Universität Hamburg
Janko Latschev
Fabian Kirchner

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Symplectic Geometry

Problem Set 11

1. In the lecture I mentioned that the Brouwer Fixpoint Theorem fails in infinite dimensions. Give an explicit counterexample in the Hilbert space $\ell^2(\mathbb{R})$ of real sequences $(x_n)_{n\geq 1}$ with $\sum_n x_n^2 < \infty$, i.e. find a continuous map of the closed unit ball of this space into itself without a fixpoint.

2. *(Packing balls)* Denote by $B^{2n}(a) \subset (\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ the ball of Gromov width a.

a) Compute the volume of $B^{2n}(a)$ in terms of a and n (the answer is a very simple formula)!

b) Argue that given any $\epsilon \in \mathbb{R}$ with $0 < \epsilon < a$ there is a symplectic (i.e. area preserving) embedding $\sigma : B^2(a-\epsilon) \to (0,a) \times (0,1)$ such that for all $\alpha \in (0, a-\epsilon]$ one has $\sigma(B^2(\alpha)) \subset (0, \alpha+\epsilon) \times (0,1)$.
Remark: A complete proof of this fact is somewhat delicate to write down. Try to describe an idea for constructing this embedding and give as much detail as possible!

c) Prove that under the map $\rho := \sigma \times \sigma : B^2(a-\epsilon) \times B^2(a-\epsilon) \to \mathbb{R}^4$ the image of the ball $B^4(a-2\epsilon) \subset B^2(a-\epsilon) \times B^2(a-\epsilon)$ is contained in

$$\triangle(a) \times \square(1) \subset \mathbb{R}^4,$$

where

$$\triangle(a) := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 > 0, x_1 + x_2 < a\}$$
$$\square(1) := \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_i < 1\}.$$

d) Construct a symplectic embedding of $\triangle(a) \times \square(1) \subset (\mathbb{R}^4, \omega_{\mathrm{st}})$ as above into $B^4(a) \subset (\mathbb{R}^4, \omega_{\mathrm{st}})$!
If you find this hard, construct a symplectic embedding $(0,a) \times (0,1) \to B^2(a)$ *first!*

e) Use these results to prove that given any integer $k \geq 1$ and any $b < \frac{a}{k}$, one can symplectically embed k^2 balls $B^4(b)$ disjointly into $B^4(a)$.
Note that as b approaches $\frac{a}{k}$, the total volume of these images approaches the volume of $B^4(a)$.

f) How do these results generalize to higher dimensions?