity argument):

Observation 6.1. *For all ground model reals $A \in [\omega]^\omega$, either $m \subseteq^* A$ or $m \cap A$ is finite.*

Also, m codes a dominating real:

Observation 6.2. *Let f be the increasing enumeration of m; i.e., $f(i)$ is the i-th element of m. Then f eventually dominates all ground model reals.*

Proof. Let $(s, A) \in \mathbb{M}$ and $g \in \omega^\omega$. Let $k = |s|$. Then find $B \subseteq A$ such that the i-th element of B is larger than $g(k + i)$. By the definition of the ordering of $\mathbb{M}$, it is clear that (s, B) forces that the $k + i$-th element of $\dot{m}$ is larger than $g(k + i)$, for all i. $\square$

If A_α, $\alpha < \mathfrak{c}$, is an almost disjoint family of infinite subsets of ω of size $\mathfrak{c}$, the conditions $(\emptyset, A_\alpha)$ are pairwise incompatible. Thus $\mathbb{M}$ is not $\mathfrak{c}$-cc but $\mathfrak{c}^+$-cc.

For $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, let $A - s = \{n \in A : n > \max(s)\}$. For $A \in [\omega]^\omega$ let $\{a^j : j \in \omega\}$ be its increasing enumeration. Also put $A^i = \{a^j : j \geq i\}$. The *fusion orderings* $\leq_i$ are given by $(t, B) \leq_i (s, A)$ if $(t, B) \leq (s, A)$, $t = s$, and the first i elements of B and A are the same. A sequence (s, A_i) is a *fusion sequence* if $(s, A_{i+1}) \leq_i (s, A_i)$ for all i. The *fusion* is (s, B) with $B = \bigcap_i A_i$. Then $(s, B) \leq_i (s, A_i)$ for all i.

Main Lemma 6.3. (fusion for Mathias forcing) *Let $N \prec H(\chi)$ be countable, and let $(s, A) \in \mathbb{M} \cap N$. Let D_n list (some of) the open dense sets of $\mathbb{M}$ which belong to N. Then there is $(s, B) \leq (s, A)$ such that for all i, whenever $n < i$, $t \subseteq \{b^j : j < i\}$, and $C \subseteq B$ are such that $(s \cup t, C) \in D_n$, then $(s \cup t, B^i) \in D_n$. If the D_n list all open dense sets of N, (s, B) is $(N, \mathbb{M})$-generic. In particular, it follows that $\mathbb{M}$ is proper and preserves $\aleph_1$.*

Proof. In N recursively construct sets A_i such that

- $A_0 = A$,
- $(s, A_{i+1}) \leq_i (s, A_i)$,
- whenever $n < i$, $t \subseteq \{a^j_i : j < i\}$, and $C \subseteq A_i$ are such that $(s \cup t, C) \in D_n$, then $(s \cup t, A^i_i) \in D_n$. $(\star)$

Again, all finite initial segments of this construction belong to N.

The basic step is clear. Suppose A_i has been constructed. Let (n_k, t_k), $k < \ell$, list all pairs (n, t) such that $n < i+1$ and $t \subseteq \{a_i^j : j < i+1\}$. In N, by recursion on $k \leq \ell$, construct decreasing $A_{i,k}$: let $A_{i,0} = A_i$. Suppose $A_{i,k}$ has been produced. If there is $C \subseteq A_{i,k}$ such that $(s \cup t_j, C) \in D_n$, then let $A_{i,k+1} = \{a_i^j : j < i+1\} \cup C$. Otherwise, let $A_{i,k+1} = A_{i,k}$. Let $A_{i+1} = A_{i,\ell}$. Since $\{a_{i+1}^j : j < i+1\} = \{a_i^j : j < i+1\}$, A_{i+1} has the required properties. This completes the recursive construction.

Let (s, B) be the fusion. It is clear that (s, B) has the required property, but we still need to check that it is $(N, \mathbb{M})$-generic. Let $(u, C) \leq (s, B)$ and $n \in \omega$. We need to find a condition in $D_n \cap N$ which is compatible with (u, C). By further extending (u, C), if necessary, we may assume $(u, C) \in D_n$. Let $u = s \cup t$. Then, for some $i > n$, $t \subseteq \{b^j : j < i\} = \{a_i^j : j < i\}$. We made $(\star)$ above hold in N, but by elementarity it holds in $H(\chi)$ as well. Since the antecedent of $(\star)$ is true in $H(\chi)$, the conclusion holds as well, and therefore $(u, A_i^i) \in D_n$. This completes the proof because $(u, C) \leq (u, A_i^i)$ and $(u, A_i^i) \in N$. □

Here are further properties of $\mathbb{M}$. They use the previous lemma together with some additional fusion argument.

Lemma 6.4. (pure decision property) *Let φ be a sentence of the forcing language. Assume $(s, A) \in \mathbb{M}$. Then there is $B \in [A]^\omega$ such that (s, B) decides φ (i.e., either $(s, B) \Vdash \varphi$ or $(s, B) \Vdash \neg\varphi$).*

Proof. Let D be the open dense set of conditions deciding φ. By 6.3, we may assume that whenever $t \subseteq A$, and $C \subseteq A$ are such that $(s \cup t, C)$ decides φ, then $(s \cup t, A - t)$ decides φ. $(\star\star)$

We claim that if $t \subseteq A$ and $(s \cup t, A - t)$ does not decide φ, then for almost all $k \in A - t$, $(s \cup t \cup \{k\}, A - \{k\})$ does not decide φ. Indeed, if there was an infinite $B \subseteq A - t$ such that $(s \cup t \cup \{k\}, A - \{k\})$ forces φ for all $k \in B$, then $(s \cup t, B)$ forces φ. By $(\star\star)$ we see that $(s \cup t, A - t)$ also forces φ, a contradiction. Similarly, if $\neg\varphi$ is forced for all $k \in B$.

Assume now that (s, A) does not decide φ. We make a proof by contradiction. Recursively construct A_i such that

- $A_0 = A$,
- $(s, A_{i+1}) \leq_i (s, A_i)$,
- whenever $t \subseteq \{a_i^j : j < i\}$, $(s \cup t, A_i - t)$ does not decide φ.

Suppose A_i has been constructed. By the previous paragraph, we easily obtain $A_{i+1}^i \subseteq A_i^i$ such that for all $t \subseteq \{a_i^j : j < i\}$ and all $k \in A_{i+1}^i$, $(s \cup t \cup \{k\}, A_i^i - \{k\})$ does not decide φ. Clearly $A_{i+1} = \{a_i^j : j < i\} \cup A_{i+1}^i$ is as required.

Let (s, B) be the fusion of the (s, A_i). Choose $(s \cup t, C) \leq (s, B)$ deciding φ. Then $t \subseteq \{a_i^j : j < i\}$ for some i, and $(s \cup t, A_i - t)$ does not decide φ. Hence $(s \cup t, A - t)$ does not decide φ. This contradicts $(\star\star)$ and the proof is complete. □

We say a forcing notion $\mathbb{P}$ has the *Laver property* if for every condition $p \in \mathbb{P}$, every function $g \in \omega^\omega$, and every $\mathbb{P}$-name $\dot{f}$ for a function bounded by g, there are a condition $q \leq p$ and a function $F : \omega \to [\omega]^{<\omega}$ with $|F(n)| \leq 2^n$ such that q forces that $\dot{f}(n) \in F(n)$ for all n. The following is easy to see.

Observation 6.5. *$\mathbb{P}$ has the Sacks property iff $\mathbb{P}$ is ω^ω-bounding and has the Laver property.*

Now comes a consequence of pure decision.

Corollary 6.6. *Mathias forcing $\mathbb{M}$ has the Laver property.*

Proof. Let $(s, A) \in \mathbb{M}$, $g \in \omega^\omega$, and let $\dot{f}$ be an $\mathbb{M}$-name for a function bounded by g. Recursively construct A_i and $F(i)$ such that

- $A_0 \leq A$,
- $(s, A_{i+1}) \leq_i (s, A_i)$,
- for all $t \subseteq \{a_i^j : j < i\}$, $(s \cup t, A_i^i)$ decides the value of $\dot{f}(i)$ and forces this value to be a member of $F(i)$,
- (s, A_i) forces $\dot{f}(i) \in F(i)$,
- $|F(i)| \leq 2^i$.

Notice that the forth item is an immediate consequence of the third.

Since $\dot{f}(0)$ may assume only finitely many values, we may apply pure decision (6.4) finitely often to obtain $A_0 \subseteq A$ such that (s, A_0) decides the value of $\dot{f}(0)$. Let $F(0)$ be the singleton consisting of this value.

Suppose A_i and $F(i)$ have been constructed. Let $\ell = 2^{i+1}$ and let t_k, $k < \ell$, list all subsets of $\{a_i^j : j < i + 1\}$. By recursion on $k \leq \ell$, construct decreasing $A_{i,k}$: let $A_{i,0} = A_i$. Suppose $A_{i,k}$ has been produced. Using again pure decision, we obtain $A_{i,k+1} \subseteq A_{i,k}$ such that $(s \cup t_k, A_{i,k+1}^{i+1})$ forces $\dot{f}(i + 1) = x_k$ for some x_k. Let $A_{i+1} = A_{i,\ell}$. Clearly A_{i+1} and $F(i+1) = \{x_k : k < \ell\}$ have the required properties. This completes the recursive construction.

Let (s, B) be the fusion of the (s, A_i). By the forth item we see that (s, B) forces that $\dot{f}(i) \in F(i)$ for all i, as required. $\square$

6.2. $\mathcal{P}(\omega)/\mathrm{fin}$ as a forcing notion. A family $\mathcal{A} \subseteq [\omega]^\omega$ has the *finite intersection property* if the intersection of any finitely many members of $\mathcal{A}$ is infinite. $B \in [\omega]^\omega$ is a *pseudointersection* of a family $\mathcal{A} \subseteq [\omega]^\omega$ if $B \subseteq^* A$ for all $A \in \mathcal{A}$. A family with a pseudointersection obviously has the finite intersection property, but the converse is false in general. The *pseudointersection number* $\mathfrak{p}$ is the smallest size of a family with the finite intersection property which has no pseudointersection. A *tower* $\mathcal{A} \subseteq [\omega]^\omega$ is a family which is well-ordered by reverse inclusion $\supseteq^*$ (i.e., $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ and $\alpha < \beta$ implies $A_\beta \subseteq^* A_\alpha$) and which has no pseudointersection. The *tower number* $\mathfrak{t}$ is the least cardinality of a tower. It is easy to see that $\mathfrak{t}$ must be regular.

For $A, B \subseteq \omega$ say $A =^* B$ if the *symmetric difference* $A \setminus B \cup B \setminus A$ is finite. $=^*$ is an equivalence relation. The corresponding quotient structure $\mathcal{P}(\omega)/\mathrm{fin}$ consists of equivalence classes $[A]$ of subsets A of ω. Put $[A] \leq [B]$ if $A \subseteq^* B$. We can naturally identify the forcing notion $(\mathcal{P}(\omega)/\mathrm{fin}, \leq)$ with $([\omega]^\omega, \subseteq^*)$.

In this language, maximal antichains are just *maximal almost disjoint families* (*mad families* for short) of subsets of ω, and it follows that $\mathcal{P}(\omega)/\mathrm{fin}$ is not $\mathfrak{c}$-cc (and trivially $\mathfrak{c}^+$-cc).

Furthermore, $\mathcal{P}(\omega)/\mathrm{fin}$ is σ-closed. In fact, by definition of $\mathfrak{t}$, it is $< \mathfrak{t}$-closed. In particular, it does not add any new reals. The *distributivity number* $\mathfrak{h}$ is the least cardinal κ such that $\mathcal{P}(\omega)/\mathrm{fin}$ is not κ*-distributive*. In particular, $\mathfrak{h}$ is the smallest κ such that $\mathcal{P}(\omega)/\mathrm{fin}$ adds a new function $f : \kappa \to V$. It is easy to see that $\mathfrak{h}$ is regular.

Here, a p.o. $\mathbb{P}$ is λ*-distributive* if for any collection A_α, $\alpha < \lambda$, of maximal antichains on $\mathbb{P}$, there is a *common refinement*, i.e. a maximal antichain A such that for all α and $q \in A$ there is $p \geq q$ in A_α. Alternatively, $\mathbb{P}$ is λ-distributive iff any intersection of λ many open dense subsets of $\mathbb{P}$ is open dense iff $\mathbb{P}$ does not add new functions to V with domain λ.

Proposition 6.7. $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s}$ *and* $\mathfrak{h} \leq \mathfrak{b}$.

Proof. The first inequality is an easy diagonal argument, and the second is obvious.

Let $\kappa < \mathfrak{t}$ and let D_α, $\alpha < \kappa$, be open dense in $\mathcal{P}(\omega)/\mathrm{fin}$. Fix $A \in [\omega]^\omega$. Recursively produce A_α, $\alpha \leq \kappa$, with $A_0 = A$, $A_\beta \subseteq^* A_\alpha$ for $\beta > \alpha$ and $A_{\alpha+1} \in D_\alpha$. The successor step is trivial and the limit step is possible by $\kappa < \mathfrak{t}$. Then A_κ belongs to $\bigcap_{\alpha<\kappa} D_\alpha$. Hence this intersection is open dense, and $\mathfrak{t} \leq \mathfrak{h}$ follows.

For $A \in [\omega]^\omega$, $\mathcal{B}_A = \{A, \omega \setminus A\}$ is a maximal antichain in $\mathcal{P}(\omega)/\mathrm{fin}$. If a maximal antichain $\mathcal{B} \subseteq [\omega]^\omega$ refines $\mathcal{B}_A$, then no member of $\mathcal{B}$ is split by A. Hence, if $\mathcal{A} \subseteq [\omega]^\omega$ is a splitting family, then $\{\mathcal{B}_A : A \in \mathcal{A}\}$ has no common refinement. This implies $\mathfrak{h} \leq \mathfrak{s}$.

For $f \in \omega^\omega$, let $\mathcal{A}_f$ be a maximal antichain in $\mathcal{P}(\omega)/\mathrm{fin}$ such that the increasing enumeration of each $A \in \mathcal{A}_f$ eventually dominates f. If $\mathcal{B}$ refines $\mathcal{A}_f$ and $B \in \mathcal{B}$, then for some k, the increasing enumeration of $B \setminus k$ dominates f. Hence, if $\mathcal{F}$ is unbounded, $\{\mathcal{A}_f : f \in \mathcal{F}\}$ has no common refinement, and we obtain $\mathfrak{h} \leq \mathfrak{b}$. $\square$

It is also known that $\mathfrak{p}$ is regular [Bl, Theorem 7.15], and that $\mathfrak{t} \leq \mathsf{add}(\mathcal{M})$ (see [BJ, Theorem 2.2.10] or [Bl, Theorem 6.12]). It is open whether $\mathfrak{p} < \mathfrak{t}$ is consistent.

The cardinals we have defined in Lecture 1 and here may be displayed in the following diagram which is a somewhat extended version of *Van Douwen's diagram.*

$$
\begin{tikzcd}
& \mathfrak{c} \arrow[dl, dash] \arrow[ddrr, dash] & & \\
\mathsf{cof}(\mathcal{M}) \arrow[dr, dash] & & & \\
& \mathfrak{d} \arrow[dl, dash] \arrow[dr, dash] & & \mathfrak{r} \arrow[dl, dash] \\
\mathfrak{s} \arrow[dr, dash] & & \mathfrak{b} \arrow[dl, dash] \arrow[dr, dash] & \\
& \mathfrak{h} \arrow[dr, dash] & & \mathsf{add}(\mathcal{M}) \arrow[dl, dash] \\
& & \mathfrak{t} \arrow[d, dash] & \\
& & \mathfrak{p} \arrow[d, dash] & \\
& & \aleph_1 &
\end{tikzcd}
$$

Theorem 6.8. (base-matrix lemma) *There are maximal almost disjoint families $\mathcal{A}_\alpha$, $\alpha < \mathfrak{h}$, such that*

(i) *$\alpha < \beta$ implies $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$,*

(ii) *for each α and each $A \in \mathcal{A}_\alpha$, $\mathfrak{c}$ many members of $\mathcal{A}_{\alpha+1}$ are almost contained in A,*

(iii) *for all $A \in [\omega]^\omega$ there are α and $B \in \mathcal{A}_\alpha$ such that B is almost contained in A.*

Such a collection of $\mathcal{A}_\alpha$ is called a *base-matrix tree.* Note that the last condition in particular implies that $\bigcup_{\alpha<\mathfrak{h}} \mathcal{A}_\alpha$ is dense in $\mathcal{P}(\omega)/\text{fin}$.

Proof. By definition of $\mathfrak{h}$, we know there are open dense $\mathcal{D}_\alpha$, $\alpha < \mathfrak{h}$, whose intersection is not dense. By an easy *homogeneity argument*, we can in fact assume that the intersection of the $\mathcal{D}_\alpha$ is empty: let $A \in [\omega]^\omega$ be such that there is no $B \in \bigcap_\alpha \mathcal{D}_\alpha$ with $B \subseteq A$. Let $f : B \to \omega$ be a bijection. Then the f-images of the $\mathcal{D}_\alpha$, $\alpha < \mathfrak{h}$, form a family of open dense sets whose intersection is empty.

Let $\{X_\gamma : \gamma < \mathfrak{c}\}$ enumerate $[\omega]^\omega$.

By recursion on α construct $\mathcal{A}_\alpha = \{A_{\alpha,\delta} : \delta < \mathfrak{c}\}$ such that (i) and (ii) hold and we additionally have

- $\mathcal{A}_\alpha \subseteq \mathcal{D}_\alpha$,
- if $|\{\delta : A_{\alpha,\delta} \cap X_\gamma \text{ is infinite}\}| = \mathfrak{c}$ then there is $B \in \mathcal{A}_{\alpha+1}$ such that $B \subseteq X_\gamma$.

The case $\alpha = 0$ is straightforward.

Next consider the case $\alpha = \beta+1$ is a successor. We have $\mathcal{A}_\beta = \{A_{\beta,\delta} : \delta < \mathfrak{c}\}$ and construct $\mathcal{A}_\alpha$ by recursion on δ. At stage δ, let γ be the smallest ordinal which has not been considered yet such that $A_{\beta,\delta} \cap X_\gamma$ is infinite. Choose a mad family $\mathcal{B}_\delta$ below $A_{\beta,\delta}$ of size continuum such that $\mathcal{B}_\delta \subseteq \mathcal{D}_\alpha$ and one member of $\mathcal{B}_\delta$ is almost contained in $A_{\beta,\delta} \cap X_\gamma$. Finally, let $\mathcal{A}_\alpha$ be the union of the $\mathcal{B}_\delta$, $\delta < \mathfrak{c}$.

Finally, let α be a limit ordinal. By $\alpha < \mathfrak{h}$, we can find a common refinement $\mathcal{A}_\alpha$ of the previous $\mathcal{A}_\beta$, and we can also assume $\mathcal{A}_\alpha \subseteq \mathcal{D}_\alpha$.

This completes the recursive construction.

We need to check property (iii): Fix $A \in \omega^\omega$. Let γ be such that $A = X_\gamma$. Since the intersection of the $\mathcal{D}_\alpha$ is empty, there must be α_0 such that X_γ intersects at least two members of $\mathcal{A}_{\alpha_0}$. Similarly, there must be $\alpha_1 > \alpha_0$ such that both of these intersections intersect at least two members of $\mathcal{A}_{\alpha_1}$. Continuing this way, produce a strictly increasing sequence α_n. Let α be their limit. Since $cf(\mathfrak{h}) \geq \aleph_1$, we have $\alpha < \mathfrak{h}$, and it is now easy to see that X_γ must intersect continuum many members of $\mathcal{A}_\alpha$. By construction, this means that there is $B \in \mathcal{A}_{\alpha+1}$ almost contained in $A = X_\gamma$, as required. □

Corollary 6.9. $\mathcal{P}(\omega)/\mathrm{fin}$ *preserves all cardinals* $\leq \mathfrak{h}$ *and* $\geq \mathfrak{c}^+$*, and collapses* $\mathfrak{c}$ *to* $\mathfrak{h}$*. In particular,* $\mathcal{P}(\omega)/\mathrm{fin}$ *preserves all cardinals iff* $\mathfrak{h} = \mathfrak{c}$*.*

Proof. Since $\mathcal{P}(\omega)/\mathrm{fin}$ is $< \mathfrak{h}$-distributive, all cardinals up to and including $\mathfrak{h}$ are preserved. By the $\mathfrak{c}^+$-cc, all cardinals $\geq \mathfrak{c}^+$ are preserved.

Let $\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ be a base-matrix tree (see Theorem 6.8). For each $B \in \mathcal{A}_\alpha$, let $\{A_{B,\gamma} : \gamma < \mathfrak{c}\}$ enumerate $\{C \in \mathcal{A}_{\alpha+1} : C \subseteq^* B\}$. Define a $\mathcal{P}(\omega)/\mathrm{fin}$-name for a function $\dot{f} : \mathfrak{h} \to \mathfrak{c}$ by stipulating that for $B \in \mathcal{A}_\alpha$, $A_{B,\gamma}$ forces that $\dot{f}(\alpha) = \gamma$. We need to argue that $\dot{f}$ is forced to be an onto function. Let $A \in [\omega]^\omega$ be a condition and let $\gamma < \mathfrak{c}$. By 6.8, there are $\alpha < \mathfrak{h}$ and $B \in \mathcal{A}_\alpha$ such that $B \subseteq^* A$. Then $A_{B,\gamma} \subseteq^* A$ forces that $\dot{f}(\alpha) = \gamma$, as required. □

What kind of generic object does $\mathcal{P}(\omega)/\mathrm{fin}$ add? An ultrafilter $\mathcal{U}$ on ω is called a *Ramsey ultrafilter* if for every partition $(X_n : n \in \omega)$ of ω such that $X_n \notin \mathcal{U}$ for all n, there is $U \in \mathcal{U}$ such that $U \cap X_n$ has at most one element for all n. Such U is called a *selector* for the X_n.

Observation 6.10. $\mathcal{P}(\omega)/\mathrm{fin}$ *generically adds a Ramsey ultrafilter* $G = \mathcal{U}$*.*

Proof. Since no new reals are added, the generic filter $G = \mathcal{U}$ clearly is an ultrafilter. So it suffices to check the Ramsey property. Again, as no reals are added, it suffices to consider partitions $(X_n : n \in \omega)$ in the ground model. Let $A \in [\omega]^\omega$. It is easy to see that there is $B \subseteq A$ such that either $B \subseteq X_n$ for some n or B is a selector for the X_n. Thus $\dot{G} = \dot{\mathcal{U}}$ is forced to be Ramsey. □

6.3. Mathias forcing with an ultrafilter. Let $\mathcal{U}$ be an ultrafilter on ω. *Mathias forcing with* $\mathcal{U}$, $\mathbb{M}_\mathcal{U}$, consists of all $(s, A) \in \mathbb{M}$ such that $A \in \mathcal{U}$, with the inherited ordering. Unlike $\mathbb{M}$, $\mathbb{M}_\mathcal{U}$ is a σ-centered forcing notion. Indeed, since $\mathcal{U}$ is an ultrafilter, any finitely many conditions of the form (s, A_i), $i < n$, are compatible, with common extension $(s, \bigcap_i A_i)$. In particular, $\mathbb{M}_\mathcal{U}$ is ccc and thus preserves cardinals.

The generic real $m_{\mathcal{U}}$ (which is defined as for $\mathbb{M}$) is a *pseudointersection* of the ultrafilter $\mathcal{U}$, i.e. for all $A \in \mathcal{U}$, we have $m_{\mathcal{U}} \subseteq^* A$. In particular, $m_{\mathcal{U}}$ is also an unsplit real (as m, see 6.1). For most $\mathcal{U}$, $m_{\mathcal{U}}$ also codes a dominating real over V (as m, see 6.2), but it is consistent there are ultrafilters $\mathcal{U}$ such that $\mathbb{M}_{\mathcal{U}}$ does not add a dominating real (certain P-points).

An interesting feature of Mathias forcing is that it naturally decomposes as a two-step iteration:

Proposition 6.11. *$\mathbb{M} \cong \mathcal{P}(\omega)/\mathrm{fin} \star \mathbb{M}_{\dot{\mathcal{U}}}$ where $\dot{\mathcal{U}}$ is the $\mathcal{P}(\omega)/\mathrm{fin}$-name for the generic Ramsey ultrafilter (see 6.10). That is, $\mathbb{M}$ is forcing equivalent to forcing first with $\mathcal{P}(\omega)/\mathrm{fin}$ and then with $\mathbb{M}_{\dot{\mathcal{U}}}$.*

Proof. Let $(s, A) \in \mathbb{M}$. Then $(A, (s, A)) \in \mathcal{P}(\omega)/\mathrm{fin} \star \mathbb{M}_{\dot{\mathcal{U}}}$, and we need to check the mapping sending (s, A) to $(A, (s, A))$ is a dense embedding. To see this, let $(A, (\dot{s}, \dot{B})) \in \mathcal{P}(\omega)/\mathrm{fin} \star \mathbb{M}_{\dot{\mathcal{U}}}$. Since $\mathcal{P}(\omega)/\mathrm{fin}$ does not add reals, there is $A' \subseteq A$ deciding what $(\dot{s}, \dot{B})$ is, say, $A' \Vdash (\dot{s}, \dot{B}) = (s, B)$. Since we also have $A' \Vdash B \in \dot{\mathcal{U}}$, we obtain $A' \subseteq^* B$. Let $B' = A' \cap B$. Then $(B', (s, B')) \leq (A, (\dot{s}, \dot{B}))$ as required. $\square$

Corollary 6.12. *$\mathbb{M}$ preserves all cardinals $\leq \mathfrak{h}$ and $\geq \mathfrak{c}^+$ and collapses $\mathfrak{c}$ to $\mathfrak{h}$. $\mathbb{M}$ preserves all cardinals iff $\mathfrak{h} = \mathfrak{c}$.*

Proof. By 6.9 and 6.11. $\square$

6.4. The countable support iteration of Mathias forcing. Like for the Sacks property (Theorem 5.8) we have:

Theorem 6.13. *Csi of proper forcing notions with the Laver property have the Laver property.*

This is similar to 4.9. See [Go, Section 6, Application 4, Corollary 6.33].

Lemma 6.14. *A forcing notion $\mathbb{P}$ with the Laver property does not add random or Cohen reals. In particular, the family of meager sets of the ground model stays covering in the extension. Similarly for the family of null sets of the ground model.*

Proof. Let $\dot{f}$ be a $\mathbb{P}$-name for an element of 2^ω. Also let $p \in \mathbb{P}$. By the Laver property there are $q \leq p$ and a function $F : \omega \to [2^{<\omega}]^{<\omega}$ with $F(n) \subseteq 2^{2n}$ and $|F(n)| \leq 2^n$ such that q forces that $\dot{f}\restriction 2n \in F(n)$ for all n.

For F as above put $A_F = \{f : \forall n \;\; f\restriction 2n \in F(n)\}$. Since $\frac{|F(n)|}{2^{2n}} = \frac{1}{2^n}$, A_F is a closed measure zero set. In particular, A_F is nowhere dense. Furthermore, q forces that $\dot{f}$ belongs to the set A_F which is coded in the ground model. By 2.6, $\dot{f}$ is neither random nor Cohen over the ground model V. Also, the family of sets A_F from V is a covering family of closed measure zero sets in the extension. $\square$

Theorem 6.15. *Assume CH holds in the ground model V. Let $\mathbb{M}_{\omega_2}$ be the csi of $\mathbb{M}$. In $V^{\mathbb{M}_{\omega_2}}$, $\mathsf{cov}(\mathcal{M}) = \mathsf{cov}(\mathcal{N}) = \aleph_1$ and $\mathfrak{h} = \mathfrak{s} = \mathfrak{b} = \mathfrak{c} = \aleph_2$.*

Proof. By 6.3 and 4.9, $\mathbb{M}_{\omega_2}$ is proper and preserves $\aleph_1$, and by CH and 4.10, it preserves $\aleph_2$. Moreover, by 6.6 and 6.13, $\mathbb{M}_{\omega_2}$ has the Laver property, and $\mathsf{cov}(\mathcal{M}) = \mathsf{cov}(\mathcal{N}) = \aleph_1$ follow from 6.14 and CH.

It is obvious that $\mathbb{M}_{\omega_2}$ adds exactly $\aleph_2$ many reals so that $\mathfrak{c} = \aleph_2$ follows. Next, $\mathfrak{b} = \mathfrak{s} = \aleph_2$ can easily be proved directly using 6.1 and 6.2 (and iteration). However, this also follows from $\mathfrak{h} = \aleph_2$ (see below) and 6.7.

To establish $\mathfrak{h} = \aleph_2$, let $\mathcal{D}_\alpha$, $\alpha < \omega_1$, be open dense in $\mathcal{P}(\omega)/\text{fin}$ in the extension $V^{\mathbb{M}_{\omega_2}}$. By homogeneity (see also the proof of 6.8), it suffices to show that the intersection of the $\mathcal{D}_\alpha$ is non-empty. A reflection argument shows that there is $\beta < \omega_2$ with cofinality ω_1 such that all $\mathcal{D}_\alpha \cap V^{\mathbb{M}_\beta}$ are open dense in $V^{\mathbb{M}_\beta}$.

Indeed, since all $V^{\mathbb{M}_\gamma}$, $\gamma < \omega_2$, satisfy CH (see 4.10), the $\aleph_2$-cc (see again 4.10) implies that for all γ there is $\delta = \delta_\gamma < \omega_2$ such that for all $X \in V^{\mathbb{M}_\gamma}$ and all $\alpha < \omega_1$ there is a $Y \in \mathcal{D}_\alpha \cap V^{\mathbb{M}_\delta}$ with $Y \subseteq X$. Let $\beta < \omega_2$ with $cf(\beta) = \omega_1$ be such that $\delta_\gamma < \beta$ for all $\gamma < \beta$. Since each $X \in [\omega]^\omega$ in $V^{\mathbb{M}_\beta}$ belongs to some $V^{\mathbb{M}_\gamma}$ for $\gamma < \beta$ by 4.11, we see that all $\mathcal{D}_\alpha \cap V^{\mathbb{M}_\beta}$ are open dense in $V^{\mathbb{M}_\beta}$.

By genericity, it is easy to see that the generic Mathias real m_β added at stage β is almost contained in a member of $\mathcal{D}_\alpha \cap V^{\mathbb{M}_\beta}$, for all α (exercise!). Hence $m_\beta \in \bigcap_\alpha \mathcal{D}_\alpha$, and $\bigcap_\alpha \mathcal{D}_\alpha \neq \emptyset$ follows, thus establishing $\mathfrak{h} = \aleph_2$. $\square$

6.5. Fragments of Martin's Axiom. Statements about some of the cardinals in this and earlier sections can be reformulated in terms of Martin-axiom like statements.

Theorem 6.16. (Bell's Theorem) *MA holds for σ-centered posets iff $\mathfrak{p} = \mathfrak{c}$.*

See [Bl, Theorem 7.12].

Proposition 6.17. *MA holds for countable posets iff $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$.*

Proof. Exercise! (This uses 1.4 and 2.6.) See also [BJ, Theorem 2.4.5] or [Bl, Theorem 7.13]. $\square$

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