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Symplectic Geometry

Problem Set 3

1. A diffeomorphism $\varphi : Q \to Q'$ between manifolds lifts to a diffeomorphism $\Phi : T^*Q \to T^*Q'$ given by the formula

$$\Phi(x, \alpha) := \left(\varphi(x), \alpha \circ (\varphi_{*,x})^{-1}\right),$$

where $\varphi_{*,x} : T_xQ \to T_{\varphi(x)}Q'$ is the differential of φ at $x \in Q$.

a) Prove that $\Phi^*(\lambda'_{\mathrm{can}}) = \lambda_{\mathrm{can}}$, and so Φ is a symplectomorphism from T^*Q to T^*Q'!

b) Let $Y : Q \to TQ$ be a complete vector field, and denote by ψ_t its flow. Let $X : T^*Q \to T(T^*Q)$ be the vector field generating the corresponding flow Ψ_t on T^*Q. Prove that X is the Hamiltonian vector field associated to the function $H : T^*Q \to \mathbb{R}$ defined as

$$H(x, \alpha) := \alpha(Y(x)).$$

2. Show that if $\varphi : M \to M$ is any symplectomorphism of (M, ω) and $H : M \to \mathbb{R}$ is smooth, then the Hamiltonian vector fields of the functions H and $H \circ \varphi^{-1}$ are related by

$$X_{H\circ\varphi^{-1}}(\varphi(x)) = \varphi_*(X_H(x)),$$

where $\varphi_* : TM \to TM$ is the differential of φ.

3. *(Hamiltonian diffeomorphisms)*
Let $\varphi_t : (M, \omega) \to (M, \omega)$ be the family of diffeomorphisms determined by the time-dependent Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$ via

$$\dot{\varphi}_t = X_{H_t} \circ \varphi_t.$$

a) For each $t \in (0, 1)$, write φ_t as time one map of a family of diffeomorphisms determined by a new Hamiltonian function built from H.

b) Find a time-dependent Hamiltonian function whose time one map is $(\varphi_1)^{-1}$.

Please turn!

c) Now suppose ψ is the time one map of a second family ψ_t determined by $F : [0,1] \times M \to \mathbb{R}$. Find a time-dependent Hamiltonian function with time one map $\psi \circ \varphi$.

In summary, you have shown that Hamiltonian diffeomorphisms form a connected subgroup $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Symp}_0(M,\omega)$ of the identity component of the group of symplectomorphisms.

4. *(This exercise implements a suggestion by D. Salamon.)*
Consider $(\mathbb{R}^2, \omega_{\mathrm{st}} = dx \wedge dy)$.

a) Find explicit autonomous Hamiltonian functions $H_i : \mathbb{R}^2 \to \mathbb{R}$ such that the time-one-maps of the corresponding Hamiltonian flows φ_t^i are

$$\varphi_1^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} \qquad \text{and} \qquad \varphi_1^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

b) Prove that $\psi = \varphi_2^1 \circ \varphi_1^1$ cannot be generated by an autonomous Hamiltonian function (and in fact is not the time-one-map of any flow!).
Hint: Assume the contrary and first argue that 0 must be a fixpoint of the flow, then consider the linearization of the flow at this fixpoint to obtain a contradiction.

This clearly illustrates the need for time-dependent Hamiltonians in the definition of $\mathrm{Ham}(M,\omega)$. In fact, general Hamiltonian diffeomorphisms often behave very differently from those generated by an autonomous function - for example they could have dense orbits, which clearly cannot happen in the autonomous case (why?).

5. The goal of this exercise is to gain a little bit of geometric intuition about compactly supported Hamiltonian diffeomorphisms by looking at a specific case.
Consider a Hamiltonian function $H : B^2(0,10) \to \mathbb{R}$ of the form $H(x,y) = y \cdot \rho(r^2)$, where $r^2 = x^2 + y^2$ and $\rho : [0,\infty) \to [0,1]$ is a smooth function with the following properties:

$$\rho(t) \equiv 1 \text{ for } 0 \leq t \leq 5, \quad \rho(t) \equiv 0 \text{ for } 8 \leq t, \quad \text{and} \quad \rho'(t) \leq 0 \text{ for all } t \in [0,\infty).$$

a) Compute the Hamiltonian vector field X_H with respect to the symplectic form $\omega = dx \wedge dy$ in terms of x, y, ρ and ρ', and make a rough sketch of it.

b) Give a qualitative description (e.g. give a rough sketch) of the image of the ball $B^2(0,1)$ under the time-t-map φ_t of the Hamiltonian flow of H for $t = 1$, $t = 10$ and $t = 10^5$!