Banach Algebras

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Bachelor Thesis

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April 18, 2005

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Foreword

The study of Banach algebras began in the twentieth century and originated from the observation that some Banach spaces show interesting properties when they can be supplied with an extra multiplication operation. A standard example was the space of bounded linear operators on a Banach space, but another important one was function spaces (of continuous, bounded, vanishing at infinity etc. functions as well as functions with absolutely convergent Fourier series). Nowadays Banach algebras is a wide discipline with a variety of specializations and applications.

This particular paper focuses on Gelfand theory — the relation between multiplicative linear functionals on a commutative Banach algebra and its maximal ideals, as well as with the spectra of its elements. Most of the content of chapters 1 thorough 3 is meant, in one way or another, to lead towards this theory. The central ingredient of Gelfand theory is the well-known Gelfand-Mazur theorem which says that if a Banach algebra is a division algebra then it is isomorphic to $\mathbb{C}$.

The first chapter is a purely algebraic one and provides us with all the necessary algebraic techniques, particularly concerning algebras without identity. The second and third chapters introduce normed algebras and Banach algebra and other concepts like the spectrum, and prove several important results among which the Gelfand-Mazur theorem. The fourth chapter is the pivotal one — where Gelfand theory is developed. In the fifth chapter several examples of Banach algebras are discussed in detail, together with their Gelfand representations. Some practical applications of the theory are also mentioned, among which Wiener's famous theorem about zeroes of functions with absolutely Fourier series, proven entirely from the context of Banach algebras.

1. Algebraic Concepts

In this chapter we introduce a number of algebraic ideas and techniques required for the study of Banach algebras. The chapter seeks to be entirely self-contained and is purely algebraic — it has no reference to normed algebras and can be studied independently of any topological or analytical considerations. However, all the concepts introduced in this chapter are strictly necessary for the further development.

1.1. Preliminaries

Algebras are, roughly speaking, combinations of vector spaces (internal addition and scalar multiplication) and rings (internal addition and internal multiplication). Examples are $\mathbb{R}$, $\mathbb{C}$, spaces of functions with point-wise operations and many more. We will work with vector spaces over $\mathbb{C}$, since that is what gives rise to many of the interesting properties we wish to study. The following definitions and results are more or less self-explanatory.

1.1.1. Definition A (complex) *algebra* A is a $\mathbb{C}$-vector space as well as a ring, such that both addition-operations agree and such that

$$\lambda(xy) = (\lambda x)y = x(\lambda y) \ \ \forall x, y \in A, \ \forall \lambda \in \mathbb{C}$$

It is called an *algebra with identity*, *commutative algebra* or *division algebra* if, as a ring, A is with identity, commutative, or division ring (skew field), respectively.

Moreover, *left, right* and *two-sided inverses* in A are defined as for rings. An element is called *regular* if it has a two-sided inverse and *singular* otherwise. The set of regular elements is denoted by A^{-1}.

A *homomorphism of algebras* is a homomorphism of rings and a linear map of vector-spaces.

Many other concepts defined for rings can directly be carried over to algebras without change of definition (e.g. *nilpotent*, *zero divisor* etc.)

1.1.2. Definition A subset $I \subset A$ is a *(left, right* or *two-sided) ideal* of A if it is such an ideal of A as a ring as well as a linear subspace of A as a vector space.

The concepts *proper* and *maximal* are defined as for rings.

1.1.3. Definition A *subalgebra* $B \subset A$ is a linear subspace such that $\forall x, y \in B$, $xy \in B$. An ideal is a subalgebra but not vice versa.

1.1.4. Remark If I is a two-sided ideal of A, then A/I is also an algebra, since it is both a ring and a vector space and $\lambda(\bar{x}\bar{y}) = \overline{\lambda(xy)} = \overline{(\lambda x)y} = (\lambda\bar{x})\bar{y}$ and similarly $\lambda(\bar{x}\bar{y}) = \bar{x}(\lambda\bar{y})$.

To avoid constantly repeating the same things about left-, right- and two-sided cases we will keep to the following convention:

1.1.5. Convention Unless explicitly stated otherwise we will speak simply of inverses, ideals, invertible elements etc. whenever the left-, right- and two-sided cases are formulated and proved analogously, and we will generally prove only the *left* case.

The following Lemma, of which we will later see variants, shows some basic but important properties of ideals, probably familiar to the reader from the theory of rings. The notation (x) stands for the ideal generated by $x \in A$.

1.1.6. Lemma Let A an algebra with identity e, and I an ideal. Then

1. $e \in I \Longrightarrow I = A$
2. $x \in I$ and x is invertible $\Longrightarrow I = A$
3. When A is commutative: x is invertible $\Longleftrightarrow (x) = A$
4. When A is commutative: x is *not* invertible $\Longleftrightarrow$ x is contained in a maximal ideal $M \subset A$.

Proof

1. Let $x \in A$. Then $x = xe \in I$ by definition. Hence $A \subset I$.
2. If $x \in I$ it follows that $e = x^{-1}x \in I$ and hence from (1), $A \subset I$.
3. If $e \in (x)$ then by definition $\exists y \in A$ s.t. $yx = e$, so x is invertible. The converse follows from (2).
4. If x is not invertible, by (3) (x) is a proper ideal, which, simply by the Lemma of Zorn, can be extended to a maximal ideal M s.t. $x \in (x) \subset M$. The converse follows from (3). $\square$

Much of the following theory (spectra, Gelfand theory etc.) will require the existence of an identity e in the algebra. However, not all naturally occurring algebras have identities so we'll need some artificial way of adding them: the remaining three sections of this chapter are devoted to precisely that concept.

The main tool is called *adjoining an identity* and will be presented in section 3, but first we present the related concept of regular ideals.

Before proceeding it is good to note that the remaining sections of this chapter may be omitted by anyone interested exclusively in algebras with identity. In this case the rest of the text must be read with the adequate adjustments. Anything concerning algebras without identities may be ignored and regular ideals are to be understood just as ideals — in particular, the space $\mathfrak{M}(A)$ of all regular maximal ideals would just be the space of all maximal ideals.

1.2. Regular Ideals

1.2.1. Definition A left ideal I is called *regular* if $\exists u \in A$ s.t. $\forall x \in A : xu \equiv x \mod I$. In that case u is called an *identity modulo* I.

Analogously for right ($ux \equiv x \mod I$) and two-sided ($xu \equiv x \equiv ux \mod I$) ideals.

Some trivial remarks: if A has an identity e, obviously every ideal is regular and e is an identity modulo any ideal. Also, every element of A is an identity modulo the regular ideal A.

If I is regular and u is an identity modulo I, then any $J \supset I$ is regular and u is also an identity modulo J.

1.2.2. Remark Consider $I \subset A$ a two-sided ideal. If u and u' are two identities modulo I then $u \equiv u' \mod I$, since $u \equiv uu' \equiv u' \mod I$. More generally, A/I is an algebra with identity $\bar{u}$, since $\forall \bar{x} \in A/I : \bar{x}\bar{u} = \overline{xu} = \bar{u}$ and $\bar{u}\bar{x} = \overline{ux} = \bar{u}$. Conversely, if A/I has an identity $\bar{u}$ then u is an identity modulo I so I is regular.

1.2.3. Definition Consider I a regular two-sided ideal. An $x \in A$ is called *left-invertible modulo* I if $\exists y \in A$ s.t. $yx \equiv u \mod I$, for u an identity modulo I. Then y is called a *left inverse of x modulo I*. Right and two-sided cases are analogous.

The definition is independent of the choice for u since if u and u' are identities modulo I, then $u \equiv u' \mod I$ and then $yx - u \in I \Leftrightarrow yx - u' \in I$.

Of course, the definition is equivalent to saying: "$\bar{y}$ is a left inverse of $\bar{x}$ in A/I ".

The following Lemma is a variant of (1.1.5.):

1.2.4. Lemma Let $I \subset A$ a regular ideal and u an identity modulo I. Then

1. $u \in I \Longrightarrow I = A$.

2. When I is two-sided: $x \in I$ and x invertible modulo $I \Longrightarrow I = A$

Proof

1. Let $x \in A$. Then $xu \in I$ and hence $x = x + xu - xu = xu + (-1) \cdot (xu - x) \in I$. So $A \subset I$.

2. If $x \in I$ and y is its inverse modulo I, it follows that $yx \in I$ and hence $u = u + yx - yx = yx + (-1) \cdot (yx - u) \in I$. Then from (1) follows $I \subset A$ $\square$

Slightly out of context for now, the next Lemma will turn out crucial when studying Gelfand theory.

1.2.5. Lemma If A is commutative and M is a regular maximal ideal of A, then A/M is a division algebra.

Proof Let u be the identity modulo M, and let $\bar{x} \neq 0$, i.e. $x \in A \setminus M$. Since $x \notin M$, the ideal $M + (x) \neq M$. Since M is maximal, this immediately implies $M + (x) = A$.

Thus, there exist $y \in A$ and $m \in M$ s.t. $m + yx = u$ and hence $yx - u = m \in M$. So y is an inverse of x modulo M, i.e. $\bar{y}$ is an inverse of $\bar{x}$ $\square$

1.3. Adjoining an Identity

Here is the main tool for dealing with algebras without identity!

1.3.1. Definition If A is without identity, we consider the direct product space $A[e] := A \oplus \mathbb{C}$ with multiplication defined by

$$(x, \alpha) \cdot (y, \beta) := (xy + \beta x + \alpha y,\ \alpha\beta)$$

It is a direct verification that this is indeed an algebra.

Next, we define $e := (0, 1)$ and identify A with the subspace $\{(x, \alpha) \in A[e] \mid \alpha = 0\}$ of $A[e]$. Clearly, this is an algebra isomorphism (since $(x, 0) \cdot (y, 0) = (xy + 0 + 0, 0) = (xy, 0)$ etc.)

Since $(x, 0) \cdot (0, 1) = (x, 0)$ and $(0, 1) \cdot (x, 0) = (x, 0)$, e is indeed an identity in $A[e]$.

This procedure is called *adjoining an identity to A* and we usually write $x + \alpha e$ instead of (x, α)

Note that if A has its own identity e' then $e \neq e'$. Moreover, an $x \in A \subset A[e]$ can't possibly be invertible in $A[e]$, because $(x + 0e)(y + \beta e) = \cdots + 0e \neq e$, for any y and any β.

If I is an ideal in A it is also an ideal in $A[e]$ (if $y \in I$ then $(x + \alpha e)y = xy + \alpha y \in I + I = I$).

Moreover, the ideal $A \subset A[e]$ is maximal (as vector space and hence also as ideal) because A has codimension 1 as a linear subspace of $A[e]$. It is two-sided because $A \subset A$ is a two-sided ideal.

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So far we've seen two two types of 'identities' related to algebras which don't have them: identities modulo ideals and adjoined identities. In fact, these concepts are closely related. We present this relation in the following three theorems.

1.3.2. Theorem Let I_e an ideal of $A[e]$ such that $I_e \not\subseteq A$. Then $I := I_e \cap A$ is a regular ideal of A.

Proof It is obvious that I is an ideal of A: take any $j \in I = I_e \cap A$ and $x \in A$, then $xj \in A$ because $x, j \in A$ and $xj \in I_e$ because $j \in I_e$. Hence $xj \in I_e \cap A = I$.

Since $I_e \not\subseteq A$, we can take $(x + \alpha e) \in I_e \setminus A$, with $\alpha \neq 0$. Since I_e is also a vector space, $(-\frac{1}{\alpha}(x + \alpha e) = (-\frac{x}{\alpha} - e)$ is also in I_e. We define $u := -\frac{x}{\alpha} \in A$ (so that $(u - e) \in I_e$), and show that u is an identity modulo I (in A).

Take any $y \in A$. Then $yu - y = (y + 0 \cdot e) \cdot (u - e)$, and since $(u - e) \in I_e$, also $(yu - y) \in I_e$. This completes the proof. $\square$

Note: if A has its own identity e_A then $e_A \neq e$ but the proposition still holds since every ideal of A is regular. What's more, the u found in this proof is still an identity modulo I.

1.3.3. Theorem If $I \subset A$ is a regular ideal with u an identity modulo I, then there exists an I_e, ideal of $A[e]$ so that $I_e \not\subseteq A$ and $I = I_e \cap A$.

Proof First of all we note that $\forall x \in A$ one has $xu \in I \iff x \in I$. Hence we may write $I = \{x \in A \mid xu \in I\}$.

We use this characterization of I to extend it to an ideal $I_e \subset A[e]$, defined analogously: $I_e := \{y \in A[e] \mid yu \in I\}$. By definition $I = I_e \cap A$.

I_e is indeed an ideal of $A[e]$: Take any $y \in I_e$ and $z \in A[e]$. To show $zy \in I_e$ we just see that $(zy)u = z(yu) = (z_0 + \zeta e) \cdot (yu) = (z_0 yu + \zeta yu) \in I$. So $(zy)u \in I$ so by definition $zy \in I_e$.

Now it only remains to show that I_e is a proper extension of I, i.e. we need to find an element in $I_e \setminus A$. But such an element is $(u-e)$, which is obviously not in A and $(u-e)\cdot u = u^2 - u \in I$, so $(u-e) \in I_e$.

This completes the proof. $\square$

1.3.4. Theorem Moreover, if I is two-sided then I_e from the previous proposition is unique.

Proof Let I_e and J_e be two ideals satisfying the conditions of (1.3.3.) In both cases we can use reasoning similar to the one in the proof of Proposition (1.3.2.) in order to obtain a $(u-e) \in I_e$ and a $(v-e) \in J_e$ such that both u and v are identities modulo I.

We will prove that $I_e \subset J_e$. Since I is two-sided, $uv - v \in I$ and $uv - u \in I$ and hence $u - v = (uv - v) - (uv - u) \in I$.

Now take any $y = (z + \lambda e) \in I_e$. Then:

$$y = (z+\lambda e) = z + (zu - zu) + (\lambda u - \lambda u) + (\lambda v - \lambda v) + \lambda e =$$
$$= z - zu + zu + \lambda u + \lambda v - \lambda u + \lambda e - \lambda v =$$
$$= z(e-u) + (z+\lambda e)u + \lambda(v-u) + \lambda(e-v) =$$
$$= z(e-u) + yu + \lambda(v-u) + \lambda(e-v) =$$

Now let's see:

1. $z \in A$ and $(e-u) \in I_e$ so $z(e-u) \in I_e \cap A = I$.
2. $y \in I_e$ and $u \in A$ so $zu \in I$.
3. $(v-u) \in I$ shown above, hence $\lambda(v-u) \in I$.
4. $(e-v) \in J_e$, hence $\lambda(e-v) \in J_e$.

Hence $y \in I + I + I + J_e$ and since $I \subset J_e$ we get $y \in J_e$.

So we have proven $I_e \subset J_e$. Analogously one can prove $J_e \subset I_e$. $\square$

1.3.5. Summary These three results can be summarized as follows: let $\Im_e$ be the set of all ideals of $A[e]$ not included in A, and let $\Im$ be the set of all regular ideals of A. Then the map:

$$\phi : \begin{array}{ccc} \Im_e & \longrightarrow & \Im \\ I_e & \longmapsto & (I := I_e \cap A) \end{array}$$

is well-defined (1.3.2.) and surjective (1.3.3.) In the case of two-sidedness, it is also bijective (1.3.4.)

Now we wish to have an analog of these results for *maximal ideals* — we want to have a bijection between the two-sided maximal ideals of $A[e]$ not contained in A, and the two-sided maximal regular ideals of A.

We use $\mathfrak{M}(A)$ to denote the set of all two-sided maximal regular ideals of A. Since in $A[e]$ every ideal is regular, and the condition $M \not\subseteq A$ is equivalent to $M \neq A$, (because A is an ideal and M a maximal ideal in $A[e]$), the bijection we wish to find is in fact

$$\mathfrak{M}(A[e]) \setminus \{A\} \cong \mathfrak{M}(A)$$

It then seems natural to look at the restriction $\phi|_{\mathfrak{M}(A[e])\setminus\{A\}}$. Because we are dealing with two-sided ideals, bijection is immediate. The only thing still left to prove is that maximal ideals carry over to maximal ideals under ϕ and ϕ^{-1}. This is the subject of the next theorem.

1.3.6. Theorem Let M_e be an ideal in A and $M_e \neq A$. Then M_e is maximal in $A[e]$ if and only if $\phi(M_e) = M_e \cap A$ is maximal in A.

Proof The notations are as in the preceding proofs.

From the proof of (1.3.3.) we see that ϕ^{-1} has the following property:

$$\text{if } I \subset J \text{ then } \phi^{-1}(I) \subset \phi^{-1}(J)$$

because $y \in I_e \Rightarrow yu \in I \subset J \Rightarrow y \in J_e$.

So, suppose M_e is maximal. Suppose $M_e \cap A \subset I$. Then $M_e \subset I_e$, and by maximality: $I_e = M_e$ or $I_e = A[e]$. The former implies $I = I_e \cap A = M_e \cap A$ whereas the latter implies $I = I_e \cap A = A[e] \cap A = A$. This proves that $M_e \cap A$ is maximal in A.

Conversely, suppose M_e is *not* maximal. Then $\exists I_e \supset M_e$ and $I_e \neq M_e$. But then $M_e \cap A \subset I_e \cap A$ and they are not equal because ϕ is injective. Hence $M_e \cap A$ isn't maximal. This proves the claim. $\square$

So we indeed have:

$$\phi|_{\mathfrak{M}(A[e])\setminus\{A\}} : \mathfrak{M}(A[e]) \setminus \{A\} \quad \xrightarrow{\simeq} \quad \mathfrak{M}(A)$$

1.3.7. Remark We could, in fact, extend the preceding bijection to a $\phi_0 : \mathfrak{M}(A[e]) \longrightarrow \mathfrak{M}(A) \cup \{A\}$, by letting it send A to A. Obviously, this is still a bijection.

1.4. Quasi-inverses

We have seen that if A is without identity we can adjoin an $e \in A[e]$. But how does this help? If $x \in A$ it is still not invertible in $A[e]$. And if A does have it's own identity, say, e', then invertibility in A and in $A[e]$ have little to do with each other. As a matter of fact, so far we haven't at all seen what invertibility in $A[e]$ has to say about A itself. Surely that must change, else there would be little point in introducing adjoined identities in the first place.

The relation is the following: though invertibility of x in A and in $A[e]$ are unrelated, that of $(e' - x)$ in A and of $(e - x)$ in $A[e]$ apparently coincide for all $x \in A$. Moreover, this property may be defined in terms of A alone.

That is the principal reason for introducing quasi-inverses (another one being a practical connection with inverses in Banach algebras, studied in (2.1.))

1.4.1. Definition Let $x \in A$. It is called *left quasi-invertible* if $\exists y \in A$ s.t. $x + y - yx = 0$, *right quasi-invertible* if $x + y - xy = 0$ and *quasi-regular* if it is both left- and right quasi-invertible. The y is then called a *(left-, right-) quasi-inverse of* x, denoted by x_{-1}. Later we shall see that quasi-inverses are unique and that if left- and right quasi-inverses exist then they coincide. The set $\{x \in A \mid x \text{ is quasi-regular}\}$ is denoted by A_{-1} (note that $0 \in A_{-1}$)

The idea behind this seemingly arbitrary definition becomes clear in the following Lemma:

1.4.2. Lemma Let A be an algebra without identity. Then the following are equivalent (left-, right- and two-sided cases included):

1. x is quasi-invertible
2. $(e - x)^{-1} = (e - x_{-1})$
3. $(e - x)$ is invertible in $A[e]$

Proof

"(1) ⇒ (2)" We directly check that

$$(e - x_{-1}) \cdot (e - x) = e - x_{-1} - x + x_{-1}x = e - 0 = e$$

"(2) ⇒ (3)" Directly

"(3) ⇒ (1)" If $(e - x)$ is invertible in $A[e]$ then there exists a $(y + \alpha e) \in A[e]$ s.t. $(y + \alpha e) \cdot (e - x) = e$. Writing out we get $-yx + y - \alpha x + \alpha e = e$ from which it directly follows that $\alpha = 1$ and $-yx + y - x = 0$. Hence $x_{-1} := -y$ is the quasi-inverse of x. $\square$

As expected, we now see that quasi-inverses are unique: let y and z be two quasi-inverses of x, then $(e-y)(e-x) = (e-z)(e-x) = e$ so $(e-y) = (e-z)$ (by uniqueness of inverses) so $y = z$. Similarly we can show: if y is a left quasi-inverse whereas z is a right quasi-inverse, then $y = z$ is a quasi-inverse.

Now it is interesting to note that an analog of (1.4.2.) exists if we replace e by e', for $e' \in A$ the identity of A, if there is one. The last implication is then proven as follows: since $(e' - x)$ is invertible, $\exists y \in A$ s.t. $y(e' - x) = e'$. Let $z := e' - y$, then $(e' - z)(e' - x) = e' - z - x + zx = e'$ and thus $x + z - zx = 0$, hence z is the quasi-inverse of x. This gives us the desired property:

1.4.3. Main property of quasi-regularity

$$x \text{ is quasi-invertible} \iff (e-x)^{-1} = (e - x_{-1}) \iff (e'-x)^{-1} = (e' - x_{-1})$$

for e the identity in $A[e]$, e' the identity in A and x_{-1} the *unique* quasi-inverse of x. $\square$

We will sometimes need a kind of turned-around version of (1.4.3.):

if $(e-x)_{-1} = (e-y)$ then $y^{-1} = (e-(e-y))^{-1} = (e-(e-y)_{-1}) = (e-(e-x)) = x$.

Now follows a Lemma similar to (1.1.6.) and (1.2.4.), concerning ideals and quasi-inverses:

1.4.4. Lemma $\forall x \in A : x$ has (left) quasi-inverse $\iff$ the ideal $I_x := \{wx - w \mid w \in A\} = A$

Proof

"$\Longrightarrow$" $x + y - yx = 0 \Rightarrow x = yx - y \Rightarrow x \in I_x$. Hence $\forall w \in A : w = w - wx + wx = (-1) \cdot (wx - w) + wx \in I_x$ and so $A \subset I_x$.

"$\Longleftarrow$" Since $x \in I_x$, $\exists y \in A$ s.t. $x = yx - y$. Hence $y = x_{-1}$. $\square$

Herewith we conclude this (rather tedious) algebraic chapter. More of this (and related) type of theory can be found in [4] and [9], from where this presentation is partially taken. In particular, [9] pp 155–168 offers a very good overview and is a recommended reference for interested readers.

2. Banach Algebras

In this chapter we will introduce normed algebras and Banach algebras. We shall prove basic properties and discuss inversion in Banach algebras. We will also give important examples, to which we shall return in chapter 5.

2.1. Preliminaries of Normed and Banach Algebras

Normed algebras are, as the definition suggests, algebras whose underlying vector space is a normed vector space. Moreover, the norm must be *sub-multiplicative*.

2.1.1. Definition A *normed algebra* $(A, \|.\|)$ is a normed vector space and an algebra, satisfying

$$\|xy\| \leq \|x\|\|y\| \ \ \forall x, y \in A$$

Some authors extend the definition by saying that if A has an identity e, then $\|e\| = 1$. Here we don't explicitly do this, however, we remark that firstly $\|x\| = \|ex\| \leq \|e\|\|x\|$ implies $\|e\| \geq 1$, and secondly, for any normed algebra $(A, \|.\|)$ whose underlying normed vector space is a Banach space (complete relative to $\|.\|$) there exists an equivalent norm $\|.\|_e$ such that $\|e\|_e = 1$. The proof of this can be found in [4] pp 23–26. Since in the future we shall only deal with such complete algebras, $\|e\| = 1$ may be assumed without loss of generality.

We will also write x^n for $x \cdot \ldots \cdot x$ n times, so that $\|x^n\| \leq \|x\|^n$. If A has identity e, we agree on $x^0 = e$.

If A is without an identity, the algebra $A[e]$ can be made into a normed algebra by setting $\|x + \alpha e\| = \|x\| + |\alpha|$. The verification that this is indeed a normed algebra is straightforward.

If A is a normed algebra and $B \subset A$ a subalgebra with the induced norm, obviously B is also a normed algebra.

The following Lemma shows what is so nice about sub-multiplicativity:

2.1.2. Lemma

1. If $x_n \to x$ then $yx_n \to yx$ and $x_n y \to xy$, $\forall y \in A$
2. If $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$
3. If $I \subset A$ is an ideal, then the topological closure of I, denoted by $\overline{I}$, is also an ideal.
4. If $A_0 \subset A$ is a subalgebra, then $\overline{A_0}$ is also a subalgebra.

Proof

1. Since $x_n \to x$, $\|x_n - x\| \to 0$ in $\mathbb{C}$, and hence $\|yx_n - yx\| = \|y(x_n - x)\| \leq \|y\|\|x_n - x\| \to \|y\| \cdot 0$ (in $\mathbb{C}$). Other case similarly.
2. A bit more work: Since $x_n \to x$ also $\|x_n\| \to \|x\|$, hence $\|x_n\|$ is bounded, say by M.

   Given ϵ, let N_x be s.t. $n \geq N_x$ implies $\|x_n - x\| < \frac{\epsilon}{2\|y\|}$ if $y \neq 0$ and arbitrary otherwise, so that in any case $\|x_n - x\|\|y\| < \frac{\epsilon}{2}$. Let N_y s.t. $n \geq N_y$ implies $\|y_n - y\| < \frac{\epsilon}{2M}$ (choose $M > 0$). Then for $N := \max(N_x, N_y)$ holds: if $n \geq N$ then

   $$\|x_n y_n - xy\| = \|x_n y_n - x_n y + x_n y - xy\| \leq \|x_n y_n - x_n y\| + \|x_n y - xy\| \leq$$

   $$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| < M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   So $x_n y_n \to xy$.
3. We know that topological closures of linear sub-spaces are linear sub-spaces. Moreover, suppose $x \in \overline{I}$. Then there is a sequence $x_n \to x$ with $x_n \in I$. Then $yx_n \in I$ $\forall y \in A$, and $yx_n \to yx$ by (1), so $yx \in \overline{I}$.
4. Let $x, y \in \overline{A_0}$. Then there are sequences $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in A_0$. Then $x_n y_n \in A_0$ $\forall n$ and from (2.) $x_n y_n \to xy$, so $xy \in \overline{A_0}$. (And obviously $x_n + y_n \to x + y$ and $\lambda x_n \to \lambda x$). $\square$

Note that item (1) actually says that multiplication is *separately continuous* whereas (2) says that multiplication as a function $A \times A \longrightarrow A$ is continuous. Note also that, for each $x \in A$, left- and right-multiplication define continuous linear operators $T_x^l := [y \mapsto yx]$ and $T_x^r := [y \mapsto xy]$, and moreover we even have a measure for their continuity: $\|T_x^l(y)\| = \|yx\| \leq \|y\|\|x\|$ $\forall y \in A$ so $\|T_x^l\| \leq \|x\|$ and analogously $\|T_x^r\| \leq \|x\|$.

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2.1.3. Definition If A is a normed algebra that is complete relative to the norm $\|.\|$, (i.e. A is a Banach space) then A is called a *Banach algebra*

If A is a Banach algebra without identity, then $A[e]$ is also a Banach algebra, since $\mathbb{C}$ is Banach and $A[e] = A \oplus \mathbb{C}$.

The following Lemma concerns closed ideals of A and will be an important tool in Gelfand theory.

2.1.4. Lemma If A is a normed algebra and I a closed two-sided ideal, then A/I with the quotient norm $\|\overline{x}\| := \inf\{\|x'\| \mid x' \in \overline{x}\})$ is also a normed algebra. Moreover, if A is Banach A/I is also Banach.

Proof By (1.1.4.) A/I is an algebra. Since I is closed it is also known that A/I is a normed vector space (see e.g. [7] pp 51–53 or [6] pg. 36). It remains only to show that $\|\bar{x}\bar{y}\| \leq \|\overline{x}\|\|\overline{y}\|$. This boils down to the following condition: $\forall \epsilon > 0$, $\exists z \in \overline{xy}$ s.t. $\|z\| \leq \|\overline{x}\|\|\overline{y}\| + \epsilon$.

So, for given ϵ, define $\delta := \min(1, \frac{\epsilon}{\|\overline{x}\|+\|\overline{y}\|+1})$. Then, by definition, we can choose $x' \in \overline{x}$ s.t. $\|x'\| < \|\overline{x}\| + \delta$ and $y' \in \overline{y}$ s.t. $\|y'\| < \|\overline{y}\| + \delta$. Then $x'y' \in \overline{xy}$ and

$$\begin{aligned}\|x'y'\| \leq \|x'\|\|y'\| < (\|\overline{x}\| + \delta) \cdot (\|\overline{y}\| + \delta) = \\ = \|\overline{x}\|\|\overline{y}\| + \|\overline{x}\|\delta + \|\overline{y}\|\delta + \delta^2 \leq \\ \leq \|\overline{x}\|\|\overline{y}\| + (\|\overline{x}\| + \|\overline{y}\| + 1)\delta \quad (\text{since } \delta \leq 1) \\ \leq \|\overline{x}\|\|\overline{y}\| + \epsilon\end{aligned}$$

Moreover, it is also a known fact that for A a Banach space and I a closed linear subspace, A/I is a Banach space [7, 6]. So, as a normed algebra, A/I is a Banach algebra. $\square$

Without great attention to detail we give some examples of Banach algebras. In chapter 5 we will take a closer look at some of them.

2.1.5. Some examples

1. If X is a normed vector space then from basic functional analysis (e.g. [7, 6, 13]) we know that $\mathcal{L}(X)$ is also a normed vector space and that it is Banach if X is Banach. With the multiplication operation being composition $\mathcal{L}(X)$ in fact becomes an algebra. Moreover, since $\forall T, S \in \mathcal{L}(X), \forall x \in X:$

   $$\|(T \circ S)(x)\| = \|T(S(x))\| \leq \|T\|\|S(x)\| \leq \|T\|\|S\|\|x\|,$$

   we have $\|T \circ S\| \leq \|T\|\|S\|$ and so $\mathcal{L}(X)$ is also a normed algebra.

2. Let X be a Hausdorff space. Then we define

   $$C(X) := \{f : X \to \mathbb{C} \mid f \text{ continuous and bounded }\}$$

and

$$C_0(X) := \{f : X \to \mathbb{C} \mid f \text{ continuous and vanish at infinity}\}$$

where "vanishing at infinity" means that $\forall \epsilon > 0\ \exists K \subset X$ compact s.t. $|f(t)| < \epsilon \quad \forall t \in X \setminus K$. An easy verification shows that $C(X)$ and $C_0(X)$ are both commutative normed algebras if equipped with point-wise operations and the supremum norm $\|.\|_\infty$. In fact they are also Banach algebras, as another simple verification (using the completeness of $\mathbb{C}$) will show.

Note that $C_0(X) \subset C(X)$ because any f vanishing at infinity is automatically bounded (within K because K is compact and f continuous and outside K by some ϵ). If X itself is compact, $C_0(X) = C(X)$.

$C(X)$ has an identity, namely the constant 1-function. $C_0(X)$ is with identity if and only if X is compact.

3. Let D be the closed unit disc in $\mathbb{C}$, i.e. $D := \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ and $A(D)$ the subset of $C(D)$ containing those functions which are analytic in the interior of D, D°. It is a normed algebra because it is a subalgebra of $C(D)$ (since addition, scalar multiplication and point-wise multiplication preserve "analyticity"). To see that $A(D)$ is also a Banach algebra, consider a sequence $f_n \to f$ in $C(X)$ (i.e. uniformly convergent) and $f_n \in A(D)$. Since $f_n \to f$ uniformly on every compact subset of D°, by a result in the theory of functions of complex variables ([3] pg. 147), we get that f is analytic. So $A(D)$ is closed in $C(D)$ which was Banach, hence $A(D)$ is Banach.

4. Now let H^∞ be the subset of $C(D^\circ)$ containing those functions which are analytic on D°. H^∞ is also a normed algebra (being a subalgebra of $C(D^\circ)$) and $A(D) \subset H^\infty$. Moreover, H^∞ is Banach by an analogous argument as in (3).

5. Let Γ be the unit circle in $\mathbb{C}$, i.e. $\Gamma := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then we denote by $AC(\Gamma)$ the set of all continuous functions which have absolutely convergent Fourier series, that is, all $f : \Gamma \longrightarrow \mathbb{C}$ s.t. $f(\lambda) = \sum_{n=-\infty}^{\infty} a_n \lambda^n$ with $\sum_{n=-\infty}^{\infty} |a_n|$ convergent. We set the norm to be $\|f\|_1 := \sum_{n=-\infty}^{\infty} |a_n|$. With point-wise operations we get that $AC(\Gamma)$ is a Banach algebra — $\|fg\|_1 \leq \|f\|_1 \|g\|_1$ follows from Fubini's theorem applied to the discrete measure on $\mathbb{Z}$ and completeness follows from the completeness of $l_1(\mathbb{Z})$. For details, we refer e.g. to [12] pp. 268–269.

2.2. Inversion and Quasi-inversion in Banach Algebras

In this section we will be dealing with inversion in Banach algebras. The completeness of the algebra is crucial — indeed none of the theory would get very far without that assumption. The particular property of completeness that we will be using is: *in a Banach space absolutely convergent series are convergent.* [6, 13]

We will show that A^{-1} is an open subset of A containing $B(e,1)$ (the open ball in e with radius 1) and that inversion is continuous. The following three propositions elaborate this.

2.2.1. Proposition Let A be a Banach algebra with identity. If $x \in A$ s.t. $\|e-x\| < 1$ then x is regular (two-sided invertible).

Proof We define

$$x^{-1} := \sum_{n=0}^{\infty} (e-x)^n$$

Since $\|e-x\| < 1$ the series $\sum_{n=0}^{\infty} \|(e-x)^n\|$ is absolutely convergent, and since A is a Banach space the sum itself is convergent so x^{-1} is well-defined.

Now,

$$xx^{-1} = (e-(e-x))x^{-1} = (e-(e-x))\sum_{n=0}^{\infty}(e-x)^n =$$

$$= (e-(e-x)) \cdot \lim_{N\to\infty}\left[\sum_{n=0}^{N}(e-x)^n\right] = \lim_{N\to\infty}\left[\sum_{n=0}^{N}(e-x)^n - \sum_{n=1}^{N+1}(e-x)^n\right] =$$

$$= \lim_{N\to\infty}\left[e-(e-x)^{N+1}\right] = e$$

Analogously we can show $x^{-1}x = e$. □

2.2.2. Proposition If x is regular and y is s.t. $\|x-y\| < \frac{1}{\|x^{-1}\|}$ then y is also regular. (Hence: A^{-1} is open).

Proof It holds $\|e - x^{-1}y\| = \|x^{-1}(x-y)\| \leq \|x^{-1}\|\|x-y\| < 1$ and thus by (2.2.1.) $x^{-1}y$ is invertible. Then we set $y^{-1} := (x^{-1}y)^{-1}x^{-1}$ and get: $y^{-1}y = (x^{-1}y)^{-1}(x^{-1}y) = e$ □

2.2.3. Proposition The map $\begin{array}{ccc} A^{-1} & \longrightarrow & A^{-1} \\ x & \longmapsto & x^{-1} \end{array}$ is continuous.

Proof Suppose $x \in A^{-1}$ and ϵ given. We define $\delta := \min(\frac{1}{\|x^{-1}\|}, \frac{1}{2\|x^{-1}\|}, \frac{\epsilon}{2\|x^{-1}\|^2})$. We will refer to the three corresponding estimates by $(*)$, $(**)$ and $(***)$, respectively.

Then for all y s.t. $\|x-y\| < \delta$ holds that $y \in A^{-1}$ because of $(*)$ and

$$\|e - x^{-1}y\| = \|x^{-1}(x-y)\| \leq \|x^{-1}\|\|x-y\| \overset{(**)}{<} \|x^{-1}\|\frac{1}{2\|x^{-1}\|} = \frac{1}{2}$$

and hence:

$$\|y^{-1}x\| = \|(x^{-1}y)^{-1}\| = \|\sum_{n=0}^{\infty}(e - x^{-1}y)^n\| \leq \sum_{n=0}^{\infty}\|e - x^{-1}y\|^n \leq \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

Therefore:

$$\|x^{-1} - y^{-1}\| = \|(y^{-1}x)(x^{-1})(y-x)(x^{-1})\| \leq \|y^{-1}x\|\|x^{-1}\|^2\|y-x\| <$$

$$< 2\|x^{-1}\|^2\|y-x\| \overset{(***)}{<} 2\|x^{-1}\|^2\frac{\epsilon}{2\|x^{-1}\|^2} = \epsilon$$

□

It is interesting to note that actually A^{-1} is a topological group, with multiplication being the group-operation. The fact that it is a group is trivial and continuity of multiplication follows from the sub-multiplicativity of the norm (see 2.1.2.) The only requirement left is the continuity of the inversion and that we have just proven! In the specific case that $A = \mathcal{L}(X)$, A^{-1} is often denoted by $\mathcal{GL}(X)$.

*

How do these results apply to Banach algebras without identity? The concept of quasi-inversion developed in chapter 1 enables us to translate the three preceding propositions into their counterparts for algebras without identity. The e refers to the adjoined identity, i.e. $e \in A[e]$.

2.2.4. Proposition Let A be a Banach algebra not necessarily with identity. If $x \in A$ s.t. $\|x\| < 1$ then x is quasi-regular.

Proof By (2.2.1.) $\|x\| = \|e - (e-x)\| < 1$ implies that $(e-x)$ is regular, hence x is quasi-regular. □

2.2.5. Proposition If x quasi-regular and y s.t. $\|x-y\| < \frac{1}{1+\|x_{-1}\|}$ then y is quasi-regular. (Hence: A_{-1} is open).

Proof For e the adjoined identity one has $\|(e-y)-(e-x)\| = \|x-y\| \leq \frac{1}{1+\|x_{-1}\|} = \frac{1}{\|e-x_{-1}\|} = \frac{1}{\|(e-x)^{-1}\|}$. Moreover, since x is quasi-regular $(e-x)$ is regular. Then from (2.2.2.) it follows that $(e-y)$ is regular and hence y is quasi-regular. □

2.2.6. Proposition The map $\begin{array}{ccc} A_{-1} & \longrightarrow & A_{-1} \\ x & \longmapsto & x_{-1} \end{array}$ is continuous.

Proof Use the fact that $x_{-1} = e-(e-x_{-1}) = e-(e-x)^{-1}$ and that subtractions and inverses are continuous. $\square$

Summarizing, we see that A_{-1} is an open subset of A containing $B(0,1)$ and that quasi-inversion is continuous.

2.2.7. Lemma If I is a proper regular ideal of A and u an identity modulo I, then $B(u,1) \subset I^c$.

Proof. From (1.2.4.) we know that $u \notin I$. Now take any $x \in I$. Suppose $\|x-u\| < 1$, then by (2.2.4.) $(x-u)$ would be quasi-regular so $\exists y$ s.t. $x - u + y - (x-u)y = 0$. Then $u = x - xy + (y + uy) \in I + I + I = I$, contradicting $u \notin I$.

Hence $\|x - u\| \geq 1$ $\forall x \in I$ and so $B(u,1) \subset I^c$. $\square$

A consequence of this is the following important corollary.

2.2.8. Corollary If I is a proper regular ideal then $\overline{I}$ (the topological closure of I) is also a proper regular ideal. Hence, maximal regular ideals are closed.

Proof From (2.1.2; 3.) $\overline{I}$ is an ideal and it is regular because $I \subset \overline{I}$. The only thing left to show is that $\overline{I} \neq A$ and that follows from the previous Lemma! $\square$

*

Finally we present a result which can strengthen (2.2.1.) and will turn out important in the future anyway. Instead of the norm $\|x\|$, it can be useful to look at the limit $\lim_{n\to\infty}(\|x^n\|^{\frac{1}{n}})$. It is not evident why this limit exists. We shall postpone the proof of that claim till chapter 3, and for now simply assume it does exists. Then we do know that since $\forall n \in \mathbb{N}$ $\|x^n\|^{\frac{1}{n}} \leq (\|x\|^n)^{\frac{1}{n}} = \|x\|$, the limit is $\leq \|x\|$.

2.2.9. Proposition If $x \in A$ s.t. $\lim_{n\to\infty}(\|(e-x)^n\|^{\frac{1}{n}}) < 1$ then x is regular. This strengthens Proposition (2.2.1.) Hence, if $\lim_{n\to\infty}(\|x^n\|^{\frac{1}{n}}) < 1$ then x is quasi-regular; this strengthens Proposition (2.2.4.)

Proof The proof is analogous to the proof of (2.2.1.), except that now the convergence of $\sum_{n=0}^{\infty}(e-x)^n$ can be concluded from the root test, applied to $\sum_{n=0}^{\infty}\|(e-x)^n\|$ $\square$

2.2.10. Definition An element $x \in A$ is *topologically nilpotent* if $\lim_{n\to\infty}(\|x^n\|^{\frac{1}{n}}) = 0$.

2.2.11. Lemma Let A be a Banach algebra with identity. If x is topologically nilpotent then it is singular (i.e. *not* two-sided invertible — it *may* be left- or right-invertible).

Proof Suppose x is regular, i.e. $xx^{-1} = x^{-1}x = e$. Then $\forall n : 1 = \|e\| = \|e^n\|^{\frac{1}{n}} = \|(x^{-1}x)^n\|^{\frac{1}{n}} = \|(x^{-1})^n(x)^n\|^{\frac{1}{n}} \leq \|x^{-1}\|\|x^n\|^{\frac{1}{n}}$, so that $\forall n$ $\|x^n\|^{\frac{1}{n}} \geq \frac{1}{\|x^{-1}\|}$ which contradicts the topological nil-potency. $\square$

3. Spectra

In this chapter we introduce the spectrum of an element in a Banach algebra. Besides proving two general results — the Polynomial Spectral Mapping Theorem and the Spectral Radius Formula, the spectrum will also be used to prove the Gelfand-Mazur Theorem, one of the fundamental ingredients of Gelfand Theory.

3.1. Preliminaries

For some readers the spectrum might be familiar in the more specific context of linear *operators*, on normed vector-spaces, Banach spaces or Hilbertspaces. The definition here is analogous and valid for general Banach algebras.

3.1.1. Definition Let A be a Banach algebra with identity and $x \in A$. The *spectrum* of x is defined as

$$\sigma(x) := \{\lambda \in \mathbb{C} \mid (x - \lambda e) \text{ is singular (not two-sided invertible) } \}$$

The complement of $\sigma(x)$ is denoted by $\Re_x$ and called the *resolvent set of* x.

The *spectral radius* $\rho(x)$ is defined as $\sup_{\lambda \in \sigma(x)} |\lambda|$.

We define the mapping $r_x : \begin{array}{ccc} \Re_x \subset \mathbb{C} & \longrightarrow & \mathbb{C} \\ \lambda & \longmapsto & (x - \lambda e)^{-1} \end{array}$ for each $x \in A$.

If A is without identity, we consider x as an element of $A[e]$.

If the underlying Banach algebra A is not clear from the context, we write $\sigma_A(x)$.

In the case when $A = \mathcal{L}(X)$ (see example 2.1.5;1.), the spectrum of an operator T may indeed be understood as a generalization of Eigenvalues from finite-dimensional linear algebra. Intuitively this doesn't have so much to do with the theory we are developing. To get the flavor of the spectrum's importance for Gelfand theory, consider A a commutative Banach algebra with identity: since by (1.1.6;4.) an element is not invertible precisely if it is contained in a maximal ideal, an equivalent formulation of the spectrum would be:

$$\sigma(x) := \{\lambda \in \mathbb{C} \mid x \equiv \lambda e \mod M, \text{ for some maximal ideal } M \subset A\}$$

It's a useful formulation, and intuitively good to hold on to when studying Gelfand theory.

Now we give some basic properties following directly from the definition, for A with identity:

3.1.2. Basic properties

1. r_x is continuous since $[\lambda \mapsto \lambda e]$ and the 'minus' are continuous by definition of topological vector-spaces and inversion is continuous in A by (2.2.3.)
2. $0 \in \sigma(x) \iff x$ is singular.
3. For all $\lambda \neq 0$: $\lambda \in \sigma(x) \Leftrightarrow (x - \lambda e)$ is singular $\Leftrightarrow (\frac{x}{\lambda} - e)$ is singular $\Leftrightarrow \frac{x}{\lambda}$ is quasi-singular.

If A is without identity, $0 \in \sigma(x)\ \forall x \in A \subset A[e]$. Also, if A is without identity, by definition we have $\sigma_A(x) = \sigma_{A[e]}(x)\ \forall x \in A$. However, if A is with identity then 0 is not in $\sigma_A(x)$ if x is regular in A whereas $0 \in \sigma_{A[e]}(x)\ \forall x$, so in general $\sigma_A(x) \neq \sigma_{A[e]}(x)$. However, the non-zero part, as we saw in (3.1.2;3.), can be described in terms of quasi-regularity in A and that is a concept that coincides for A and $A[e]$. Therefore

$$\sigma_{A[e]}(x) = \sigma_A(x) \cup \{0\}\ \forall x \in A \quad (A \text{ with or without identity})$$

In the future we will normally consider spectra of algebras with identity.

Next we wish to show two important properties of the spectrum. The first (rather elementary) result says that $\sigma(x)$ is a closed subset of $\overline{B(0, \|x\|)}$ (the closed ball of radius $\|x\|$), $\forall x \in A$. Therefore $\sigma(x)$ is compact. The second result, that $\sigma(x)$ is *non-empty*, is much more sophisticated and fundamental. It's proof relies heavily on the theory of functions of complex variables and indeed works only because we are working in complex algebras. It is also the main ingredient in the proof of the famous Gelfand-Mazur Theorem which says that complex Banach algebras which are division algebras are isomorphic to $\mathbb{C}$.

3.1.3. Theorem Let A be a Banach algebra with identity e and let $x \in A$. Then $\forall \lambda \in \sigma(x) : |\lambda| \leq \|x\|$ and $\sigma(x)$ is closed in $\mathbb{C}$. (Hence: $\sigma(x)$ is compact.)

Proof Suppose $|\lambda| > \|x\|$. Then $\|\frac{x}{\lambda}\| = \frac{\|x\|}{|\lambda|} < 1$ and, from (3.3.4.) and (1.4.3.) $(e - \frac{x}{\lambda})$ is invertible and hence $(x - \lambda e)$ is also invertible, contradicting $\lambda \in \sigma(x)$.

For the second statement note that the mapping $\lambda \mapsto (x - \lambda e)$ is continuous, $\Re_x$ is the inverse image of A^{-1} under this continuous map and A^{-1} is open. Hence $\Re_x$ is open and hence $\sigma(x) = \Re_x^c$ is closed. $\square$

3.1.4. Theorem $\sigma(x) \neq \emptyset$.

Proof Suppose $\sigma(x) = \emptyset$. Then $(x - \lambda e)$ is regular for all λ, including 0, so that x would also be regular.

Let $\Lambda \in A^*$ be a bounded linear functional such that $\Lambda(x^{-1}) = 1$, the existence of which is due to the Hahn-Banach Theorem. Define

$$g : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \lambda & \longmapsto & \Lambda((x - \lambda e)^{-1}) \end{array}$$

Then g is well-defined on all $\mathbb{C}$ by the assumption and it is continuous because it is $\Lambda \circ r_x$.

We claim that g is analytic and bounded.

Let $\lambda, \mu \in \mathbb{C}$. Then

$$\frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \frac{\Lambda[(x - \lambda e)^{-1} - (x - \mu e)^{-1}]}{\lambda - \mu} =$$

$$= \frac{\Lambda[(x - \mu e)^{-1}((x - \mu e) - (x - \lambda e))(x - \lambda e)^{-1}]}{\lambda - \mu} =$$

$$= \Lambda[(x - \mu e)^{-1}(x - \lambda e)^{-1}]$$

Now, since $\lambda \longmapsto \frac{g(\mu)-g(\lambda)}{\mu-\lambda}$ is continuous, $\lim_{\lambda \to \lambda_0} \left[\frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0}\right]$ exists and is equal to $\Lambda\left[((x - \lambda_0 e)^{-1})^2\right]$. Hence g is analytic.

Moreover, $\lim_{|\lambda|\to\infty} g(\lambda) = \lim_{|\lambda|\to\infty} \Lambda((x - \lambda e)^{-1}) = \Lambda\left[\lim_{|\lambda|\to\infty}\left(\frac{(\frac{x}{\lambda} - e)^{-1}}{\lambda}\right)\right] = \Lambda(0) = 0$.

So g is analytic and bounded, so, by Liouville's Theorem of Complex Function Theory, ([3] pg. 146) g is constant and since it vanishes at infinity that constant must be 0. But then $g(0) = 0$ which is a contradiction to $g(0) = \Lambda(x^{-1}) = 1$.

Therefore $\sigma(x)$ must be nonempty. □

As an immediate corollary we get the following:

3.1.5. Gelfand-Mazur Theorem. Let A be a Banach algebra with identity. If A is a division algebra, then the map $\begin{array}{ccc} \mathbb{C} & \longrightarrow & A \\ \lambda & \longmapsto & \lambda e \end{array}$ is an isometric algebra isomorphism. (As always, we assume $\|e\| = 1$; otherwise, $\lambda \longmapsto \frac{\lambda}{\|e\|}e$ would be the isometric algebra isomorphism).

Proof The given map is obviously an injective isometric algebra homomorphism, so it only remains to show that it is surjective. Well, let $x \in A$. Since $\sigma(x) \neq \emptyset$, let $\lambda \in \sigma(x)$. Suppose $x - \lambda e \neq 0$. Then, since $\mathbb{C}$ is a division algebra, $x - \lambda e$ is regular, contradicting $\lambda \in \sigma(x)$. Hence $x = \lambda e$. □

The Gelfand-Mazur Theorem is generally considered as one of the cornerstones of the theory of Banach algebras in general and Gelfand Theory in particular. It was first proven by S. Mazur in 1932 (though the result was left unpublished at the time).

The proof we gave is due to R. Arens and it works, actually, in a context slightly more general than that of Banach algebras — in any algebra that is a Banach space and in which the inverse is continuous. For interested readers, this general theory is very well elaborated in [9] pp.170–175.

Mazur's original proof, which *does* depend on the sub-multilplicativity of the norm (or, in fact, just on the *separate* continuity of multiplication), can be found in [14] pp. 18–20.

To conclude this section we give one application of Gelfand-Mazur. We don't directly need it but it's a nice illustration of the power of the theorem.

3.1.6. Theorem Let A be a Banach algebra with identity. If $\forall x \in A^{-1}$, $\|x^{-1}\| \leq \frac{1}{\|x\|}$ then A is isometrically isomorphic to $\mathbb{C}$.

Proof In view of the Gelfand-Mazur theorem it is sufficient to prove that A is a division algebra, i.e. $A^{-1} = A \setminus \{0\}$.

We note that $A \setminus \{0\}$ is connected (any two points of A lie in a 2-dimensional linear sub-space of A, and $\mathbb{C}^2 \setminus \{0\}$ is connected). Next, since A^{-1} is open in A it is obviously also open in $A \setminus \{0\}$. We show that A^{-1} is also closed in $A \setminus \{0\}$.

Let $x_n \to x$ be a convergent sequence in $A \setminus \{0\}$ with $x_n \in A^{-1}$ $\forall n$. Consequently, there exists an $\epsilon > 0$ s.t. $\|x_n\| > \epsilon$ $\forall n$ as well as $\|x\| > \epsilon$.

But then $\|x_n^{-1}\| \leq \frac{1}{\|x_n\|} < \frac{1}{\epsilon}$. Therefore

$$\|x_n^{-1} - x_m^{-1}\| \leq \|x_m^{-1}(x_n - x_m)x_n^{-1}\| \leq$$
$$\leq \|x_m^{-1}\| \|(x_n - x_m)\| \|x_n^{-1}\| \leq \frac{\|x_n - x_m\|}{\epsilon^2}$$

and since $\{x_n\}$ is Cauchy, $\{x_n^{-1}\}$ is also Cauchy, and hence convergent to a $y \in A$.

But then

$$xy = \lim_{n\to\infty}(x_n) \lim_{n\to\infty}(x_n^{-1}) = \lim_{n\to\infty}(x_n x_n^{-1}) = \lim_{n\to\infty}(e) = e$$

and similarly $yx = e$, so $y = x^{-1}$ and hence $x \in A^{-1}$.

So we see that A^{-1} is an open and closed subset of the connected set $A \setminus \{0\}$, hence it is either empty or $A \setminus \{0\}$ itself, but it isn't empty since $e \in A^{-1}$. This proves the theorem. □

3.1.7. Corollary Let A be with identity. If $\forall x, y \in A$ we have $\|xy\| = \|x\|\|y\|$ then A is isometrically isomorphic to $\mathbb{C}$.

Proof If $x \in A^{-1}$ then $1 = \|e\| = \|xx^{-1}\| = \|x\|\|x^{-1}\|$ and hence $\|x^{-1}\| \leq \frac{1}{\|x\|}$ $\square$

3.2. Polynomial Spectral Mapping Theorem and the Spectral Radius Formula

This section introduces two other important results regarding spectra: the Polynomial Spectral Mapping Theorem and the Spectral Radius Formula.

3.2.1. Definition Let $p \in \mathbb{C}[X]$. Then, obviously, for each $x \in A$ we can define a ring-homomorphism

$$\varphi_x : \begin{array}{ccc} \mathbb{C}[X] & \longrightarrow & A \\ p & \longmapsto & p(x) \end{array}$$

In this way the *polynomial* p can be interpreted as a continuous mapping

$$p : \begin{array}{ccc} A & \longrightarrow & A \\ x & \longmapsto & p(x) \end{array}$$

3.2.2. Polynomial Spectral Mapping Theorem Let A a Banach algebra with identity, and p a polynomial (in A and $\mathbb{C}$ resp.) Then

$$\sigma(p(x)) = p\left[\sigma(x)\right] \quad \forall x \in A.$$

"$\supset$" Let $\lambda \in \sigma(x)$. Then $p(X) - p(\lambda) \in \mathbb{C}[X]$ is a polynomial with a zero at λ. Hence $p(X) - p(\lambda) = (X - \lambda)q(X)$ for some other polynomial $q(X) \in \mathbb{C}[X]$. Then (using the homomorphism φ_x):

$$p(x) - p(\lambda)e = (x - \lambda e)q(x)$$

But since $x - \lambda e$ is singular, so is $p(x) - p(\lambda)e$. Thus $p(\lambda) \in \sigma(p(x))$.

"$\subset$" Let $\mu \in \sigma(p(x))$. We need to prove that $\exists \lambda \in \sigma(x)$ s.t. $p(\lambda) = \mu$. Let $q(X) := p(X) - \mu \in \mathbb{C}[X]$. In $\mathbb{C}[X]$ this can be factored as $q(X) = a(X - \lambda_1) \cdot \dots \cdot (X - \lambda_n)$. Then (again using the homomorphism φ_x): $p(x) - \mu e = a(x - \lambda_1 e) \cdot \dots \cdot (x - \lambda_n e)$. Since $p(x) - \mu e$ is singular, one of the factors $(x - \lambda_k e)$ must be singular, i.e. $\lambda_k \in \sigma(x)$. But (returning to $\mathbb{C}[X]$) $0 = q(\lambda_k) = p(\lambda_k) - \mu$ i.e. $p(\lambda_k) = \mu$. $\square$

3.2.3. Spectral Radius Formula Let A be a Banach algebra and $x \in A$. Then $\lim_{n\to\infty}(\|x^n\|^{\frac{1}{n}})$ exists and is equal to $\rho(x)$, the spectral radius of x.

Proof Without loss of generality we may assume A to have an identity e (if it does not, $\rho(x)$ is defined in terms of $A[e]$, but $\lim_{n\to\infty}(\|x^n\|^{\frac{1}{n}})$ in A and in $A[e]$ coincide for all $x \in A$.)

Also, if $x = 0$ one trivially has $\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda e \text{ is singular}\} = \{0\}$, i.e. $\rho(0) = 0 = \lim_{n\to\infty}(\|0^n\|^{\frac{1}{n}})$, so we only need to consider $x \neq 0$.

Now, for any $\lambda \in \sigma(x)$, from the preceding theorem follows $\lambda^n \in \sigma(x^n)$, for all n, and thus, by (3.1.3.), $|\lambda|^n = |\lambda^n| \leq \|x^n\|$ and hence $|\lambda| \leq \|x^n\|^{\frac{1}{n}}$. Consequently $|\lambda| \leq \liminf_{n\to\infty}(\|x^n\|^{\frac{1}{n}})$ and thus $\rho(x) \leq \liminf_{n\to\infty}(\|x^n\|^{\frac{1}{n}})$.

Conversely, let $\lambda \in \mathbb{C}$ s.t. $|\lambda| < \frac{1}{\rho(x)}$. We wish to show that $(e - \lambda x)$ is regular.

If $\lambda = 0$ that is definitely the case, and if $\lambda \neq 0$ then $|\frac{1}{\lambda}| > \rho(x)$, so that $\frac{1}{\lambda} \notin \sigma(x)$ and hence $(x - \frac{1}{\lambda}e)$ is regular and so $(e - \lambda x) = (-\lambda)(x - \frac{1}{\lambda})$ is also regular.

Let $\Lambda \in A^*$ be any continuous functional. Similarly as in the proof of (3.1.4.) we can now define

$$g: \begin{array}{ccc} B(0, \frac{1}{\rho(x)}) & \longrightarrow & \mathbb{C} \\ \lambda & \longmapsto & \Lambda((e - \lambda x)^{-1}) \end{array}$$

and, by an analogous method, see that g is analytic on $B(0, \frac{1}{\rho(x)})$.

Now suppose λ is such that $|\lambda| < \frac{1}{\|x\|} \leq \frac{1}{\rho(x)}$. Then $\|\lambda x\| < 1$ and so by (2.2.1.) $(e - \lambda x)^{-1} = \sum_{n=0}^{\infty}(\lambda x)^n$, so that we get:

$$g(\lambda) = \Lambda((e - \lambda x)^{-1}) = \Lambda\left[\sum_{n=0}^{\infty}(\lambda x)^n\right] = \sum_{n=0}^{\infty}\Lambda(x^n)\lambda^n$$

Since this is the power series of g on $B(0, \frac{1}{\|x\|}) \subset B(0, \frac{1}{\rho(x)})$ and g was analytic on the whole of $B(0, \frac{1}{\rho(x)})$, g also has the same power series on the whole of $B(0, \frac{1}{\rho(x)})$ (see e.g. [3] pg. 91).

In particular, this means that the sequence $\{\Lambda((\lambda x)^n)\}_{n=0}^{\infty}$ is bounded. Moreover, Λ was chosen arbitrarily, therefore the sequence is bounded for all $\Lambda \in A^*$. Then by the uniform-boundedness principle (e.g. [7] pp. 199–201) $\{(\lambda x)^n\}_{n=0}^{\infty}$ itself is bounded.

Then $\exists M$ s.t. $\|(\lambda x)^n\| = |\lambda|^n\|x^n\| \leq M \; \forall n$. Thus we get:

$$\forall n: \quad \|x^n\|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|\lambda|}$$

Thus $\limsup_{n\to\infty}\|x^n\|^{\frac{1}{n}} \leq \frac{1}{|\lambda|}$, for all λ s.t. $\frac{1}{|\lambda|} > \rho(x)$ and so: $\limsup_{n\to\infty}\|x^n\|^{\frac{1}{n}} \leq \rho(x)$. Hence

$$\limsup_{n\to\infty}\|x^n\|^{\frac{1}{n}} \leq \rho(x) \leq \liminf_{n\to\infty}\|x^n\|^{\frac{1}{n}}$$

Therefore $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ is well-defined and equals to $\rho(x)$. $\square$

In particular, this completes what we had left unfinished in section (2.2.) about the existence of this limit!

4. Gelfand Representation Theory

In this crucial chapter, we shall develop Gelfand theory for commutative Banach algebras. We will introduce the maximal ideal space, Gelfand topology and Gelfand representations. Afterwards we shall discuss the radical, generators of Banach algebras and their relation with Gelfand theory.

4.1. Multiplicative Linear Functionals and the Maximal Ideal Space

To begin with, we introduce concepts not strictly limited to Gelfand theory. The following definitions and properties are self-explanatory.

4.1.1. Definition Let A be a Banach algebra. A non-zero (hence surjective) linear functional $\tau : A \longrightarrow \mathbb{C}$ which is also an algebra homomorphism (i.e. $\tau(xy) = \tau(x)\tau(y)$) is called a *multiplicative linear functional* (or *complex homomorphism* or *character*). The space of these is denoted by $\Delta(A)$.

4.1.2. Basic Properties

1. If A has identity e then $\tau(e) = 1$
2. If x is invertible then $\tau(x^{-1}) = \frac{1}{\tau(x)}$
3. $\Delta(A)$ is *not* a linear space, and what's more:
   - If $\tau \in \Delta(A)$ then $\lambda\tau \notin \Delta(A)\ \forall\lambda \neq 1$
   - If $\tau, \varsigma \in \Delta(A)$ then $\tau + \varsigma \notin \Delta(A)$
4. $K := \ker(\tau)$ is a regular maximal 2-sided ideal.

Proof

1. True for all non-zero algebra homomorphisms ($\tau(e) = \tau(ee) = \tau(e)\tau(e)$).
2. $1 = \tau(e) = \tau(x^{-1}x) = \tau(x^{-1})\tau(x)$ and idem. for right inverses.

3. If $\lambda = 0$ then $\lambda\tau \notin \Delta(A)$ by definition, so suppose $\lambda \neq 0$. Let $u \in A$ s.t. $\tau(u) = 1$. Then $\lambda\tau(uu) = \lambda\tau(u)\tau(u)$ whereas $\lambda\tau(u) \cdot \lambda\tau(u) = \lambda^2\tau(u)\tau(u)$. But $\lambda \neq \lambda^2$ for $\lambda \neq 0, \lambda \neq 1$ so $\lambda\tau$ is not multiplicative.

   Similarly, suppose $(\tau + \varsigma)$ is multiplicative. Then $\forall x, y \in A$, $(\tau + \varsigma)(xy) = (\tau + \varsigma)(x) \cdot (\tau + \varsigma)(y)$. In particular, for an x_0 s.t. $\tau(x_0) = 1$, we get:

   $$\tau(x_0 y) + \varsigma(x_0 y) = (\tau + \varsigma)(x_0 y) = (\tau + \varsigma)(x_0) \cdot (\tau + \varsigma)(y) =$$

   $$= \tau(x_0 y) + \varsigma(x_0)\tau(y) + 1 \cdot \varsigma(y) + \varsigma(x_0 y)$$

   Therefore

   $$\varsigma(x_0)\tau(y) + \varsigma(y) = 0, \quad \forall y \in A$$

   i.e.

   $$\varsigma = \lambda\tau$$

   and from the previous result this implies $\varsigma = \tau$, since both τ and ς must be in $\Delta(A)$.

   But then $(\tau+\varsigma) = (2\tau) \in \Delta(A)$ which isn't possible (again, by the previous result).

4. If $k \in K$ then $\forall x : \tau(xk) = \tau(x)\tau(k) = \tau(x) \cdot 0 = 0$, hence $xk \in K$, and similarly $kx \in K$. Moreover, the kernel of a linear functional is a linear subspace so K is an ideal. Also, since τ is surjective the subspace has codimension 1 and therefore it's maximal (as linear subspace and hence also as ideal).

   To show that K is regular, choose a $u' \in K^c$. Then $u := \frac{u'}{\tau(u')} \in K^c$ and $\tau(u) = 1$. Therefore $\tau(xu - x) = \tau(x)\tau(u) - \tau(x) = 0$ and so $xu - x \in K$. Similarly, $ux - x \in K$. So u is an identity modulo K. $\square$

4.1.3. Remark From (4.1.2;4.) it follows that $K = \ker(\tau)$ is a maximal regular ideal, so by (2.2.8.) it is closed. Then, from a theorem of functional analysis, τ is continuous ([7] pg. 57). Hence $\Delta(A) \subset A^*$.

4.1.4. Proposition $\forall x \in A : |\tau(x)| \leq \|x\|$. Hence $\|\tau\| \leq 1$.

Proof. Suppose that for some $x \quad |\tau(x)| > \|x\|$.

Then $\|\frac{x}{\tau x}\| < 1$ and from (2.2.4.) $\frac{x}{\tau(x)}$ is quasi-regular. So, there must exist a $y \in A$ s.t. $\frac{x}{\tau(x)} + y - \frac{xy}{\tau(x)} = 0$. But then $0 = \tau(0) = \tau\left(\frac{x}{\tau(x)} + y - \frac{xy}{\tau(x)}\right) = 1 + \tau(y) - \tau(y) = 1$ and we have a contradiction. $\square$

If A has an identity e then $|\tau(e)| = |1| = 1$ and hence $\|\tau\| \geq \frac{1}{\|e\|} = 1$, hence $\|\tau\| = 1$. (If we don't assume $\|e\| = 1$ then we can only say $\frac{1}{\|e\|} \leq \|\tau\| \leq 1$.)

4.1.5. Definition We denote the space of all two-sided maximal regular ideals of A by $\mathfrak{M}(A)$. Define the mapping

$$\Phi : \begin{array}{ccc} \Delta(A) & \longrightarrow & \mathfrak{M}(A) \\ \tau & \longmapsto & \ker(\tau) \end{array}$$

4.1.6. Lemma Φ is injective.

Proof Let $\tau, \varsigma \in \Delta(A)$ be s.t. $\ker(\tau) = \ker(\varsigma)$. Let u be such that $\tau(u) = 1$.

Then for all $x \in A : \tau(x - \tau(x)u) = 0$, so $(x - \tau(x)u) \in \ker(\tau) = \ker(\varsigma)$. So $\varsigma(x - \tau(x)u) = \varsigma(x) - \varsigma(u)\tau(x) = 0$, hence $\varsigma(x) = \varsigma(u)\tau(x)$. Therefore $\varsigma \equiv \varsigma(u)\tau$ and so from (4.1.2; 3.) follows $\tau = \varsigma$. □

*

Having established these general properties of multiplicative linear functionals we now turn to the specifics of Gelfand theory — namely, that Φ is bijective. That is, not only are the kernels of multiplicative linear functionals maximal regular ideals but also conversely: every maximal ideal is the kernel of a multiplicative linear functional.

Due to the dependence on (1.2.5.) the theory works only for *commutative* Banach algebras. Therefore from now on we shall always assume that A is commutative.

The next theorem, though fundamental in nature, is in fact nothing more than a clever combination of all our previous results.

4.1.7. Theorem Let A be a commutative Banach algebra (not necessarily with identity) and Φ defined as in (4.1.5.) Then Φ is bijective.

Proof Only the surjectivity remains to be shown. Let K be a two-sided maximal regular ideal. Then by (1.2.5.) A/K is a division algebra, since A is commutative. By (4.1.3.) K is closed, thus, by (2.1.4.), it is a Banach algebra. Therefore, by Gelfand-Mazur (3.1.5.), it is isomorphic to $\mathbb{C}$. Let that isomorphism be γ and the canonical quotient $A \to A/K$ be q. We then simply define

$$\tau := \gamma \circ q$$

Since both q and γ are algebra homomorphisms τ also is, and q is non-zero because $K \neq A$, therefore τ is non-zero, so $\tau \in \Delta(A)$. Also, since γ is an isomorphism, $\ker(\tau) = \ker(q) = K$ and so $\Phi(\tau)$ is indeed K. □

Now, in case A is without identity we can look at $\Delta(A[e])$. As a direct corollary of (1.3.6.) (concerning the bijection $\mathfrak{M}(A[e]) \setminus \{A\} \cong \mathfrak{M}(A)$) here we get:

4.1.8. Lemma If $\tau_e \in \Delta(A[e])$ and $\tau_e[A] \neq 0$ then $\tau_e|_A \in \Delta(A)$ and, conversely, $\forall \tau \in \Delta(A)$ there is a unique extension $\tau_e \in \Delta A[e]$ s.t. $\tau_e|_A = \tau$ (hence: $\tau_e[A] \neq 0$).

Proof The condition that $\tau_e[A] \neq 0$ is obviously equivalent to $\ker(\tau_e) \neq A$, since both $\ker(\tau_e)$ and A are maximal ideals in $A[e]$. The rest follows by setting

$$\psi := \Phi_A^{-1} \circ \phi \circ \Phi_{A[e]} : \Delta(A[e]) \setminus \{A\} \xrightarrow{\simeq} \Delta(A)$$

for the ϕ from (1.3.6.) $\square$

Similarly to (1.3.7.) we can extend the preceding bijection to a

$$\psi_0 : \Delta(A[e]) \longrightarrow \Delta(A) \cup \{0\}$$

which sends τ with $\ker(\tau) = A$ to the zero-functional. Also, we can extend Φ to Φ_0 which sends 0 to A. Then $\psi_0 = \Phi_0 \circ \phi_0 \circ \Phi_0^{-1}$ for the ϕ_0 from (1.3.7.) Clearly, all the components are still bijections and so is ψ_0.

This also shows that there is precisely one multiplicative functional on $A[e]$, let it be called τ_∞, such that $\tau_\infty[A] = 0$ — since A is maximal in $A[e], \ker(\tau_\infty) = A$ and τ_∞ is precisely $\Phi^{-1}(A)$. Of course, $\tau(x + \alpha e) = \alpha \ \ \forall x \in A, \alpha \in \mathbb{C}$.

*

In (4.1.7.) we have established the identification between $\Delta(A)$ and $\mathfrak{M}(A)$. The proof was based on the Gelfand-Mazur theorem which, in turn, was based on the spectrum. It is therefore not surprising that there is a straightforward relationship between $\Delta(A)$ and spectra. To understand this let us take a closer look at the *explicit* construction of multiplicative linear functionals from maximal ideals.

If a maximal regular ideal K is given, τ_K is given by $[x \mapsto x \bmod K \mapsto \lambda]$, for the $\lambda \in \mathbb{C}$ which, according to Gelfand-Mazur, definitely exists, such that $\bar{x} = \lambda\bar{e}$, i.e. $x \equiv \lambda e \bmod K$, i.e. $x - \lambda e \in K$. This implies that $x - \lambda e$ is singular and thus $\lambda = \tau_K(x) \in \sigma(x)$.

In fact, the set of $\tau(x)$ for all $\tau \in \Delta(A)$ is exactly the spectrum $\sigma(x)$, for Banach algebras with identity. This result is known as the Beurling-Gelfand Theorem, of which we now prove the full version:

4.1.9. Beurling-Gelfand Theorem Let A be a commutative Banach algebra with identity, and $x \in A$. Then $\sigma(x) = \{\tau(x) \mid \tau \in \Delta(A)\}$.

In particular: x is regular $\Longleftrightarrow \tau(x) \neq 0 \ \forall \tau \in \Delta(A)$.

Proof

"$\supset$" Suppose $\lambda = \tau(x)$ for some $\tau \in \Delta(A)$. Then $\tau(x - \lambda e) = \lambda - \lambda = 0$ and so $(x - \lambda e) \in \ker(\tau)$. But since $\ker(\tau)$ is a proper ideal $(x - \lambda e)$ can't be regular, so $\lambda \in \sigma(x)$.

"$\subset$" Suppose $\lambda \in \sigma(x)$. Then $(x - \lambda e)$ is singular, from which it follows that $(x - \lambda e)$ can be extended to a maximal regular ideal K and then $\tau := \Phi^{-1}(K)$ is in $\Delta(A)$ and $\tau(x - \lambda e) = \tau(x) - \lambda = 0$, so $\lambda = \tau(x)$ □

If A is without identity:

$$\sigma(x) = \sigma_{A[e]}(x) = \{\tau(x) \mid \tau \in \Delta(A[e])\} =$$

$$= \{\tau(x) \mid \tau \in \Delta(A) \cup \{0\}\} = \{0\} \cup \{\tau(x) \mid \tau \in \Delta(A)\}$$

so that even here the theorem works "up to a 0".

The previous reformulation enables us to generalize the concept of a spectrum to that of a *joint spectrum* of a finite set of elements $\{x_1, \ldots, x_n\}$.

4.1.10. Definition Let A be a commutative Banach algebra with identity and $x_1, \ldots, x_n \in A$. Then the *joint spectrum* is defined by

$$\sigma(x_1, \ldots, x_n) := \{(\tau(x_1), \ldots, \tau(x_n)) \mid \tau \in \Delta(A)\}$$

Since $\sigma(x_1, \ldots, x_n) \subset \sigma(x_1) \times \cdots \times \sigma(x_n)$, the joint spectrum is a subset of $\overline{B_{\mathbb{C}}(0, \|x_1\|)} \times \cdots \times \overline{B_{\mathbb{C}}(0, \|x_n\|)}$.

Obviously, for a single $x \in A$ the concept of the joint spectrum coincides with that of the usual spectrum.

Now, what can we say about $0 \in \sigma(x_1, \ldots, x_n)$? For a single x we had that $0 \in \sigma(x) \Leftrightarrow x$ is singular, which is equivalent to: $(x) \neq A$ (1.1.6; 3.) In fact, we now have a generalized version:

4.1.11. Proposition Let A be a Banach algebra with identity. Then

$$0 \in \sigma(x_1, \ldots, x_n) \Longleftrightarrow (x_1, \ldots, x_n) \neq A.$$

Proof

"$\Longrightarrow$" Suppose $0 \in \sigma(x_1, \ldots, x_n)$. Then $\exists \tau \in \Delta(A)$ s.t. $\tau(x_1) = \cdots = \tau(x_n) = 0$, hence $x_1, \ldots, x_n \in \ker(\tau)$. But then also $(x_1, \ldots, x_n) \subset \ker(\tau) \neq A$.

"$\Longleftarrow$" Suppose $(x_1, \ldots, x_n) \neq A$. Then there exists a maximal ideal K s.t. $(x_1, \ldots, x_n) \subset K$. But then let $\tau := \Phi^{-1}(K)$. Then $\tau \in \Delta(A)$ is s.t. $\tau[K] = 0$ so in particular $\tau(x_1) = \ldots \tau(x_n) = 0$. □

Before concluding this section, one final remark:

4.1.12. Remark $\Delta(A)$ is empty precisely when A has no proper regular ideals. In particular, if A is with identity and $A \neq \{0\}$, then $\{0\}$ is a proper regular ideal of A, therefore it can be extended to a maximal regular ideal. Thus $\Delta(A) \neq \emptyset$.

4.2. The Gelfand Topology

The space $\Delta(A) \subset A^*$ could easily be equipped with the induced norm topology of A^*. However, we wish $\Delta(A)$ to be locally compact and compact when A is with identity and in the induced norm topology of A^* that is not always the case. So we can try another topology — however, when dealing with Gelfand representations, it is also a prerequisite that all linear functionals in A^{**} are continuous. It turns out that the weak* topology fits both requirements and, as we shall show in this section, is a very natural choice for the topology on $\Delta(A)$.

The theory surrounding the weak* topology is assumed to be familiar to the reader. An introduction about it can be found e.g. in [7] pp. 223–235 or [13] pp. 27–34.

4.2.1. Definition The *Gelfand topology* is the topology on $\Delta(A)$ induced by the weak* topology on A^*. It can also be considered a topology on the maximal ideal space $\mathfrak{M}(A)$ by identifying it with $\Delta(A)$ via Φ.

Now we will show that in the Gelfand topology $\Delta(A)$ is locally compact in general and compact if A has an identity. We reach this result by first considering $\Delta(A) \cup \{0\}$, i.e. the space of *all* (not necessarily surjective) multiplicative linear functionals.

4.2.2. Theorem In the weak* topology $\Delta(A) \cup \{0\}$ is closed.

Proof $\Delta(A) \cup \{0\}$ is the set of *all* multiplicative linear functionals on A, including the zero-functional. So, suppose $\tau_\alpha \to \tau$ is a net in A^* convergent in the weak* topology s.t. $\tau_\alpha \in \Delta(A) \cup \{0\}\ \forall \alpha$. We need to show that τ is multiplicative.

By definition of the weak* topology, $\forall x \in A\ \tau_\alpha(x) \to \tau(x)$. Then, $\forall x, y \in A$:

$$\tau(xy) = \lim_{\alpha\to\infty} \tau_\alpha(xy) = \lim_{\alpha\to\infty} \tau_\alpha(x)\tau_\alpha(y) = \lim_{\alpha\to\infty} \tau_\alpha(x) \cdot \lim_{\alpha\to\infty} \tau_\alpha(y) = \tau(x)\tau(y)$$

□

4.2.3. Theorem If A has an identity then $\Delta(A)$ itself is closed (in the weak* topology). Therefore $\{0\}$ is an isolated point of $\Delta(A) \cup \{0\}$.

Proof In view of the previous proposition it just remains to show that if all the $\tau_\alpha \neq 0$ then $\tau \neq 0$. But since $\tau_\alpha \neq 0$, $\tau_\alpha \in \Delta(A)$ and hence $\tau_\alpha(e) \to \tau(e)$ and $\tau_\alpha(e) = 1\ \forall \alpha$, so $\tau(e) = 1$ and $\tau \neq 0$. $\square$

4.2.4. Corollary $\Delta(A) \cup \{0\}$ is compact in the weak* topology. Therefore $\Delta(A)$ is locally compact, with $\{0\}$ being the point needed for the one-point compactification of $\Delta(A)$ if $\Delta(A)$ is not compact itself. If A is with identity then $\Delta(A)$ itself is always compact.

Proof This follows from the fact that $\forall \tau \in \Delta(A) \cup \{0\} : \|\tau\| \leq 1$ (4.1.4.) so $\Delta(A) \cup \{0\}$ is contained in the closed unit ball of A^* which, by the Banach-Alaoglu theorem ([7] pg. 229), is compact. Closed subsets of a compact set are also compact. $\square$

So we have established the nature of $\Delta(A)$ in the weak* topology of A^*. However, compactness and local compactness of a space are properties that don't depend on the space surrounding it, therefore the results also hold in the Gelfand topology which is defined on $\Delta(A)$ alone.

4.3. The Gelfand Representation

From general functional analysis we know that we can injectively embed any Banach space X in its bi-dual X^{**} by the map:

$$j : \begin{array}{ccc} X & \longrightarrow & X^{**} \\ x & \longmapsto & \delta_x = [\Lambda \mapsto \Lambda(x)] \end{array}$$

where the δ_x are point-evaluations (see e.g. [6] pp. 52–53, or [7, 13]). The results show that j is even an isometric linear operator.

By restricting the point-evaluations $\delta_x : A^* \longrightarrow \mathbb{C}$ to $\Delta(A) \subset A^*$ we denote the new point-evaluations by $\hat{x}$, i.e. $\hat{x} := \delta_x \mid_{\Delta(A)}$, i.e.

$$\hat{x}(\tau) = \tau(x) \qquad \forall x \in A,\ \forall \tau \in \Delta(A)$$

The $\hat{x}$ are continuous because the δ_x are continuous in the weak* topology of A^* by definition and $\hat{x}$ and the Gelfand topology are both restrictions to $\Delta(A)$. Moreover, since the δ_x are bounded linear functionals with $\|\delta_x\| = \|x\|$ and $\Delta(A)$ is a set bounded by 1, $\hat{x}$ is bounded by $\|x\|$, i.e. $\|\hat{x}\|_\infty \leq \|x\|$. So, $\hat{x} \in C(\Delta(A))\ \forall x \in A$ (continuous and bounded).

4.3.1. Definition We define the map $\wedge$ by

$$\wedge : \begin{array}{ccc} A & \longrightarrow & C(\Delta(A)) \\ x & \longmapsto & \hat{x} \end{array}$$

This map is called the *Gelfand transformation* and $\hat{x}$ the *Gelfand transform of* x. The image of A under $\wedge$ is denoted by $\hat{A}$ and called the *Gelfand representation of* A.

4.3.2. Basic properties

1. $\wedge$ is an algebra homomorphism.
2. $\hat{A}$ is a normed algebra (with the sup-norm inherited from $C(\Delta(A))$).
3. $\hat{A}$ separates the points of $\Delta(A)$.
4. If A has an identity then $\hat{A}$ contains all constant functions.
5. $\wedge$ is norm-decreasing so $\| \wedge \| \leq 1$ (operator norm) and $\wedge \in \mathcal{L}(A, \hat{A})$. If A has an identity, $\| \wedge \| = 1$.
6. $\|\hat{x}\|_\infty = \rho(x)$ (the spectral radius).

Proof

1. We check that, for example, $\forall \tau : (xy)^\wedge(\tau) = \tau(xy) = \tau(x)\tau(y) = \hat{x}(\tau)\hat{x}(\tau)$ and so $\wedge(xy) = \wedge(x) \wedge (y)$. Similarly we can check addition and scalar multiplication.
2. Since $\hat{A}$ is the image of an algebra homomorphism it is itself an algebra. Therefore it is a subalgebra of the normed algebra $C(\Delta(A))$ and hence it is itself a normed algebra.
3. If $\tau \neq \sigma$ then $\exists x \in A$ s.t. $\tau(x) \neq \sigma(y)$, i.e. $\hat{x}(\tau) \neq \hat{x}(\sigma)$.
4. Let $\lambda \in \mathbb{C}$. Then $\forall \tau \in \Delta(A) : (\hat{\lambda e})(\tau) = \tau(\lambda e) = \lambda$. So the $(\hat{\lambda e})$ are constant λ-functions.
5. We already showed that $\| \wedge (x)\|_\infty = \|\hat{x}\|_\infty \leq \|x\|$ and thus $\| \wedge \| \leq 1$. Moreover, if A has an identity e, $\| \wedge (e)\|_\infty = \|\hat{e}\|_\infty = 1$ since $\hat{e}$ is the constant 1 function. So in that case $\| \wedge \| = 1$
6. It is a direct corollary of the Beurling-Gelfand Theorem (4.1.9.) that $\sigma(x) = \text{range}(\hat{x})$ if A is with identity and $\sigma(x) = \text{range}(\hat{x}) \cup \{0\}$ if A is without identity. In either case we immediately get $\rho(x) = \|\hat{x}\|_\infty$. $\square$

Gelfand representation theory has numerous applications, many of which we will look at in greater or lesser detail in chapter 5. In the next sections we will present two other concepts, the radical and generators of Banach algebras, and relate them to Gelfand representation theory.

4.4. The Radical and Semi-simplicity

4.4.1. Definition If A is a Banach algebra, the *radical of* A, denoted by $\mathrm{Rad}(A)$, is defined as the intersection of all maximal regular ideals of A:

$$\mathrm{Rad}(A) := \bigcap_{M \in \mathfrak{M}(A)} M$$

If $\mathrm{Rad}(A) = \{0\}$ we say that A is *semisimple*.

If $\mathfrak{M}(A) = \emptyset$ (i.e. there are no proper regular ideals in A) we define $\mathrm{Rad}(A) := A$ and say that A is a *radical algebra*. In particular, algebras with identity are not radical algebras (see 4.1.12.) We shall not come across radical algebras in this text.

Since all $M \in \mathfrak{M}(A)$ are closed, so is $\mathrm{Rad}(A)$.

The preceding theory immediately provides us with the obvious properties, presented in the next Lemma.

4.4.2. Lemma The following are equivalent:

1. $x \in \mathrm{Rad}(A)$
2. $\tau(x) = 0 \ \forall \tau \in \Delta(A)$
3. $\hat{x} = 0$
4. $x \in \ker(\wedge)$, for $\wedge$ the Gelfand transformation
5. $\sigma(x) = \{0\}$
6. $\rho(x) = 0$
7. x is topologically nilpotent.

Proof First of all: $x \in \mathrm{Rad}(A) \iff \forall M \in \mathfrak{M}(A)$: $x \in M \iff \forall \tau \in \Delta(A)$: $\tau(x) = 0$, due to the bijection $\Phi : \Delta(A) \xrightarrow{\simeq} \mathfrak{M}(A)$. This proves (1) $\Leftrightarrow$ (2).

The equivalence (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is trivial whereas (2) $\Leftrightarrow$ (5) follows from the Beurling-Gelfand Theorem (4.1.9.)

Moreover, (5) $\Leftrightarrow$ (6) is trivial as well whereas (6) $\Leftrightarrow$ (7) follows from the spectral radius formula (3.2.3.) $\square$

4.4.3. Remark So in particular: $\ker(\wedge) = \mathrm{Rad}(A)$.

As an immediate corollary we get:

4.4.4. Corollary The following are equivalent:

1. A is semisimple
2. $\tau(x) = 0\ \forall \tau \in \Delta(A)$ implies $x = 0$. In other words (since the τ are linear): $\Delta(A)$ separates points of A.
3. The Gelfand transformation $\wedge$ is injective. ($\hat{A} \cong A$.)
4. $\sigma(x) = \{0\} \iff x = 0$
5. $\rho(x) = 0 \iff x = 0$
6. x is topologically nilpotent $\iff x = 0$.

4.5. Generators of Banach algebras

4.5.1. Definition Let A be a commutative Banach algebra with identity and $E \subset A$. Then the *algebra generated by* E, notation $\langle E \rangle$, is defined as the smallest closed subalgebra of A (equiv.: smallest Banach subalgebra of A) which contains $E \cup \{e\}$. If $\langle E \rangle = A$ we say that A is generated by E. We say that A (or equivalently, a subalgebra A_0 of A) is *finitely generated* if it is generated by a finite subset E.

The following Lemma gives an equivalent description of generators.

4.5.2. Lemma A is finitely generated, say by $E = \{e_1, \ldots, e_r\}$, if and only if the set of polynomials in $(e, e_1, \ldots, e_r)$ is dense in A — in other words, if $\forall x \in A : x = \lim_{n\to\infty} p_n$ where each p_n is of the form

$$\sum_{i_0 i_1 \ldots i_r} \lambda_{i_0 i_1 \ldots i_r} e^{i_0} \cdot e_1^{i_1} \cdot \ \ldots \ \cdot e_r^{i_r}$$

Proof Obviously the set of all polynomials in $(e, e_1, \ldots, e_n)$ is a subalgebra of A. Let it be denoted by P_0. From (2.1.2; 3.) it then follows that $\overline{P_0}$ is a closed subalgebra of A. Then by the minimality in the definition, $A = \langle E \rangle = \overline{P_0}$. $\square$

Now we shall see why generators are important in the sense of Gelfand theory — the next lemma shows that every multiplicative linear functional is *uniquely determined* by its action on the generators.

4.5.3. Lemma If A (commutative, with identity) is finitely generated by $E = \{e_1, \ldots, e_r\}$, then $\Delta(A)$ and the joint spectrum $\sigma(e_1, \ldots, e_r)$ are homeomorphic.

Proof We define the following map:

$$\begin{array}{ccc} \Delta(A) & \longrightarrow & \sigma(e_1, \ldots, e_r) \\ \tau & \longmapsto & (\tau(e_1), \ldots, \tau(e_r)) \end{array}$$

The map is surjective by definition and it is injective because if $\tau(e_i) = \varsigma(e_i)\ \forall i$ (and $\tau(e) = \varsigma(e) = 1$ holds anyway) we get:

$$\tau(x) = \tau(\lim \left[\sum_{i_0 i_1 \ldots i_r} (\lambda_{i_0 i_1 \ldots i_r} e^{i_0} \cdot e_1^{i_1} \cdot \cdots \cdot e_r^{i_r}) \right]) =$$

$$= \lim \left[\sum_{i_0 i_1 \ldots i_r} (\lambda_{i_0 i_1 \ldots i_r} \tau(e)^{i_0} \cdot \tau(e_1)^{i_1} \cdot \cdots \cdot \tau(e_r)^{i_r}) \right] =$$

$$= \lim \left[\sum_{i_0 i_1 \ldots i_r} (\lambda_{i_0 i_1 \ldots i_r} \varsigma(e)^{i_0} \cdot \varsigma(e_1)^{i_1} \cdot \cdots \cdot \varsigma(e_r)^{i_r}) \right] = \varsigma(x)$$

using the fact that τ and ς are both continuous, linear and multiplicative.

So the map is bijective. Moreover, it is continuous because if $\tau_\alpha \to \tau$ is a convergent net then $\tau_\alpha(e_i) = \hat{e}_i(\tau_\alpha) \to \hat{e}_i(\tau) = \tau(e_i)\ \forall e_i$ by definition of the weak* topology. Now, since $\Delta(A)$ is compact (because A is with identity) and $\sigma(e_1, \ldots, e_n)$ is Hausdorff, the mapping is a homeomorphism. $\square$

5. Examples of Gelfand Representations

In the last chapter we give several examples of Banach algebras, all of them function algebras, explicitly find the space of multiplicative linear functionals, $\Delta(A)$, and relate it to the spaces on which the functions themselves are defined. In some cases these will be homeomorphic whereas in others homeomorphic to a dense subspace. Each section can be considered as a small introduction to a specialized field of functional analysis, and as a whole they form a nice illustration of the relation between analysis and algebra that Gelfand theory has created for us.

5.1. $C(X)$ for X compact and Hausdorff

We begin this chapter with a little digression: concerning the concept of a *pullback* of mappings, something that we shall repeatedly need. Though probably familiar to most readers, we will make precise what we mean and prove certain elementary properties.

5.1.1. Definition Let X, Y and Z be arbitrary sets, $F(X) := \{f : X \to Z\}$ and $F(Y) := \{f : Y \to Z\}$ sets of maps and $\alpha : X \to Y$ a mapping. Then the *pullback* of α, denoted by α^*, is defined as follows:

$$\alpha^* : \begin{array}{ccc} F(Y) & \longrightarrow & F(X) \\ f & \longmapsto & f \circ \alpha \end{array}$$

5.1.2. Basic properties The following properties hold in general:

1. If α is surjective then α^* is injective.
2. If α is injective then α^* is surjective.
3. If α is bijective then $(\alpha^*)^{-1} = (\alpha^{-1})^*$

Proof

1. Let $f, g \in F(Y), f \neq g$. Then $\exists y \in Y$ s.t. $f(y) \neq g(y)$. Since α is surjective $\exists x \in X$ s.t. $y = \alpha(x)$, and then $\alpha^* f(x) = f(\alpha(x)) = f(y) \neq g(y) = g(\alpha(x)) = \alpha^* g(x)$. Therefore $\alpha^* f \neq \alpha^* g$

2. Let $f \in F(X)$. Define $g \in F(Y)$ as follows: if $y \in \text{Range}(\alpha)$, set it to be $f(x)$, for the *unique* x s.t. $\alpha(x) = y$; otherwise set it arbitrarily. Thus: $\forall x \in X : \alpha^* g(x) = f(\alpha(x)) = g(y) = f(x)$. So $\alpha^* g = f$.

3. We need to prove: $(\alpha^{-1})^* \circ \alpha^* = \text{id}_{F(Y)}$ and $\alpha^* \circ (\alpha^{-1})^* = \text{id}_{F(X)}$. But if $f \in F(Y)$, $(\alpha^{-1})^* \circ \alpha^*(f) = f \circ \alpha \circ \alpha^{-1} = f$, whereas if $f \in F(X)$, $\alpha^* \circ (\alpha^{-1})^*(f) = f \circ \alpha^{-1} \circ \alpha = f$. This completes the proof. $\square$

Although the general case of pullbacks was defined for arbitrary sets of maps, we can limit them to, for instance, continuous functions, bounded functions etc. In this case we normally don't specify the fact that the pullback is in fact a *restriction* but simply write α^* (e.g. if we are dealing with $C(X)$ and $C(Y)$ and $\alpha : X \longrightarrow Y$ then by α^* we mean $\alpha^*|_{C(Y)}$).

Also, even though parts 1 and 3 of the preceding properties always hold, for part 2 one must be very careful that the g, defined as in the proof, is indeed the right type of function, i.e. that it is in the subset of $F(Y)$ which we are considering. The verification of this is not always that trivial.

In the situation we will actually come across, we shall consider $C(X)$ and $C(Y)$ (or $C_0(X)$ and $C_0(Y)$) and $\alpha : X \longrightarrow Y$ a homeomorphism of topological spaces: then, the "g" from the proof of (5.1.2;2) can be $f \circ \alpha^{-1}$ which is continuous because f and α^{-1} are continuous.

*

Now let us actually turn to $C(X)$, X being a compact Hausdorff topological space and $C(X)$ the commutative Banach algebra of continuous (and hence bounded) complex-valued functions (see also 2.1.5;2.) Since $C(X)$ has an identity (the constant 1 function) $\Delta(C(X)) \neq \emptyset$.

Before we proceed, let us quickly prove a characterization of regularity in $C(X)$. A trivial observation is that $f \in C(X)$ is regular if and only if f has no zeroes on X. We generalize this notion to a finite set of functions $f_1, \ldots, f_n$ in the following Lemma (clearly the former observation follows from it).

5.1.3. Lemma Let $f_1, \ldots, f_n \in C(X)$, for X compact, Hausdorff. Then, for $(f_1, \ldots, f_n)$ denoting the ideal generated by the functions, we have:

$$(f_1, \ldots, f_n) = C(X) \Longleftrightarrow \text{the } f_i \text{ have no common zeroes}$$

Proof

"$\Longrightarrow$" If t is a common zero of all f_i's, it is also a zero of any function in the ideal, so definitely $1 \notin (f_1, \ldots, f_n)$.

"$\Longleftarrow$" If $f_1, \ldots, f_n$ have no common zeroes, then the function

$$f := f_1\overline{f_1} + \cdots + f_n\overline{f_n} = |f_1|^2 + \cdots + |f_n|^2$$

is in the ideal $(f_1, \ldots, f_n)$ by definition and has no zeroes on X. Therefore, f has the well-defined inverse $\frac{1}{f}(t) := \frac{1}{f(t)}$. Since f is invertible and in $(f_1, \ldots, f_n)$, $(f_1, \ldots, f_n) = C(X)$ $\square$

Now let's turn to the "Gelfand" aspect of $C(X)$. For each $t \in X$ we can define τ_t by

$$\tau_t(f) := f(t) \quad \forall f \in C(X)$$

Since $\tau_t(\lambda f + \mu g) = (\lambda f + \mu g)(t) = \lambda f(t) + \mu g(t) = \lambda \tau_t(f) + \mu \tau_t(g)$, as well as $\tau_t(fg) = fg(t) = f(t) \cdot g(t) = \tau_t(f) \cdot \tau_t(g)$, we see that τ_t is a multiplicative linear functional. Moreover, $\forall t \in X\ \exists f \in C(X)$ s.t. $f(t) \neq 0$. Hence τ_t is non-zero and therefore it is in $\Delta(C(X))$.

This immediately shows that in this case $\wedge$ is injective: suppose $f \neq g$, then $\exists t \in X$ s.t. $f(t) \neq g(t)$. Then $\hat{f}(\tau_t) = \tau_t(f) = f(t) \neq g(t) = \tau_t(g) = \hat{g}(\tau_t)$, and so $\hat{f} \neq \hat{g}$. Thus $\widehat{C(X)} \cong C(X)$, and $C(X)$ is semisimple.

Much more interesting is the fact that the converse holds: for every $\tau \in \Delta(C(X))\ \exists t \in X$ s.t. $\tau = \tau_t$. Moreover, this identification is a homeomorphic.

5.1.4. Theorem The map $\theta : \begin{array}{ccccc} X & \longrightarrow & \Delta(C(X)) & \overset{\Phi}{\cong} & \mathfrak{M}(C(X)) \\ t & \longmapsto & \tau_t & \longleftrightarrow & \ker(\tau_t) \end{array}$ is bijective.

Proof We have already shown that θ is well-defined. Furthermore, it is injective, because, as $C(X)$ separates points of X, $t \neq s$ implies $\exists f \in C(X)$ s.t. $f(t) \neq f(s)$, so that $\tau_t(f) \neq \tau_s(f)$ and hence $\tau_t \neq \tau_s$.

To show that it is surjective, we need to reason with $\mathfrak{M}(C(X))$ instead of $\Delta(C(X))$ (unsurprisingly, since algebra plays a leading role in Gelfand theory!).

So, suppose $\exists K \in \mathfrak{M}(C(X))$ s.t. $\forall t \in X : K \neq \ker(\tau_t)$. Then, from the maximality of both ideals, also $K \not\subseteq \ker(\tau_t)$, i.e. $\exists f_t \in K$ s.t. $f_t \notin \ker(\tau_t)$ i.e. $f_t(t) \neq 0$. But since f_t is continuous there is an open neighbourhood U_t of t s.t. f_t has no zero on U_t. In this way we get an open cover of X using $\{U_t\}\ \forall t$, and because X is compact there exist a finite number of points $t_1, \ldots t_n \in X$ s.t. $\bigcup_{j=1\ldots n} U_{t_j}$ is an open cover of X.

So, in particular, there exists a finite set of functions $f_{t_1}, \ldots, f_{t_n} \in C(X)$ which have no common zero, so by (5.1.3.) $(f_{t_1}, \ldots, f_{t_n}) = C(X)$. But all the f_{t_i} were in K by definition, so $(f_{t_1}, \ldots, f_{t_n}) \subset K$, and thus $K = C(X)$, which is a contradiction to $K \in \mathfrak{M}(C(X))$.

Conclusion: θ is surjective. $\square$

5.1.5. Remark We immediately see that for each $f \in C(X)$, $\theta^* \hat{f} = f$, since $\left(\theta^* \hat{f}\right)(t) = \hat{f}(\tau_t) = \tau_t(f) = f(t)$, $\forall t \in X$. So, $\theta^* = \wedge^{-1}$ and is bijective. This enables an identification between f and $\hat{f}$.

5.1.6. Theorem The mapping θ is a homeomorphism.

Proof Let $t_\alpha \to t$ be a convergent net in X. Since all $f \in C(X)$ are continuous, $\forall f \in C(X) : f(t_\alpha) \to f(t)$ is a convergent net in $\mathbb{C}$. Then, $\tau_{t_\alpha}(f) \to \tau_t(f)$ $\forall f \in C(X)$, so in the weak* topology $\theta(t_\alpha) = \tau_{t_\alpha} \to \tau_t = \theta(t)$ which shows that θ is continuous.

Now, because X is compact and $\Delta(C(X))$ Hausdorff, θ is a homeomorphism. $\square$

Summarizing, we have shown that the spaces X and $\Delta(C(X))$ are homeomorphic topological spaces.

In particular, we see that $\widehat{C(X)}$ is the whole space $C(\Delta(C(X)))$. Namely, if $g \in C(\Delta(C(X)))$ then $f := \theta^* g = g \circ \theta$ is continuous and hence in $C(X)$. But then, just like before, $(\theta^* \hat{f})(t) = \hat{f}(\tau_t) = f(t) = (\theta^* g)(t)$, so $\theta^* \hat{f} = \theta^* g$ and since θ^* is bijective (because θ is) this implies $g = \hat{f}$. In other words, each function in $C(\Delta(C(X)))$ is the Gelfand transform of an element of $C(X)$, i.e. $\widehat{C(X)} = C(\Delta(C(X)))$.

If we are willing to treat θ as a real *identification* between X and $\Delta(C(X))$, f and $\hat{f}$ become the same function, so that in this case the Gelfand transformation $\wedge$ is the identity mapping.

As a simple application of this theory we prove the following proposition:

5.1.7. Proposition Every compact subset K of $\mathbb{C}^n$ is the joint spectrum of elements of some commutative Banach algebra A.

Proof Let $A := C(K)$ and $f_i \in C(K)$ be the i-th projection, for $i \in \{1, \dots, n\}$. Then

$$\sigma(f_1, \dots, f_n) = \{(\tau(f_1), \dots, \tau(f_n)) \mid \tau \in \Delta(C(K))\} =$$
$$= \{(f_1(z), \dots, f_n(z)) \mid z \in K\} = \{(z_1, \dots, z_n) \mid z \in K\} = K$$

by the preceding results. $\square$

*

Another interesting application is the following result: two compact Hausdorff spaces X and Y are homeomorphic precisely when $C(X)$ and $C(Y)$ are isomorphic as Banach algebras. We split this up into two theorems.

5.1.8. Theorem Let X and Y be compact Hausdorff spaces. Suppose $\alpha : X \longrightarrow Y$ is a continuous map. Then $\alpha^* : C(Y) \longrightarrow C(X)$ is an algebra homomorphism. If α is a homeomorphism, α^* is an algebra isomorphism.

Proof Since if f is continuous, $f \circ \alpha$ is continuous, the image of α^* is indeed a subset of $C(X)$.

Clearly we have:

$$\alpha^*(\lambda f + \mu g) = \lambda f \circ \alpha + \mu g \circ \alpha = \lambda(f \circ \alpha) + \mu(g \circ \alpha) = \lambda(\alpha^* f) + \mu(\alpha^* g)$$

and similarly

$$\alpha^*(fg) = fg \circ \alpha = (f \circ \alpha) \cdot (g \circ \alpha) = \alpha^* f \cdot \alpha^* g$$

so α^* is an algebra homomorphism.

Now if α is bijective and continuous in both directions it follows from (5.1.2; 1 & 2) and the remark afterwards that α^* is bijective. $\square$

5.1.9. Theorem If $\gamma : C(Y) \longrightarrow C(X)$ is an algebra homomorphism there exists a continuous $\alpha : X \to Y$ such that $\alpha^* = \gamma$. If γ is an isomorphism then α is a homeomorphism.

Proof We know, from (5.1.1.), that any non-zero algebra homomorphism $\gamma : C(Y) \to C(X)$ defines the pullback $\gamma^* : \Delta(C(X)) \to \Delta(C(Y))$ (the image of γ^* is in $\Delta(C(Y))$ because if $\tau \in \Delta(C(Y))$ then $\gamma^* \tau = \tau \circ \gamma$ is a non-zero algebra homomorphism.)

Now, from (5.1.9.) X is identifiable with $\Delta(C(X))$ and Y with $\Delta(C(Y))$. Thus we obtain $\alpha : X \to Y$ as simply γ^* taking into account the identifications (so formally: $\alpha = \theta_Y^{-1} \circ \gamma^* \circ \theta_X$). It follows immediately that if γ is bijective, so is γ^* and hence α.

Now we show that $\alpha^* = \gamma$. Let $t \in X$. By definition, $\alpha(t) \in Y$ is such that $\tau_t \circ \gamma = \tau_{\alpha(t)}$ for any t. Thus the following holds:

$$\forall t \in X, \;\; \forall f \in C(Y) : (\tau_t \circ \gamma)(f) = (\tau_{\alpha(t)})(f)$$

$$\forall t \in X, \;\; \forall f \in C(Y) : (\gamma f)(t) = f(\alpha(t)) = (\alpha^* f)(t)$$

Hence

$$\forall f \in C(Y) : \gamma f = \alpha^* f,$$

So

$$\alpha^* = \gamma$$

Now it only remains to show that α is continuous. So let $U \subset Y$ be an open subset, upon which we must show that $\alpha^{-1}[U]$ is open in X. For that purpose, choose $t_0 \in \alpha^{-1}[U]$. Then, since U is open, $\alpha(t_0) \in U$ and Y is compact and Hausdorff, there exists an $f \in C(Y)$ s.t. $f(\alpha(t_0)) = 1$ and $f[Y \setminus U] = 0$. (This is due to Urysohn's Lemma, [10] pp. 75–77; if Y is metric, this can, for instance, be a tent function with radius ϵ where $B(t_0, \epsilon) \subset U$).

Let $V := (\gamma f)^{-1}[\mathbb{C} \setminus \{0\}]$ which is open in X since γf is continuous. What is V? We have

$$\forall t \in X: \quad t \in V \Longleftrightarrow \gamma f(t) \neq 0 \Longleftrightarrow \alpha^* f(t) = f(\alpha(t)) \neq 0$$

i.e. $V = \{t \in X \mid f(\alpha(t)) \neq 0\}$ and therefore $t_0 \in V$. Furthermore, $t \in V$ implies $f(\alpha(t)) \neq 0$ which implies $\alpha(t) \in U$. Hence $\alpha[V] \subset U$ and so $t_0 \in V \subset \alpha^{-1}[U]$. This proves that α is continuous.

If γ was an isomorphism, α^{-1} is continuous by an analogous argument so that in that case $\alpha : X \xrightarrow{\simeq} Y$ is a homeomorphism. □

Summarizing, we have indeed proven: If X and Y are compact, Hausdorff spaces then X and Y are homeomorphic if and only if the Banach algebras $C(X)$ and $C(Y)$ are isomorphic algebras.

5.2. $C_0(X)$ for X locally compact and Hausdorff.

Here we consider a locally compact Hausdorff space X. In this instance we have two distinct function algebras we wish to consider: $C_0(X)$ and $C(X)$, for which we have $C_0(X) \subset C(X)$ (see also (2.1.5;2.))

We shall prove that X and $\Delta(C_0(X))$ are homeomorphic spaces, by using the results from the previous section applied to the one-point-compactification of X. It is a wonderful illustration of how the theory of algebras *without* identity (in particular (4.1.8.)) can be applied.

5.2.1. Lemma Consider the one-point-compactification of X, denoted by X_∞. Then $C_0(X)$ can be identified with $\{f \in C(X_\infty) \mid f(\infty) = 0\}$

Proof Let $f \in C_0(X)$ and extend it to a function $\tilde{f}$ on X_∞ by setting $\tilde{f}(\infty) := 0$. We have to show that $\tilde{f} \in C(X_\infty)$ i.e. that $\tilde{f}$ is continuous. Well, at any $t \in X$ $\tilde{f}$ is continuous because it is an extenstion of f, so consider $f(\infty)$. Given $\epsilon > 0$ there is a compact subset K of X such that $\forall t \in X \setminus K : |\tilde{f}(t)| = |f(t)| < \epsilon$. But by definition of the point-compactification $U := X_\infty \setminus K$ is an open neighbourhood of ∞, and we have $|\tilde{f}(t)| < \epsilon\ \forall t \in U$ so $\tilde{f}$ is continuous.

Conversely, if $g \in C(X_\infty)$ s.t. $g(\infty) = 0$, then, by continuity, $\forall \epsilon$ there exists an open neighbourhood U of ∞ s.t. $|g(t)| < \epsilon\ \forall t \in U$. But then $K := U^c$ is compact by definition and $\forall t \in K^c : |g_{|X}(t)| < \epsilon$, so $g_{|X} \in C_0(X)$. □

5.2.2. Lemma Furthermore, $C(X_\infty)$ can in fact be identified with $C_0(X)[e]$.

Proof We identify the adjoined e from $C_0(X)[e]$ with the constant 1-function from $C(X_\infty)$. Obviously, if $f \in C(X_\infty)$ (s.t. $f(\infty) = 0$), then $(f + \lambda \cdot 1) \in C(X_\infty)$, $\forall \lambda$. Conversely, every $f \in C(X_\infty)$ can be written as $(f - f(\infty)) + f(\infty) \cdot 1$ where $(f - f(\infty))$ is a function in $C(X_\infty)$ which is 0 at ∞. The result follows from this. $\square$

5.2.3. Proposition X and $\Delta(C_0(X))$ are canonically homeomorphic.

Proof From (5.1.9.) we know that $\theta : \begin{array}{ccc} X_\infty & \longrightarrow & \Delta(C(X_\infty)) \\ t & \longmapsto & \tau_t \end{array}$ is a homeomorphism. Now the previous results show that $\Delta(C(X_\infty)) \cong \Delta(C_0(X)[e]) \cong \Delta(C_0(X)) \cup \{0\}$, so $\theta_0 : X_\infty \longrightarrow \Delta(C_0(X)) \cup \{0\}$ is a bijection. An argument similar to (5.1.8.) shows that it is a homeomorphism. Moreover, by (5.2.2.) we see that $\theta_0(\infty)(f) = \tau_\infty(f) = f(\infty) = 0$, $\forall f \in C_0(X)$, i.e. $\theta_0(\infty) = \tau_\infty$ is the 0-functional.

Thus θ_0 is a homeomorphism between X_∞ and $\Delta(C_0(X)) \cup \{0\}$ which sends ∞ to 0. But $\Delta(C_0(X)) \cup \{0\}$ is the one-point compactification of $\Delta(C_0(X))$ and X_∞ that of X, so X and $\Delta(C_0(X))$ are also homeomorphic. $\square$

We can again identify X and $\Delta(C_0(X))$ via θ_0, so that the Gelfand transformation is the identity mapping.

5.3. Stone-Čech compactification

So the situation is clear for $C_0(X)$. But what about $C(X)$? It is indeed *not* the case that $X \cong \Delta(C(X))$. But, apparently, X *can* be considered as a dense subset of $\Delta(C(X))$. This observation leads to the idea of the *Stone-Čech compactification*.

Before proceeding any further, we will give a characterization of regularity in $C(X)$, a kind of "topological counterpart" of (5.1.3.)

5.3.1. Lemma Let $f_1, \ldots, f_n \in C(X)$ for X locally compact, Hausdorff. Then:

$$(f_1, \ldots, f_n) = C(X) \Longleftrightarrow \exists \delta > 0 \text{ s.t. } \forall t \in X : \exists i \text{ s.t. } |f_i(t)| \geq \delta$$

Or, as contraposition:

$$(f_1, \ldots, f_n) \neq C(X) \Longleftrightarrow \forall \delta > 0 : \exists t \in X \text{ s.t. } \forall i : |f_i(t)| < \delta$$

Proof

"$\Longrightarrow$" Suppose $1 \in (f_1, \dots, f_n)$. Then $\exists g_1, \dots, g_n \in C(X)$ s.t. $1 \equiv f_1 g_1 + \dots + f_n g_n$, i.e. $\forall t \in X : f_1(t)g_1(t) + \dots + f_n(t)g_n(t) = 1$. But all g_i's are bounded by M_i, i.e. all of them are bounded by $M := \max(M_1, \dots, M_n)$. Let $\delta := \frac{1}{M \cdot n}$. If $\exists t \in X$ s.t. $|f_i(t)| < \delta\ \forall f_i$, then $|f_1(t)g_1(t) + \dots + f_n(t)g_n(t)| \leq |f_1(t)g_1(t)| + \dots + |f_n(t)g_n(t)| < \frac{1}{M \cdot n}M + \dots \frac{1}{M \cdot n}M = \frac{1}{n} + \dots + \frac{1}{n} = 1$ and so $f_1 g_1 + \dots + f_n g_n \neq 1$.

"$\Longleftarrow$" Let $f := f_1\overline{f_1} + \dots + f_n\overline{f_n}$, so that $f \in (f_1, \dots, f_n)$ by definition. Then the assumption implies that $\forall t \in X : |f(t)| = |f_1(t)|^2 + \dots + |f_n(t)|^2 > \delta^2$. Then we can define f^{-1} by $f^{-1}(t) := \frac{1}{f(t)}$, and f^{-1} is continuous and bounded by $\frac{1}{\delta^2}$, hence $f^{-1} \in C(X)$. So f is invertible from which follows $(f_1, \dots, f_n) = C(X)$. $\square$

5.3.2. Definition In general, suppose X is a locally compact Hausdorff space. If there is a compact space βX such that X can be embedded in βX and moreover every bounded continuous complex function on X can be uniquely extended to a continuous function on βX, then we say that βX is the *Stone-Čech compactification of* X.

We introduce the Stone-Čech compactification from the point of view of Banach-algebras only.

In a manner completely analogous to the preceding section it is obvious that $\theta : t \longmapsto \tau_t$ is an injective map from X to a subset of $\Delta(C(X))$ ($C(X)$ is semisimple). Let us denote this subset by θX. In this instance it is not necessarily the case that $\theta X = \Delta(C(X))$.

However, we still do have that for all $\hat{f} \in \Delta(C(X))$, $\theta^*(\hat{f}\mid_{\theta X}) = f$ and by an argument like (5.1.6.), θ is a homeomorphism between X and θX.

Furthermore, we claim that θX is dense in $\Delta(C(X))$. Once this claim is proven, clearly $f \in C(X)$ can be extended to the compact set $\Delta(C(X))$ using the identification $f \leftrightarrow \hat{f}$ (via θ^*), so $\Delta(C(X))$ is a Stone-Čech compactification of X.

5.3.3. Theorem θX is dense in $\Delta(C(X))$

Proof Let $\tau \in \Delta(C(X))$ and N a neighbourhood of τ. We will deduce the existence of an $x \in X$ s.t. $\tau_x \in N$.

Since $\Delta(C(X))$ has the weak* topology we may assume N to be of the form

$$N := N(f_1, \dots, f_r; \epsilon) = \{\varsigma \in \Delta(C(X)) \mid \quad |\varsigma(f_i) - \tau(f_i)| < \epsilon, \ \ \forall i = 1, \dots, r\}$$

for given $f_1, \dots, f_r \in C(X)$ and $\epsilon > 0$.

For each $k \in \{1, \dots, r\}$ we define $u_k \in C(X)$ by

$$u_k(x) := f_k(x) - \tau(f_k)$$

Then, $\forall k \in \{1, \ldots, r\} : \tau(u_k) = \tau(f_k) - \tau(f_k)\tau(1) = 0$ so $u_k \in \ker(\tau)$.

Therefore $(u_1, \ldots, u_r) \subset \ker(\tau) \neq C(X)$, and from the contraposition of (5.3.1.) it follows that $\exists x \in X$ s.t. $|u_k(x)| < \epsilon$, $\forall k$.

But then $\forall k : |\tau_x(f_k) - \tau(f_k)| = |f_k(x) - \tau(f_k)| = |u_k(x)| < \epsilon$, so $\tau_x \in N$. This completes the proof. $\square$

Somewhat more can be said. The Stone-Čech compactification $\Delta(C(X))$ of X is a 'maximal compactification' in the following sense: Suppose $j : X \hookrightarrow Y$ is an embedding such that $j(X)$ is dense in Y. Then there is a surjective map $\tilde{j} : \Delta(C(X)) \to Y$ s.t. $j = \tilde{j} \circ \theta$. We shall not go further into details of this.

5.4. $A(D)$

Here we consider $A(D)$, the subalgebra of $C(D)$ for D the closed unit disc, containing those functions that are analytic on D°. See also (2.1.5;3).

The following results hold analogously as for $C(X)$:

1. Each $\lambda \in D$ defines a point-evaluation $\tau_\lambda \in \Delta(A(D))$ and the mapping $\theta : \lambda \longmapsto \tau_\lambda$ is injective (since $\Delta(A(D))$ separates points of D).
2. $\wedge$ is bijective and $A(D)$ is semisimple.
3. $\theta^*(\hat{f}\mid_{\theta A(D)}) = f$ and $\theta^* = \wedge^{-1}$.
4. θ is continuous.

We *will* indeed see that the identification $D \cong \Delta(A(D))$ goes through in this case as well. However, an argument like the one for $C(X)$ cannot be repeated (we would have to show analyticity in parts of the proof).

Instead, the identification is based on the observation that $A(D)$ is generated by the identity function id, where $id(\lambda) = \lambda\ \forall \lambda \in D$. (Note that this is *not* the identity element of the Banach algebra).

5.4.1. Theorem $A(D) = \langle id \rangle$.

Proof Let $f \in \Delta(A(D))$. Define the sequence $\{f_n\}$ by setting $f_n(\lambda) := f(\lambda \cdot (1 - \frac{1}{n}))$. We claim that the f_n converge to f uniformly on D.

Because D is compact, f is not only continuous but also *uniformly* continuous on D. Thus, if an $\epsilon > 0$ is given:

$$\exists \delta > 0 \text{ s.t. } \forall \lambda, \mu \in D : \quad |\lambda - \mu| < \delta \Longrightarrow |f(\lambda) - f(\mu)| < \epsilon$$

Let $N > \frac{1}{\delta}$. Then for all $\lambda \in D$:

$$\forall n \geq N: \quad |(1-\frac{1}{n})\lambda - \lambda)| = |\frac{1}{n}\lambda| < \delta \cdot |\lambda| \leq \delta$$

and therefore

$$\forall n \geq N: \quad |f_n(\lambda) - f(\lambda)| = |f((1-\frac{1}{n})\lambda) - f(\lambda)| < \epsilon$$

This holds for all $\lambda \in D$ so $f_n \to f$ uniformly on D.

Furthermore, since f is analytic in D°, all f_n are analytic in an open disc with radius $R = \frac{n}{n-1} > 1$ and hence have power series that, on D, converge uniformly to f_n.

Given $\epsilon > 0$ we can first choose an N s.t. $\|f_N - f\|_\infty < \frac{\epsilon}{2}$ due to uniform convergence. Now we define the polynomial function

$$p := \sum_{k=0}^{K} a_k id^k$$

where $\sum_{k=0}^{\infty} a_k id^k$ is the power series uniformly converging to f_N on D and K is chosen s.t. $\|p - f_N\|_\infty < \frac{\epsilon}{2}$. Then we see:

$$\|p - f\|_\infty \leq \|p - f_N\|_\infty + \|f_N - f\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since p is a polynomial in id the result follows from (4.5.2.) $\square$

5.4.2. Corollary $\Delta(A(D)) \cong D$.

Proof From (4.5.3.) and the previous proposition we know that $\Delta(A(D)) \cong \sigma(id)$. Therefore, it remains to show that $\sigma(id) = \{\tau(id) \mid \tau \in \Delta(A(D))\} = D$.

"$\subset$" Since $A(D)$ is with identity, $\|\tau\| = 1 \ \forall \tau \in \Delta(A(D))$. Since $\|id\|_\infty = 1$ we have $|\tau(id)| \leq \|\tau\|\|id\| = 1$, so $\tau(id) \in D$.

"$\supset$" Suppose $\lambda \in D$. Then $\tau_\lambda(id) = id(\lambda) = \lambda \in \sigma(id)$. $\square$

Note that this actually defines a bijection:

$$\begin{array}{cccccc} & D & = & \sigma(id) & \xrightarrow{\sim} & \Delta(A(D)) \\ \theta: & \lambda & = & \tau_\lambda(id) & \longmapsto & \tau_\lambda \\ & & & \tau(id) & \longleftarrow\!\shortmid & \tau \end{array}$$

where the last mapping was defined in (4.5.3.) In particular, this means $\tau = \tau_{\tau(id)} \ \forall \tau$ and hence $\tau(f) = \tau_{\tau(id)}(f) = f(\tau(id)), \forall \tau \ \forall f$.

These results have many practical applications and we shall now give one of those. Recall that for $C(X)$ for compact X we had a nice characterization of the ideal generated by a finite set of functions in terms of the common zeroes (5.1.3.) which we had used to prove the identification of X with $\Delta(C(X))$. Here we have reached the identification by another road but can now directly show that an analog of (5.1.3.) holds:

5.4.3. Proposition Let $f_1, \ldots, f_n \in A(D)$. Then:

$$(f_1, \ldots, f_n) = A(D) \Longleftrightarrow \text{the } f_i \text{ have no common zeroes}$$

Proof The proof is of the contraposition:

$(f_1, \ldots, f_n) \neq A(D)$

$\overset{(4.1.14.)}{\Longleftrightarrow}$

$0 \in \sigma(f_1, \ldots, f_n)$

$\Longleftrightarrow$

$\exists \tau \in \Delta(A(D))$ s.t. $\tau(f_1) = \cdots = \tau(f_n) = 0$

$\overset{(*)}{\Longleftrightarrow}$

$\exists \lambda \in D$ s.t. $f_1(\lambda) = \cdots = f_n(\lambda) = 0$

where $(*)$ refers to the preceding theory. □

5.5. $AC(\Gamma)$

The algebra $AC(\Gamma)$ of absolutely convergent Fourier series on the unit circumference (see (2.1.5;5.) is another example of a semisimple function algebra for which $\Gamma \cong \Delta(AC(\Gamma))$. Obviously $\theta : \lambda \longmapsto \tau_\lambda$ is an injective continuous mapping. Just as with $A(D)$, $AC(\Gamma)$ is also generated by the identity — but here this observation follows directly from the definition. So it remains only to show that $\sigma(id) = \Gamma$.

The "$\supset$" part is obvious and for "$\subset$" we note the following: since $AC(\Gamma)$ is with identity, $\|\tau(id)\| \leq \|\tau\|\|id\| = 1$. Furthermore, on Γ, id^{-1} is $\overline{id}$ (complex conjugation) and so $\|id^{-1}\| = 1$, too. Therefore $\|\tau(id)\| = \|\frac{1}{\tau(id^{-1})}\| \geq \frac{1}{\|\tau\|\|id^{-1}\|} = 1$, so $\|\tau(id)\| = 1$ and hence $\tau(id) \in \Gamma\ \forall \tau \in \Delta(AC(\Gamma))$.

An application of this identification is Wiener's theorem. Historically, this theorem had a certain practical value (part of the so-called General Tauberian theorem) and the fact that it's proof was so simple when using Banach algebra techniques, came as something of a surprise. In fact, this success was one of the early motivations for mathematicians to study Banach algebras in greater detail.

5.5.1. Wiener's Theorem $f \in AC(\Gamma)$ is invertible $\Longleftrightarrow$ f has no zeroes on Γ.

Proof Analogously to the proof of (5.4.3.) we in fact have an even stronger result, namely that

$$(f_1, \dots, f_n) = AC(\Gamma) \iff f_i \text{ have no common zero}$$

Wiener's theorem follows as a specific case. $\square$

5.6. H^∞

Our final example is H^∞, the subalgebra of $C(D^\circ)$ consisting of those functions that are analytic on D°, see also (2.1.5;4.) This is, just like $C(X)$ for X not compact, an example of a function algebra whose underlying space is not homeomorphic to $\Delta(H^\infty)$. However, we do have $\theta : \lambda \mapsto \tau_\lambda$ which is injective and continuous; once again H^∞ is semisimple and $\theta^* = \wedge^{-1}$.

Moreover, θD° is dense in $\Delta(H^\infty)$ — this result is commonly known as the "Corona Theorem", due to Carleson. It is based, in turn, on the so-called "Reduced Corona Theorem", a highly technical theorem based (as should be expected) on the theory of functions of complex variables. Therefore its proof is omitted here and instead we refer to [2] pp. 202–218.

5.6.1. Reduced Corona Theorem Let $f_1, \dots, f_n \in H^\infty$. If $\exists \delta > 0$ s.t. $|f_1(\lambda)| + \dots + |f_n(\lambda)| \geq \delta \; \forall \lambda \in D^\circ$, then $\exists g_1, \dots, g_n \in H^\infty$ s.t. $f_1 g_1 + \dots + f_n g_n = 1$. $\square$

Of course, it should be clear that the Reduced Corona Theorem is a variation of (5.3.1.) for $C(X)$ when X is locally compact (that being a topological variant of (5.1.3.) for $C(X)$ when X is compact). Now, similarly as the Stone-Čech compactification, here we get the following corollary:

5.6.2. Corona Theorem θD° is dense in $\Delta(H^\infty)$

Proof The proof proceeds along the same (familiar) lines as that of (5.3.3.) Let $\tau \in \Delta(H^\infty)$ and N a neighbourhood of τ. As always we may assume N to be of the form

$$N := N(f_1, \dots, f_r; \epsilon) = \{\varsigma \in \Delta(H^\infty) \mid \quad |\varsigma(f_i) - \tau(f_i)| < \epsilon, \;\; \forall i = 1, \dots, r\}$$

for given $f_1, \dots, f_r \in H^\infty$ and $\epsilon > 0$.

For each $k \in \{1, \dots, r\}$ let $u_k \in H^\infty$ be

$$u_k(\lambda) := f_k(\lambda) - \tau(f_k)$$

Then, $\forall k \in \{1, \ldots, r\} : \tau(u_k) = 0$ so $u_k \in \ker(\tau)$.

Now suppose $|u_1| + \cdots + |u_r|$ was uniformly positive for all $\lambda \in H^\infty$, i.e. $\exists \delta > 0$ s.t. $\forall \lambda \in D^\circ : |u_1(\lambda)| + \cdots + |u_r(\lambda)| \geq \delta > 0$. Then by the Reduced Corona Theorem (5.6.1.) $\exists g_1 \ldots g_r \in H^\infty$ s.t. $f_1 g_1 + \cdots + g_r f_r = 1$. But since all the f_i's are in $\ker(\tau)$, $f_1 g_1 + \cdots + g_r f_r = 1$ is in $\ker(\tau)$ which is a contradiction.

It follows that there must exist a $\lambda \in D^\circ$ s.t. $|u_1(\lambda)| + \cdots + |u_r(\lambda)| < \epsilon$, i.e. $u_k(\lambda) < \epsilon \; \forall k$.

But then $\forall k : |\tau_\lambda(f_k) - \tau(f_k)| = |f_k(\lambda) - \tau(f_k)| = |u_k(\lambda)| < \epsilon$, so $\tau_\lambda \in N$. This completes the proof. $\square$

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