The constructible universe L

Idea: instead of $V_{\alpha+1} := P(V_\alpha)$ we take only the definable subsets at each step.

F.O. def. with param, evaluated in the model.

Def: M a set, $X \subseteq M$ is definable over M if there exists formula ϕ and param. $a_1 \ldots a_n \in M$ such that

$$X = \{x \in M : M \models \phi(x, a_1 \ldots a_n)\}$$

$\omega \in M$ bigger e.g. $M = V_{\omega+7}$

E.g: $\omega \subseteq M$, $X = \{0, 2, 4, 6, 8, \ldots\}$ all even numbers.

Then X is definable by $\phi(x) \equiv \exists n (\in \omega) (n+n = x)$

Def: $D(M) := \{X \subseteq M : X \text{ is definable over } M\}$

Def:

$$L_0 = \emptyset$$

$$L_{\alpha+1} = D(L_\alpha)$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$$

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha \quad \leftarrow \text{The constructible universe}$$

NB: This def. takes place in ZF. Is that ok?

$X \in D(M) \iff \underbrace{\exists \phi}_{\text{Quantifying over formulas?}} \exists a_1 \ldots a_n$ s.t. $\forall x \; x \in X \leftrightarrow M \models \phi(x, \ldots)$

A: This is ok because M is a set and we can code formulas: $\ulcorner \phi \urcorner$, $D(M) = \{X \subseteq M : \exists \ulcorner \phi \urcorner \exists a_1 \ldots a_n \ldots\ldots\ldots\}$

Model-theoretic $\models$

Let's look at what we have:

$$L_0 = \emptyset = V_0$$
$$L_n = V_n$$
$$L_\omega = V_\omega$$
$$L_{\omega+1} \neq V_{\omega+1}$$

In part: $|V_{\omega+1}| = 2^{\aleph_0}$ but $|L_{\omega+1}| = \omega$

Actually the same argument gives us:

$$|L_\alpha| = |\alpha| \quad \text{for every } \alpha \geq \omega$$

NB: L_{ω_1} is unctbl. You want that every $\{x_0\}$ for $x \in L_{\omega_1}$ is in L_{ω_1+1}. $\phi(x) \equiv$ " $x = x_0$ "

$$|L_\alpha| = |\alpha| < \omega_1 \quad \text{for } \alpha < \omega_1$$
$$|L_{\omega_1}| = \Big|\bigcup_{\alpha<\omega_1} L_\alpha\Big| = \omega_1$$

Theorem: $L \models ZF$

(most are easy ... but for Comprehension, you need reflection)

Theorem (ZF) : $L \models ZFC$

(Gödel 1938)

Corollary : $\mathrm{Con}(ZF) \to \mathrm{Con}(ZFC)$

Proof: In fact, L satisfies " Global Choice " : there is $<_L$ which well-orders the whole universe L.

To do that, wellorder all L_α by $<_\alpha$, recursively. Suppose $(L_\alpha, <_\alpha)$ is a w.o. Then $<_\alpha^{lex}$ is also a well order.

Take $x, y \in L_{\alpha+1}$.

1. If $x, y \in L_\alpha$: $x <_{\alpha+1} y \iff x <_\alpha y$ ($<_{\alpha+1}$ extends $<_\alpha$)

2. If $x \in L_\alpha$, $y \in L_{\alpha+1} \setminus L_\alpha$: $x <_{\alpha+1} y$ (and vice versa)
(new sets come after old sets)

3. $x, y \in L_{\alpha+1} \setminus L_\alpha$:

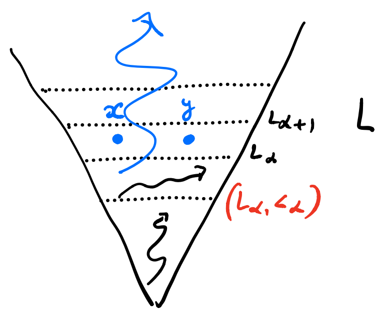
$x <_{\alpha+1} y$ iff

recursively code φ by nat. numbers.

the $<_\alpha$-least φ and $<_\alpha^{lex}$-least $a_1 \dots a_n$ defining x

are $<_\alpha^{lex}$-less than

the $<_\alpha$-least ψ and $<_\alpha^{lex}$-least $b_1 \dots b_n$ defining y.



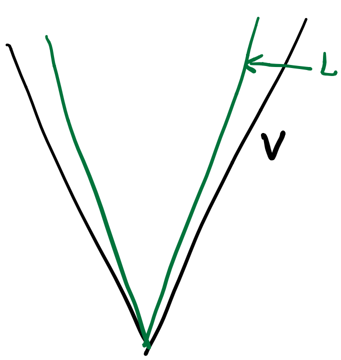
$\square$

<u>Now we turn to CH:</u>

<u>Def.</u>: The <u>Axiom of Constructibility</u> is the statement

$$\forall x \; \exists \alpha \;\; x \in L_\alpha$$

usually abbreviated by "$V = L$"



<u>Q</u> : $L \models (V = L)$?

Lemma: The L_α-hierarchy is absolute for trans. models.

($\alpha \mapsto L_\alpha$ is absolute) $\qquad \alpha \in M \Rightarrow L_\alpha \in M$

Reason: $L_{\alpha+1} := D(L_\alpha)$ involves recursive definitions from basic atomic statements.

$M \models \phi$

Corollary 1: $L \models (V=L)$

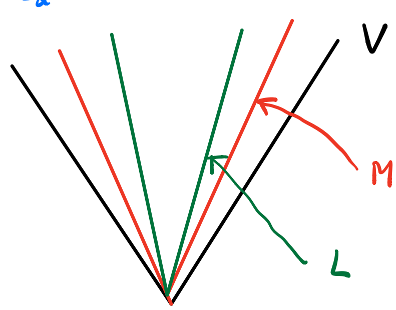
Proof: $(V=L)^L = (\forall x \; \exists \alpha \; x \in L_\alpha)^L = \forall x \in L \; \exists \alpha \in L \; x \in L_\alpha$ **TRUE**.

Corollary 2: L is the minimal model of ZF. (proper class, transitive model)

Proof: If $M \models ZF$ trans, prop. class, $\mathrm{Ord} \subseteq M$.

$\alpha \in M \Rightarrow L_\alpha \in M \Rightarrow L_\alpha \subseteq M$.

Then $L \subseteq M$

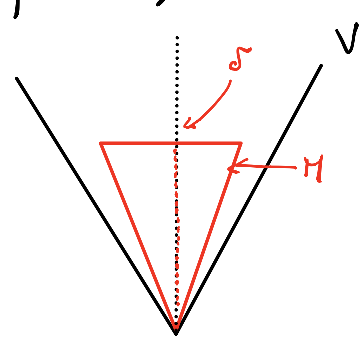


Set-version of above:

Def: M trans. model of ZF^*, M is a set.

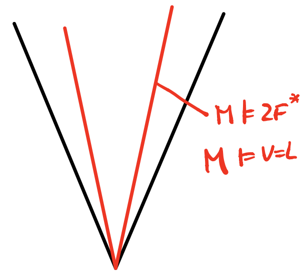
The height of M, denoted by $o(M)$ = least ordinal $\delta \notin M$. (δ limit)

Equiv: $o(M) = \mathrm{Ord} \cap M$.



Lemma (*): Suppose $M \models ZF^* + V=L$ (M transitive)

1. If M proper class then $M = L$
2. If M is a set, then $M = L_{o(M)}$



Proof: Same in both cases!

"$\supseteq$" we already showed.

"$\subseteq$" $M \models (V=L)$

$\Rightarrow M \models (\forall x \, \exists \alpha \; x \in L_\alpha)$

$\Rightarrow \forall x \in M \; \exists \alpha \in M \; (x \in L_\alpha)^M$

$\Rightarrow \forall x \in M \; \boxed{\exists \alpha < o(M)} \; x \in L_\alpha$ if M class, then "$\exists \alpha$"

$\Rightarrow M \subseteq L_{o(M)}$

Theorem: $V=L \rightarrow GCH$.

Corollary: $L \models GCH$ (since $L \models (V=L)$)

Corollary: $Con(ZF) \rightarrow Con(ZFC + GCH)$

Proof of Theorem: I will prove CH (GCH is the same).

It suffices to show

$$P(\omega) \subseteq L_{\omega_1}$$

because then $2^{\aleph_0} = |P(\omega)| \leq |L_{\omega_1}| = \omega_1$.

Take $x \subseteq \omega$. Let $a := \{x\} \cup \omega$ (just to make it transitive and $|a| = \omega$).

By Reflection #3: there is M s.t.

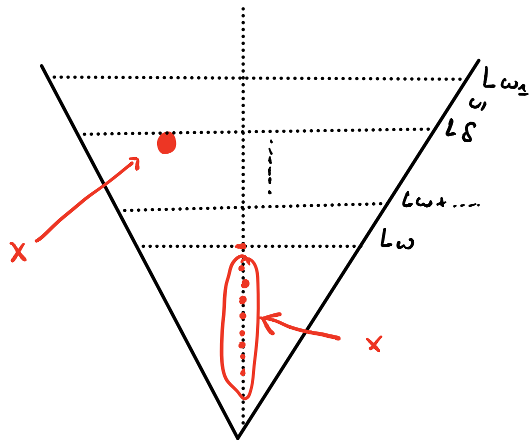
1. M transitive
2. $|M| = \omega$
3. $a \subseteq M$
4. $M \models ZF^* + V=L$ (because $V \models ZF + V=L$)

But by Lemma $(*)$ $M = L_{o(M)}$.

But $o(M) = \mathrm{Ord} \cap M$ is a ctbl. ordinal!

So $o(M) < \omega_1$.

So $x \in a \subseteq M = L_{o(M)} \subseteq L_{\omega_1}$ $\square$



Condensation Lemma.

Idea: $V=L \to$ the only trans. set models reflecting $ZF^* + V=L$ are the L_δ's themselves.

NB: Same for G.C.H.

Replace ω by κ, and ω_1 by κ^+.