Universität Hamburg
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A linear algebra lemma

In the lecture, I stated and used the following assertion, giving an incorrect proof. Here is a correction.

Lemma. *Let (V,ω) be a symplectic vector space and $W \subset V$ any linear subspace, and set $N := W \cap W^{\perp_\omega}$. Then there is an isomorphism of symplectic vector spaces*

$$\Phi : (V,\omega) \xrightarrow{\cong} (W/N,\omega) \oplus (W^{\perp}/N,\omega) \oplus (N \oplus N^*, \omega_{\text{can}}).$$

Proof. Let J be an ω-compatible complex structure on V. We define $V_3 := JN$.

Claim 1: *We have $(W + W^{\perp_\omega}) \cap V_3 = \{0\}$, and the map*

$$\begin{aligned} N &\to V_3^* \\ n &\mapsto (\iota(n)\omega)|_{V_3} \end{aligned}$$

is an isomorphism.

To prove the first assertion, observe that if $\overline{n} = Jn \in V_3$ is some nonzero vector, then since J is ω-compatible we have $\omega(n,\overline{n}) > 0$. This proves that $\overline{n} \notin (W + W^{\perp_\omega})$, because N is ω-orthogonal to $W + W^{\perp_\omega}$ by definition. The second statement follows similarly, since by compatibility the given map is injective. As both spaces have the same dimension, the map must be an isomorphism.

Now we define

$$\begin{aligned} V_1 &:= \{w \in W \mid \omega(w,\overline{n}) = 0 \text{ for all } \overline{n} \in V_3\} \\ V_2 &:= \{w \in W^{\perp_\omega} \mid \omega(w,\overline{n}) = 0 \text{ for all } \overline{n} \in V_3\} \end{aligned}$$

Claim 2. *We have $W = V_1 \oplus N$ and $W^{\perp_\omega} = V_2 \oplus N$.*

It clearly suffices to prove one of the statements, so we prove the first one. Observe that $V_1 \cap N = \{0\}$, as follows directly from Claim 1 and the definition of V_1. Now given any $w \in W$, consider the element $\varphi \in (JN)^*$ defined as

$$\varphi(\overline{n}) := \omega(w,\overline{n}).$$

By Claim 1, there exists some $n \in N$ such that $\varphi = \iota(n)\omega|_{V_3}$, and so it follows that $w - n \in V_1$. This proves that $W = V_1 + N$, and since we already proved that $V_1 \cap N = \{0\}$, the sum is direct.

Now it follows from Claims 1 and 2 that

$$V \cong V_1 \oplus N \oplus V_2 \oplus V_3.$$

Indeed, by Claim 2 we have $V_1 \oplus N \oplus V_2 \cong W + W^{\perp_\omega}$, and since by Claim 1 the subspace V_3 has trivial intersection with this space and moreover it has the right dimension, it is a complement.

Finally, using this decomposition of V, we can define the isomorphism Φ as

$$\Phi(v_1 + n + v_2 + v_3) := ([v_1], [v_2], (n, \iota(v_3)\omega)).$$

One readily checks that

$$\omega(v_1 + n + v_2 + v_3, w_1 + m + w_2 + w_3) = \omega(v_1, w_1) + \omega(v_2, w_2) + \omega(n, w_3) + \omega(v_3, m)$$

since for example v_1 is ω-orthogonal to $m + w_2 \in W^{\perp_\omega}$, and it is also ω-orthogonal to w_3 by the definition of V_1. As the right hand side agrees with the symplectic structure on the target vector space, we have proven the lemma. $\square$