Universität Hamburg
Janko Latschev
Pavel Hajek

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Symplectic Geometry

Problem Set 2

1. Let $W_0 \subseteq (\mathbb{R}^{2n}, \omega_{\text{st}})$ be a Lagrangian subspace, and set $W_1 := JW_0 \subseteq \mathbb{R}^{2n}$. Let $(e_1, \ldots, e_n)$ be an orthonormal (with respect to the standard euclidean inner product g_{st}) basis for W_0, and set $f_k = Je_k$, so that $(f_1, \ldots, f_k)$ is an orthonormal basis for W_1 and $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is a symplectic basis for $\mathbb{R}^{2n}$.

Prove that the graph of a linear map $B : W_0 \to W_1$ is a Lagrangian subspace of $\mathbb{R}^{2n}$ if and only if the matrix of B with respect to the basis $(e_1, \ldots, e_n)$ of W_0 and $(f_1, \ldots, f_n)$ of W_1 is symmetric.

2. As in the lecture we denote by $\mathcal{L}(n)$ the space of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{\text{st}})$

a) Prove that the loop $\Psi : \mathbb{R}/\mathbb{Z} \to \text{Sp}(4, \mathbb{R})$ defined in the lecture by setting

$$\Psi(t) := e^{\pi i t} \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix} \in U(2) \subset \text{Sp}(4, \mathbb{R})$$

has Maslov index 1.

b) Prove that with $\Lambda_0(t) = e^{\pi i t} \cdot \mathbb{R} \in \mathcal{L}(1)$ and $\Lambda(t) := \Lambda_0(t) \oplus \Lambda_0(t) \in \mathcal{L}(2)$ we have

$$\Lambda(t) = \Psi(t) \cdot (\mathbb{R}^2 \oplus \{0\}).$$

As discussed in the lecture, this proves that $\mu(\Lambda_0) = 1$.

c) Prove that the Maslov index for Lagrangian subspaces is characterized uniquely by the (homotopy), (product), (direct sum) and (zero) axioms.

d) Prove that the Maslov index for Lagrangian loops has the concatenation property: If Λ_1 and Λ_2 are two loops in $\mathcal{L}(n)$ with $\Lambda_1(0) = \Lambda_2(0)$, then

$$\mu(\Lambda_1 \star \Lambda_2) = \mu(\Lambda_1) + \mu(\Lambda_2).$$

Please turn!

e) The space $\mathcal{L}^{\mathrm{or}}(n)$ of oriented Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ is a double cover (two-sheeted covering space) of $\mathcal{L}(n)$. It can be identified with $U(n)/SO(n)$. Prove that if $p : \mathcal{L}^{\mathrm{or}}(n) \to \mathcal{L}(n)$ is the covering projection map, then $p_*(\pi_1(\mathcal{L}^{\mathrm{or}}(n)) = 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_1(\mathcal{L}(n))$. In other words: the Maslov index of a loop of oriented Lagrangian subspaces is even.

3. **a)** Prove that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies $A^2 = -\mathbb{1}$ if and only if $d = -a$ and $ad - bc = 1$.

b) Deduce that the subset $\mathcal{J} \subseteq \mathrm{SL}(2, \mathbb{R}) \cong \mathrm{Sp}(2, \mathbb{R})$ of all such maps has two connected components, one containing J_{st} and the other containing $-J_{\mathrm{st}}$.

c) What is the condition on a map A as above to be tamed by ω_{st}? To be compatible with ω_{st}?

4. Let g be any euclidean inner product on $\mathbb{R}^{2n}$.

a) Prove that there exists a basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ which is both symplectic with respect to ω_{st} and g-orthogonal. Moreover, one can require $g(e_k, e_k) = g(f_k, f_k)$ (but this need not be equal to 1).
Hint: Write $g(u, v) = \omega(u, Av)$ for a linear map $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Prove that $B = iA \in \mathrm{Mat}(2n, \mathbb{C})$ is hermitian (i.e. satisfies $\overline{B}^T = B$) and so it has purely imaginary eigenvalues and can be diagonalized. Now build the required basis from real and imaginary parts of suitable eigenvectors of B.

In a finite dimensional real vector space V, any euclidean inner product g determines an open *ellipsoid* via

$$E_g = \{v \in V \ : \ g(v, v) < 1\}.$$

b) Prove that for any ellipsoid $E \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ there exists a symplectic linear map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\Phi(E)$ is a *standard symplectic ellipsoid*, meaning it is of the form

$$E(r_1, \ldots, r_n) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \cong \mathbb{R}^{2n} \ : \ \sum_j \frac{|z_j|^2}{r_j^2} < 1\}.$$

Here the numbers $0 < r_1 \leq r_2 \leq \cdots \leq r_n$ are uniquely determined by E.

c) What does this mean geometrically for $n = 1$?