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Symplectic Geometry

Problem Set 1

1. Prove that a 2-form ω on a $2n$-dimensional real vector space V is symplectic if and only if

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}} \neq 0.$$

2. Recall that for a linear subspace W of a symplectic vector space (V, ω) we defined the ω-orthogonal complement as

$$W^\perp := \{v \in V \,:\, \omega(u, v) = 0 \text{ for all } u \in W\}.$$

a) Prove that $\dim W + \dim W^{\perp_\omega} = \dim V$!

b) Prove that W is a symplectic subspace if and only if $(W, \omega|_W)$ is a symplectic vector space if and only if $V = W \oplus W^{\perp_\omega}$!

c) More generally, prove that the quotient space $W/(W \cap W^\perp)$ always inherits a symplectic structure from V.

3. Prove that a linear subspace W of codimension 1 in a symplectic vector space (V, ω) is always coisotropic!

4. Prove that the standard euclidean scalar product $g(\,.\,,\,.\,)_{\mathrm{st}}$, the standard complex structure J_{st} and the standard symplectic form ω_{st} on $\mathbb{R}^{2n}$ are related by

$$g(u, v)_{\mathrm{st}} = \omega_{\mathrm{st}}(u, J_{\mathrm{st}} v).$$

5. Prove that given two Lagrangian subspaces $L_0, L_1 \subset (V, \omega)$ of a symplectic vector space such that $L_0 \cap L_1 = \{0\}$ (i.e. L_0 and L_1 are transverse), there exists a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ for V such that $L_0 = \mathrm{span}(e_1, \ldots, e_n)$ and $L_1 = \mathrm{span}(f_1, \ldots, f_n)$.

Please turn!

6. Give examples of elements of $\mathrm{SL}(4, \mathbb{R})$ which are not elements of $\mathrm{Sp}(4, \mathbb{R})$!

7. We have seen in class that the eigenvalues of a symplectic matrix come in quadrupels λ, λ^{-1}, $\bar{\lambda}$ and $\bar{\lambda}^{-1}$. Since a 2×2 symplectic matrix has at most two eigenvalues, these four values cannot all be distinct in this case. Prove that for $A \in \mathrm{Sp}(2, \mathbb{R})$

- $\operatorname{Tr} A > 2$ if and only if both eigenvalues are real, positive and different from 1,
- $-2 \leq \operatorname{Tr} A \leq 2$ if and only if both eigenvalues are on the unit circle, and
- $\operatorname{Tr} A < -2$ if and only if both eigenvalues are real, negative and different from -1.

8. Prove that $\mathrm{Sp}(2, \mathbb{R})$ is diffeomorphic to the open solid torus $S^1 \times \mathbb{R}^2$!