Solutions for Exercise 3 on Problem Set 5

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be given by $H(x, y) := f(x)$. Then the Hamiltonian vector field of H is defined by

$$\omega_{\mathrm{st}}(X_H, \cdot) = -dH = -\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

so

$$X_H = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i}.$$

It follows that the flow of X_H is given by

$$\phi_t^{X_H}((x, y)) = \left(x, y + t \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right) \right).$$

In particular, if one identifies $\mathbb{R}^{2n}$ with $T^*\mathbb{R}^n$ by mapping $(x, y) \in \mathbb{R}^{2n}$ to $\sum_{i=1} y_i dx_i \in T_x^*\mathbb{R}^n$, then $\phi_1^{X_H}(\mathbb{R}^n \times \{0\})$ is identified with the Graph of df.
Now let $L_0, L_1 \subseteq \mathbb{R}^{2n}$ be Lagrangian submanifolds intersecting transversely in the point $p = (x, y) \in \mathbb{R}^{2n}$. W. l. o. g. assume that $p = 0 = (0, 0)$. By the Lagrangian neighbourhood theorem, there exist $\epsilon' > 0$ and a symplectomorphism $\varphi_0 : B^{2n}(0, \epsilon') \to U_0 \subseteq \mathbb{R}^{2n}$ to a neighbourhood of 0 s. t. $\varphi_0(L_1) \cap B^{2n}(0, \epsilon') = \{0\} \times \mathbb{R}^n \cap B^{2n}(0, \epsilon')$. Because $T_0 L_0 \pitchfork T_0 L_1$ there exists $0 < \epsilon < \epsilon'$ s. t. $\varphi_0(L_0) \cap B^{2n}(0, \epsilon) = \{(x', \alpha(x')) \mid x' \in B^n(0, \epsilon)\}$ for a function $\alpha : \mathbb{R}^n \to \mathbb{R}^n$. Interpreting α as a section of $T^*\mathbb{R}^n$ over $B^n(0, \epsilon)$ (i. e. as a 1-form) under the identification $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ as before, by an example from the lecture, the graph of α is Lagrangian if and only if α is closed. Since $B^n(0, \epsilon)$ is simply connected, by the Poincaré-Lemma there exists $f : B^n(0, \epsilon) \to \mathbb{R}$ with $\alpha = df$. By what was shown above, for the Hamiltonian $H(x, y) = f(x)$ on $T^*B^n(0, \epsilon) \cong B^n(0, \epsilon) \times \mathbb{R}^n$, $\phi_1^{X_H}(B^n(0, \epsilon) \times \{0\}) = \mathrm{Graph}(\alpha)$, so $(\phi_1^{X_H})^{-1}$ maps $\varphi_0(L_0) \cap B^{2n}(0, \epsilon)$ to $B^n(0, \epsilon) \times \{0\}$. Furthermore, because $\alpha_0 = df_0 = 0$, $(X_H)_{(0,y)} = 0$, so $(\phi_1^{X_H})^{-1}$ leaves $\varphi_0(L_1) = \{0\} \times B^n(0, \epsilon)$ invariant. Hence $\varphi := (\phi_1^{X_H})^{-1} \circ \varphi_0$ has the desired properties.
For the final claim, define

$$\begin{aligned} f : \mathbb{R}^n \setminus \{0\} &\to \mathbb{R} \\ x &\mapsto \ln(\|x\|). \end{aligned}$$

Then $df = \frac{1}{\|x\|^2} \sum_{i=1}^{n} x_i dx_i$. Considered as a function $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, this maps $x \mapsto \frac{x}{\|x\|^2}$ and is its own inverse. Its graph Γ in $(\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R} \times S^{n-1}$ and since $g^{-1} = g$, it can be considered either as the graph of g over $(\mathbb{R}^n \setminus \{0\}) \times \{0\}$ or over $\{0\} \times (\mathbb{R}^n \setminus \{0\})$. Also, consider

Γ as a subset of $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Pick a smooth cutoff function $\rho : \mathbb{R}^n \to [0,1]$ with $\rho(x) = 1$ for $\|x\| \geq 1$ and $\rho(x) = 0$ for $\|x\| \leq \frac{1}{2}$. Define the Hamiltonian $H_1 : \mathbb{R}^{2n} \to \mathbb{R}$, $(x,y) \mapsto \rho(x)f(x)$. This is well defined on all of $\mathbb{R}^{2n}$ by choice of ρ. The Hamiltonian flow of H_1 then is the identity on $B^n(0,\frac{1}{2})\times\mathbb{R}^n$ and as above, $(\phi_1^{X_{H_1}})^{-1}$ maps $\Gamma \cap (\mathbb{R}^n \setminus B^n(0,1)) \times \mathbb{R}^n$ to $\mathbb{R}^n \times \{0\} \cap (\mathbb{R}^n \setminus B^n(0,1)) \times \mathbb{R}^n$. Analogously, define $H_2 : \mathbb{R}^{2n} \to \mathbb{R}$, $(x,y) \mapsto \rho(y)f(y)$ with Hamiltonian flow $\phi_t^{X_{H_2}}$. The image L of Γ under $(\phi_1^{X_{H_2}})^{-1} \circ (\phi_1^{X_{H_1}})^{-1}$ (which is the time-1-map of a Hamiltonian flow by a previous exercise) then has the desired properties.