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Full-splitting Miller trees and infinitely often equal reals

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ABSTRACT

We investigate two closely related partial orders of trees on ω^ω: the *full-splitting Miller trees* and the *infinitely often equal trees*, as well as their corresponding σ-ideals. The former notion was considered by Newelski and Rosłanowski while the latter involves a correction of a result of Spinas. We consider some Marczewski-style regularity properties based on these trees, which turn out to be closely related to the property of Baire, and look at the dichotomies of Newelski–Rosłanowski and Spinas for higher projective pointclasses. We also provide some insight concerning a question of Fremlin whether one can add an infinitely often equal real without adding a Cohen real, which was recently solved by Zapletal.

1. Introduction

A common theme in descriptive set theory and forcing theory on the reals are perfect-set-style *dichotomy theorems*—statements asserting that all Borel (or analytic) sets are either in a σ-ideal $\mathfrak{I}$ on ω^ω, or else contain the branches of a certain kind of tree. When $\mathbb{P}$ denotes the partial order of these trees ordered by inclusion, such a theorem guarantees that there is a dense embedding

$$\mathbb{P} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$$

from $\mathbb{P}$ to the partial order of Borel sets positive with respect to $\mathfrak{I}$ (also ordered by inclusion), and hence that the two posets are forcing-equivalent. The most famous result of this kind is the original *perfect set theorem*, showing that the Sacks partial order (perfect trees ordered by inclusion) densely embeds into the partial order of uncountable Borel sets. Jindřich Zapletal [18,19] developed an extensive theory of *idealized forcing*,

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i.e., forcing with $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ for various σ-ideals $\mathfrak{I}$ on the reals. In Zapletal's framework, properties of the forcing can be studied directly using properties of the σ-ideal. On the other hand, there is a long-established tradition of studying forcing properties using combinatorics on trees. A dichotomy theorem provides the best of both worlds, since it allows us to freely switch back and forth between the "idealized" and the "tree" framework, depending on which suits the situation better.

In this paper we consider two closely related dichotomies. The following two definitions are due to Newelski and Rosłanowski [14].

Definition 1.1. A tree $T \subseteq \omega^{<\omega}$ is called a *full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that s is *full-splitting*, i.e., $s^\frown \langle n \rangle \in T$ for every n. Let $\mathbb{FM}$ denote the partial order of full-splitting Miller trees ordered by inclusion.

Definition 1.2. For $f : \omega^{<\omega} \to \omega$, let

$$D_f := \{x \in \omega^\omega \mid \forall^\infty n \, (x(n) \neq f(x{\restriction}n))\}.$$

Then $\mathfrak{D}_\omega := \{A \subseteq \omega^\omega \mid A \subseteq D_f \text{ for some } f\}$.

The original motivation of [14] was the connection to infinite games of the same kind as used by Morton Davis in [4] in the proof of the perfect set theorem from determinacy, but played on ω^ω instead of 2^ω. Let $G^*(A)$ be the game in which Player I chooses $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and Player II chooses $n_i \in \omega$, and I wins iff $s_0{}^\frown \langle n_0 \rangle^\frown s_1{}^\frown \langle n_1 \rangle^\frown \cdots \in A$. It is easy to see (cf. [15]) that Player I wins $G^*(A)$ if and only if there exists a tree $T \in \mathbb{FM}$ such that $[T] \subseteq A$, and Player II wins $G^*(A)$ if and only if $A \in \mathfrak{D}_\omega$. General properties of so-called *Mycielski ideals* (i.e., ideals of sets for which II wins a corresponding game) imply that $\mathfrak{D}_\omega$ is a σ-ideal on ω^ω. Using Solovay's "unfolding" method (see e.g. [9, Exercise 27.14]) it follows from the determinacy of closed games that analytic sets are either $\mathfrak{D}_\omega$-small or contain $[T]$ for some $T \in \mathbb{FM}$.

The next concept is due to Spinas [16].

Definition 1.3. For every $x \in \omega^\omega$ let $K_x := \{y \in \omega^\omega \mid \forall^\infty n \; (x(n) \neq y(n))\}$, and let $\mathfrak{I}_{\text{ioe}}$ be the σ-ideal generated by K_x, for $x \in \omega^\omega$.

In [16], $\mathfrak{I}_{\text{ioe}}$-positive sets were called "countably infinitely often equal families", since a set A is $\mathfrak{I}_{\text{ioe}}$-positive if and only if for every countable sequence of reals $\{x_i \mid i < \omega\}$ there exists $a \in A$ which hits every x_i infinitely often. The following result was claimed in [16, Theorem 3.3]: "every analytic set is either $\mathfrak{I}_{\text{ioe}}$-small or contains $[T]$ for some $T \in \mathbb{FM}$". This dichotomy is clearly in error, as the simple example below shows:

Example 1.4. Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\text{succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then $\text{succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if } s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if } s(|s|-1) = 1 \end{cases}$

where $\text{succ}_T(s) := \{n \mid s^\frown \langle n \rangle \in T\}$. Clearly T is $\mathfrak{I}_{\text{ioe}}$-positive but cannot contain a full-splitting subtree.

The correct dichotomy for the ideal $\mathfrak{I}_{\text{ioe}}$ involves a subtle modification of the concept of a full-splitting tree, suggested by Spinas in private communication.

Definition 1.5 *(Spinas).* A tree $T \subseteq \omega^\omega$ is called an *infinitely often equal tree*, or simply *ioe-tree*, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$. Let $\mathbb{IE}$ denote the partial order of ioe-trees ordered by inclusion.

Clearly $\mathbb{FM} \subseteq \mathbb{IE}$ while the converse is false by Example 1.4. It is not hard to see that the proof of [16, Theorem 3.3] does yield the following correct dichotomy theorem: "every analytic set is either $\mathfrak{I}_{\mathrm{ioe}}$-small or contains $[T]$ for some $T \in \mathbb{IE}$." This dichotomy, like the one of Newelski–Rosłanowski, also allows for an easy analysis in terms of infinite games, see Definition 4.5 and Theorem 4.6. Moreover, an argument as in Theorem 4.2 provides an alternative, arguably more elementary, proof of the dichotomy. The partial order $\mathbb{IE}$ has been considered in unpublished work of Goldstern and Shelah [7], but hasn't been studied elsewhere to our knowledge.

Summarizing the situation, we have two closely related perfect-set-style dichotomy theorems leading to the following dense embeddings:

$$\mathbb{FM} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{D}_\omega$$
$$\mathbb{IE} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{I}_{\mathrm{ioe}}$$

We will study these objects from various points of view. In Section 2 we look at some general properties of these two forcings/ideals, relating them to one another as well as to Cohen forcing and the meager ideal. In Section 3 we consider regularity properties generated by these forcings/ideals which are closely related to the property of Baire, and in Section 4 we focus on the dichotomies themselves, but for projective classes above analytic. Section 5 is devoted to an interesting problem concerning the forcing $\mathbb{IE}$ and Cohen reals.

We use standard set-theoretic notation; for a tree $T \subseteq \omega^{<\omega}$ and $t \in T$, we write $\mathrm{succ}_T(t) = \{n \mid t^\frown \langle n\rangle \in T\}$ and $T{\uparrow}t$ to denote $\{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$. We will frequently use the notation D_f and K_x to refer to the generators of the σ-ideals $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\mathrm{ioe}}$, as in Definitions 1.2 and 1.3. Also, we will say that two reals $x, y \in \omega^\omega$ are *infinitely often equal* (*ioe*) if $\exists^\infty n\, (x(n) = y(n))$ and *eventually different* (*evd*) if $\forall^\infty n\, (x(n) \neq y(n))$.

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2. Some general properties

The first easy observations involve the relationships between $\mathfrak{D}_\omega$, $\mathfrak{I}_{\mathrm{ioe}}$ and the ideal $\mathcal{M}$ of meager subsets of ω^ω.

Lemma 2.1. $\mathfrak{I}_{\mathrm{ioe}} \subsetneq \mathfrak{D}_\omega \subsetneq \mathcal{M}$.

Proof. For every $x \in \omega^\omega$ let $f_x : \omega^{<\omega} \to \omega$ be defined by $f_x(s) := x(|s|)$. Then it is easy to see that $K_x \subseteq D_{f_x}$, and since $\mathfrak{D}_\omega$ is a σ-ideal $\mathfrak{I}_{\mathrm{ioe}} \subseteq \mathfrak{D}_\omega$ follows. On the other hand, if T is the tree from Example 1.4 then $[T] \in \mathfrak{D}_\omega$ (because $[T]$ does not contain an $\mathbb{FM}$-subtree) but $[T] \notin \mathfrak{I}_{\mathrm{ioe}}$.

To see $\mathfrak{D}_\omega \subsetneq \mathcal{M}$ notice that for $f : \omega^{<\omega} \to \omega$ the sets $D_{f,N} := \{y \mid \forall n > N\, (y(n) \neq f(y{\restriction}n))\}$ are nowhere dense, so $D_f = \bigcup_{N\in\omega} D_{f,N}$ is meager. On the other hand $\{x \mid \forall n\, (x(2n) = 0)\}$ is meager but contains a full-splitting Miller tree, hence it is not in $\mathfrak{D}_\omega$. $\square$

Lemma 2.2 *(Newelski–Rosłanowski). There exists a continuous function $\varphi : \omega^\omega \to \omega^\omega$ such that for all meager A, $\varphi^{-1}[A] \in \mathfrak{D}_\omega$.*

Proof. Given a fixed enumeration $\{s_i \mid i < \omega\}$ of $\omega^{<\omega}$, let φ be defined by

$$\varphi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown s_{x(2)}{}^\frown \dots$$

Also let $\varphi' : \omega^{<\omega} \to \omega^{<\omega}$ be a function on initial segments such that $\varphi(x) = \bigcup_{n\in\omega} \varphi'(x\restriction n)$. Then, given a nowhere dense set $X \subseteq \omega^\omega$, define $f : \omega^{<\omega} \to \omega$ as follows: given $s \in \omega^{<\omega}$ find t such that $[\varphi'(s)^\frown t] \cap X = \varnothing$, and let i be such that $t = s_i$. Then set $f(s) = i$. One can easily verify that $\varphi^{-1}[X] \subseteq D_f$, which is sufficient since $\mathfrak{D}_\omega$ is a σ-ideal.

An alternative way to view this is as follows: given an arbitrary $\mathbb{FM}$-tree T, φ“$[T]$ is non-meager. □

The relationship between $\mathfrak{I}_{\mathrm{ioe}}$, $\mathfrak{D}_\omega$ and $\mathcal{M}$ is also apparent by considering cardinal invariants. Recall the definitions of $\mathrm{cov}(\mathfrak{I}), \mathrm{add}(\mathfrak{I}), \mathrm{cof}(\mathfrak{I})$ and $\mathrm{non}(\mathfrak{I})$ for σ-ideals on ω^ω (see e.g. [1, Section 1.3]). The following result is an easy generalization of [14, Theorem 3.1 and Corollary 3.3].

Theorem 2.3.

1. $\mathrm{cov}(\mathfrak{I}_{\mathrm{ioe}}) = \mathrm{cov}(\mathfrak{D}_\omega) = \mathrm{cov}(\mathcal{M})$ *and* $\mathrm{non}(\mathfrak{I}_{\mathrm{ioe}}) = \mathrm{non}(\mathfrak{D}_\omega) = \mathrm{non}(\mathcal{M})$.
2. $\mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}) = \mathrm{add}(\mathfrak{D}_\omega) = \omega_1$ *and* $\mathrm{cof}(\mathfrak{I}_{\mathrm{ioe}}) = \mathrm{cof}(\mathfrak{D}_\omega) = 2^{\aleph_0}$.

Proof.

1. Since $\mathfrak{I}_{\mathrm{ioe}} \subseteq \mathfrak{D}_\omega \subseteq \mathcal{M}$ it immediately follows that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathfrak{D}_\omega) \leq \mathrm{cov}(\mathfrak{I}_{\mathrm{ioe}})$. For the other direction, we recall Bartoszyński's characterization [1, Theorem 2.4.1] saying that $\mathrm{cov}(\mathcal{M})$ is the least size of an *eventually different family*, i.e., a family $F \subseteq \omega^\omega$ such that for every $x \in \omega^\omega$ there exists $y \in F$ which is eventually different from x. From this it easily follows that $\mathrm{cov}(\mathfrak{I}_{\mathrm{ioe}}) \leq \mathrm{cov}(\mathcal{M})$. The proof for non is dual.
2. This follows from the following claim, proved by Newelski and Rosłanowski in [14, Theorem 3.2].

   Claim 2.4 *(Newelski–Rosłanowski). Let $\{x_\alpha \mid \alpha < 2^{\aleph_0}\}$ be a collection of reals such that $\forall\alpha \neq \beta\ \exists^\infty n\ (x_\alpha(n) \neq x_\beta(n))$, and for each α put $X_\alpha := \{x \mid \forall n\ (x(n) \neq x_\alpha(n))\}$. Then for every uncountable $F \subseteq 2^{\aleph_0}$, $\bigcup_{\alpha\in F} X_\alpha \notin \mathfrak{D}_\omega$.*

   Since each $A_\alpha \in \mathfrak{I}_{\mathrm{ioe}} \subseteq \mathfrak{D}_\omega$ while $\bigcup_{\alpha\in F} X_\alpha \notin \mathfrak{D}_\omega \supseteq \mathfrak{I}_{\mathrm{ioe}}$, the above claim implies the result for both $\mathfrak{D}_\omega$ and $\mathfrak{I}_{\mathrm{ioe}}$. □

Turning to forcing properties, let us recall some results of Zapletal.

Definition 2.5. A σ-ideal $\mathfrak{I}$ on ω^ω is *σ-generated by closed sets* if every set in $\mathfrak{I}$ is contained in an F_σ-set in $\mathfrak{I}$.

Theorem 2.6 *(Zapletal). If $\mathfrak{I}$ is a σ-ideal on ω^ω σ-generated by closed sets then the forcing $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ is proper and preserves Baire category (non-meager ground-model sets remain non-meager in the extension).*

Proof. See [19, Theorem 4.1.2]. □

Corollary 2.7. $\mathbb{FM}$ *and* $\mathbb{IE}$ *are proper and preserve Baire category. In particular, they do not add dominating or random reals.*

Proof. The generators D_f and K_x are clearly F_σ sets, so the results follows by Zapletal's theorem. It is not too hard to provide direct Axiom A-style proofs for this, in fact for $\mathbb{FM}$ it was already done in [14, Section 2]. □

The following concept is very practical when dealing with idealized forcing notions, and was first explicitly defined in [3,8].

Definition 2.8. Let $\mathfrak{I}$ be a σ-ideal on the reals, and assume that membership of Borel sets in the ideal is a $\mathbf{\Sigma}^1_2$ predicate (on Borel codes). Let M be a model of set theory containing all countable ordinals and the defining parameter of $\mathfrak{I}$. Then a real x is called $\mathfrak{I}$-*quasigeneric* over M if and only if for every Borel set $B \in \mathfrak{I}$ with Borel code in M, $x \notin B$.

The importance of $\mathbf{\Sigma}^1_2$-definability is that the statement $B \in \mathfrak{I}$ should be absolute between M and V. In general, being an $\mathfrak{I}$-quasigeneric real is much weaker then being a $(B(\omega^\omega) \setminus \mathfrak{I})$-generic real. For example, a real is Sacks-quasigeneric (i.e., $\mathfrak{I}_{\mathsf{ctbl}}$-quasigeneric, where $\mathfrak{I}_{\mathsf{ctbl}}$ is the ideal of countable subsets of ω^ω) over M if and only if $x \notin M$; and it is Miller-quasigeneric (i.e., K_σ-quasigeneric, where K_σ is the ideal of σ-*compact* subsets of ω^ω) over M if and only if it is unbounded over $\omega^\omega \cap M$. However, a $(\mathcal{B}(\omega^\omega) \setminus \mathfrak{I})$-generic real is always $\mathfrak{I}$-quasigeneric. When $\mathfrak{I}$ is a ccc ideal then the two notions are equivalent.

Definition 2.9. A real x is called *infinitely often equal* (*ioe*) over a model M, iff $\forall y \in \omega^\omega \cap M\, \exists^\infty n\, (x(n) = y(n))$. A real x is called *infinitely often following* (*iof*) over a model M, iff $\forall f \in \omega^{(\omega^{<\omega})} \cap M\, \exists^\infty n\, (x(n) = f(x{\restriction}n))$.

Lemma 2.10. *Let M be a model of set theory with $\omega_1 \subseteq M$ and x a real. Then:*

1. *x is $\mathfrak{I}_{\mathrm{ioe}}$-quasigeneric over M iff it is ioe over M, and*
2. *x is $\mathfrak{D}_\omega$-quasigeneric over M iff it is iof over M.*

Proof. The proofs of both statements are analogous so let us only show the first. If x avoids $\mathfrak{I}_{\mathrm{ioe}}$-small Borel sets coded in M, then for any $y \in \omega^\omega \cap M$, K_y is a Borel $\mathfrak{I}_{\mathrm{ioe}}$-small set coded in M, so $x \notin K_y$, so x is ioe to y. Conversely, suppose x is ioe over M and $B \in \mathfrak{I}_{\mathrm{ioe}}$ is a Borel set coded in M. Since "$B \in \mathfrak{I}_{\mathrm{ioe}}$" is a Σ^1_2 statement on the code of B, by absoluteness $M \models B \in \mathfrak{I}_{\mathrm{ioe}}$. Therefore there are $x_i \in M$ such that $B \subseteq \bigcup_{i<\omega} K_{x_i}$ (this statement is Π^1_1, hence absolute). But x is ioe to all x_i, so by definition $x \notin K_{x_i}$ for all i, hence $x \notin B$. □

Therefore, $\mathbb{IE}$ canonically adds an ioe real, whereas $\mathbb{FM}$ canonically adds an iof real. From Lemma 2.1 it immediately follows that a Cohen real is an iof real, and an iof real is an ioe real. Also, from Lemma 2.2 it follows that if x is an iof real then $\varphi(x)$ is a Cohen real (so $\mathbb{FM}$ adds a Cohen real). Moreover, the following is well-known:

Fact 2.11 *(Bartoszyński/Folklore). If $V_0 \subseteq V_1 \subseteq V_2$ are models of set theory, in V_1 there is an ioe real over V_0 and in V_2 there is an ioe real over V_1, then in V_2 there is a Cohen real over V_0.*

Corollary 2.12. $\mathbb{IE} * \mathbb{IE}$ *adds a Cohen real.*

So $\mathbb{C}$, $\mathbb{FM}$ and $\mathbb{IE}$ all have a very similar effect on the structure of the real line. For example, an ω_2-iteration of $\mathbb{FM}$ or $\mathbb{IE}$ with countable supports yields the same values for the cardinal invariants in Cichoń's diagram as an ω_2-iteration/product of Cohen forcing, namely $\omega_1 = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \omega_2$. Newelski and Rosłanowski observed that "it seems that forcing $\mathbb{FM}$ is the best one for adding Cohen reals in countable support iterations." In fact, we do not know the answer to the following basic question:

Question 2.13. *What is a "natural" forcing property (e.g. adding or not adding certain types of reals) which distinguishes* $\mathbb{FM}$ *from Cohen forcing?*

The situation with $\mathbb{IE}$ and adding Cohen reals is much more subtle, and is closely related to issues of homogeneity, which we will now describe.

Definition 2.14 *(Zapletal).* A σ-ideal $\mathfrak{I}$ on ω^ω is *homogeneous* if for every $\mathfrak{I}$-positive Borel set B, there exists a Borel function $f : \omega^\omega \to B$ such that f-preimages of $\mathfrak{I}$-small sets are $\mathfrak{I}$-small.

Remark 2.15. The meager ideal is well-known to be homogeneous, and for $\mathfrak{D}_\omega$, observe that for every full-splitting Miller tree T there exists a natural homeomorphism ψ between ω^ω and $[T]$ (generated by the identification of $\omega^{<\omega}$ with the split-nodes of T), with the additional property that for every full-splitting tree S, $\psi``[S]$ generates a full-splitting sub-tree of T. It follows that $\mathfrak{D}_\omega$ is homogeneous.

On the other hand, $\mathfrak{I}_{\text{ioe}}$ fails to be homogeneous—this will indirectly follow both from Corollary 3.6 and from Lemma 5.2. The crucial point is the following idea from unpublished work of Goldstern and Shelah [7]:

Definition 2.16. A tree $T \subseteq \omega^{<\omega}$ is called an *almost full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that $\forall^\infty n \, (s^\frown \langle n \rangle \in T)$.

Lemma 2.17 *(Goldstern–Shelah). There exists a $T^{\text{GS}} \in \mathbb{IE}$ such that* every *$\mathbb{IE}$-subtree of T^{GS} is an almost full-splitting Miller tree.*

Proof. Construct T^{GS} in such a way that:

1. Every splitting note $t \in T^{\text{GS}}$ is full-splitting.
2. If $s \neq t$ are splitting nodes of T^{GS} then $|s| \neq |t|$.
3. If $t \in T^{\text{GS}}$ is a non-splitting node of T then $t(|t| - 1) = 0$.

Such a tree can easily be constructed inductively after fixing some bijection $f : \omega^{<\omega} \cong \omega$. It is not hard to see that if S is any sub-tree of T^{GS} which is an ioe-tree, then it has to be an almost full-splitting, since the only way that a node of S can be extended to "hit" an arbitrary $k > 0$ on some fixed level, is to extend that node to a $t \in S$ such that $t^\frown \langle k \rangle \in S$ for all $k > 0$. $\square$

An argument just as in the proof of Lemma 2.2 easily extends to show that if T is an almost full-splitting Miller tree then $\varphi``[T]$ is non-meager, implying that $T^{\text{GS}} \Vdash_{\mathbb{IE}}$ "$\varphi(\dot{x}_G)$ is a Cohen real". Nevertheless, since $\mathfrak{I}_{\text{ioe}}$ is not homogeneous, there is no a priori reason why there could not be some other $\mathbb{IE}$-condition forcing that no Cohen reals are added. We shall return to this question in Section 5.

3. Marczewski-type regularity properties

A vast array of "Marczewski-type" regularity properties have been considered in the literature, where a set $A \subseteq \omega^\omega$ is considered "measurable" if every set in a certain partial order can be shrunk to a smaller set in the same partial order, which is completely contained in, or disjoint from, the given set A, possibly modulo a suitable ideal. Polish mathematicians had a strong interest in such properties for a long time, e.g. [17]. More modern treatments include [2,3,5], while [8,12,11] provide more abstract treatments in the setting of *forcing with trees* or *idealized forcing*. See also [13] for a related treatment in terms of *category bases*. The Baire property, Lebesgue measurability, the Ramsey property and many other properties can be formulated as Marczewski-type properties. Following this setting we define:

Definition 3.1. A set $A \subseteq \omega^\omega$ is called

- $\mathbb{FM}$-*measurable* iff $\forall T \in \mathbb{FM} \, \exists S \in \mathbb{FM} \, (S \leq T$ and $[S] \subseteq A$ or $[S] \cap A = \varnothing)$.
- $\mathbb{IE}$-*measurable* iff $\forall T \in \mathbb{IE} \, \exists S \in \mathbb{IE} \, (S \leq T$ and $[S] \subseteq A$ or $[S] \cap A = \varnothing)$.

We also define *weak* (local) versions of the above.

Definition 3.2. A set $A \subseteq \omega^\omega$ is

- *weakly* $\mathbb{FM}$*-measurable* iff $\exists S \in \mathbb{FM}$ ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).
- *weakly* $\mathbb{IE}$*-measurable* iff $\exists S \in \mathbb{IE}$ ($[S] \subseteq A$ or $[S] \cap A = \varnothing$).

If $\mathbf{\Gamma}$ is some pointclass of sets (e.g. Borel, projective etc.) we follow standard practice and use the notation $\mathbf{\Gamma}(\mathbb{FM}), \mathbf{\Gamma}(\mathbb{IE})$, $\mathbf{\Gamma}(\mathrm{w}\mathbb{FM})$ and $\mathbf{\Gamma}(\mathrm{w}\mathbb{IE})$ to refer to the statements "all sets in $\mathbf{\Gamma}$ are $\mathbb{FM}$-measurable", "... are $\mathbb{IE}$-measurable", "... are weakly $\mathbb{FM}$-measurable" and "... are weakly $\mathbb{IE}$-measurable", respectively. $\mathbf{\Gamma}$(Baire) refers to "all sets in $\mathbf{\Gamma}$ have the property of Baire".

Usually, the homogeneity of the ideal/partial order of trees (in the sense of Definition 2.14) ensures that for sufficiently nice pointclasses $\mathbf{\Gamma}$, the "weak" notion of measurability is equivalent to the strong one.

Observation 3.3. Let $\mathbf{\Gamma}$ be a pointclass closed under continuous pre-images. Since $\mathfrak{D}_\omega$ is homogeneous (in fact witnessed by a continuous reduction) it is easy to see that $\mathbf{\Gamma}(\mathbb{FM}) \Leftrightarrow \mathbf{\Gamma}(\mathrm{w}\mathbb{FM})$.

Theorem 3.4. *Let $\mathbf{\Gamma}$ be a pointclass closed under continuous pre-images. Then the following are equivalent:*

1. $\mathbf{\Gamma}$(*Baire*)
2. $\mathbf{\Gamma}(\mathbb{FM})$
3. $\mathbf{\Gamma}(\mathbb{IE})$

Proof.

- $1 \Rightarrow 2$. Let $A \subseteq \omega^\omega$ be a set in $\mathbf{\Gamma}$. By $\mathbf{\Gamma}$(Baire) we can find a basic open set $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ stand for "modulo a meager set". Without loss of generality, assume the former. Then there is a G_δ set $B \subseteq A$ which is co-meager in $[s]$. Since $\mathfrak{D}_\omega \subseteq \mathcal{M}$, B cannot be $\mathfrak{D}_\omega$-small, hence it contains an $\mathbb{FM}$-tree. By Observation 3.3 this is sufficient.
- $2 \Rightarrow 3$. We say that an ioe-tree T is *in strict form* if it can be written as follows:
  - for every $\sigma \in \omega^{<\omega}$, there exists $N_\sigma \subseteq \omega^n$, for some $n \geq 1$, such that
    - $\forall k\, \exists! s \in N_\sigma\, (s(n-1) = k)$, and
    - for $m < (n-1)$, there is some k such that $s(m) \neq k$ for all $s \in N_\sigma$.

    We use $\mathrm{len}(N_\sigma) = n$ to denote the length of N_σ, and we canonically enumerate N_σ as $\{s^\sigma_k \mid k < \omega\}$, in such a way that $s^\sigma_k(n-1) = k$.
  - T is the tree generated by sequences of the form

$$s^{\varnothing}_{n_0}{}^\frown s^{\langle n_0\rangle}_{n_1}{}^\frown s^{\langle n_0,n_1\rangle}_{n_2}{}^\frown \dots {}^\frown s^{\langle n_0,n_1,n_2,\dots,n_{\ell-1}\rangle}_{n_\ell}$$

    for some sequence $\langle n_0, n_1, \dots, n_\ell\rangle$.

  Ioe-trees in strict form are somewhat easier to visualize and deal with. If T is in strict form, $t \in T$, and we need to find the first N such that $\forall k\, \exists s \supseteq t$ with $s(N) = k$, we only need to find the (unique) shortest sequence σ such that $t \subseteq s^{\varnothing}_{\sigma(0)}{}^\frown s^{\sigma\restriction 1}_{\sigma(1)}{}^\frown \dots {}^\frown s^{\sigma\restriction|\sigma|-1}_{\sigma(|\sigma|-1)}$, and then we have $N = \sum_{i=0}^{|\sigma|} N_{\sigma\restriction i}$. Every ioe-tree T can be pruned to an ioe-subtree $S \leq T$ in strict form.

  So, let $A \in \mathbf{\Gamma}$ and let T be an ioe-tree, assuming, without loss of generality, that T is in strict form. Define a function $\psi' : \omega^{<\omega} \to T$ inductively by $\psi'(\varnothing) = \varnothing$ and $\psi'(\sigma^\frown \langle n\rangle) := \psi'(\sigma)^\frown s^\sigma_n$. This gives rise to a natural homeomorphism $\psi : \omega^\omega \cong [T]$. Since $\psi^{-1}[A]$ is also in $\mathbf{\Gamma}$ we can find a full-splitting tree S such that $[S] \subseteq \psi^{-1}[A]$ or $[S] \cap \psi^{-1}[A] = \varnothing$. So we will be done if we can show that ψ“$[S]$ generates an ioe-subtree of T. But this follows from the definition of ψ' and the fact that the last digit of every s^σ_n is n.

- $3 \Rightarrow 1$. Recall the function φ from Lemma 2.2. Let $A \in \mathbf{\Gamma}$ and let $A' := \varphi^{-1}[A]$, also in $\mathbf{\Gamma}$. Now recall the Goldstern–Shelah tree T^{GS} from Lemma 2.17. Since A' is $\mathbb{IE}$-measurable, there exists $S \leq T^{\mathrm{GS}}$ such that $[S] \subseteq A$ or $[S] \cap A = \varnothing$, without loss of generality the former. But since S is an almost full-splitting tree, $\varphi``[S]$ is not meager, but it is analytic, so it is comeager in some basic open $[s]$. Then $[s] \subseteq^* A$. This is sufficient because $\mathbf{\Gamma}$(Baire) is equivalent to the assertion that for all $A \in \mathbf{\Gamma}$ there exists a basic open $[s]$ such that $[s] \subseteq^* A$ or $[s] \cap A =^* \varnothing$. $\square$

In the “$3 \Rightarrow 1$”-direction of the above proof, the Golstern–Shelah tree was used in a quintessential way; this suggests that the property called *weak* $\mathbb{IE}$-*measurability* behaves substantially different. Indeed, the following theorem is the most surprising result of this section.

Theorem 3.5. $\mathbf{\Delta}^1_2$(*Baire*) $\Rightarrow \mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$.

Corollary 3.6. *It is consistent that* $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ *is true while* $\mathbf{\Sigma}^1_2(\mathbb{IE})$ *is false; in particular* $\mathbb{IE}$- *and weak* $\mathbb{IE}$-*measurability are not classwise equivalent.*

Proof. We know that the ω_1-Cohen model $L^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathrm{Baire}) + \neg\mathbf{\Sigma}^1_2(\mathrm{Baire})$ ([1, Sections 9.3 and 9.3]). Therefore, by Theorems 3.4 and 3.5 we have $L^{\mathbb{C}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE}) + \neg\mathbf{\Sigma}^1_2(\mathbb{IE})$. $\square$

This is the first instance we know of where such a situation occurs in the context of a very naturally defined ideal.

Proof of Theorem 3.5. Assume $\mathbf{\Delta}^1_2(\mathrm{Baire})$. Let $A \subseteq \omega^\omega$ be a $\mathbf{\Sigma}^1_2$ set. We have to find an $\mathbb{IE}$-tree T such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

We may assume that for some r, $\omega_1^{L[r]} = \omega_1$, since otherwise $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ follows easily (for example from $\mathbf{\Sigma}^1_2(\mathrm{Baire})$). We may also assume, without loss of generality, that the parameters in the definition of A are in $L[r]$. Using the Borel decomposition of $\mathbf{\Sigma}^1_2$ sets we can write $A = \bigcup_{\alpha<\omega_1} B_\alpha$, where B_α are Borel sets coded in $L[r]$. If there exists at least one α such that $B_\alpha \notin \mathfrak{I}_{\mathrm{ioe}}$, then there is an $\mathbb{IE}$-tree T with $[T] \subseteq B_\alpha \subseteq A$ and we are done. So suppose that all B_α are $\mathfrak{I}_{\mathrm{ioe}}$-small. For each α, since $L[r] \models B_\alpha \in \mathfrak{I}_{\mathrm{ioe}}$, we can fix a sequence $\langle x^\alpha_i \mid i < \omega\rangle$ of reals in $L[r]$ such that $B_\alpha \subseteq \bigcup_{i<\omega} K_{x^\alpha_i}$.

Let $\rho : \omega^\omega \to \omega^\omega$ be defined by $\rho(x) := \langle x(0), x(2), x(4), \dots\rangle$. By $\mathbf{\Delta}^1_2(\mathrm{Baire})$, we know that in V there is a Cohen real c over $L[r]$ ([1, Theorem 9.2.1]). Then c is infinitely often equal over $L[r]$, and in particular, infinitely often equal to $\rho(x^\alpha_i)$ for all $\alpha < \omega_1, i < \omega$. Let T_c be the $\mathbb{FM}$-tree such that

$$[T_c] = \{y \mid \rho(y) = c\}.$$

We claim that $[T_c] \cap A = \varnothing$. Let $a \in A$, then there is some $\alpha < \omega_1$ such that $a \in B_\alpha$. By absoluteness of “$B_\alpha \subseteq \bigcup_{i<\omega} K_{x^\alpha_i}$”, there is some $i < \omega$ such that a is eventually different from x^α_i. Let $N \in \omega$ be such that $\forall n > N\ (a(n) \neq x^\alpha_i(n))$. But since c is ioe to $\rho(x^\alpha_i)$, we can easily find $n > N$ such that $c(n) = x^\alpha_i(2n) \neq a(2n)$. By definition this implies that $a \notin [T_c]$. $\square$

4. The dichotomy for higher projective sets

Definition 4.1. Let us say that a set $A \subseteq \omega^\omega$ satisfies the $\mathbb{FM}$-*dichotomy* if A is either $\mathfrak{D}_\omega$-small or contains $[T]$ for an $\mathbb{FM}$-tree T, and that is satisfies the $\mathbb{IE}$-*dichotomy* if A is either $\mathfrak{I}_{\mathrm{ioe}}$-small or contains $[T]$ for some $\mathbb{IE}$-tree T. We use $\mathbf{\Gamma}(\mathbb{FM}\text{-dich})$ and $\mathbf{\Gamma}(\mathbb{IE}\text{-dich})$ to abbreviate “all sets in $\mathbf{\Gamma}$ have the $\mathbb{FM}$-dichotomy” and “all sets in $\mathbf{\Gamma}$ have the $\mathbb{IE}$-dichotomy”, respectively.

So we know that $\boldsymbol{\Sigma}^1_1(\mathbb{FM}\text{-dich})$ and $\boldsymbol{\Sigma}^1_1(\mathbb{IE}\text{-dich})$ are true, but we can also ask to which higher projective levels the dichotomies can be extended. Note that these properties are stronger then their Marczewski-counterparts, i.e., $\boldsymbol{\Gamma}(\mathbb{FM}\text{-dich}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{FM})$ and $\boldsymbol{\Gamma}(\mathbb{IE}\text{-dich}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{IE})$ for all projective pointclasses $\boldsymbol{\Gamma}$. The main result of this section concerns the consistency strength of $\boldsymbol{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ and $\boldsymbol{\Sigma}^1_2(\mathbb{IE}\text{-dich})$. In general, statements of this kind have a rather unpredictable behavior: for example, K_σ-regularity (see [10]) for $\boldsymbol{\Sigma}^1_2$ sets is equiconsistent with ZFC, while the perfect set property for $\boldsymbol{\Sigma}^1_2$ (even $\boldsymbol{\Pi}^1_1$) sets has the strength of an inaccessible. In yet other cases, it is actually inconsistent (Zapletal, see [11, Proposition 2.4.4]). The properties considered here will fall into the second category.

First we prove a "Mansfield–Solovay-style" theorem for $\mathbb{FM}$ and $\mathbb{IE}$. Its proof uses a completely classical Cantor-Bendixson analysis and it is worth noting that an analogous argument replacing trees on $\omega \times \omega_1$ by trees on $\omega \times \omega$ provides alternative (arguably more elementary) proofs of the Newelski–Rosłanowski and the Spinas dichotomy theorems, i.e., $\boldsymbol{\Sigma}^1_1(\mathbb{FM}\text{-dich})$ and $\boldsymbol{\Sigma}^1_1(\mathbb{IE}\text{-dich})$.

Lemma 4.2.

1. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{FM}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{D}_\omega$-small Borel sets coded in $L[r]$.*
2. *For any $\Sigma^1_2(r)$ set A, either there exists an $\mathbb{IE}$-tree $U \in L[r]$, such that $[U] \subseteq A$, or A can be covered by $\mathfrak{I}_{\text{ioe}}$-small Borel sets coded in $L[r]$.*

Proof. 1. Let $T \in L[r]$ be a tree on $\omega \times \omega_1$ such that $A = p[T]$ (where p denote the projection to the first coordinate). For any tree S on $\omega \times \omega_1$ define

$$S' := \{(s,h) \in S \mid \exists s' \supseteq s\, \forall n\, \exists h' \supseteq h\, ((s'^\frown \langle n\rangle, h') \in S)\}.$$

Next define inductively $T^{(0)} := T$, $T^{(\alpha+1)} := (T^{(\alpha)})'$ and $T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$. As the above construction is absolute, it follows that all $T^{(\alpha)}$ are in $L[r]$.

Let α be least such that $T^{(\alpha)} = T^{(\alpha+1)}$, and consider two cases:

- $T^{(\alpha)} \neq \varnothing$. Then every $(s,h) \in T^{(\alpha)}$ has the property that there exists s' such that for every n there is h' extending h with $(s'^\frown \langle n\rangle, h') \in T^{(\alpha)}$. Using this it is easy to inductively construct a tree $U \subseteq \omega^{<\omega}$, such that every $s \in U$ has a full-splitting extension $s' \in U$, i.e., U is an $\mathbb{FM}$-tree, and moreover, such that given any branch $x \in [U]$, there is a corresponding branch $g \in {\omega_1}^\omega$ such that $(x,g) \in [T^{(\alpha)}] \subseteq [T]$. Therefore $x \in p[T] = A$.
- $T^{(\alpha)} = \varnothing$. In this case, for every $\gamma < \alpha$ and every $h \in (\omega_1)^{<\omega}$ we define a function $f_{\gamma,h} : \omega^{<\omega} \to \omega$ by:

$$f_{\gamma,h}(s) = n \;:\Longleftrightarrow\; \exists h' \supseteq h\, ((s,h') \in T^{(\gamma)}) \text{ and}$$
$$n \text{ is least such that } \forall h'' \supseteq h\, ((s^\frown \langle n\rangle, h'') \notin T^{(\gamma)})$$

  if such an n exists, and $f_{\gamma,h}(s) = 0$ otherwise.
  Since each $f_{\gamma,h}$ is explicitly constructed from $T^{(\gamma)}$ and h, clearly it is in $L[r]$. Also let $D_{f_{\gamma,h}} := \{x \mid \forall^\infty n\, (x(n) \neq f_{\gamma,h}(x{\restriction}n))\}$ be the Borel $\mathfrak{D}_\omega$-small sets corresponding to $f_{\gamma,h}$, clearly also coded in $L[r]$. We will finish the proof by concluding that $A \subseteq \bigcup\{D_{f_{\gamma,h}} \mid \gamma < \alpha, h \in (\omega_1)^{<\omega}\}$.
  Take any $x \in A$, and let $g \in {\omega_1}^\omega$ be such that $(x,g) \in [T]$. Let $\gamma < \alpha$ be least such that $(x,g) \in [T^{(\gamma)}] \setminus [T^{(\gamma+1)}]$, and let $s \subseteq x$ and $h \subseteq g$ be such that $(s,h) \in T^{(\gamma)} \setminus T^{(\gamma+1)}$. By definition of $T^{(\gamma+1)} := (T^{(\gamma)})'$ we know that for any s' extending s there exists n such that $(s'^\frown \langle n\rangle, h') \notin T^{(\gamma)}$ for any h' extending h. Take any $k > |s|$. Then $(x{\restriction}k, g{\restriction}k) \in T^{(\gamma)}$, and let n be least such that $(x{\restriction}k^\frown \langle n\rangle, h') \notin T^{(\gamma)}$ for any h' extending h. But then, the definition of $f_{\gamma,h}$ implies that $f_{\gamma,h}(x{\restriction}k) = n$. On the other hand,

$(x\restriction(k+1), g\restriction(k+1))$ is also in $T^{(\gamma)}$, and so $x(k)$ cannot have value n. In particular $x(k) \neq f_{\gamma,h}(x\restriction k)$. Since this argument applies for all $k > |s|$, this proves $x \in D_{f_{\gamma,h}}$.

2. The argument is completely analogous, so we only mention the changes that need to be made. Here, the pruning operation for a tree S on $\omega \times \omega_1$ is defined as follows:

$$S' := \{(s, h) \in S \mid \exists N > |s|\, \forall k\, \exists s' \supseteq s\, \exists h' \supseteq h \\ ((s', h') \in S, |s'| = N + 1 \text{ and } s'(N) = k)\}.$$

Then, in the case that $T^{(\alpha)} = \varnothing$ proceed as follows: for every $\gamma < \alpha$, $s \in \omega^{<\omega}$ and $h \in (\omega_1)^{<\omega}$ define a real $x_{\gamma,s,h} \in \omega^\omega$ by:

$$x_{\gamma,s,h}(N) = k \;:\Longleftrightarrow\; (s, h) \in T^{(\gamma)} \text{ and } N > |s| \text{ and} \\ k \text{ is least s.t. } \forall s' \supseteq s\, \forall h' \supseteq h\, ((s', h') \in T^{(\gamma)} \wedge |s'| = N + 1 \rightarrow s'(N) \neq k)$$

if such a k exists, and 0 otherwise. The proof is completed by showing that $A \subseteq \bigcup\{K_{(x_{\gamma,s,h})} \mid \gamma < \alpha, s \in \omega^{<\omega}, h \in (\omega_1)^{<\omega}\}$; details are left to the reader. □

We are now ready to prove the main theorem of this section. Recall that by Lemma 2.10 iof reals are $\mathfrak{D}_\omega$-quasigeneric and ioe reals are $\mathfrak{I}_{\mathrm{ioe}}$-quasigeneric, which will be frequently used in the proof.

Theorem 4.3. *The following are equivalent:*

1. $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$
2. $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$
3. $\forall r \in \omega^\omega\ \{x \mid x \textit{ is not iof over } L[r]\} \in \mathfrak{D}_\omega$
4. $\forall r \in \omega^\omega\ \{x \mid x \textit{ is not ioe over } L[r]\} \in \mathfrak{I}_{\mathrm{ioe}}$
5. $\forall r \in \omega^\omega\ (\omega_1^{L[r]} < \omega_1)$

Proof. First we prove $1 \Leftrightarrow 3 \Leftrightarrow 5$.

- $1 \Rightarrow 3$. Fix an arbitrary r and let $X := \{x \mid x \text{ is not iof over } L[r]\}$. It is not hard to see that X is a $\Sigma^1_2(r)$ set, so by assumption either $X \in \mathfrak{D}_\omega$ or there is some $T \in \mathbb{FM}$ such that $[T] \subseteq X$. We will show that the second option is impossible.
  From $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ we have $\mathbf{\Sigma}^1_2(\mathbb{FM})$, and by Theorem 3.4 also $\mathbf{\Sigma}^1_2(\text{Baire})$. In particular, there is a Cohen real c, which is an iof real, over $L[r, T]$. Let $T \in \mathbb{FM}$ and recall that there is a homeomorphism $\psi : \omega^\omega \cong [T]$ such that ψ-preimages of $\mathfrak{D}_\omega$-small sets are $\mathfrak{D}_\omega$-small (Remark 2.15). Since being an iof real is the same as being $\mathfrak{D}_\omega$-quasigeneric, it easily follows that $\psi(c)$ is an iof real in $[T]$. This contradicts $[T] \subseteq X$.
- $3 \Rightarrow 1$. Notice that Lemma 4.2(1) actually says: every $\mathbf{\Sigma}^1_2$ set A either contains $[T]$ for $T \in \mathbb{FM}$ or $A \subseteq \{x \mid x \text{ is not } \mathfrak{D}_\omega\text{-quasigeneric over } L[r]\} = \{x \mid x \text{ is not iof over } L[r]\}$, from which the result follows.
- $5 \Rightarrow 3$. If $\omega_1^{L[r]} < \omega_1$ then $\{x \mid x \text{ is not iof over } L[r]\} = \bigcup\{B \mid B \text{ is a Borel } \mathfrak{D}_\omega\text{-small set coded in } L[r]\}$ is a countable union of $\mathfrak{D}_\omega$-small sets.
- $3 \Rightarrow 5$. Recall Claim 2.4 used in the proof of $\mathrm{add}(\mathfrak{D}_\omega) = \omega_1$, which, in particular, implies that for any family $F = \{x_\alpha \mid \alpha < \omega_1\}$ of reals satisfying $\forall \alpha \neq \beta\, \exists^\infty n\, (x_\alpha(n) \neq x_\beta(n))$, and letting $X_\alpha := \{x \mid \forall n\, (x(n) \neq x_\alpha(n))\}$, we have
  - $X_\alpha \in \mathfrak{I}_{\mathrm{ioe}} \subseteq \mathfrak{D}_\omega$ for all $\alpha < \omega_1$, and
  - $\bigcup_{\alpha<\omega_1} X_\alpha \notin \mathfrak{D}_\omega$.

If $\omega_1^{L[r]} = \omega_1$ for some r, then we have an F as above satisfying $F \subseteq \omega^\omega \cap L[r]$. But then $\{x \mid x$ is not iof over $L[r]\} = \bigcup\{B \mid B$ is a Borel $\mathfrak{D}_\omega$-small coded in $L[r]\} \supseteq \bigcup\{X_\alpha \mid \alpha < \omega_1\}$ cannot be $\mathfrak{D}_\omega$-small.

The proofs of $2 \Leftarrow 4 \Leftrightarrow 5$ are analogous. For $2 \Rightarrow 4$ we must be more careful since the ideal $\mathfrak{I}_{\mathrm{ioe}}$ is not homogeneous, so we cannot conclude that there is an ioe real inside $[T]$ for every $T \in \mathbb{IE}$, just from the existence of ioe reals. However, using the same trick as in the "$2 \Rightarrow 3$"-direction of the proof of Theorem 3.4 we can argue as follows: given a $T \in \mathbb{IE}$ in strict form, find a homeomorphism ψ between ω^ω and $[T]$ such that ψ-preimages of $\mathfrak{I}_{\mathrm{ioe}}$-small sets are $\mathfrak{D}_\omega$-small. Then, if c is an iof real, $\psi(c)$ is an ioe real. □

The consistency of the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomies beyond $\mathbf{\Sigma}^1_2$ sets can be established in the Solovay model and from determinacy hypotheses. Here we should note that, while the remaining results of this section are not particularly surprising, they are nevertheless not "trivial" results, since, as we already mentioned, there are dichotomies which are true for analytic sets but are inconsistent for $\mathbf{\Sigma}^1_2$ sets, so there is no a priori reason to believe that our dichotomies for higher projective sets are consistent.

In the next theorem we will assume familiarity with the Solovay model (see e.g. [9, Section 11] for details). $\mathrm{Col}(\omega, {<}\kappa)$ will denote the standard Lévy partial order for collapsing an inaccessible κ to ω_1.

Theorem 4.4. *Let κ be inaccessible and let G be $\mathrm{Col}(\omega, {<}\kappa)$-generic over V. Then in $V[G]$ all sets definable from countable sequences of ordinals satisfy the $\mathbb{FM}$- and the $\mathbb{IE}$-dichotomy, and in $L(\mathbb{R})^{V[G]}$ all sets of reals satisfy the $\mathbb{FM}$- and $\mathbb{IE}$-dichotomy.*

Proof. The proofs of both dichotomies are similar; we prove the $\mathbb{FM}$-case in detail and leave the $\mathbb{IE}$-case to the reader.

Let $A \subseteq \omega^\omega$ be a set in $V[G]$, defined by ϕ and a countable sequence of ordinals $\vec{a}$. By well-known properties of the Lévy collapse, there is a formula $\tilde{\phi}$ such that for all x:

$$V[G] \models \phi(\vec{a}, x) \iff V[\vec{a}][x] \models \tilde{\phi}(\vec{a}, x).$$

Assume that $A \notin \mathfrak{D}_\omega$. In particular, A cannot be covered by Borel $\mathfrak{D}_\omega$-small sets coded in $V[\vec{a}]$, since $V[\vec{a}]$ only contains countably many reals. So there is an $x \in A$ which is $\mathfrak{D}_\omega$-quasigeneric over $V[\vec{a}]$, i.e., iof over $V[\vec{a}]$. By another standard property of the Lévy collapse, there is a $\mathrm{Col}(\omega, {<}\kappa)$-generic H such that $V[G] = V[\vec{a}][H]$, and moreover, a complete suborder Q of $\mathrm{Col}(\omega, {<}\kappa)$ in $V[\vec{a}]$, such that $|Q| < \kappa$ and $x \in V[\vec{a}][H \cap Q]$. Then in $V[\vec{a}]$, there is a Q-name $\dot{x}$ for x and a condition $p \in Q \cap H$ satisfying:

$$p \Vdash_Q \tilde{\phi}(\vec{a}, \dot{x}) \ \wedge\ \text{“}\dot{x} \text{ is iof over } V[\vec{a}]\text{”}.$$

Since $\omega_1^{V[G]}$ is inaccessible in $V[\vec{a}]$, in $V[G]$ there are only countably many Q-dense sets in $V[\vec{a}]$. Let $\{D_i \mid i < \omega\}$ enumerate all of them.

In $V[G]$, by induction we will construct $U \subseteq \omega^{<\omega}$, and for every $t \in U$ a corresponding Q-condition p_t, such that

1. $s \subseteq t \iff p_t \leq p_s$,
2. for every t, $p_t \Vdash t \subseteq \dot{x}$,
3. for every t, $p_t \in D_i$, where $i = |\{s \in U \mid s \subseteq t\}|$, and
4. the downward-closure of U is an $\mathbb{FM}$-tree.

Let $p_\varnothing \leq p$ be any condition in D_0. Suppose $s \in U$ has been constructed, $p_s \in Q$ satisfies $p_s \Vdash s \subseteq \dot{x}$, and $i := |\{s' \in U \mid s' \subseteq s\}|$. Extend p_s to $p'_s \in D_i$.

Claim. *There exists $t \supseteq s$ such that for all n, there is $q \leq p'_s$ such that $q \Vdash t^\frown \langle n \rangle \subseteq \dot{x}$.*

Proof. Suppose not: so for any $t \supseteq s$ there is n such that no $q \leq p'_s$ forces $t^\frown \langle n \rangle \subseteq \dot{x}$, so that $p'_s \Vdash t^\frown \langle n \rangle \not\subseteq \dot{x}$. Define a function $f : \omega^{<\omega} \to \omega$ in $V[\vec{a}]$ by letting $f(t)$ be such n as above, for all $t \supseteq s$, and $f(t) = 0$ for other t. Then we have

$$p'_s \Vdash \forall t \supseteq s \; (t^\frown \langle f(t) \rangle \not\subseteq \dot{x}).$$

Since also $p'_s \Vdash s \subseteq \dot{x}$, in particular we have

$$p'_s \Vdash \forall n > |s| \; (\dot{x}{\restriction}n^\frown \langle f(\dot{x}{\restriction}n) \rangle \not\subseteq \dot{x})$$

and so

$$p'_s \Vdash \forall n > |s| \; (\dot{x}(n) \neq f(\dot{x}{\restriction}n)),$$

contradicting the fact that $p'_s \Vdash$ "$\dot{x}$ is iof over $V[\vec{a}]$". $\square$ (Claim).

By the claim, we can fix such a t, and for every n, a condition $p_{t^\frown \langle n \rangle} \leq p'_s$ forcing $t^\frown \langle n \rangle \subseteq \dot{x}$. Finally we add all these $t^\frown \langle n \rangle$ to the set U, and this completes the inductive construction.

Let $T(U)$ be the tree generated by U. It is clear that $T(U) \in \mathbb{FM}$, so it only remains to show that $[T(U)] \subseteq A$. In $V[G]$, let y be any real in $[T(U)]$. We have to show that $V[G] \models \phi(\vec{a}, y)$. By construction, y can be viewed as the limit of some $\{t_n \mid n < \omega\}$, where all $t_n \in U$. Let $G_y := \{q \in Q \mid \exists n \, (p_{t_n} \leq q)\}$. Since G_y meets every D_i, it is Q-generic over $V[\vec{a}]$, and since $p_{t_n} \Vdash_Q t_n \subseteq \dot{x}$ for every n, we know that $\dot{x}[G_y] = y$. Also, since $p \Vdash_Q V[\vec{a}][\dot{x}] \models \tilde{\phi}(\vec{a}, \dot{x})$, it follows that $V[\vec{a}][y] \models \tilde{\phi}(\vec{a}, y)$, and therefore $V[G] \models \phi(\vec{a}, y)$. This completes the proof of the $\mathbb{FM}$-dichotomy.

The proof of the $\mathbb{IE}$-dichotomy is analogous, replacing $\mathfrak{D}_\omega$ by $\mathfrak{I}_{\mathrm{ioe}}$, $\mathbb{FM}$ by $\mathbb{IE}$ and "iof reals" by "ioe reals". The corresponding claim must read as follows: "*There is $N > |s|$ such that for all $k \in \omega$ there are t and $q \leq p'_s$ such that $t \supseteq s$, $|t| = N + 1$, $t(N) = k$ and $q \Vdash t \subseteq \dot{x}$.*" The claim is proved by assuming the contrary and producing a real $z \in V[\vec{a}]$ such that $p'_s \Vdash \forall N > |s| \; (\dot{x}(N) \neq z(N))$, contradicting $p'_s \Vdash$ "$\dot{x}$ is ioe over $V[\vec{a}]$". $\square$

Another way to extend the dichotomy beyond $\mathbf{\Sigma}^1_2$ sets is by the use of infinite games; the $\mathbb{FM}$-dichotomy was originally motivated by a Morton Davis-like game. The following game corresponds to the $\mathbb{IE}$-dichotomy.

Definition 4.5. Let $G^{\mathbb{IE}}(A)$ be the game in which players I and II play as follows:

I:	N_0		s_0 N_1		s_1 N_2		$\dots$	$\dots$	
II:		k_0		k_1		k_2			$\dots$

where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$, $N_i \geq 1$, $k_i \in \omega$, and the following rules must be obeyed for all i:

- $|s_i| = N_i$,
- $s_i(N_i - 1) = k_i$.

Then player I wins iff $z := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$. For technical reasons, we formalize the game as if Player I makes two consecutive moves rather than a pair (s_i, N_{i+1}).

Theorem 4.6.

1. *Player I has a winning strategy in $G^{\mathbb{IE}}(A)$ iff there is an $\mathbb{IE}$-tree T such that $[T] \subseteq A$.*
2. *Player II has a winning strategy in $G^{\mathbb{IE}}(A)$ iff $A \in \mathfrak{I}_{\mathrm{ioe}}$.*

Proof. We will only show the left-to-right direction of 2. Let τ be a winning strategy for player II. Suppose $p = \langle N_0, k_0, s_0, N_1, \ldots, k_{\ell-1}, s_{\ell-1}\rangle$ is a *position of the game* of length 3ℓ (and $p = \varnothing$ if $\ell = 0$). Then we define $p^* := s_0{}^\frown \ldots {}^\frown s_{\ell-1}$ (and $p^* = \varnothing$ if $p = \varnothing$), and for an $x \in \omega^\omega$ we say

- p is *compatible* with x iff $p^* \subseteq x$, and
- p *rejects* x iff p is compatible with x but for any N_ℓ, $k_\ell := \tau(p^\frown \langle N_\ell\rangle)$ is such that $p^\frown \langle N_\ell, k_\ell, s_\ell\rangle$ is incompatible with x for any s_ℓ satisfying $s_\ell(N_\ell) = k_\ell$—in other words, for any N_ℓ, $x(\sum_{i=0}^{\ell} N_i - 1) \neq k_\ell$.

Let $H_p := \{x \mid p \text{ rejects } x\}$. For any $x \in A$ there must be a p which rejects it, otherwise x would be a play according to τ. So $A \subseteq \bigcup_p H_p$, and we shall be done if we can show that $H_p \in \mathfrak{I}_{\mathrm{ioe}}$ for any p.

Fix p of length 3ℓ as before, let $M := \sum_{i=0}^{\ell-1} N_i$, and define a real z as follows: for $N < M$ let $z(N) = 0$, and for $N \geq M$, let $N_\ell := N - M + 1$ and let $z(N) := \tau(p^\frown \langle N_\ell\rangle)$. Suppose p rejects x, which by definition means that for any N_ℓ, $x(\sum_{i=0}^{\ell} N_i - 1) \neq \tau(p^\frown \langle N_\ell\rangle)$. In particular, for any $N \geq M$ we have

$$x(N) = x(M + N_\ell - 1) = x(\sum_{i=0}^{\ell} N_i - 1) \neq \tau(p^\frown \langle N_\ell\rangle) = z(N).$$

Hence $H_p \subseteq K_z = \{x \mid \forall^\infty n\, (x(n) \neq z(n)\}$ which completes the proof. $\square$

5. Half a Cohen real

Recall Fact 2.11, which says that if we iteratively add two ioe reals to a model of set theory then we add a Cohen real; for that reason, an ioe real has sometimes received the name “half a Cohen real”. A natural questions which appeared in Fremlin’s list of open problems [6] is:

Question 5.1 *(Fremlin). Is it possible to add an ioe real without adding a Cohen real?*

This question was recently answered in the positive by Zapletal [20] using rather unorthodox methods.

Theorem 5.2 *(Zapletal 2013). Let X be a compact metrizable space which is infinite-dimensional, and all of its compact subsets are either infinite-dimensional or zero-dimensional. Let $\mathfrak{I}$ be the σ-ideal σ-generated by the compact zero-dimensional subsets of X. Then $\mathcal{B}(X) \setminus \mathfrak{I}$ adds an ioe real but not a Cohen real.*

In spite of the beauty of this result, Zapletal himself mentions: “as the usual approach towards forcing problems includes a direct combinatorial construction of a suitable poset, the following question is natural: ... is there a combinatorial description of a forcing satisfying [Theorem 5.2] which does not mention topological dimension?”

As $\mathbb{IE}$ seems, in a sense, to be a “minimal” forcing for adding ioe reals, we may wonder whether $\mathbb{IE}$ does not add Cohen reals below some condition, thus providing an alternative solution to Fremlin’s problem. The main purpose of this section is to prove the following property for $\mathbb{IE}$:

Theorem 5.3. *For every continuous function $f : \omega^\omega \to \omega^\omega$ there exists a $T \in \mathbb{IE}$ such that $f“[T]$ is meager.*

The relation between this result and Fremlin’s problem is given by the following:

Fact 5.4 *(Zapletal). If $\mathfrak{I}$ is a σ-ideal generated by closed sets then $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ has the* continuous reading of names*: for every $\dot{x}$ and B such that $B \Vdash \dot{x} \in \omega^\omega$, there exists $C \leq B$ and a continuous $f : C \to \omega^\omega$ (in the ground model) such that $C \Vdash \dot{x} = f(\dot{x}_{\text{gen}})$.*

Since $\mathfrak{I}_{\text{ioe}}$ is σ-generated by closed sets, the above fact can be applied to $\mathbb{IE}$. So if we can find an $\mathbb{IE}$-condition T_0 and strengthen Theorem 5.3 to "for every $S \leq T_0$ and every continuous $f : [S] \to \omega^\omega$, there exists $T \leq S$ such that $f``[T]$ is meager", it will follow that $\mathbb{IE}$ does not add Cohen reals below $T_0$: for any $\dot{x}$, find $f$ and $S \leq T_0$ such that $S \Vdash \dot{x} = f(\dot{x}_{\text{gen}})$, then find $T \leq S$ such that $f``[T]$ is meager, implying that $T \Vdash$ "$\dot{x}$ is not Cohen" since $T \Vdash \dot{x} \in f``[T]$.

Proof of Theorem 5.3. The proof is quite unusual in the following sense: first we prove that it holds under the assumption that $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$, and then argue that the assumption can be dropped by absoluteness.

Lemma 5.5. *Assume* $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$. *Then for every continuous function $f : \omega^\omega \to \omega^\omega$ there exists a $T \in \mathbb{IE}$ such that $f``[T]$ is meager.*

Proof. Towards contradiction, assume that the theorem is false and fix an $f : \omega^\omega \to \omega^\omega$ such that $f``[T]$ is non-meager for every $T \in \mathbb{IE}$. This is equivalent to saying that f-preimages of meager sets are $\mathfrak{I}_{\text{ioe}}$-small. Let $\{X_\alpha \mid \alpha < \text{add}(\mathcal{M})\}$ be a collection of meager sets such that $\bigcup_\alpha X_\alpha$ is non-meager. We will derive a contradiction by showing that for every basic open $[s]$ there is a basic open $[t] \subseteq [s]$ such that $[t] \cap \bigcup_\alpha X_\alpha =^* \varnothing$.

Fix $[s]$ and a homeomorphism $\psi : \omega^\omega \cong [s]$. Every $X'_\alpha := \psi^{-1}[X_\alpha]$ is still meager, so every $Y_\alpha := f^{-1}[X'_\alpha]$ is $\mathfrak{I}_{\text{ioe}}$-small. For each α let $\langle x_i^\alpha \mid i < \omega \rangle$ be such that $Y_\alpha \subseteq \bigcup_i K_{x_i^\alpha}$. Now, letting ρ be the function defined by $\rho(x) := \langle x(0), x(2), \dots \rangle$, using $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$ and Bartoszyński's characterization of $\text{cov}(\mathcal{M})$, we find that $\{\rho(x_i^\alpha) \mid i < \omega, \alpha < \text{add}(\mathcal{M})\}$ is not an eventually different family, hence there exists c which is infinitely often equal to all $\rho(x_i^\alpha)$.

Construct T_c such that $[T_c] = \{y \mid \rho(y) = c\}$, and by exactly the same argument as in the proof of Theorem 3.5 we know that $[T_c] \cap Y_\alpha = \varnothing$ for every α. But then, by assumption, $f``[T_c]$ is non-meager, and then also $\psi``f``[T_c]$ is non-meager; but it is analytic, hence comeager in a basic open $[t]$. This completes the proof since $[t]$ avoids $\bigcup_\alpha X_\alpha$ modulo meager. □ (Lemma)

To conclude the theorem from the lemma we use a simple absoluteness argument, i.e., we check the complexity of the statement "for all continuous $f : \omega^\omega \to \omega^\omega$ there is $T \in \mathbb{IE}$ such that $f``[T]$ is meager". Note the following:

1. "$f : \omega^\omega \to \omega^\omega$ is a continuous function" can be expressed as "$f' : \omega^{<\omega} \to \omega^{<\omega}$ is monotone and unbounded along each real", which is $\mathbf{\Pi}^1_1$ on (the code of) f'.
2. "$T \in \mathbb{IE}$" is arithmetic on the code of T.
3. $f``[T]$ is an analytic set whose code is recursive in f' and T.
4. For an analytic set to be meager is $\mathbf{\Pi}^1_1$.

Then the statement in question can be expressed as:

$$\forall f' \, (f' \text{ is continuous } \to \exists T \, (T \in \mathbb{IE} \wedge f``[T] \in \mathcal{M}))$$

which is a $\mathbf{\Pi}^1_3$ sentence, hence downward absolute between forcing extensions of V and V itself. So the proof is completed by going to any forcing extension $V^{\mathbb{P}}$ satisfying $\text{add}(\mathcal{M}) < \text{cov}(\mathcal{M})$ (e.g., add ω_2 Cohen reals), applying the lemma and then applying absoluteness to conclude that the statement was already true in V. □

Unfortunately, we do not know whether the proposed strengthening of Theorem 5.3 is valid below some condition T_0, i.e., whether there is T_0 such that for any $S \leq T_0$ and any continuous $f : [S] \to \omega^\omega$ there is $T \leq S$ such that $f``[T]$ is meager. Certainly, a sufficient condition for this would be a T_0 such that for every $S \leq T_0$, $\mathbb{E}{\restriction}S$ is homogeneous (in the sense of Definition 2.14).

On the other hand, by Lemma 2.17 we know that such a T_0 certainly cannot be the trivial condition, since the tree T^{GS} forces that Cohen reals are added. It is not so hard to find other $\mathbb{E}$-conditions, aside of T^{GS}, which also add Cohen reals, but it is not clear whether such conditions are dense.

Notice that the question of $\mathbb{E}$ adding Cohen reals can also be formulated in the setting of the following (closed) game:

Definition 5.6. Let $G^{\mathbb{E}}_{\neg\mathbb{C}}$ be the game defined as follows:

$$\begin{array}{r||lcr}
\text{I:} & & S \leq T_0, f : [S] \to \omega^\omega \text{ continuous} & \dots \\
\hline
\text{II:} & T_0 \in \mathbb{E} & & T \leq S
\end{array}$$

$$\begin{array}{ccccc}
 & (s_0, x(0)) & & (s_1, x(1)) & \dots \\
\hline
\dots & & t_0 & & t_1 \quad \dots
\end{array}$$

where $s_i, t_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and $x(i) \in \omega$ are such that $x \in [T]$. Assuming all the rules are followed, Player I wins iff $f(x) = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$.

Lemma 5.7. *If Player I wins $G^{\mathbb{E}}_{\neg\mathbb{C}}$ then every $\mathbb{E}$-condition forces that Cohen reals are added. If Player II wins $G^{\mathbb{E}}_{\neg\mathbb{C}}$ then, letting T_0 be II's first move, $T_0 \Vdash$ "there are no Cohen reals".*

Proof. After the first three moves have been played and $f : [S] \to \omega^\omega$ and $T \leq S$ have been chosen, the rest of the game is essentially Solovay's unfolded version of the Banach–Mazur game, and by a standard argument (see, e.g., [9, Exercise 27.14]) it follows that if Player I wins that game, then $f``[T]$ is comeager in a basic open set, whereas if Player II wins that game, then $f``[T]$ is meager. The rest is clear. $\square$

6. Questions

The most interesting question seems to be the following:

Question 6.1. *Is there an $\mathbb{E}$-condition forcing that no Cohen reals are added, or does $\mathbb{E}$ always add Cohen reals? This can be formulated as "who wins the game $G^{\mathbb{E}}_{\neg\mathbb{C}}$?".*

In Section 3 we completely solved the question of projective regularity for $\mathbb{FM}$- and $\mathbb{E}$-measurability, but not yet for the (arguably more interesting) *weak* $\mathbb{E}$-measurability. We have the following implications:

$$\boldsymbol{\Delta}^1_2(\mathbb{E}) \Leftrightarrow \boldsymbol{\Delta}^1_2(\text{Baire}) \Rightarrow \boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E}) \Rightarrow \boldsymbol{\Delta}^1_2(\text{w}\mathbb{E}),$$

where the first equivalence is due to Theorem 3.4, the second implication due to Theorem 3.5 and the third one trivial. But we do not know anything about the reverse implications. In particular

Question 6.2. *Are $\boldsymbol{\Delta}^1_2$(Baire) and $\boldsymbol{\Delta}^1_2(\text{w}\mathbb{E})$ equivalent? If not, then are $\boldsymbol{\Delta}^1_2$(Baire) and $\boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E})$ equivalent, or are $\boldsymbol{\Sigma}^1_2(\text{w}\mathbb{E})$ and $\boldsymbol{\Delta}^1_2(\text{w}\mathbb{E})$ equivalent?*

A related question is:

Question 6.3. *Can* $\mathbf{\Delta}^1_2(\mathrm{w}\mathbb{IE})$ *and* $\mathbf{\Sigma}^1_2(\mathrm{w}\mathbb{IE})$ *be characterized in terms of the existence of quasi-generic reals over* $L[r]$*?*

Finally, in Theorem 4.3 we characterized $\mathbf{\Sigma}^1_2(\mathbb{FM}\text{-dich})$ and $\mathbf{\Sigma}^1_2(\mathbb{IE}\text{-dich})$, but did not talk about the $\mathbf{\Delta}^1_2$- and $\mathbf{\Pi}^1_1$-levels.

Question 6.4. *Can* $\mathbf{\Pi}^1_1(\mathbb{FM}\text{-dich})$ *and* $\mathbf{\Pi}^1_1(\mathbb{IE}\text{-dich})$ *be added to the list of equivalent statements in Theorem 4.3?*

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