19 Computable Isomorphism

By the way, Yurii has opened a website accessible to everyone at

http://staff.science.uva.nl/~jkhomski/recursion.html

19.1 Definition. A property $\mathcal{P}$ of sets is *computably invariant* if for every computable permutation f (of ω), $X \in \mathcal{P}$ implies $f[X] \in \mathcal{P}$.

1_ω is a computable permutation, and the computable permutations are closed under inversion and composition.

19.2 Examples. Computably invariant are:
(i) being c.e.;
(ii) cardinality;
(iii) being computable;
and *not* computably invariant are:
(iv) containing 2;
(v) containing the even numbers;
(vi) being an index set.

19.3 Definition. Sets A and B are *computably isomorphic*, notation $A \equiv B$, if there exists a computable permutation p such that $p[A] = B$.

Computable isomorphism is an equivalence relation; its equivalence classes are called *computable isomorphism types*.

The following is an effective analogue of the Schröder-Bernstein Theorem.

19.4 Myhill's Isomorphism Theorem. $A \equiv B$ if and only if $A \equiv_1 B$.

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Assume f 1-reduces A to B, and g 1-reduces B to A. We construct a suitable permutation h as the union of a chain of decidable finite functions h_s, $s \in \omega$. We set out from $h_0 = \emptyset$.

Suppose h_s has been constructed, it is 1-1, and

$$\forall u \in \mathrm{Dom}\, h_s\ (u \in A \Leftrightarrow h_s(u) \in B).$$

Let s be even, say $s = 2x$. If $x \in \mathrm{Dom}\, h_s$, $h_{s+1} = h_s$. Otherwise, $\{x\} \cup \mathrm{Dom}\, h_s$ has more elements than $\mathrm{Ran}\, h_s$, so we can define $h_{s+1}(x)$ to be the first element of the sequence $f(x)$, $f(h_s^{-1} f(x))$, $f(h_s^{-1} f(h_s^{-1} f(x)))$,… that does not belong to $\mathrm{Ran}\, h_s$.

If s be odd, say $s = 2y + 1$, we proceed in the same way, but in the other direction, doing nothing if $y \in \mathrm{Ran}\, h_s$, putting it in otherwise and using g to find a corresponding element for the domain of h_{s+1}. ☒

The proof shows more.

19.5 Corollary. Let $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ be partitions of ω into infinite sets, and f and g computable injections such that for all n, $f[A_n] \subseteq B_n$ and $g[B_n] \subseteq A_n$. Then there exists a computable permutation h such that for all n, $h[A_n] = B_n$.

20 Acceptable numberings

Let $\mathcal{P}$ be the class of p.c. functions of one variable. A *numbering* of the p.c. functions is a surjection ρ: $\omega \twoheadrightarrow \mathcal{P}$. The numbering φ used thus far is the *standard* or *canonical* numbering. A numbering ρ is *acceptable* if there are computable functions f and g such that for all x, $\varphi_{f(x)} = \rho_x$ and $\rho_{g(x)} = \varphi_x$.

Acceptable Numbering Theorem (Rogers). For any acceptable numbering ρ of the p.c. functions, there exists a computable permutation h such that for all e, $\rho_{h(e)} = \varphi_e$.

Proof. Part exercise, part promise. ☒

Homework. Make sure you understand the exercise part of the proof of the Acceptable Numbering Theorem. Hand in: 1.6.9 (inacceptable numberings).

21 Σ_1 forms

The projection of a relation $R \subseteq \omega \times \omega$ is the set $\{x \mid \exists y Rxy\}$. Σ_1 is the class of projections of computable relations. A Σ_1 (*normal*) *form* for a set is a presentation as such a projection.

21.1 Normal Form Theorem for c.e. sets. A set is c.e. if and only if it is (in) Σ_1.

Proof. ($\Rightarrow$) If A is c.e., then $A = W_e$ for some index e, and

$$x \in A \Leftrightarrow \exists s\, x \in W_{e,s}.$$

($\Leftarrow$) If R is computable, and $A = \{x \mid \exists y Rxy\}$, define: $\psi(x) \simeq \mu y Rxy$. Then A = Dom ψ. ☒

22.2 Definition. If $A = W_e$, we call e a Σ_1-*index* or a *c.e.-index* for A.

22.3 Quantifier Contraction Theorem. $A \in \Sigma_1$ iff there are $n \in \omega$ and an $(n + 2)$-ary computable relation R such that

$$x \in A \Leftrightarrow \exists y_0 \ldots y_n R x y_0 \ldots y_n.$$

Proof. Define a binary computable relation S by

$$Sxy \Leftrightarrow R(x, (y)_0 \ldots (y)_n).$$

Then $x \in A \Leftrightarrow \exists y Sxy$. ☒

An n-ary relation R is c.e. if $\{\langle y_1,\ldots, y_n\rangle \mid Ry_1\ldots y_n\}$ is c.e. By what we have just seen, this is equivalent to the existence of a computable relation S such that $Ry_1\ldots y_n \Leftrightarrow \exists z S y_1 \ldots y_n z$. We also have:

22.4 Corollary. The projection of a c.e. relation is c.e.

The Quantifier Contraction Theorem is very useful for showing sets to be c.e. For example,

(1) $K = \{x | \exists s\ x \in W_{x,s}\}$.

(2) $K_0 = \{x | \exists s\ \pi_1 x \in W_{\pi_1 x,s}\}$.

(3) $K_1 = \{x | \exists s \exists y\ y \in W_{x,s}\}$.

(4) Ran $\varphi_e = \{y | \exists s \exists x\ y = \varphi_{e,s}(x)\}$.

(5) φ_e itself, as a relation, is c.e.: it is $\{(y, x) | \exists s\ y = \varphi_{e,s}(x)\}$.

22.5 Uniformization Theorem. For every c.e. relation R, there exists a p.c. function ψ such that

$$\psi(\overline{x})\downarrow \Leftrightarrow \exists y R y \overline{x} \Leftrightarrow R(\psi(\overline{x}), \overline{x}).$$

Proof. Since R is c.e., there exists a computable relation S such that $Ry\overline{x} \Leftrightarrow \exists z S y \overline{x} z$. Define: $\psi(\overline{x}) \simeq (\mu u . S((u)_0, \overline{x}, (u)_1)_0$. ⊠

The function ψ is called a *selector function*. Observe that an index for ψ can be computed (it can be found *uniformly*) from any given c.e. index of R. Moreover, if R actually is a unary function, then $\psi = R$.