Homework 10, due Monday 9 May

1. For which of Exercise 2 (a), (b) of Homework 9 does the converse hold? Give a counterexample if you claim that it does not. [4 pts]

2. (a) Show that for any set X in a Heyting algebra $\mathfrak{A}$ the set $\{c \in A \mid b_1 \wedge \cdots \wedge b_k \leq c$ for some $b_1, \ldots, b_k \in X\}$ is the smallest filter in $\mathfrak{A}$ containing X. [2 pts]

   (b) Show that, if F is a filter in $\mathfrak{A}$ not containing $a \in \mathfrak{A}$, then there exists a prime filter G in $\mathfrak{A}$ such that $F \subseteq G$ and $a \notin G$. (For ease of writing you can assume that $\mathfrak{A}$ is countable. Note the similarity filter theory, keeping in mind (a)) [2 pts]

   (c) Show that, if X and Y are subsets of $\mathfrak{A}$ such that

      i. for no $c \in Y$ there exist $b_1, \ldots, b_k \in X$ such that $b_1 \wedge \cdots \wedge b_k \leq c$,

      ii. if $c_1, c_2 \in Y$ then $c_1 \vee c_2 \in Y$,

   then there exists a prime filter F such that $X \subseteq F$ and $F \cap Y = \varnothing$. [2 pts]

   (d) Show that, if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and F is a prime filter in $\mathfrak{A}$, then F can be extended to a prime filter G in $\mathfrak{B}$ in such a way that the intersection of G and $A \setminus F$ is empty. [2 pts]

3. * Show that, if θ has n propositional variables, then

   θ has the disjunction property for formulas in n variables (i.e. if $\theta \vdash \chi_1 \vee \chi_2$, then $\theta \vdash \chi_1$ or $\theta \vdash \chi_2$)

   iff

   for all $u, v \in \mathcal{U}(n)$ such that $u \models \theta$ and $v \models \theta$ there exists w with wRu and wRv and $w \models \theta$. [6 pts]