Proof that $M[G] \models$ Replacement.

Let $a \in M[G]$ and φ formula. leave out parameters for convenience.

Suppose $M[G] \models \forall x \in a \ \exists y \ \varphi(x,y)$

To show: There is $b \in M[G]$ such that

$$M[G] \models \forall x \in a \ \exists y \in b \ \varphi(x,y)$$

Let τ be such that $\tau_G = a$.

Claim: $\exists Q \in M[G]$ such that, for all $\sigma \in dom(\tau)$ and $p \in \mathbb{P}$

- If $\exists \theta : p \Vdash \varphi(\sigma, \theta)$, then $\exists \theta \in Q : p \Vdash \varphi(\sigma, \theta)$

Proof of Claim: In M, define the following function:

$F(\sigma, p) :=$ 1.) If $\exists \theta$ s.t. $p \Vdash \varphi(\sigma, \theta)$, pick the one with least rank to be the value of $F(\sigma, p)$

2.) Otherwise, $\emptyset$.

This argument is often used, sometimes called "Scott's Trick".

Then F is welldefined in M. Moreover $dom(\tau) \times \mathbb{P} \in M$, and $F[dom(\tau) \times \mathbb{P}] \subseteq Q$. Since $M \models$ REPL, $Q \in M$

$\square$ (Claim)

Now, let $\pi := Q \times \{1\}$. Then it is not hard to verify:

$$M[G] \models \forall x \in \tau_G \ \exists y \in \pi_G \ \varphi(x,y).$$

$\square$