Universität Hamburg
Janko Latschev
Pavel Hájek

Summer 2023

Symplectic Geometry

Problem Set 8

1. For a function $a : \mathbb{R}^4 \to \mathbb{R}$, we consider the almost complex structure J_a on the manifold $M = \mathbb{R}^4$ which in the global coordinates (x_1, x_2, y_1, y_2) has the form

$$J_a(p) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ a(p) & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -a(p) & 0 \end{pmatrix}, \text{ i.e. } J_a\left(\frac{\partial}{\partial x_1}\right) = a(p)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_1} \quad \text{etc.}$$

a) Prove that if $|a(p)| < 1$ for all $p \in \mathbb{R}^4$, then J_a is tamed by the standard symplectic form $\omega_{\text{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$!
Hint: Recall that the taming condition means that $\omega(v, Jv) > 0$ for all non-zero v, but ω need not be J-invariant, so that the bilinear form $\omega(\,.\,, J\,.\,)$ need not be symmetric.

b) Under which conditions on the function a is the almost complex structure J_a on $\mathbb{R}^4$ integrable?
Hint: Argue that in order to determine N_{J_a} on any two vectors $v, w \in T_p\mathbb{R}^4$, it suffices to know $N_{J_a}\left(\frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial x_2}(p)\right)$, and then compute this.

2. We consider an almost complex structure J on an open neighborhood U of $0 \in \mathbb{R}^{2n}$.

a) Prove that if $f : U \to \mathbb{C}$ is J-holomorphic, .i.e. we have $df \circ J = J_{\text{st}} \circ df$, then for each point $p \in U$ the differential df_p has either rank 0 or rank 2, and $\ker df \subseteq TU$ is closed under the Lie bracket.

b) Prove that the image of the Nijenhuis tensor N_J is contained in $\ker df$.

c) Consider the case $n = 2$ (i.e. $U \subseteq \mathbb{R}^4$) and construct an almost complex structure J such that there are *no nonconstant J-holomorphic functions $f : U \to \mathbb{C}$*.

Please turn!

3. Earlier in the semester, we introduced the Fubini-Study form ω_{FS} on $\mathbb{C}P^n$ as the unique 2-form satisfying

$$\pi^*\omega_{\mathrm{FS}} = \iota^*\omega_{\mathrm{st}},$$

where $\pi : S^{2n+1} \to \mathbb{C}P^n$ and $\iota : S^{2n+1} \to \mathbb{C}^{n+1}$ are the standard projection and the standard embedding.

a) Prove that the form

$$\widetilde{\omega} := \frac{1}{\|z\|^2} \cdot \omega_{\mathrm{st}}$$

on $\mathbb{C}^{n+1} \setminus \{0\}$ is invariant under the $\mathbb{C}^*$-action by rescaling, and notice that $\iota^*\widetilde{\omega} = \iota^*\omega_{\mathrm{st}}$.

b) Recall that we defined $H_z := \mathrm{span}_{\mathbb{C}}(z)^{\perp} \subset T_z\mathbb{C}^{n+1}$. Now use a coordinate expression for the orthogonal projection $T_z\mathbb{C}^{n+1} \to H_z$ and part **a)** to prove that in homogeneous coordinates $[z_0 : \ldots : z_n]$ on $\mathbb{C}P^n$ the Fubini-Study form is given by[1]

$$\omega_{\mathrm{FS}} = \frac{i}{2}\left(\sum_j \frac{dz_j \wedge d\bar{z}_j}{\|z\|^2} - \sum_{j,k} \frac{\bar{z}_j z_k dz_j \wedge d\bar{z}_k}{\|z\|^4}\right).$$

c) The open sets $U_i = \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$ cover $\mathbb{C}P^n$. Prove that the maps

$$\begin{aligned} \psi_i : U_i &\to \mathbb{C}^n \\ [z_0 : \ldots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right) \end{aligned}$$

give complex charts for $\mathbb{C}P^n$, i.e. the transition maps are holomorphic. What is $\mathbb{C}P^n \setminus U_i$ diffeomorphic to? What is $U_i \cap U_j$ diffeomorphic to?

d) Derive an expression for the Fubini-Study form in inhomogeneous coordinates $\zeta_j := \frac{z_j}{z_0}$ on the open set $U_0 \subset \mathbb{C}P^n$.

e) Compute an explicit expression for $\eta = \frac{i}{2}\partial\bar{\partial}(\log(1 + \|\zeta\|^2))$ on $\mathbb{C}^n$ and compare with the result for $(\psi_0^{-1})^*\omega_{\mathrm{FS}}$ you obtained in **d)**.

f) Compute the integral

$$\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}}.$$

[1] We did a brief version of this calculation in class earlier in the semester.