- $MA_{\mathbb{P}}(\kappa)$
- $MA(\kappa) \equiv MA_{\mathbb{P}}(\kappa)$ for all ccc. $\mathbb{P}$
- $MA \equiv \forall \kappa < 2^{\aleph_0}\ MA(\kappa)$

NB: $MA(\omega)$ is true

$MA(2^{\aleph_0})$ is false

$\mathfrak{p}$:= least cardinal κ s.t. $MA(\kappa)$ fails.

Know: $\omega < \mathfrak{p} \leq 2^{\aleph_0}$

$CH \to MA$ (trivial)

Interesting: $\neg CH + MA \Rightarrow \omega < \omega_1 < \mathfrak{p} = 2^{\aleph_0}$

Ex: $\mathcal{M}$:= ideal of meager subsets of $\mathbb{R}$

$\mathcal{M}$ is a σ-ideal.

But: $\mathcal{M}$ not closed under $\bigcup_{\alpha < 2^{\aleph_0}} X_\alpha$

$2^{\aleph_0} = \omega_2 + MA \to \mathcal{M}$ is closed under $\aleph_1$-unions

Proof: similar to Baire Cat. Theorem. ✓

?

Martin + Solovay, 1970:

Schedule (?):
1. MA
2. Basics of forcing: $\mathbb{P}$-names, gen. extensions $M \in M[G]$

semantic syntactic

(semantic) $\Vdash$ - relation (not $\Vdash^*$)

Forcing Theorem.

3. Application of forcing: $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \neg CH)$

At first leave out

technical

- proof of Forcing Theorem, syntactic $\Vdash^*$
- $M[G] \models ZFC$

F is a class function means there is a formula

$\phi(x,y)$

and $ZFC \vdash \forall x\, \exists! y\ \phi(x,y)$

Earlier we said

- To prove $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \Phi)$

  construct a proper class M and prove

  $ZFC \vdash (M \models ZFC + \Phi)$

  (that really means $ZFC \vdash \varphi^M$ for every φ in $ZFC + \Phi$)

Q: Can we do the same for $ZFC + \neg CH$?

Problem: If we could do the above, then

$ZFC + V = L \vdash (M \models ZFC + \neg CH)$

But: $ZFC + V = L \vdash L \subseteq M \subseteq V = L$

$L = M = V$

CH

Think in terms of models: $L \models$ "there are no proper class-sized models of ZFC"

Problem: $\longrightarrow$ solution is to look at set-sized models.

$M \models ZFC^{**} \leftarrow$ finite fragment $\to \perp$

$M \subseteq M[G] \models ZFC^{*} + \neg CH \vdash \perp$

$M \subseteq M[G]$ generic extension of M

$p \Vdash^{*} \Phi$