Uniform rigidity sequences for weak mixing diffeomorphisms on $\mathbb{D}^2$, $\mathbb{A}$ and $\mathbb{T}^2$

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Abstract

In the case of the disc $\mathbb{D}^2$, the annulus $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ and the torus $\mathbb{T}^2$ we will show that if a sequence of natural numbers satisfies a certain growth rate, then there is a weak mixing diffeomorphism that is uniformly rigid with respect to that sequence. The proof is based on a quantitative version of the Anosov-Katok-method with explicitly defined conjugation maps and the constructions are done in the C^∞-topology. Beyond that we can deduce a similar result in the real-analytic topology in the case of $\mathbb{T}^2$.

1 Introduction

In [GM] the notion of uniform rigidity was introduced as the topological analogue of rigidity in ergodic theory:

Definition 1.1. 1. Let T be an invertible measure-preserving transformation of a non-atomic probability space $(X, \mathcal{B}, \mu)$. T is called rigid if there exists an increasing sequence $(n_m)_{m\in\mathbb{N}}$ of natural numbers such that the powers T^{n_m} converge to the identity in the strong operator topology as $m \to \infty$, i.e. $\|f \circ T^{n_m} - f\|_2 \to 0$ as $m \to \infty$ for all $f \in L^2(X,\mu)$. So rigidity along a sequence $(n_m)_{m\in\mathbb{N}}$ implies $\mu(T^{n_m} A \cap A) \to \mu(A)$ as $m \to \infty$ for all $A \in \mathcal{B}$.

2. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, where X is a compact metric space with metric d. A measure-preserving homeomorphism $T : X \to X$ is called uniformly rigid if there exists an increasing sequence $(n_m)_{m\in\mathbb{N}}$ of natural numbers such that $d_u(T^{n_m}, id) \to 0$ as $m \to \infty$, where $d_u(S,T) = d_0(S,T) + d_0(S^{-1}, T^{-1})$ with $d_0(S,T) := \sup_{x\in X} d(S(x), T(x))$ is the uniform metric on the group of measure-preserving homeomorphisms on X.

Remark 1.2. Uniform rigidity implies rigidity. In [Ya], example 3.1, an example of a rigid, but not uniformly rigid homeomorphism of $\mathbb{T}^2$ is presented. Thus, rigidity and uniform rigidity do not coincide on $\mathbb{T}^2$.

In [JKLSS], Proposition 4.1., it is shown that if an ergodic map is uniformly rigid, then any uniform rigidity sequence has zero density. Afterwards, the following question is posed:

Question 1.3. Which zero density sequences occur as uniform rigidity sequences for an ergodic transformation?

Under some assumptions on the sequence $(n_m)_{m\in\mathbb{N}}$ measure-preserving transformations that are weak mixing and rigid along this sequence are constructed by a cutting and stacking method in [BJLR]. Recall that a measure-preserving transformation $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is called weak mixing if for all $A,B\in\mathcal{B}$: $\frac{1}{N}\sum_{n=1}^{N}\left|\mu\left(T^nA\cap B\right)-\mu\left(A\right)\cdot\mu\left(B\right)\right|\to 0$ as $N\to\infty$.
K. Yancey considered Question 1.3 in the setting of homeomorphisms on $\mathbb{T}^2$ (see [Ya]). Given a sufficient growth rate of the sequence she proved the existence of a weak mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to this sequence: Let $\psi\left(x\right)=x^{x^3}$. If $(n_m)_{m\in\mathbb{N}}$ is an increasing sequence of natural numbers satisfying $\frac{n_{m+1}}{n_m}\geq\psi\left(n_m\right)$, there exists a weak mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to $(n_m)_{m\in\mathbb{N}}$. In this paper we start to examine this problem in the smooth category.

Theorem 1. *Let* $\varphi_1(n):=4^{(n+2)^{n+2}}\cdot\left((n+2)!\right)^{11(n+2)^{n+6}}\cdot\exp\left(100n^2\right)^{11n(n+1)^{n+5}}$ *and* M *be* $\mathbb{D}^2$, $\mathbb{A}$ *or* $\mathbb{T}^2$. *If* $(\tilde{q}_n)_{n\in\mathbb{N}}$ *is a sequence of natural numbers satisfying*

$$\tilde{q}_{n+1}\geq\varphi_1\left(n\right)\cdot\tilde{q}_n^{6\cdot(n+1)^{n+1}},$$

then there exists a weak mixing C^∞*-diffeomorphism of* M *that is uniformly rigid with respect to* $(\tilde{q}_n)_{n\in\mathbb{N}}$.

We note that our requirement on the growth rate is less restrictive than the mentioned condition in [Ya], Theorem 1.5.. In fact, the proof in [Ya] shows that a condition of the form $\frac{n_{m+1}}{n_m}\geq n_m^{4n_m^2+20}$ is sufficient for her construction of a weakly mixing homeomorphism, which is uniformly rigid along $(n_m)_{m\in\mathbb{N}}$. Our requirement on the growth rate is still weaker.
The aimed diffeomorphisms are constructed with the aid of the so-called "approximation by conjugation-method" introduced in [AK]. On every smooth compact connected manifold of dimension $m\geq 2$ admitting a non-trivial circle action $\mathcal{S}=\{S_t\}_{t\in\mathbb{S}^1}$ this method enables the construction of smooth diffeomorphisms with specific ergodic properties (e.g. weak mixing ones in [AK], section 5, or [GK]) or non-standard smooth realizations of measure-preserving systems (e.g. [AK], section 6, and [FSW]). These diffeomorphisms are constructed as limits of conjugates $f_n=H_n\circ S_{\alpha_{n+1}}\circ H_n^{-1}$, where $\alpha_{n+1}=\frac{p_{n+1}}{q_{n+1}}\in\mathbb{Q}$, $H_n=H_{n-1}\circ h_n$ and h_n is a measure-preserving diffeomorphism satisfying $S_{\frac{1}{q_n}}\circ h_n=h_n\circ S_{\frac{1}{q_n}}$. See [FK] for more details and other results of this method.
Our specific constructions are inspired by the construction of weak mixing diffeomorphisms with prescribed Liouvillean rotation number by B. Fayad and M. Saprykina ([FS]): Basically, we use the same criterion for weak mixing and the conjugation maps are supposed to have the same effect on the partition elements. In [FS] the conjugation maps are based on a "quasi-rotation" φ_ε on the unit square $[-1,1]^2$, which is the rotation by $\frac{\pi}{2}$ on $[-1+2\varepsilon,1-2\varepsilon]^2$ and coincides with the identity in a neighbourhood of the boundary. It is designed with the aid of "Moser's trick" and it does not admit a norm estimate with explicit dependence on the parameter ε. Since the parameter ε depends on n and we require precise norm estimates of the conjugation maps in order to deduce a sufficient growth rate of the uniform rigidity sequence, the construction of the conjugation maps has to be modified. We give further details on these modifications and outline our constructions in section 3.

Remark 1.4. In the case of the torus, we can apply essentially the same method and obtain the following result:
Let $\rho>0$. *If* $(\tilde{q}_n)_{n\in\mathbb{N}}$ *is a sequence of natural numbers satisfying* $\tilde{q}_1\geq\rho+1$ *and*

$$\tilde{q}_{n+1}\geq 2^n\cdot 64^2\cdot\pi^4\cdot n^4\cdot\tilde{q}_n^{26}\cdot\exp\left(4\pi\cdot n\cdot\tilde{q}_n^6\cdot\exp\left(2\pi\cdot\tilde{q}_n^4\cdot\left(1+n\cdot\tilde{q}_n\right)\right)\right),$$

then there exists a weak mixing $\text{Diff}^{\omega}_{\rho}$*-diffeomorphism of* $\mathbb{T}^2$ *that is uniformly rigid with respect to* $(\tilde{q}_n)_{n\in\mathbb{N}}$.
We do not include the detailed proof because the method of reparameterized linear flows as in [Fa] seems to be a more appropriate approach.

2 Definitions and notations

In this chapter we want to introduce advantageous definitions and notations. In particular, we discuss topologies on the space of diffeomorphisms on the considered manifolds.

2.1 C^{∞}-topology

For defining explicit metrics on $\text{Diff}^k\left(\mathbb{T}^2\right)$ and in the following the subsequent notations will be useful:

Definition 2.1. 1. For a sufficiently differentiable function $f:\mathbb{R}^2\to\mathbb{R}$ and a multiindex $\vec{a}=(a_1,a_2)\in\mathbb{N}_0^2$

$$D_{\vec{a}}f := \frac{\partial^{|\vec{a}|}}{\partial x_1^{a_1}\partial x_2^{a_2}}f,$$

where $|\vec{a}|=a_1+a_2$ is the order of $\vec{a}$.

2. For a continuous function $F:(0,1)^2\to\mathbb{R}$

$$\|F\|_0 := \sup_{z\in(0,1)^2}|F(z)|.$$

For $f,g\in\text{Diff}^k\left(\mathbb{T}^2\right)$ let $F,G:\mathbb{R}^2\to\mathbb{R}^2$ denote their lifts. Furthermore, for a function $F:\mathbb{R}^2\to\mathbb{R}^2$ we denote by $[F]_i$ the i-th coordinate function.

Definition 2.2. 1. For $f,g\in\text{Diff}^k\left(\mathbb{T}^2\right)$ we define

$$\tilde{d}_0(f,g)=\max_{i=1,2}\left\{\inf_{p\in\mathbb{Z}}\left\|[F-G]_i+p\right\|_0\right\}$$

as well as

$$\tilde{d}_k(f,g)=\max\left\{\tilde{d}_0(f,g),\left\|D_{\vec{a}}[F-G]_i\right\|_0\ :\ i=1,2\ ,\ 1\leq|\vec{a}|\leq k\right\}.$$

2. Using the definitions from 1. we define for $f,g\in\text{Diff}^k\left(\mathbb{T}^2\right)$:

$$d_k(f,g)=\max\left\{\tilde{d}_k(f,g)\ ,\ \tilde{d}_k\left(f^{-1},g^{-1}\right)\right\}.$$

Obviously d_k describes a metric on $\text{Diff}^k\left(\mathbb{T}^2\right)$ measuring the distance between the diffeomorphisms as well as their inverses. As in the case of a general compact manifold the following definition connects to it:

Definition 2.3. 1. A sequence of $\text{Diff}^{\infty}\left(\mathbb{T}^2\right)$-diffeomorphisms is called convergent in $\text{Diff}^{\infty}\left(\mathbb{T}^2\right)$ if it converges in $\text{Diff}^k\left(\mathbb{T}^2\right)$ for every $k\in\mathbb{N}$.

2. On $\text{Diff}^{\infty}\left(\mathbb{T}^2\right)$ we declare the following metric

$$d_{\infty}\left(f,g\right)=\sum_{k=1}^{\infty}\frac{d_k\left(f,g\right)}{2^k\cdot\left(1+d_k\left(f,g\right)\right)}.$$

It is a general fact that $\text{Diff}^{\infty}\left(\mathbb{T}^2\right)$ is a complete metric space with respect to this metric d_{∞}. Moreover, we add the adjacent notation:

Definition 2.4. Let $f\in\text{Diff}^k\left(\mathbb{T}^2\right)$ with lift F be given. Then

$$\begin{aligned}\|Df\|_0&\coloneqq\max_{i,j\in\{1,2\}}\|D_j\left[F\right]_i\|_0\\ \|f\|_k&\coloneqq\max\left\{\inf_{p\in\mathbb{Z}}\|f_i-p\|_0\,,\|D_{\vec{a}}f_i\|_0: i=1,2,\vec{a}\text{ multiindex with }1\leq|\vec{a}|\leq k\right\}\end{aligned}$$

and

$$|||f|||_k\coloneqq\max\left\{\|f\|_k\,,\left\|f^{-1}\right\|_k\right\}.$$

Remark 2.5. Since for $h\in\text{Diff}^{\infty}\left(\mathbb{T}^2\right)$, every multiindex $\vec{a}$ with $|\vec{a}|\geq1$ and every $i\in\{1,2\}$ the derivative $D_{\vec{a}}h_i$ is $\mathbb{Z}^2$-periodic, it holds for any diffeomorphism g:

$$\sup_{z\in(0,1)^m}\left|\left(D_{\vec{a}}h_i\right)\left(g\left(z\right)\right)\right|\leq|||h|||_{|\vec{a}|}.$$

Analogously we can define the same expressions in the case of the annulus $\mathbb{A}=\mathbb{S}^1\times[0,1]$. In the case of the disc the $\text{Diff}^k\left(\mathbb{D}^2\right)$-topologies are defined in a natural way with the aid of the supremum norm on the disc.
Concerning the composition of functions the next result is useful:

Lemma 2.6. *Let M be $\mathbb{D}^2$, $\mathbb{A}$ or $\mathbb{T}^2$. Moreover, let $g,h\in\text{Diff}^{\infty}\left(M\right)$ and $k\in\mathbb{N}$. Then for the composition $g\circ h$ it holds*

$$\|g\circ h\|_k\leq(k+1)!\cdot\|g\|_k\cdot\|h\|_k^k\quad\text{ and }\quad|||g\circ h|||_k\leq(k+1)!\cdot|||g|||_k^k\cdot|||h|||_k^k.$$

Proof. By induction on $k\in\mathbb{N}$ we will prove the following observation:
Claim: *For any multiindex $\vec{a}\in\mathbb{N}_0^2$ with $|\vec{a}|=k$ and $i\in\{1,2\}$ the partial derivative $D_{\vec{a}}\left[g\circ h\right]_i$ consists of at most $(k+1)!$ summands, where each summand is the product of one derivative of g of order at most k and at most k derivatives of h of order at most k.*

- *Start:* $k=1$
  For $i_1,i\in\{1,2\}$ we compute:

  $$D_{x_{i_1}}\left[g\circ h\right]_i\left(x_1,x_2\right)=\sum_{j_1=1}^{2}\left(D_{x_{j_1}}\left[g\right]_i\right)\left(h\left(x_1,x_2\right)\right)\cdot D_{x_{i_1}}\left[h\right]_{j_1}\left(x_1,x_2\right).$$

  Hence, this derivative consists of $2!=2$ summands and each summand has the announced form.

- *Induction assumption:* The claim holds for $k\in\mathbb{N}$.

- *Induction step:* $k\to k+1$
  Let $i\in\{1,2\}$ and $\vec{b}\in\mathbb{N}_0^2$ be any multiindex of order $\left|\vec{b}\right|=k+1$. There are $j\in\{1,2\}$ and

a multiindex $\vec{a}$ of order $|\vec{a}| = k$ such that $D_{\vec{b}} = D_{x_j} D_{\vec{a}}$. By the induction assumption the partial derivative $D_{\vec{a}} [g \circ h]_i$ consists of at most $(k+1)!$ summands, at which the summand with the most factors is of the subsequent form:

$$D_{\vec{c}_1} [g]_i (h (x_1, x_2)) \cdot D_{\vec{c}_2} [h]_{i_2} (x_1, x_2) \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} (x_1, x_2) \,,$$

where each $\vec{c}_i$ is of order at most k. Using the product rule we compute how the derivative D_{x_j} acts on such a summand:

$$\left(\sum_{j_1=1}^{2} D_{x_{j_1}} D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} [h]_{j_1} D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} \right) +$$
$$D_{\vec{c}_1} [g]_i \circ h \cdot D_{x_j} D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{\vec{c}_{k+1}} [h]_{i_{k+1}} + ...+$$
$$D_{\vec{c}_1} [g]_i \circ h \cdot D_{\vec{c}_2} [h]_{i_2} \cdot ... \cdot D_{x_j} D_{\vec{c}_{k+1}} [h]_{i_{k+1}}$$

Thus, each summand is the product of one derivative of g of order at most $k+1$ and at most $k+1$ derivatives of h of order at most $k+1$. Moreover, we observe that $2+k$ summands arise out of one. So the number of summands can be estimated by $(k+2) \cdot (k+1)! = (k+2)!$ and the claim is verified.

Using this claim we obtain for $i \in \{1, 2\}$ and any multiindex $\vec{a} \in \mathbb{N}_0^2$ of order $|\vec{a}| = k$:

$$\|D_{\vec{a}} [g \circ h]_i\|_0 \le (k+1)! \cdot \|g\|_k \cdot \|h\|_k^k \,.$$

Applying the claim on $h^{-1} \circ g^{-1}$ yields:

$$\left\|D_{\vec{a}} \left[h^{-1} \circ g^{-1}\right]_i\right\|_0 \le (k+1)! \cdot \|g\|_k^k \cdot \|h\|_k \,.$$

We conclude

$$|||g \circ h|||_k \le (k+1)! \cdot |||g|||_k^k \cdot |||h|||_k^k.$$

□

2.2 Analytic topology

Real-analytic diffeomorphisms of $\mathbb{T}^2$ homotopic to the identity have a lift of type

$$F(\theta, r) = (\theta + f_1(\theta, r), r + f_2(\theta, r)) \,,$$

where the functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ are real-analytic and $\mathbb{Z}^2$-periodic for $i = 1, 2$. For these functions we introduce the subsequent definition:

Definition 2.7. For any $\rho > 0$ we consider the set of real-analytic $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$, that can be extended to a holomorphic function on $A^\rho := \left\{(\theta, r) \in \mathbb{C}^2 \ : \ |\mathrm{im}\theta| < \rho, \ |\mathrm{im} r| < \rho\right\}$.

1. For these functions let $\|f\|_\rho := \sup_{(\theta,r)\in A^\rho} |f(\theta, r)|$.
2. The set of these functions satisfying the condition $\|f\|_\rho < \infty$ is denoted by $C_\rho^\omega\left(\mathbb{T}^2\right)$.

Furthermore, we consider the space $\mathrm{Diff}_\rho^\omega\left(\mathbb{T}^2\right)$ of those diffeomorphisms homotopic to the identity, for whose lift we have $f_i \in C_\rho^\omega\left(\mathbb{T}^2\right)$ for $i = 1, 2$.

Definition 2.8. For $f, g \in \mathrm{Diff}^{\omega}_{\rho}\left(\mathbb{T}^2\right)$ we define

$$\|f\|_{\rho} = \max_{i=1,2} \|f_i\|_{\rho}$$

and the distance

$$d_{\rho}(f,g) = \max_{i=1,2} \left\{ \inf_{p\in\mathbb{Z}} \|f_i - g_i - p\|_{\rho} \right\}.$$

Remark 2.9. $\mathrm{Diff}^{\omega}_{\rho}\left(\mathbb{T}^2\right)$ is a Banach space (see [Sa] or [Ly] for a more extensive treatment of these spaces).

3 Outline of the proof

In our constructions we consider $\mathbb{D}^2$, $\mathbb{A}$ and $\mathbb{T}^2$ with the standard circle actions $\mathcal{R} = \{R_t\}_{t\in\mathbb{S}^1}$ comprising of the diffeomorphisms $R_t(\theta, r) = (\theta + t, r)$.
First of all, we will do the constructions on $\mathbb{T}^2$ and $\mathbb{A}$. Inductively, we will design a sequence of smooth measure-preserving diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ with $H_n = H_{n-1} \circ h_n$. The conjugation map h_n will be a composition $h_n = g_n \circ \phi_n$, where $g_n(\theta, r) = (\theta + [nq_n^{\sigma}] \cdot r, r)$ with some $0 < \sigma < 0.25$ and the conjugation map ϕ_n is constructed in section 5, and the sequence of rational numbers will be

$$\alpha_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \alpha_n - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}},$$

where $a_n \in \mathbb{Z}$, $1 \le a_n \le q_n$ is chosen in such a way that $\tilde{q}_{n+1} \cdot p_n \equiv a_n \mod q_n$. Therewith, we have $|\alpha_{n+1} - \alpha_n| \le \frac{1}{\tilde{q}_{n+1}}$ and $\tilde{q}_{n+1} \cdot \alpha_{n+1} = \frac{\tilde{q}_{n+1}\cdot p_n}{q_n} - \frac{a_n}{q_n} \equiv 0 \mod 1$, which implies $f_n^{\tilde{q}_{n+1}} = \mathrm{id}$. Hence, $(\tilde{q}_n)_{n\in\mathbb{N}}$ will be a uniform rigidity sequence of $f = \lim_{n\to\infty} f_n$ under some restrictions on the closeness between f_n and f (see subsection 6.3), which depend on the norms of the conjugation maps H_i and the distances $|\alpha_{i+1} - \alpha_i| \le \frac{1}{\tilde{q}_{i+1}}$ for every $i > n$. Thus, we have to estimate the norms $|||H_n|||_{n+1}$ carefully in subsection 6.1. At the end of subsection 6.2 this will yield a sufficient condition on the growth rate of the uniform rigidity sequence $(\tilde{q}_n)_{n\in\mathbb{N}}$ and in subsection 6.4 we prove that f is weak mixing using a criterion similar to that deduced in [FS] (see section 4).
Finally, we will transform these constructed diffeomorphisms into smooth diffeomorphisms on $\mathbb{D}^2$. Hereby, we will prove Theorem 1 in the case of $\mathbb{D}^2$ in section 7.

4 Criterion for weak mixing

In this section we will formulate a criterion for weak mixing on $M = \mathbb{T}^2$ or $M = \mathbb{A}$.

4.1 $(\gamma, \delta, \varepsilon)$-distribution of horizontal intervals

We recall the following definitions stated in [FS]:

Definition 4.1. Let $\hat{\eta}$ be a partial decomposition of $\mathbb{T}$ into intervals. On $M = \mathbb{T}^2$ or $M = \mathbb{A}$ we will consider a decomposition η consisting of intervals in $\hat{\eta}$ times some $r \in [0,1]$. Sets of this form will be called horizontal intervals and decompositions of this type standard partial decompositions. On the other hand, sets of the form $\{\theta\} \times J$, where J is an interval on the r-axis, are called vertical intervals.

Hereby, we can introduce the notion of $(\gamma, \delta, \varepsilon)$-distribution of a horizontal interval in the vertical direction:

Definition 4.2. A diffeomorphism $\Phi : M \to M$ $(\gamma, \delta, \varepsilon)$-distributes a horizontal interval I if the following conditions are satisfied

- $\pi_r (\Phi (I))$ is an interval J with $1 - \delta \leq \lambda (J) \leq 1$,
- $\Phi (I)$ is contained in a vertical strip $[c, c + \gamma] \times J$ for some $c \in \mathbb{S}^1$,
- for any interval $\tilde{J} \subseteq J$ we have

$$\left| \frac{\lambda\left(I \cap \Phi^{-1}\left(\mathbb{T} \times \tilde{J}\right)\right)}{\lambda (I)} - \frac{\lambda\left(\tilde{J}\right)}{\lambda (J)} \right| \leq \varepsilon \cdot \frac{\lambda\left(\tilde{J}\right)}{\lambda (J)}.$$

4.2 Statement of the criterion

The proof of the criterion is the same as in [FS], section 3. The only difference occurs in comparison to Lemma 3.5., which in our case will be stated in the subsequent way:

Lemma 4.3. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of standard partial decompositions of M into horizontal intervals of length less than q_n^{-1}. Moreover, let g_n be defined by $g_n (\theta, r) = (\theta + [nq_n^{\sigma}] \cdot r, r)$ with some $0 < \sigma < 0.25$ and let $(H_n)_{n\in\mathbb{N}}$ be a sequence of area-preserving diffeomorphisms such that for every $n \in \mathbb{N}$:*

$$\text{(C1)} \qquad \|DH_{n-1}\|_0 \leq q_n^{0.25}.$$

Consider the partitions $\nu_n := \{\Gamma_n = H_{n-1} (g_n (I_n)) \ : \ I_n \in \eta_n\}$.
Then $\eta_n \to \epsilon$ implies $\nu_n \to \epsilon$.

Proof. For every $\varepsilon > 0$ we can choose n large enough such that $\mu\left(\bigcup_{I\in\eta_n} I\right) > 1-\varepsilon$ and there is a collection of squares $\tilde{S}_n := \{S_{n,i}\}$ with side length between $q_n^{-0.6}$ and $q_n^{-0.9}$ with total measure of the union $S_n := \bigcup_i S_{n,i}$ greater than $1-\sqrt{\varepsilon}$. Then we have $\mu\left(\bigcup_{I\in\eta_n} I \cap S_n\right) \geq (1 - \sqrt{\varepsilon})\cdot \mu (S_n)$, because otherwise $\mu\left(S_n \setminus \bigcup_{I\in\eta_n} I\right) > \sqrt{\varepsilon} \cdot \mu (S_n) > \sqrt{\varepsilon} \cdot (1 - \sqrt{\varepsilon})$ and so $\mu\left(\mathbb{T}^2 \setminus \bigcup_{I\in\eta_n} I\right) > \sqrt{\varepsilon}-\varepsilon > \varepsilon$ in case of $\varepsilon < \frac{1}{4}$, which contradicts $\mu\left(\bigcup_{I\in\eta_n} I\right) > 1-\varepsilon$. Since the horizontal intervals $I \in \eta_n$ have length less than q_n^{-1}, we can approximate the squares in the above collection $\tilde{S}_n$ for n sufficiently large in such a way that $\mu\left(\bigcup_{I\in\eta_n, I\subset S_n} I\right) \geq (1 - 2\sqrt{\varepsilon}) \cdot \mu (S_n)$.

In the next step we consider the sets $C_{n,i} := H_{n-1} (g_n (S_{n,i}))$ with $S_{n,i} \in \tilde{S}_n$. For these sets $C_{n,i}$ we have:

$$\operatorname{diam} (C_{n,i}) \leq \|DH_{n-1}\|_0 \cdot \|Dg_n\|_0 \cdot \operatorname{diam} (S_{n,i}) \leq n \cdot \sqrt{2} \cdot q_n^{\sigma-0.35}$$

which goes to 0 as $n \to \infty$ because $\sigma < 0.25$. Therefore, any Borel set B can be approximated by a union of such sets $C_{n,i}$ with any prescribed accuracy if n is sufficiently large, i.e. for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for $n \geq N$ there is an index set J_n: $\mu\left(B\triangle\bigcup_{i\in J_n} C_{n,i}\right) < \varepsilon$. Now we choose the union of these elements $I \in \eta_n$ contained in the occurring cubes $S_{n,i}$ and obtain:

$$\begin{aligned}\mu\left(B\triangle\bigcup H_{n-1} \circ g_n (I)\right) &\leq \mu\left(B\triangle \bigcup_{i\in J_n} C_{n,i}\right) + \mu\left(S_n \setminus \bigcup_{I\in\eta_n, I\subset S_n} I\right)\\ &< \varepsilon + 2\sqrt{\varepsilon} \cdot \mu (S_n) < 3\sqrt{\varepsilon}.\end{aligned}$$

Thus, B gets well approximated by unions of elements of ν_n if n is chosen sufficiently large. □

Now the criterion for weak mixing can be stated in the following way (compare with [FS], Proposition 3.9.):

Proposition 4.4. *Let $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ be diffeomorphisms constructed as explained in the Outline with $\sigma < \frac{1}{4}$ and such that $\|DH_{n-1}\|_0 \leq q_n^{0.25}$ holds for all $n \in \mathbb{N}$.*
Suppose that the limit $f := \lim_{n\to\infty} f_n$ exists. If there exists a sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers satisfying $d_0\left(f_n^{m_n}, f^{m_n}\right) < \frac{1}{2^n}$ and a sequence $(\eta_n)_{n\in\mathbb{N}}$ of standard partial decompositions of M into horizontal intervals of length less than q_n^{-1} such that $\eta_n \to \epsilon$ and the diffeomorphism $\Phi_n := \phi_n \circ R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1}$ $\left(\frac{1}{nq_n^{\sigma}}, \frac{1}{n}, \frac{1}{n}\right)$-distributes every interval $I_n \in \eta_n$, then the limit diffeomorphism f is weak mixing.

Remark 4.5. In [FS] it is demanded $\|DH_{n-1}\|_0 < \ln(q_n)$ instead of requirement C1. We did this modification because the fulfilment of the original condition would lead to stricter requirements on the rigidity sequence: In particular, equation A3 would require an exponential growth rate.

5 Explicit constructions

Once again, we consider $M = \mathbb{T}^2$ or $M = \mathbb{A}$. In this section we will construct the conjugation map ϕ_n satisfying the subsequent Proposition.

Proposition 5.1. *There exists a smooth measure-preserving diffeomorphism $\phi_n : M \to M$ satisfying the following properties*

- $\phi_n \circ R_{\frac{1}{q_n}} = R_{\frac{1}{q_n}} \circ \phi_n$
- *On* $\left[\frac{1}{8nq_n}, \frac{1}{2q_n} - \frac{1}{8nq_n}\right] \times \left[\frac{1}{4n}, 1 - \frac{1}{4n}\right]$*:* $\phi_n(r,\theta) = \left(\frac{1}{2q_n} - \frac{1}{2q_n} \cdot \theta, 2q_n \cdot r\right)$.
- *On* $\left[\frac{1}{2q_n}, \frac{1}{q_n}\right] \times [0,1]$ *the map ϕ_n is equal to the identity.*
- *We have for every $s \in \mathbb{N}$, $s \geq 2$:*

$$|||\phi_n|||_s \leq 4^s \cdot ((s+1)!)^{s+1} \cdot \left(s! \cdot \exp\left(100n^2\right)\right)^{11\cdot s^5} \cdot q_n^s.$$

In order to prove this Proposition let $\delta > 0$. Our first aim is to construct a measure-preserving diffeomorphism on the square $\Delta := [-1,1]^2$ that coincides with the rotation by $\frac{\pi}{2}$ on $\Delta(5\delta) := [-1+5\delta, 1-5\delta]^2$ and with the identity in a neighbourhood of the boundary, namely $\Delta \setminus \Delta(\delta)$. As announced in the introduction, in comparison with [FS] this map has to play the role of φ_n which was constructed with the aid of "Moser's trick". Since we need precise norm estimates, we have to modify the construction.

5.1 Bump functions and map ψ_δ

We use the smooth map

$$j(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

First of all, we find norm estimates for this function j:

Lemma 5.2. *For every $s \in \mathbb{N}$:*

$$\|j\|_s := \max_{t=0,1,\ldots,s} \max_{x\in[0,1]} \left|j^{(t)}(x)\right| \leq 3^{2s} \cdot s^{1.5s} \cdot (s-1)!.$$

Proof. We consider $\tilde{j}(x) = \exp\left(-\frac{1}{x^2}\right)$ with derivative $\tilde{j}^{(1)}(x) = \frac{2}{x^3}\exp\left(-\frac{1}{x^2}\right)$. Differentiating yields $j^{(2)}(x) = \exp\left(-\frac{1}{x^2}\right)\cdot\frac{4}{x^6} - \exp\left(-\frac{1}{x^2}\right)\cdot\frac{6}{x^4}$. Continuing in this way we observe for $j^{(s)}$ that the number a_s of summands is $a_s = 2\cdot a_{s-1}$ (with $a_1 = 1$) and the exponent in the denominator is at most $3s$. Hence, for $x \in [0,1]$ an upper bound of $\left|j^{(s)}(x)\right|$ is given by

$$a_s \cdot \exp\left(-\frac{1}{x^2}\right)\cdot\frac{1}{x^{3s}}\cdot 2\cdot\prod_{i=1}^{s-1}(3i) = \exp\left(-\frac{1}{x^2}\right)\cdot\frac{1}{x^{3s}}\cdot 2^s\cdot 3^{s-1}\cdot(s-1)!$$

For $x = \sqrt{\frac{2}{3s}}$ this expression has a maximum on $[0,1]$ and takes the value

$$2^{-0.5s}\cdot 3^{2.5s-1}\cdot s^{1.5s}\cdot(s-1)!\cdot\exp(-1.5s) \leq \frac{3^{0.5s}}{2^{0.5s}\cdot\exp(1.5s)}\cdot 3^{2s}\cdot s^{1.5s}\cdot(s-1)!.$$

□

Using the map j we define the bump function

$$k_{a,b}(x) = \frac{j(b-x)}{j(x-a)+j(b-x)},$$

where $a,b \in (0,1)$. We examine this bump function $k_{a,b}$:

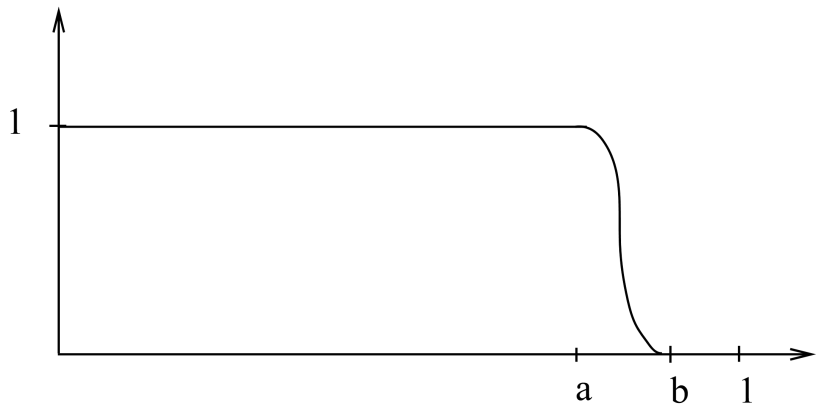

Figure 1: Qualitative shape of the bump function $k_{a,b}$

Lemma 5.3. *For every $s \in \mathbb{N}$:*

$$\|k_{a,b}\|_s \leq 2^{s-1}\cdot 3^{2s^2+2s}\cdot s^{1.5s^2+1.5s}\cdot s!^{s+2}\cdot\exp\left(\left(\frac{2}{b-a}\right)^2\cdot(s+1)\right).$$

Proof. At first, we consider the denominator $l(x) := j\,(x-a) + j\,(b-x)$: On $[0,1]$ it is minimal for $x = \frac{a+b}{2}$ and takes the value $l\left(\frac{a+b}{2}\right) = 2 \cdot \exp\left(-\frac{4}{(b-a)^2}\right)$.
In order to examine the derivatives of $k_{a,b}$ we use the quotient rule. Hereby, we observe that the denominator of $k^{(s)}_{a,b}$ is $(l(x))^{s+1}$. Moreover, each summand of the numerator is a product of a derivative of $j\,(b-x)$ of order at most s and s derivatives of l of order at most s. Hence, the derivative of this numerator consists of $(s+1)\cdot a_s$ summands, at which a_s is the number of summands of the numerator of $k^{(s)}_{a,b}$. Then, the number a_{s+1} of summands of the numerator of $k^{(s+1)}_{a,b}$ is at most $2\cdot(s+1)\cdot a_s$. Since $a_0 = 1$ we obtain $a_s = 2^s \cdot s!$. We conclude:

$$\begin{aligned}\left|k^{(s)}_{a,b}\,(x)\right| \leq & \frac{\|j\|_s \cdot \|l\|^s_s}{\left(\min_{x\in[0,1]} l(x)\right)^{s+1}} \cdot a_s \leq \frac{\|j\|_s \cdot \left(2\cdot\|j\|_s\right)^s}{\left(\min_{x\in[0,1]} l(x)\right)^{s+1}} \cdot a_s \\ \leq & 3^{2s}\cdot s^{1.5s}\cdot(s-1)!\cdot 2^s \cdot 3^{2s^2}\cdot s^{1.5s^2}\cdot(s-1)!^s \cdot \frac{1}{2^{s+1}} \exp\left(\frac{4(s+1)}{(b-a)^2}\right)\cdot 2^s\cdot s!\end{aligned}$$

□

In our constructions we use $a = 1-3\delta$ and $b = 1-2\delta$. We denote the corresponding map by k_δ. Hereby, we define the smooth diffeomorphism

$$\psi_\delta\left(\tilde{\theta},\tilde{r}\right) = \left(\tilde{\theta} + \frac{\pi}{2}\cdot k_\delta\,(\tilde{r})\,,\tilde{r}\right)$$

on $\mathbb{R}^2$ with symplectic polar coordinates $\tilde{\theta} \in \mathbb{R}/2\pi\mathbb{Z}$, $\tilde{r} \in \mathbb{R}^+$. This map coincides with the rotation by $\frac{\pi}{2}$ on $B(1-3\delta)$ and with the identity on $\mathbb{R}^2 \setminus B(1-2\delta)$. As a direct consequence of the previous results we conclude:

Lemma 5.4. *For every $s \in \mathbb{N}$:*

$$|||\psi_\delta|||_s \leq \pi \cdot 2^{s-1}\cdot 3^{2s^2+2s}\cdot s^{1.5s^2+1.5s}\cdot s!^{s+2}\cdot\exp\left(\frac{4}{\delta^2}\cdot(s+1)\right).$$

5.2 Maps κ_δ and φ_δ

In the construction of our conjugation map φ_δ there is an angle-dependent dilation. In order to make this angle-dependence smooth we use the bump functions. We define the smooth map κ_δ:

- On $\left[0,\frac{\pi}{2}\right]$:

$$\kappa_\delta\,(\theta) = k_{\frac{\pi}{4}-\frac{\delta}{2},\frac{\pi}{4}+\frac{\delta}{2}}\,(\theta)\cdot\frac{1}{\left(\cos\,(\theta)\right)^2} + \left(1 - k_{\frac{\pi}{4}-\frac{\delta}{2},\frac{\pi}{4}+\frac{\delta}{2}}\,(\theta)\right)\cdot\frac{1}{\left(\sin\,(\theta)\right)^2}$$

- On $\left[\frac{\pi}{2},\pi\right]$:

$$\kappa_\delta\,(\theta) = k_{\frac{3\pi}{4}-\frac{\delta}{2},\frac{3\pi}{4}+\frac{\delta}{2}}\,(\theta)\cdot\frac{1}{\left(\sin\,(\theta)\right)^2} + \left(1 - k_{\frac{3\pi}{4}-\frac{\delta}{2},\frac{3\pi}{4}+\frac{\delta}{2}}\,(\theta)\right)\cdot\frac{1}{\left(\cos\,(\theta)\right)^2}$$

- On $\left[\pi,\frac{3\cdot\pi}{2}\right]$:

$$\kappa_\delta\,(\theta) = k_{\frac{5\pi}{4}-\frac{\delta}{2},\frac{5\pi}{4}+\frac{\delta}{2}}\,(\theta)\cdot\frac{1}{\left(\cos\,(\theta)\right)^2} + \left(1 - k_{\frac{5\pi}{4}-\frac{\delta}{2},\frac{5\pi}{4}+\frac{\delta}{2}}\,(\theta)\right)\cdot\frac{1}{\left(\sin\,(\theta)\right)^2}$$

- On $\left[\frac{3\cdot\pi}{2}, 2\pi\right]$:

$$\kappa_\delta(\theta) = k_{\frac{7\pi}{4}-\frac{\delta}{2}, \frac{7\pi}{4}+\frac{\delta}{2}}(\theta) \cdot \frac{1}{(\sin(\theta))^2} + \left(1 - k_{\frac{7\pi}{4}-\frac{\delta}{2}, \frac{7\pi}{4}+\frac{\delta}{2}}(\theta)\right) \cdot \frac{1}{(\cos(\theta))^2}$$

Remark 5.5. We note: $\kappa_\delta\left(\theta + \frac{\pi}{2}\right) = \kappa_\delta(\theta)$.

Lemma 5.6. *For every $s \in \mathbb{N}$:*

$$\|\kappa_\delta\|_s \leq 2^{4s+2} \cdot 3^{2s^2+2s} \cdot s!^{s+3} \cdot s^{1.5s^2+1.5s} \cdot \exp\left(\frac{4}{\delta^2} \cdot (s+1)\right).$$

Proof. In a first step, we examine the map $v(x) := \frac{1}{(\cos(x))^2}$ with derivative $v^{(1)}(x) = \frac{2\cdot\sin(x)}{(\cos(x))^3}$. The next derivatives are computed with the aid of the quotient rule. The denominator of $v^{(s)}(x)$ is $(\cos(x))^{s+2}$ with degree $n_s = s+2$ and the numerator consists of z_s summands, where each summand is a product of p_s factors $\sin(x)$ and $\cos(x)$ respectively. Then, the quotient rule yields $p_{s+1} = p_s + 1$ and $z_{s+1} = (n_s + p_s) \cdot z_s$. Since $z_1 = 2$ and $p_1 = 1$ we obtain $p_s = s$ and

$$z_{s+1} = z_s \cdot (2s+2) = z_{s-1} \cdot 2 \cdot s \cdot 2 \cdot (s+1) = 2^{s+1} \cdot (s+1)!.$$

Hereby, we conclude for $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, i.e. $\cos(x) \geq 0.5$:

$$\left|v^{(s)}(x)\right| \leq \frac{z_s}{(\cos(x))^{s+2}} \leq 2^s \cdot s! \cdot 2^{s+2} = 2^{2s+2} \cdot s!.$$

By the same arguments we obtain the same bound on the derivatives of $\frac{1}{(\sin(x))^2}$ for $x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$. An estimate deduced from the product rule yields

$$\begin{aligned}\|\kappa_\delta\|_s &\leq 2^s \cdot \|k_\delta\|_s \cdot \|v\|_{s,\left[-\frac{\pi}{3},\frac{\pi}{3}\right]} + 2^s \cdot \|1-k_\delta\|_s \cdot \left\|\frac{1}{(\sin(x))^2}\right\|_{s,\left[\frac{\pi}{6},\frac{5\pi}{6}\right]} \\ &\leq 2^{s+1} \cdot \|k_\delta\|_s \cdot \|v\|_{s,\left[-\frac{\pi}{3},\frac{\pi}{3}\right]}.\end{aligned}$$

The same holds true on the other particular domains. Then we find the claimed estimate

$$\begin{aligned}\|\kappa_\delta\|_s &\leq 2^{s+1} \cdot 2^{s-1} \cdot 3^{2s^2+2s} \cdot s^{1.5s^2+1.5s} \cdot s!^{s+2} \cdot \exp\left(\left(\frac{2}{\delta}\right)^2 (s+1)\right) \cdot 2^{2s+2} \cdot s! \\ &\leq 2^{4s+2} \cdot 3^{2s^2+2s} \cdot s^{1.5s^2+1.5s} \cdot s!^{s+3} \cdot \exp\left(\frac{4\cdot(s+1)}{\delta^2}\right).\end{aligned}$$

□

Once again, we consider $\mathbb{R}^2$ equipped with symplectic polar coordinates $\left(\tilde{\theta}, \tilde{r}\right)$. For $r_1, r_2 \in (0,1)$ we define the map

$$\varphi_{r_1,r_2,\delta}\left(\tilde{\theta}, \tilde{r}\right) = \left(\tilde{\theta}, \kappa_\delta\left(\tilde{\theta}\right) \cdot r_1^2 + \tilde{r} - r_1\right) \text{ on } B(r_1, r_2),$$

where $B(r_1, r_2) = \left\{\left(\tilde{\theta}, \tilde{r}\right) : \tilde{\theta} \in \mathbb{R}/2\pi\mathbb{Z}, \tilde{r} \in [r_1, r_2]\right\}$. In our constructions we use $r_1 = 1 - 4\delta$ and $r_2 = 1 - \delta$. The corresponding map is called φ_δ.

5.3 The map ϕ_δ

With the aid of the maps introduced in the previous subsections we construct the smooth diffeomorphism $\tilde{\phi}_\delta$ on $\mathbb{R}^2$ equipped with symplectic polar coordinates $\left(\tilde{\theta}, \tilde{r}\right)$:

$$\tilde{\phi}_\delta\left(\tilde{\theta}, \tilde{r}\right) = \begin{cases} \left(\tilde{\theta} + \frac{\pi}{2}, \tilde{r}\right) & \textit{inside of } \varphi_\delta\left(\mathbb{S}^1 \times \{r_1\}\right) \\ \varphi_\delta \circ \psi_\delta \circ \varphi_\delta^{-1}\left(\tilde{\theta}, \tilde{r}\right) & \textit{on } \varphi_\delta\left(B\left(r_1, r_2\right)\right) \\ \left(\tilde{\theta}, \tilde{r}\right) & \textit{outside of } \varphi_\delta\left(\mathbb{S}^1 \times \{r_2\}\right) \end{cases}$$

Recall that the domain $\varphi_\delta\left(B\left(r_1, r_2\right)\right)$ is invariant under the rotation about arc $\frac{\pi}{2}$ due to Remark 5.5.
For $(\theta, \bar{r}) = \varphi_\delta\left(\theta, r_1\right)$ we have

$$\tilde{\phi}_\delta\left(\theta, \bar{r}\right) = \varphi_\delta \circ \psi_\delta\left(\theta, r_1\right) = \varphi_\delta\left(\theta + \frac{\pi}{2} \cdot k_\delta\left(r_1\right), r_1\right) = \left(\theta + \frac{\pi}{2}, \bar{r}\right)$$

and for $(\theta, \bar{r}) = \varphi_\delta\left(\theta, r_2\right)$ we have

$$\tilde{\phi}_\delta\left(\theta, \bar{r}\right) = \varphi_\delta \circ \psi_\delta\left(\theta, r_2\right) = \varphi_\delta\left(\theta + \frac{\pi}{2} \cdot k_\delta\left(r_2\right), r_2\right) = \left(\theta, \bar{r}\right).$$

Since $r_1 < a < b < r_2$ these equalities hold true on a neighbourhood of the points. Thus, $\tilde{\phi}_\delta$ is a smooth diffeomorphism. Furthermore, $\tilde{\phi}_\delta$ is measure-preserving because the maps φ_δ and ψ_δ are.

Lemma 5.7. *For every $s \in \mathbb{N}$:*

$$\begin{aligned} |||\tilde{\phi}_\delta|||_s \leq & \pi^s \cdot 2^{4s^3+3s^2+3s+3} \cdot 3^{2s^4+4s^3+4s^2+2s} \cdot s!^{s^3+4s^2+4s+4} \cdot s^{1.5s^4+3s^3+3s^2+1.5s} \\ & \cdot \exp\left(\frac{4}{\delta^2} \cdot \left(s^3 + 2s^2 + 2s + 1\right)\right) \end{aligned}$$

Proof. On $\varphi_\delta\left(B\left(r_1, r_2\right)\right)$ we have

$$\begin{aligned} \tilde{\phi}_\delta\left(\tilde{\theta}, \tilde{r}\right) = \Bigg(& \tilde{\theta} + \frac{\pi}{2} \cdot k_\delta\left(r_1 + \tilde{r} - \kappa_\delta\left(\tilde{\theta}\right) \cdot r_1^2\right), \\ & \tilde{r} + \kappa_\delta\left(\tilde{\theta} + \frac{\pi}{2} \cdot k_\delta\left(r_1 + \tilde{r} - \kappa_\delta\left(\tilde{\theta}\right) \cdot r_1^2\right)\right) \cdot r_1^2 - \kappa_\delta\left(\tilde{\theta}\right) \cdot r_1^2 \Bigg) \end{aligned}$$

In a first step we consider $K_1\left(\tilde{\theta}, \tilde{r}\right) := k_\delta\left(r_1 + \tilde{r} - \kappa_\delta\left(\tilde{\theta}\right) \cdot r_1^2\right)$. With the aid of the chain rule we compute

$$\left\|K_1\right\|_s \leq s! \cdot \left\|k_\delta\right\|_s \cdot \left\|\kappa_\delta\right\|_s^s.$$

Again using the chain rule we obtain for $K_2\left(\tilde{\theta}, \tilde{r}\right) := \kappa_\delta\left(\tilde{\theta} + \frac{\pi}{2} \cdot K_1\left(\tilde{\theta}, \tilde{r}\right)\right)$:

$$\left\|K_2\right\|_s \leq s! \cdot \left\|\kappa_\delta\right\|_s \cdot \left(2 \cdot \frac{\pi}{2} \cdot \left\|K_1\right\|_s\right)^s.$$

By the previous norm estimates we conclude

$$\begin{aligned}\left\|\tilde{\phi}_\delta\right\|_s \leq & 2\cdot\left\|K_2\right\|_s \leq 2\cdot s!\cdot\left\|\kappa_\delta\right\|_s\cdot\pi^s\cdot\left\|K_1\right\|_s^s \\ \leq & 2\cdot s!\cdot\left\|\kappa_\delta\right\|_s\cdot\pi^s\cdot\left(s!\cdot\left\|k_\delta\right\|_s\cdot\left\|\kappa_\delta\right\|_s^s\right)^s \\ \leq & 2\cdot s!^{s+1}\cdot\pi^s\cdot\left\|\kappa_\delta\right\|_s^{s^2+1}\cdot\left\|k_\delta\right\|_s^s \\ \leq & \pi^s\cdot 2^{4s^3+3s^2+3s+3}\cdot 3^{2s^4+4s^3+4s^2+2s}\cdot s!^{s^3+4s^2+4s+4}\cdot s^{1.5s^4+3s^3+3s^2+1.5s} \\ & \cdot\exp\left(\frac{4}{\delta^2}\cdot\left(s^3+2s^2+2s+1\right)\right)\end{aligned}$$

Since $\tilde{\phi}_\delta^{-1}$ is of the same form, we obtain the claim. □

The coordinate change from symplectic polar coordinates to cartesian coordinates is given by:

$$P\left(\tilde{\theta},\tilde{r}\right)=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\sqrt{\tilde{r}}\cdot\cos\left(\tilde{\theta}\right)\\\sqrt{\tilde{r}}\cdot\sin\left(\tilde{\theta}\right)\end{pmatrix}$$

A direct computation yields $|\det\left(JP\right)|=\frac{1}{2}$ except at the origin. Hereby, we consider the area-preserving map $\phi_\delta := P\circ\tilde{\phi}_\delta\circ P^{-1}$. By our choice of r_1 the map ϕ_δ is the rotation about the angle $\frac{\pi}{2}$ on $\left[-1+5\delta,1-5\delta\right]^2$. Moreover, it coincides with the identity outside of $\left[-1+\delta,1-\delta\right]^2$. In order to obtain norm estimates of ϕ_δ we examine the coordinate change P on $B\left(r_1,r_2\right)$:

Lemma 5.8. *For every $s\in\mathbb{N}$:*

$$\|P\|_{s,B(r_1,r_2)}\leq 2^{s-1.5}\cdot(s-1)!.$$

Proof. The norm $\|P\|_{s,B(r_1,r_2)}$ is determined by the derivatives of $p(r)=\sqrt{r}$. Direct computation shows

$$p^{(s)}(r)=r^{-\frac{2s-1}{2}}\cdot(-1)^s\cdot\prod_{i=0}^{s-1}\frac{2i-1}{2}=r^{-\frac{2s-1}{2}}\cdot(-1)^{s+1}\cdot\frac{1}{2^s}\cdot\prod_{i=1}^{s-1}(2i-1).$$

Since $r_1\geq 0.5$ we obtain

$$\|P\|_{s,B(r_1,r_2)}\leq r_1^{-\frac{2s-1}{2}}\cdot\frac{1}{2^s}\cdot 2^{s-1}\cdot(s-1)!\leq 2^{s-1.5}\cdot(s-1)!.$$

□

For the inverse $P^{-1}|_{P(B(r_1,r_2))}$ we deduce the subsequent estimate:

Lemma 5.9. *For every $s\in\mathbb{N}$:*

$$\left\|P^{-1}\right\|_{s,P(B(r_1,r_2))}\leq 2^{3s-2}\cdot(s-1)!.$$

Proof. The inverse coordinate transformation is given by

$$P^{-1}\left(x,y\right)=\begin{pmatrix}\tilde{\theta}\\\tilde{r}\end{pmatrix}=\begin{pmatrix}\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\\x^2+y^2\end{pmatrix}\text{ for } y\geq 0$$

$$P^{-1}\left(x,y\right)=\begin{pmatrix}\tilde{\theta}\\\tilde{r}\end{pmatrix}=\begin{pmatrix}-\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\\x^2+y^2\end{pmatrix}\text{ for } y< 0$$

The norm estimate is determined by the derivatives of $\pm \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)$ with respect to x. The first derivative is $q^{(1)}(x) = -\frac{y}{x^2+y^2}$. The further derivatives are found with the aid of the quotient rule: The denominator of $q^{(s)}(x)$ is $\left(x^2+y^2\right)^{n_s}$ with exponent n_s and the numerator consists of z_s summands with p_s factors x or y. Then we have $n_{s+1} = n_s + 1$, $p_{s+1} = p_s + 1$ and $z_{s+1} = 2 \cdot z_s \cdot (p_s + n_s)$. Since $n_1 = 1$, $p_1 = 1$ and $z_1 = 1$ we conclude $n_{s+1} = s + 1$, $p_{s+1} = s + 1$ as well as

$$z_{s+1} = 2 \cdot z_s \cdot 2 \cdot s = 4s \cdot z_s = 4s \cdot 4 \cdot (s-1) \cdot z_{s-1} = 4^s \cdot s!,$$

i.e. $z_s = 4^{s-1} \cdot (s-1)! = 2^{2s-2} \cdot (s-1)!$. Due to $x^2 + y^2 \geq r_1 \geq 0.5$ we obtain

$$\left\|P^{-1}\right\|_{s,P(B(r_1,r_2))} \leq \frac{z_s}{\left(x^2+y^2\right)^{n_s}} \leq \frac{2^{2s-2} \cdot (s-1)!}{r_1^s} \leq 2^{3s-2} \cdot (s-1)!.$$

□

Let $s \geq 2$. Lemma 2.6 yields for $\bar{\phi} := \tilde{\phi}_\delta \circ P^{-1}$:

$$\left\|\bar{\phi}\right\|_s \leq (s+1)! \cdot \left\|\tilde{\phi}_\delta\right\|_s \cdot \left\|P^{-1}\right\|^s_{s,P(B(r_1,r_2))}.$$

Again using Lemma 2.6 we obtain

$$\begin{aligned}
\|\phi_\delta\|_s &\leq (s+1)! \cdot \|P\|_{s,B(r_1,r_2)} \cdot \left\|\bar{\phi}\right\|^s_s \\
&\leq ((s+1)!)^{s+1} \cdot \|P\|_{s,B(r_1,r_2)} \cdot \left\|\tilde{\phi}_\delta\right\|^s_s \cdot \left\|P^{-1}\right\|^{s^2}_{s,P(B(r_1,r_2))} \\
&\leq ((s+1)!)^{s+1} \cdot \pi^{s^2} \cdot 2^{4s^4+6s^3+s^2+4s-1.5} \cdot 3^{2s^5+4s^4+4s^3+2s^2} \cdot s!^{s^4+4s^3+4s^2+4s} \\
&\quad \cdot s^{1.5s^5+3s^4+3s^3+1.5s^2} \cdot \exp\left(\frac{4}{\delta^2} \cdot \left(s^4+2s^3+2s^2+s\right)\right) \cdot (s-1)!^{s^2+1} \\
&\leq ((s+1)!)^{s+1} \cdot \pi^{s^2} \cdot 2^{4s^4+6s^3+s^2+4s-1.5} \cdot 9^{s^5+2s^4+2s^3+s^2} \cdot \\
&\quad s!^{s^4+4s^3+5s^2+4s+1} \cdot s^{1.5s^5+3s^4+3s^3+1.5s^2} \cdot \exp\left(\frac{4 \cdot \left(s^4+2s^3+2s^2+s\right)}{\delta^2}\right)
\end{aligned}$$

Since $2 \leq s!$, $s \leq s!$ and $\pi \leq 9 \leq \exp\left(\frac{1}{\delta^2}\right)$ we continue in the following mater:

$$\begin{aligned}
&\|\phi_\delta\|_s \\
\leq &(s+1)!^{s+1} \cdot s!^{1.5s^5+8s^4+13s^3+7.5s^2+8s-0.5} \cdot \exp\left(\frac{s^5+6s^4+10s^3+10s^2+4s}{\delta^2}\right)
\end{aligned}$$

Due to $s \geq 2$ we have $1.5s^5 + 8s^4 + 13s^3 + 7.5s^2 + 8s - 0.5 \leq 11s^5$ as well as $s^5 + 6s^4 + 10s^3 + 10s^2 + 4s \leq 8s^5$. Thus, we proved the following statement:

Lemma 5.10. *For every $s \in \mathbb{N}$, $s \geq 2$:*

$$|||\phi_\delta|||_s \leq ((s+1)!)^{s+1} \cdot \left(s! \cdot \exp\left(\frac{1}{\delta^2}\right)\right)^{11 \cdot s^5}.$$

5.4 Conjugation map ϕ_n

In the first instance, we construct the conjugation map ϕ_n on the fundamental sector $\left[0, \frac{1}{q_n}\right] \times [0,1]$ by the same approach as in [FS], section 5.2.2. On $D_n^1 \coloneqq \left[0, \frac{1}{2q_n}\right] \times [0,1]$ we use the affine transformation $C_n(\theta, r) = (4q_n \cdot \theta - 1, 2r - 1)$ sending D_n^1 onto Δ. Hereby, we set

$$\phi_n = C_n^{-1} \circ \phi_{\frac{1}{10n}} \circ C_n$$

On $D_n^2 \coloneqq \left[\frac{1}{2q_n}, \frac{1}{q_n}\right] \times [0,1]$ we define $\phi_n = \mathrm{id}$. We observe that ϕ_n is smooth, area-preserving and coincides with the the identity in a neighbourhood of the boundary of the fundamental sector. Hence, we can extend it equivariantly by the formula $\phi_n\left(\theta + \frac{k}{q_n}, r + l\right) = \left(\frac{k}{q_n}, l\right) + \phi_n(\theta, r)$ for every $k, l \in \mathbb{Z}$. Then ϕ_n becomes a diffeomorphism on $\mathbb{T}^2$ as well as $\mathbb{A}$.

6 The smooth case on $\mathbb{T}^2$ and $\mathbb{A}$

6.1 Properties of the conjugation maps h_n and H_n

We aim for precise requirements on the growth rate of the uniform rigidity sequence to guarantee convergence of the sequence of diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$. For this purpose, we need norm estimates on the conjugation maps.

Lemma 6.1. *We have for every $s \in \mathbb{N}$, $s \geq 2$:*

$$|||h_n|||_s \leq 2 \cdot 4^s \cdot ((s+1)!)^{s+1} \cdot \left(s! \cdot \exp\left(100n^2\right)\right)^{11 \cdot s^5} \cdot n^s \cdot q_n^{s \cdot (1+\sigma)}.$$

Proof. Obviously, we have for $\phi_n = C_n^{-1} \circ \phi_{\frac{1}{10n}} \circ C_n$:

$$|||\phi_n|||_s \leq (4q_n)^s \cdot |||\phi_{\frac{1}{10n}}|||_s.$$

Using the explicit definitions of the maps g_n and $h_n = g_n \circ \phi_n$ we can compute

$$h_n(\theta, r) = \left([\phi_n]_1 + [nq_n^\sigma] \cdot [\phi_n]_2, [\phi_n]_2\right)$$

as well as

$$h_n^{-1}(\theta, r) = \left([\phi_n]_1 - [nq_n^\sigma] \cdot [\phi_n]_2, [\phi_n]_2\right).$$

Then we obtain with the aid of Lemma 5.10

$$|||h_n|||_s \leq 2 \cdot [nq_n^\sigma]^s \cdot |||\phi_n|||_s \leq 2 \cdot 4^s n^s \cdot (s+1)!^{s+1} \cdot \left(s! \cdot \exp\left(100n^2\right)\right)^{11s^5} \cdot q_n^{s \cdot (1+\sigma)}.$$

□

In the next step we want to deduce norm estimates for the conjugation map $H_n = H_{n-1} \circ h_n$ under some assumptions on the growth rate of the numbers q_n:

Lemma 6.2. *Let $k, n \in \mathbb{N}$ and $k \geq 2$. Assume*

(A2) $$q_{n+1} \geq n \cdot q_n^{2+\sigma}.$$

Then we have

$$|||H_n|||_k \leq 4^{n \cdot (k+1)^n} \cdot ((k+1)!)^{n \cdot (k+1)^n} \cdot \left(k! \cdot \exp\left(100n^2\right)\right)^{11n \cdot k^{n+4}} \cdot \left(n \cdot q_n^{2+\sigma}\right)^{k^n}.$$

Proof. Let $k \in \mathbb{N}$, $k \geq 2$. We proof the claim by induction on n:
Start $n = 1$: Lemma 6.1 yields the statement for $H_1 = h_1$.
Induction assumption: The claim holds true for $n \in \mathbb{N}$.
Induction step $n \rightarrow n+1$ *:* We apply Lemma 2.6, Lemma 6.1 and the induction assumption on the composition $H_{n+1} = H_n \circ h_{n+1}$:

$$\begin{aligned}
&|||H_{n+1}|||_k \leq (k+1)! \cdot |||H_n|||_k^k \cdot |||h_{n+1}|||_k^k \\
\leq &(k+1)! \cdot 4^{n\cdot(k+1)^{n+1}} \cdot ((k+1)!)^{n\cdot(k+1)^n \cdot k} \cdot \left(k! \cdot \exp\left(100n^2\right)\right)^{11n\cdot k^{n+5}} \cdot q_{n+1}^{k^{n+1}} \\
&\cdot 4^{(k+1)\cdot k} \cdot ((k+1)!)^{(k+1)\cdot k} \cdot \left(k! \cdot \exp\left(100(n+1)^2\right)\right)^{11\cdot k^6} \cdot \left(n \cdot q_{n+1}^{1+\sigma}\right)^{k^2} \\
\leq &4^{(n+1)(k+1)^{n+1}} (k+1)!^{(n+1)(k+1)^{n+1}} \left(k! \exp\left(100(n+1)^2\right)\right)^{11(n+1)k^{n+5}} \left(nq_{n+1}^{2+\sigma}\right)^{k^{n+1}}
\end{aligned}$$

□

Remark 6.3. As a special case of Lemma 2.6 we observe that $\|DH_n\|_0 \leq 2! \cdot \|DH_{n-1}\|_0 \cdot \|Dh_n\|_0$. With the aid of Lemma 6.1 we can estimate:

$$\|DH_n\|_0 \leq 2! \cdot q_n^{0.25} \cdot 32 \cdot \exp\left(100n^2\right)^{11} \cdot n \cdot q_n^{1+\sigma},$$

where we used condition C1, i.e. $\|DH_{n-1}\|_0 \leq q_n^{0.25}$. In order to guarantee this property for DH_n we demand:

$$\text{(A3)} \qquad q_{n+1} \geq \|DH_n\|_0^4 \geq 64^4 \cdot \exp\left(100n^2\right)^{44} \cdot n^4 \cdot q_n^{5+4\sigma}.$$

6.2 Proof of Convergence

In the proof of convergence the following result, which is more precise than [FS], Lemma 5.6., is useful:

Lemma 6.4. *Let $k \in \mathbb{N}_0$ and $h \in \text{Diff}^\infty(M)$. Then for all $\alpha, \beta \in \mathbb{R}$ we obtain:*

$$d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) \leq C_k \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|,$$

where $C_k = (k+1)!$.

Proof. As an application of the claim in the proof of Lemma 2.6 we observe
Fact: *For any $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| = k$ and $i \in \{1,2\}$ the partial derivative $D_{\vec{a}}\left[h \circ R_\alpha \circ h^{-1}\right]_i$ consists of at most $(k+1)!$ summands, where each summand is the product of one derivative of h of order at most k and at most k derivatives of h^{-1} of order at most k.*
Furthermore, with the aid of the mean value theorem we can estimate for any multiindex $\vec{a} \in \mathbb{N}_0^2$ with $|\vec{a}| \leq k$ and $i \in \{1,2\}$:

$$\left|D_{\vec{a}}[h]_i\left(R_\alpha \circ h^{-1}(x_1,x_2)\right) - D_{\vec{a}}[h]_i\left(R_\beta \circ h^{-1}(x_1,x_2)\right)\right| \leq |||h|||_{k+1} \cdot |\alpha - \beta|$$

Since $\left(h_n \circ R_\alpha \circ h_n^{-1}\right)^{-1} = h_n \circ R_{-\alpha} \circ h_n^{-1}$ is of the same form, we obtain in conclusion:

$$\begin{aligned}
d_k\left(h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1}\right) &\leq (k+1)! \cdot |||h|||_{k+1} \cdot |||h|||_k^k \cdot |\alpha - \beta| \\
&\leq (k+1)! \cdot |||h|||_{k+1}^{k+1} \cdot |\alpha - \beta|.
\end{aligned}$$

□

Under some conditions on the proximity of α_n and α_{n+1} we can prove convergence:

Lemma 6.5. *We assume*

$$|\alpha_{n+1}-\alpha_n| \leq \frac{1}{2^n \cdot (n+1)! \cdot q_n \cdot |||H_n|||_{n+1}^{n+1}}. \tag{A1}$$

Then the diffeomorphisms $f_n = H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}$ *satisfy:*

- *The sequence* $(f_n)_{n\in\mathbb{N}}$ *converges in the* Diff$^\infty(M)$*-topology to a measure-preserving diffeomorphism* f.
- *We have for every* $n \in \mathbb{N}$ *and* $m \leq q_{n+1}$*:*

$$d_0\left(f^m, f_n^m\right) < \frac{1}{2^n}.$$

Proof. 1. According to our construction it holds $h_n \circ R_{\alpha_n} = R_{\alpha_n} \circ h_n$ and hence we can apply Lemma 6.4 for every $k, n \in \mathbb{N}$:

$$\begin{aligned} d_k\left(f_n, f_{n-1}\right) &= d_k\left(H_n \circ R_{\alpha_{n+1}} \circ H_n^{-1}, H_n \circ R_{\alpha_n} \circ H_n^{-1}\right) \\ &\leq C_k \cdot |||H_n|||_{k+1}^{k+1} \cdot |\alpha_{n+1}-\alpha_n|. \end{aligned}$$

By the assumptions of this Lemma it follows for every $k \leq n$:

$$d_k\left(f_n, f_{n-1}\right) \leq d_n\left(f_n, f_{n-1}\right) \leq C_n \cdot |||H_n|||_{n+1}^{n+1} \cdot \frac{1}{2^n C_n q_n \cdot |||H_n|||_{n+1}^{n+1}} < \frac{1}{2^n}. \tag{1}$$

In the next step we show that for arbitrary $k \in \mathbb{N}$ $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Diff$^k(M)$, i.e. $\lim_{n,m\to\infty} d_k\left(f_n, f_m\right) = 0$. For this purpose, we calculate:

$$\lim_{n\to\infty} d_k\left(f_n, f_m\right) \leq \lim_{n\to\infty} \sum_{i=m+1}^{n} d_k\left(f_i, f_{i-1}\right) = \sum_{i=m+1}^{\infty} d_k\left(f_i, f_{i-1}\right). \tag{2}$$

We consider the limit process $m \to \infty$, i.e. we can assume $k \leq m$ and obtain from equations 1 and 2:

$$\lim_{n,m\to\infty} d_k\left(f_n, f_m\right) \leq \lim_{m\to\infty} \sum_{i=m+1}^{\infty} \frac{1}{2^i} = 0.$$

Since Diff$^k(M)$ is complete, the sequence $(f_n)_{n\in\mathbb{N}}$ converges consequently in Diff$^k(M)$ for every $k \in \mathbb{N}$. Thus, the sequence converges in Diff$^\infty(M)$ by definition.

2. Again with the help of Lemma 6.4 we compute for every $i \in \mathbb{N}$:

$$\begin{aligned} d_0\left(f_i^m, f_{i-1}^m\right) &= d_0\left(H_i \circ R_{m\cdot\alpha_{i+1}} \circ H_i^{-1}, H_i \circ R_{m\cdot\alpha_i} \circ H_i^{-1}\right) \\ &\leq |||H_i|||_1 \cdot m \cdot |\alpha_{i+1}-\alpha_i|. \end{aligned}$$

Since $m \leq q_{n+1} \leq q_i$ we conclude for every $i > n$:

$$d_0\left(f_i^m, f_{i-1}^m\right) \leq |||H_i|||_1 \cdot m \cdot \frac{1}{2^i \cdot (i+1)! \cdot q_i \cdot |||H_i|||_{i+1}^{i+1}} < \frac{m}{q_i} \cdot \frac{1}{2^i} \leq \frac{1}{2^i}.$$

Thus, for every $m \leq q_{n+1}$ we get the claimed result:

$$d_0\left(f^m, f_n^m\right) \leq \lim_{k\to\infty} \sum_{i=n+1}^{k} d_0\left(f_i^m, f_{i-1}^m\right) < \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \left(\frac{1}{2}\right)^n.$$

□

By Lemma 6.2 we have

$$2^n \cdot (n+1)! \cdot q_n \cdot |||H_n|||_{n+1}^{n+1} \leq \varphi_1(n) \cdot q_n^{3\cdot(n+1)^{n+1}},$$

at which $\varphi_1(n) := 4^{(n+2)^{n+2}} \cdot ((n+2)!)^{11\cdot(n+2)^{n+6}} \cdot \exp\left(100n^2\right)^{11n(n+1)^{n+5}}$. Since $|\alpha_{n+1} - \alpha_n| = \frac{a_n}{q_n \cdot \tilde{q}_{n+1}} \leq \frac{1}{\tilde{q}_{n+1}}$ the requirement A1 can be met if we demand

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot q_n^{3\cdot(n+1)^{n+1}}.$$

Hereby, the other conditions A3 and A2 are fulfilled.
Using $q_n = q_{n-1} \cdot \tilde{q}_n < \tilde{q}_n^2$ this yields the condition

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot \tilde{q}_n^{6\cdot(n+1)^{n+1}}.$$

This condition is satisfied by the assumptions of Theorem 1. Hence, we can apply Lemma 6.5 and obtain convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ in the $\mathrm{Diff}^\infty(M)$-topology to a measure-preserving diffeomorphism f. In the following subsections we will prove that f is the aimed diffeomorphism as asserted in Theorem 1, namely uniformly rigid with respect to $(\tilde{q}_n)_{n\in\mathbb{N}}$ and weak mixing.

6.3 Proof of uniform rigidity along the sequence $(\tilde{q}_n)_{n\in\mathbb{N}}$

By definition $\tilde{q}_{n+1} \leq q_{n+1}$. Hence, the second statement of Lemma 6.5 implies $d_0\left(f_n^{\tilde{q}_{n+1}}, f^{\tilde{q}_{n+1}}\right) < \frac{1}{2^n}$. Since the number α_{n+1} was chosen in such a way that $f_n^{\tilde{q}_{n+1}} = \mathrm{id}$, we have $d_0\left(\mathrm{id}, f^{\tilde{q}_{n+1}}\right) < \frac{1}{2^n}$, which goes to zero as $n \to \infty$. Thus, $(\tilde{q}_n)_{n\in\mathbb{N}}$ is an uniform rigidity sequence of f.

6.4 Proof of weak mixing

By the same approach as in [FS] we want to apply Proposition 4.4. For this purpose, we introduce a sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers $m_n \leq q_{n+1}$ in subsection 6.4.1 and a sequence $(\eta_n)_{n\in\mathbb{N}}$ of standard partial decompositions in subsection 6.4.2. Finally, we show that the map $\Phi_n := \phi_n \circ R_{\alpha_{n+1}}^{m_n} \circ \phi_n^{-1}$ $\left(0, \frac{1}{2n}, 0\right)$-distributes the elements of this partition.

6.4.1 Choice of the mixing sequence $(m_n)_{n\in\mathbb{N}}$

By condition A3 our chosen sequence $(q_n)_{n\in\mathbb{N}}$ satisfies

$$q_{n+1} \geq n \cdot q_n^5. \tag{C2}$$

Define

$$\begin{aligned} m_n &= \min\left\{ m \leq q_{n+1} \ : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2\cdot q_n} + \frac{k}{q_n} \right| \leq \frac{q_n}{q_{n+1}} \right\} \\ &= \min\left\{ m \leq q_{n+1} \ : \ m \in \mathbb{N}, \ \inf_{k\in\mathbb{Z}} \left| m \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\} \end{aligned}$$

Lemma 6.6. *The set* $\left\{ m \leq q_{n+1} : m \in \mathbb{N}, \inf_{k\in\mathbb{Z}} \left| m \frac{q_n \cdot p_{n+1}}{q_{n+1}} - \frac{1}{2} + k \right| \leq \frac{q_n^2}{q_{n+1}} \right\}$ *is nonempty for every* $n \in \mathbb{N}$, *i.e.* m_n *exists.*

Proof. The number α_{n+1} was constructed by the rule $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n}{q_n} - \frac{a_n}{q_n \cdot \tilde{q}_{n+1}}$, where $a_n \in \mathbb{Z}$, $1 \leq a_n \leq q_n$, i.e. $p_{n+1} = p_n \tilde{q}_{n+1} - a_n$ and $q_{n+1} = q_n \tilde{q}_{n+1}$. So $\frac{q_n \cdot p_{n+1}}{q_{n+1}} = \frac{p_{n+1}}{\tilde{q}_{n+1}}$ and the set $\left\{ j \frac{q_n \cdot p_{n+1}}{q_{n+1}} : j = 1, 2, ..., q_{n+1} \right\}$ contains $\frac{\tilde{q}_{n+1}}{\gcd(p_{n+1}, \tilde{q}_{n+1})}$ different equally distributed points on $\mathbb{S}^1$. Hence, there are at least $\frac{\tilde{q}_{n+1}}{q_n} = \frac{q_{n+1}}{q_n^2}$ different such points and so for every $x \in \mathbb{S}^1$ there is a $j \in \{1, ..., q_{n+1}\}$ such that

$$\inf_{k \in \mathbb{Z}} \left| x - j \cdot \frac{q_n \cdot p_{n+1}}{q_{n+1}} + k \right| \leq \frac{q_n^2}{q_{n+1}}.$$

In particular, this is true for $x = \frac{1}{2}$. □

Remark 6.7. We define

$$\Delta_n := \left(m_n \cdot \frac{p_{n+1}}{q_{n+1}} - \frac{1}{2 \cdot q_n} \right) \mod \frac{1}{q_n}.$$

By the above construction of m_n it holds: $|\Delta_n| \leq \frac{q_n}{q_{n+1}}$. By C2 we get: $|\Delta_n| \leq \frac{1}{q_n^4}$.

6.4.2 Application of the criterion

The partition η_n is defined to be the standard partial decomposition of M consisting of the horizontal intervals

$$\begin{aligned} I_{n,j} \times \{r\} &:= \left[\frac{j}{q_n} + \frac{1}{8nq_n}, \frac{j}{q_n} + \frac{1}{2q_n} - \frac{1}{8nq_n} \right] \times \{r\} \text{ and} \\ \bar{I}_{n,j} \times \{r\} &:= \left[\frac{j}{q_n} + \frac{1}{2q_n} + \frac{1}{8nq_n} - a_n, \frac{j+1}{q_n} - \frac{1}{8nq_n} - a_n \right] \times \{r\}, \end{aligned}$$

where $j \in \mathbb{Z}$ and $r \in \left[\frac{1}{4n}, 1 - \frac{1}{4n}\right]$. Obviously, we have $\eta_n \to \varepsilon$ and the length of the horizontal intervals is at most q_n^{-1}. In order to apply the criterion for weak mixing we prove

Lemma 6.8. *Let $I_n \in \eta_n$. Then Φ_n $\left(0, \frac{1}{2n}, 0\right)$-distributes I_n.*

Proof. Since $C_n (I_{n,j} \times \{r\})$ is located in the domain where $\phi^{-1}_{\frac{1}{10n}}$ acts as the rotation on it, $\phi_n^{-1} (I_{n,j} \times \{r\}) = \{\theta\} \times \left[\frac{1}{4n}, 1 - \frac{1}{4n}\right]$ for some $\theta \in I_{n,j}$. By definition of the number m_n and Remark 6.7 we have $R^{m_n}_{\alpha_{n+1}} \circ \phi_n^{-1} (I_{n,j} \times \{r\}) \subset R_{\frac{j'}{q_n}} \left(D_n^2\right)$ for some $j' \in \mathbb{Z}$. Hence, ϕ_n acts as the identity on it and we have $\Phi_n (I_{n,j} \times \{r\}) = \{\theta'\} \times \left[\frac{1}{4n}, 1 - \frac{1}{4n}\right]$ for some $\theta' \in \mathbb{S}^1$.
Again, by definition of the number m_n and the bound on a_n we have

$$\begin{aligned} \Phi_n \left(\bar{I}_{n,j} \times \{r\}\right) &= \phi_n \circ R^{m_n}_{\alpha_{n+1}} \left(\bar{I}_{n,j} \times \{r\}\right) = \phi_n (I_{n,j'} \times \{r\}) \\ &= \{\theta\} \times \left[\frac{1}{4n}, 1 - \frac{1}{4n}\right] \end{aligned}$$

for some $j' \in \mathbb{Z}$ and $\theta \in \mathbb{S}^1$.
In both cases, $\Phi_n (I_n)$ is a vertical interval (hence, $\gamma = 0$) and the projection on the r-axis is $\left[\frac{1}{4n}, 1 - \frac{1}{4n}\right]$ (hence, $\delta = \frac{1}{2n}$). Finally, we can take $\varepsilon = 0$ because the restriction of Φ_n to I_n is an affine map. □

By Lemma 6.5, 2. we have $d_0 \left(f^{m_n}, f_n^{m_n}\right) < \frac{1}{2^n}$ because $m_n \leq q_{n+1}$ by definition. Because of the requirement A3 on the number q_n we have C1 (see Remark 6.3). Thus, we can apply Proposition 4.4 and conclude that the constructed diffeomorphisms are weak mixing.

7 The case of the disc $\mathbb{D}^2$

Using the polar coordinate change

$$P(\theta, r) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{r} \cdot \cos(2\pi\theta) \\ \sqrt{r} \cdot \sin(2\pi\theta) \end{pmatrix}$$

transforming the annulus into the disc (with $|\det(JP)| = \pi$ except at the origin) we can define a sequence of smooth area-preserving diffeomorphisms $\tilde{f}_n = P \circ f_n \circ P^{-1}$ on $\mathbb{D}^2 \setminus \{(0,0)\}$, where f_n is constructed as in the previous section. Since these diffeomorphisms satisfy $f_n = R_{\alpha_{n+1}}$ on $\mathbb{S}^1 \times \left[0, \frac{1}{10n}\right]$, we observe for any $k \in \mathbb{N}$

$$d_k\left(\tilde{f}_n, \tilde{f}_{n-1}\right) \leq (k+1)! \cdot |||P \circ H_n|||^{k+1}_{k+1, \mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]} \cdot |\alpha_{n+1} - \alpha_n|.$$

Under the condition $|\alpha_{n+1} - \alpha_n| < \frac{1}{2^n \cdot (n+1)! \cdot q_n \cdot |||P \circ H_n|||^{n+1}_{n+1, \mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]}}$ we can prove convergence of the sequence $\left(\tilde{f}_n\right)_{n \in \mathbb{N}}$ in $\mathrm{Diff}^\infty\left(\mathbb{D}^2\right)$ as before and the limit diffeomorphism $\tilde{f}$ can be extended to the origin smoothly. This diffeomorphism is weak mixing because the coordinate transformation is area-preserving (up to a multiplicative constant).
In order to find estimates on $|||P \circ H_n|||_{n+1, \mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]}$ we use the same techniques and estimates as in the previous section. Additionally, we recall from the proof of Lemma 5.8 and Lemma 5.9 respectively:

$$\|P\|_{s, B(r_1, r_2)} \leq \pi^s \cdot (s-1)! \cdot r_1^{-s} \text{ and } \left\|P^{-1}\right\|_{s, PB(r_1, r_2)} \leq 2^{2s-2} \cdot (s-1)! \cdot r_1^{-s}$$

With the aid of Lemma 2.6 and Lemma 6.2 we have

$$\begin{aligned}
&2^n \cdot (n+1)! \cdot q_n \cdot \|P \circ H_n\|_{n+1, \mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]} \\
\leq &2^n \cdot (n+1)! \cdot q_n \cdot (n+2)! \cdot \|P\|_{n+1, \mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]} \cdot \|H_n\|^{n+1}_{n+1} \\
\leq &4^{(n+2)^{n+2}} \cdot ((n+2)!)^{11 \cdot (n+2)^{n+6}} \cdot \exp\left(100n^2\right)^{11n(n+1)^{n+5}} \cdot q_n^{3 \cdot (n+1)^{n+1}}
\end{aligned}$$

as well as

$$\begin{aligned}
&2^n \cdot (n+1)! \cdot q_n \cdot \left\|H_n^{-1} \circ P^{-1}\right\|_{n+1, P\left(\mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]\right)} \\
\leq &2^n \cdot (n+1)! \cdot q_n \cdot (n+2)! \cdot |||H_n|||_{n+1} \cdot \left\|P^{-1}\right\|^{n+1}_{n+1, P\left(\mathbb{S}^1 \times \left[\frac{1}{10n}, 1\right]\right)} \\
\leq &4^{(n+2)^{n+2}} \cdot ((n+2)!)^{11 \cdot (n+2)^{n+5}} \cdot \exp\left(100n^2\right)^{11n(n+1)^{n+4}} \cdot q_n^{3 \cdot (n+1)^n}.
\end{aligned}$$

By the same arguments as above we find the sufficient condition on the growth rate

$$\tilde{q}_{n+1} \geq \varphi_1(n) \cdot \tilde{q}_n^{6 \cdot (n+1)^{n+1}},$$

at which $\varphi_1(n) := 4^{(n+2)^{n+2}} \cdot ((n+2)!)^{11 \cdot (n+2)^{n+6}} \cdot \exp\left(100n^2\right)^{11n(n+1)^{n+5}}$.
Since this condition is fulfilled due to our assumptions of Theorem 1, we obtain convergence of the sequence $\left(\tilde{f}_n\right)_{n \in \mathbb{N}}$ in $\mathrm{Diff}^\infty\left(\mathbb{D}^2\right)$ to a limit diffeomorphism $\tilde{f}$. As argued above, $\tilde{f}$ is weak mixing and uniformly rigid with respect to $(\tilde{q}_n)_{n \in \mathbb{N}}$. Hence, Theorem 1 is also proven in the case of the disc $\mathbb{D}^2$.

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