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Symplectic Geometry

Problem Set 9

1. Consider a Kähler manifold (M,ω,J) and suppose that $\varphi : M \to M$ is an isometric involution ($\varphi^2 = \mathrm{id}$) of the corresponding Kähler metric $g_J = \omega(\,.\,, J\,.\,)$ which is antiholomorphic, i.e. such that $\varphi_* \circ J = -J \circ \varphi_*$.

a) Prove that φ is antisymplectic, i.e. $\varphi^*\omega = -\omega$.

b) Prove that the fixed point set is a Lagrangian submanifold of (M,ω).

c) What is the fixed point set of $\varphi : \mathbb{C}P^n \to \mathbb{C}P^n$, given in homogeneous coordinates as complex conjugation

$$\varphi([z_0 : \ldots : z_n]) = [\bar{z}_0 : \ldots : \bar{z}_n]?$$

Remark: Note that if $X \subset \mathbb{C}P^n$ is a smooth complex submanifold given as the zero set of finitely many homogeneous polynomials with ***real*** *coefficients, then φ also induces an antiholomorphic and antisymplectic involution on X. This gives many interesting examples.*

2. a) Find a sequence $\{u_k\}_{k\geq 1}$ of holomorphic maps

$$u_k : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

with the following properties:

- The composition of each u_k with each of the two projections to the factors $\mathbb{C}P^1$ is a biholomorphic map.
- The point $([0:1],[0:1]) \in \mathbb{C}P^1 \times \mathbb{C}P^1$ is contained in $u_k(\mathbb{C}P^1)$ for each $k \geq 1$.
- The images $u_k(\mathbb{C}P^1)$ converge as subsets to a subset of $\mathbb{C}P^1 \times CP^1$ of the form $\{p\} \times \mathbb{C}P^1 \cup \mathbb{C}P^1 \times \{q\}$, for suitable points p and q in $\mathbb{C}P^1$.

b) Now find sequences of Möbius tranformations $\varphi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ and $\psi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that the maps $v_k := u_k \circ \varphi_k$ converge to a holomorphic parametrization of $(\mathbb{C}P^1 \setminus \{p\}) \times \{q\}$ and the maps $w_k := u_k \circ \psi_k$ converge to a holomorphic parametrization of $\{p\} \times (\mathbb{C}P^1 \setminus \{q\})$, both in C^∞_{loc} in the complement of suitable points in $\mathbb{C}P^1$.

3. A holomorphic vector bundle on a complex manifold X is a complex vector bundle $E \to X$ whose transition functions with respect to an atlas of trivializations over open subsets $U_i \subset X$ are given by holomorphic maps $\varphi_{ij} : U_i \cap U_j \to GL(r, \mathbb{C})$. Prove the following assertions:

a) The cotangent bundle $K_\Sigma = T^*\Sigma$ of a Riemann surface (Σ, j) is a holomorphic line bundle. It is called the *canonical bundle* of Σ.

b) We define $U \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$ as the subset

$$U := \{ ([z_0 : z_1], w) \mid w \in \mathbb{C} \cdot \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \}.$$

Then, with the obvious projection $\pi : U \to \mathbb{C}P^1$, this is a holomorphic line bundle over $\mathbb{C}P^1$, called the *universal line bundle* over $\mathbb{C}P^1$.

c) Let $\mathcal{U}_i := \{ [z_0 : z_1] \mid z_i \neq 0 \} \subseteq \mathbb{C}P^1$ be the two open subsets giving the standard covering of $\mathbb{C}P^1$ by charts For every $k \in \mathbb{Z}$ we can define a holomorphic line bundle $E_k \to \mathbb{C}P^1$ by gluing the trivial bundles $E^0 = \mathcal{U}_0 \times \mathbb{C}$ and $E^1 = \mathcal{U}_1 \times \mathbb{C}$ via the transition map

$$\begin{aligned} \psi_k : E^0|_{\mathcal{U}_0 \cap \mathcal{U}_1} &\to E^1|_{\mathcal{U}_0 \cap \mathcal{U}_1} \\ ([z_0 : z_1], v) &\mapsto \left([z_0 : z_1], \left(\frac{z_0}{z_1} \right)^k \cdot v \right). \end{aligned}$$

Then the bundle $E_k \to \mathbb{C}P^1$ admits nonzero holomorphic sections $s : \mathbb{C}P^1 \to E_k$ if and only if $k \geq 0$, in which case the dimension of the $\mathbb{C}$-vector space of holomorphic sections is $k + 1$.

d) Every holomorphic vector bundle over $\mathbb{C}P^1$ is isomorphic to one of the E_k (you do not need to prove that). To which values of k do the canonical bundle $K_{\mathbb{C}P^1}$ and the universal bundle U correspond?

Remark: One can show that $\langle c_1(E_k), [\mathbb{C}P^1] \rangle = k$.