

- $MA_p(\kappa)$
- $MA(\kappa) \equiv MA_{IP}(\kappa)$ for all ccc. IP
- $MA \equiv \forall \kappa < 2^{\aleph_0} MA(\kappa)$

NB: $MA(\omega)$ is true
 $MA(2^{\aleph_0})$ is false

p := least cardinal κ s.t. $MA(\kappa)$ fails.

Know: $\omega < p \leq 2^{\aleph_0}$

CH \rightarrow MA (trivial)

Interesting: $\neg CH + MA \Rightarrow \omega < \omega_1 < p = 2^{\aleph_0}$

Ex: \mathcal{M} := ideal of meager subsets of \mathbb{R}
 \mathcal{M} is a σ -ideal.

But: \mathcal{M} not closed under $\bigcup_{\alpha < 2^{\aleph_0}} X_\alpha$

$2^{\aleph_0} = \omega_2 + MA \rightarrow \mathcal{M}$ is closed under \aleph_1 -unions

Proof: similar to Baire Cat. Theorem. ✓

?
Martin + Solovay, 1970:

Schedule (?): 1. MA
 2. Basics of forcing: P -names, gen. extensions $M \in M[G]$
 (semantic) \Vdash -relation (not \Vdash^*)
 Forcing Theorem.

semantic \swarrow
 syntactic \swarrow

3. Application of forcing : $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \neg\text{CH})$

At first leave out
technical → - proof of Forcing Theorem, syntactic \Vdash^*
- $M[G] \models \text{ZFC}$

F is a class function means there is a formula

$$\phi(x, y)$$

and $\text{ZFC} \vdash \forall x \exists! y \phi(x, y)$

Earlier we said

• To prove $\text{Con}(\text{ZFC}) \rightarrow \text{Con}(\text{ZFC} + \Phi)$

construct a proper class M and prove

$$\text{ZFC} \vdash (M \models \text{ZFC} + \Phi)$$

(that really means $\text{ZFC} \vdash \varphi^M$ for every φ in $\text{ZFC} + \Phi$)

Q: Can we do the same for $\text{ZFC} + \neg\text{CH}$?

Problem: If we could do the above, then

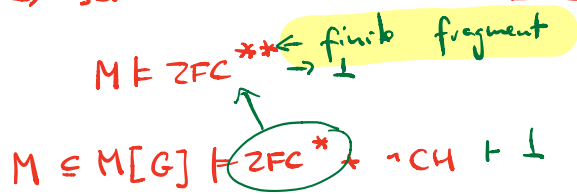
$$\text{ZFC} + \text{V=L} \vdash (M \models \text{ZFC} + \neg\text{CH})$$

But: $\text{ZFC} + \text{V=L} \vdash L \models M \models \text{V=L}$
 $L = M = V$

CH

Think in terms of models: $L \models$ "there are no proper class-sized models of ZFC"

Problem: \rightarrow solution is to look at set-sized models.



$M \in M[G]$ generic extension of M

$$p \Vdash^* \dot{\Phi}$$