Transversality

There are various ways in which one can motivate the concept of transversality. Let us start from

Question 1. Suppose we are given a submanifold $Z \subseteq N$ of a smooth manifold N. Which conditions on a map $f : M \to N$ will ensure that the preimage $f^{-1}(Z) \subseteq M$ is necessarily a submanifold?

To start analyzing this question, note that for every $q \in Z$, we can find a submanifold chart, i.e. a diffeomorphism $\psi : V \to \psi(V) \subseteq \mathbb{R}^n$ such that $\psi(Z \cap V) = \psi(V) \cap \mathbb{R}^k \times \{0\}$, where $n = \dim N$ and $k = \dim Z$. Now consider the open subset $U := f^{-1}(V) \subseteq M$. For a point $x \in U$ we have

$$\begin{aligned} f(x) \in Z &\iff \psi \circ f(x) \in \mathbb{R}^k \times \{0\} \\ &\iff \pi_{\mathbb{R}^{n-k}} \circ \psi \circ f(x) = 0. \end{aligned}$$

So one way to ensure that $f^{-1}(Z) \cap U$ is a submanifold of M is to require that 0 is a regular value of the map

$$h := \pi_{\mathbb{R}^{n-k}} \circ \psi \circ f : U \to \mathbb{R}^{n-k}.$$

Note that for $x \in h^{-1}(0)$ the differential $(\pi_{\mathbb{R}^{n-k}})_{*,0} \circ \psi_{*,f(x)} = \pi_{\mathbb{R}^{n-k}} \circ \psi_{*,f(x)}$ vanishes on $T_{f(x)}Z$, so the differential $h_{*,x} : T_xM \to T_0\mathbb{R}^{n-k}$ will be surjective if and only if $f_{*,x} : T_xM \to T_{f(x)}N$ maps T_xM onto a subspace containing a complement of $T_{f(x)}Z \subseteq T_{f(x)}N$. This is one way to arrive at

Definition 1. *A map $f : M \to N$ is called* transverse *to a submanifold $Z \subseteq N$ if for all points $x \in f^{-1}(Z)$ one has*

$$f_{*,x}(T_xM) + T_{f(x)}Z = T_{f(x)}N. \tag{1}$$

We use the notation $f \pitchfork Z$ to denote the fact that f is transverse to Z.

The above discussion is now summarized by

Theorem 1. *If $f : M \to N$ is transverse to the submanifold $Z \subseteq N$, then $f^{-1}(Z)$ is a submanifold of M of dimension*

$$\dim M + \dim Z - \dim N. \qquad \square$$

In the discussion so far, we implicitly assumed that M, N and Z have no boundary. For later reference, we now state two more general versions.

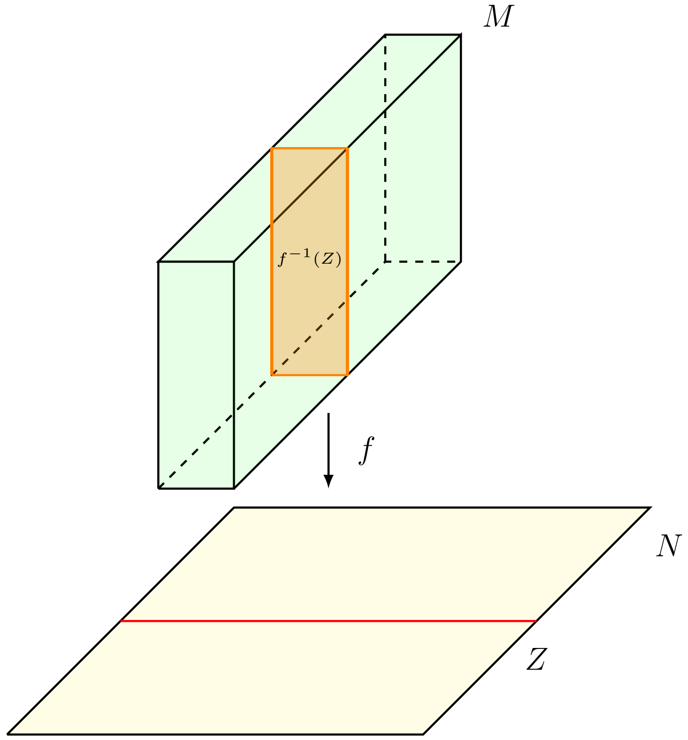
Theorem 2. *Suppose M is a manifold with boundary and $Z \subseteq N$ has no boundary, and $f : M \to N$ is a smooth map. If both f and $f|_{\partial M}$ are transverse to Z, then $f^{-1}(Z) \subseteq M$ is a submanifold with boundary $f^{-1}(Z) \cap \partial M$.* □

Theorem 3. *Suppose M has no boundary, but $Z \subseteq N$ is a submanifold with boundary. If $f : M \to N$ is transverse to both Z and ∂Z, then $f^{-1}(Z)$ is a submanifold with boundary $f^{-1}(\partial Z)$.* □

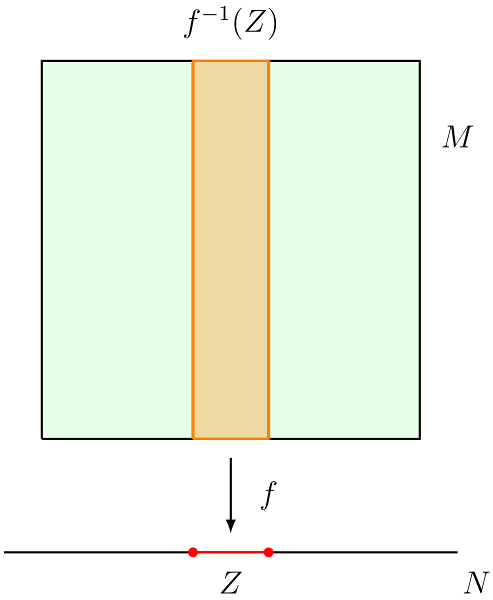
The proof of Theorem 2 directly follows from our earlier discussion of preimages of regular values in submanifolds with boundary, and the situation of Theorem 3 can be interpreted as a special case of a sublevel set of a regular value, which we also treated earlier.

Here are two simple examples illustrating these situations:

Example 1. Let $M := [-1, 1] \times \mathbb{R}^2$, let $N = \mathbb{R}^2$ and let $Z = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Consider $f : M \to N$ given by $f(x, y, z) = (x, y)$. Both f and $f|_{\partial M}$ are transverse to Z, and the preimage $f^{-1}(Z) \subseteq M$ is the strip $S = \{(x, 0, z) \in M\}$.



Example 2. Let $M = \mathbb{R}^2$, $N = \mathbb{R}$ and $Z = [-1, 1] \subseteq \mathbb{R}$. This time, we consider the map $f : M \to N$ given by $f(x, y) = x$. This is a submersion, so it is transverse to any submanifold of the target N, in particular also to Z and ∂Z. The preimage $f^{-1}(Z) \subseteq M$ is the strip $S = [-1, 1] \times \mathbb{R}$.



Question 2. What happens when both M and Z are allowed to have boundary?

Before we continue, we make a few further remarks on the definition of transversality.

Remarks 1.

- If $Z = \{q\} \subseteq N$ is a single point, the transversality of $f : M \to N$ just means that the point $q \in N$ is a regular value.
- For $Z = N$, *every* map is transverse to Z. Note that the above theorems are not very interesting in this case.
- A submersion $f : M \to N$ is transverse to *every* submanifold $Z \subseteq N$.
- A map $f : M \to N$ with $f(M) \cap Z = \varnothing$ trivially satisfies the conditions for transversality to Z.
- If $\dim M + \dim Z < \dim N$, then the condition (1) cannot be achieved for dimension reasons. So in this case transversality of f to Z means $f(M) \cap Z = \varnothing$.
- An interesting situation arises when $f : M \to N$ is an embedding of a submanifold. In this case, if f is transverse to another submanifold $Z \subseteq N$, then the intersection $M \cap Z$ is a submanifold of M, of Z and of N.

The results so far in some sense explain why transversality is desirable. Our next goal is to prove that it is also typical, in the sense that starting from an arbitrary map f it can be achieved by a suitable small perturbation.

Remark 2. The subset of maps transverse to a given submanifold is dense in both the strong and weak topologies on $C^\infty(M, N)$[1]. We will not get into the details of this statement, as a discussion of these topologies (which agree when M is compact) would lead us too far from the main aims of this course. The interested reader is invited to consult chapters 2 and 3 of the book [1] on this topic.

[1] It is also open if the domain M is a closed manifold.

The next fact is very simple, but so useful that we state it as a separate lemma.

Lemma 4. *Suppose the submanifold $Z \subseteq N$ is embedded as a* closed subset *of N, and $f : M \to N$ is a smooth map. Then the subset $T_f \subseteq M$ of points $p \in M$ at which transversality of f to Z holds is an open subset of M.*

Proof. The set T_f is the union of two sets, namely the set T_1 of points whose images under f avoid Z, and the set T_2 of points that get mapped to Z under f and where condition (1) holds.
Under the assumption of the lemma, the set $T_1 = f^{-1}(N \setminus Z) = M \setminus f^{-1}(Z)$ is clearly open in M. Therefore it remains to show that all points in T_2 are interior points of T_f. To prove this, let $p \in T_2$ be given, and let $q = f(p)$. Pick a submanifold chart $\psi : V \to \psi(V) \subseteq \mathbb{R}^n$ for Z near q with $\psi(q) = 0$ and $\psi(Z \cap V) = \psi(V) \cap \mathbb{R}^k \times \{0\}$. As f satisfies the condition (1) at p, we know that $\pi_{\mathbb{R}^{n-k}}$ maps $\psi_{*,f(p)} \circ f_{*,p}(T_pM)$ surjectively onto $\mathbb{R}^{n-k}$. As surjectivity of a linear map is equivalent (in coordinates) to the nonvanishing of the determimant of a suitable minor, this is an open condition. So there is an open neighborhood $W \subseteq M$ of p such that for any $x \in W$ either $f(x) \notin Z$ or (1) is satisfied at x. This shows that p is also an interior point of T_f. As $p \in T_2$ was arbitrary, this completes the proof of the lemma. □

Remark 3. A compact submanifold is automatically a closed subset. The condition excludes examples like $Z = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}^2$.

The main tool in proving that transversality is very common is provided by the following

Theorem 5 (Transversality for families)**.** *Suppose M, X, and N are manifolds (we assume that $\partial X = \varnothing$), and*

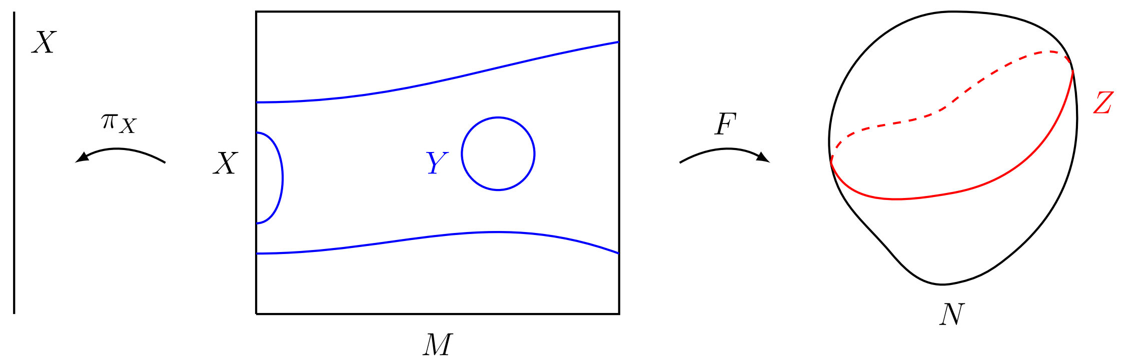
$$F : M \times X \to N$$

*is a smooth map transverse to the submanifold $Z \subseteq N$ without boundary. If $\partial M \neq \varnothing$, we also require that $F|_{\partial M \times X}$ is transverse to Z.
Then for almost all $x \in X$ the map*

$$\begin{aligned} F_x : M &\longrightarrow N \\ p &\mapsto F(p, x) \end{aligned}$$

*is transverse to Z.
Moreover, if M is compact and $Z \subseteq N$ is a* closed subset*, then the set of such points $x \in X$ is* open *and dense.*

Proof. By our assumption, the subset $Y := F^{-1}(Z) \subseteq M \times X$ is a submanifold, which has boundary $\partial Y = Y \cap \partial(M \times X) = Y \cap ((\partial M) \times X)$ if $\partial M \neq \varnothing$. Schematically, the situation might look like this:



We consider the projection $\pi = (\pi_X)|_Y : Y \to X$. By Sard's theorem, almost all points $x \in X$ are regular values of π and $\pi|_{\partial Y}$. Now the first part of the theorem follows from the following observation.

Claim: If $x \in X$ is a regular value of π, the $F_x \pitchfork Z$, and if $x \in X$ is a regular value of $\pi|_{\partial Y}$, then $(F_x)|_{\partial M} \pitchfork Z$.

As the two parts of the claim are proved by identical arguments, we only consider the first one. Note that

$$\begin{aligned} x \in X \text{ is a regular value for } \pi &\iff \forall\,(p,x) \in Y : \pi_{*,(p,x)} : T_{(p,x)}Y \to T_xX \text{ is surjective} \\ &\iff \forall\,(p,x) \in Y : T_{(p,x)}M \times X = T_{(p,x)}Y + T_{(p,x)}M \times \{x\}. \end{aligned}$$

Here the first equivalence is just the definition, and the second one follows from the fact that $T_{(p,x)}M \times \{x\} = \ker \pi_{*,(p,x)}$, so that surjectivity of $\pi_{*,(p,x)}$ is indeed equivalent to $T_{(p,x)}Y$ being a complement to it. Now by assumption we know that $F \pitchfork Z$, so that for $(p,x) \in Y$ we have

$$F_{*,(p,x)}(T_{(p,x)}M \times X) + T_{F(p,x)}Z = T_{F(p,x)}N.$$

Since at points $(p,x) \in Y$ we have $F_{*,(p,x)}T_{(p,x)}Y \subseteq T_{(F(p,x)}Z$, we conclude using the above equivalences that

$$\begin{aligned} &x \in X \text{ is a regular value for } \pi \\ &\implies \forall\,(p,x) \in Y : F_{*,(p,x)}(T_{(p,x)}M \times \{x\}) + T_{F(p,x)}Z = T_{F(p,x)}N, \end{aligned}$$

which in turn is equivalent to $F_x \pitchfork Z$.

It remains to prove the last part of the theorem. So suppose $Z \subseteq N$ is a closed subset. Then $F^{-1}(Z) \subseteq M \times X$ is also a closed subset. Let $U \subseteq X$ be an open subset such that the closure $K = \overline{U} \subseteq X$ is compact. If M is compact, then $Y \cap (M \times K)$ is also compact. Because the set $C \subset Y$ of critical points of π is closed, its intersection $C_K := C \cap (M \times K)$ is compact, and hence so is its image $\pi(C_K) \subseteq K$. But then the set $U \setminus C_K$ of regular values of π in U is open. As X can be covered by open sets U of this type, we conclude that the set of regular values of F is open.

The argument for $F|_{\partial Y}$ is completely analogous, and so the theorem is proven. □

Here is a first application of this result.

Example 3. Suppose that $f : M \to \mathbb{R}^n$ is any smooth map and $Z \subseteq \mathbb{R}^n$ is any submanifold. Then we apply the theorem to the map

$$F : M \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(p, \lambda) \mapsto f(p) + \lambda,$$

which is clearly a submersion, to conclude that there are arbitrarily small $\lambda \in \mathbb{R}^n$ such that the translation $f + \lambda$ is transverse to Z.
Note that a countable union of sets of measure 0 is still a set of measure 0, so we could even find an arbitrarily small $\lambda \in \mathbb{R}^n$ such that the translation $f + \lambda$ is transverse to any countable collection of submanifolds $Z_i \subseteq \mathbb{R}^n$, with no condition on their relative positions.

Our next goal is to generalize the result in the example to an approximation result for maps between manifolds by a transverse ones. In the statement we use

Definition 2. *Suppose $A \subseteq M$ is a closed subset. Two maps $f_0, f_1 : M \to N$ with $f_0(a) = f_1(a)$ for all $a \in A$ are called* smoothly homotopic relative to A *if there exists a smooth map*

$$F : M \times [0, 1] \to N$$

such that $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ and $F(a, t) = f_0(a)$ for all $(a, t) \in A \times [0, 1]$.

One easily sees that homotopy relative to A is an equivalence relation on maps that agree on A. For $A = \varnothing$, we get the usual notion of smooth homotopy.

Theorem 6. *Let $f : M \to M$ be a smooth map, where M is a compact manifold (possibly with boundary), and let $Z \subseteq N$ be a submanifold which is a closed subset. If $U \subseteq M$ is open such that $f|_{M \setminus U}$ and $f|_{\partial M \setminus U}$ are transverse to Z, then there exists a map $g : M \to N$ which is homotopic to f relative to $M \setminus U$ such that $g \pitchfork Z$ and $g|_{\partial M} \pitchfork Z$.*

Proof. By assumption, the image $f(M) \subseteq N$ is compact, so we can find finitely many vector fields $X_1, \ldots, X_r$ on N with compact support such that for all $q \in f(M)$ their values $X_1(q), \ldots, X_r(q)$ span T_qN. As the vector field have compact support, each of them has a complete flow

$$\phi^{X_i} : N \times \mathbb{R} \to N,$$

which is characterized by $\phi^{X_i}(q, 0) = q$ and $\frac{d}{dt}\phi^{X_i}(q, t) = X_i(\phi^{x_i}(q, t))$.
We also choose a smooth function $\beta : M \to \mathbb{R}$ such that $\beta|_{M \setminus U} = 0$ and $\beta|_U > 0$. We then define

$$F : M \times \mathbb{R}^r \to N$$

by

$$F(p, \lambda_1, \ldots, \lambda_r) := \phi^{X_1}_{\beta(p)\lambda_1} \circ \ldots \phi^{X_r}_{\beta(p)\lambda_r} \circ f(p).$$

We now make the following obserations:

(a) For all $p \in M \setminus U$ and all $\lambda \in \mathbb{R}^r$ we have $F(p, \lambda) = f(p)$. In particular, by assumption at all such points the maps F and $F|_{\partial M \times \mathbb{R}^r}$ are transverse to Z.

(b) For all $p \in U$ we have
$$\frac{\partial F}{\partial \lambda_i}(p,0) = \beta(p) \cdot X_i(p),$$
so that at all points $(p,0) \in U \times \{0\}$ the differential $F_{*,(p,0)}$ is surjective already on the second factor $\{0\} \times \mathbb{R}^r$. So at all points of $U \times \{0\} \subset M \times \mathbb{R}^r$ the maps F and $F|_{\partial M \times \mathbb{R}^r}$ are also transverse to Z.

By Lemma 4, the set of points $(p,\lambda) \in M \times \mathbb{R}^r$ for which transversality holds for F is open in $M \times \mathbb{R}^r$, and the set of points at which transversality holds for $F|_{\partial M \times \mathbb{R}^r}$ is open in $\partial M \times \mathbb{R}^r$. Combining this with observation (b) above and the fact that $\overline{U}$ is compact, we conclude that there is an $\varepsilon > 0$ such that the transversality conditions for both F and $F|_{\partial M \times \mathbb{R}^r}$ hold for all points $(p,\lambda) \in \overline{U} \times B(0,\varepsilon)$. Together with observation (a), this implies that the maps
$$F|_{M \times B(0,\varepsilon)} \qquad \text{and} \qquad F|_{\partial M \times B(0,\varepsilon)}$$
are transverse to Z. Picking a common regular value $\lambda_0 \in B(0,\varepsilon)$ of the projection to $\mathbb{R}^r$ from $F^{-1}(Z) \cap (M \times B(0,\varepsilon)$ and from $F^{-1}(Z) \cap (\partial M \times B(0,\varepsilon))$, we see that the map $g : M \to N$ defined as $g(p) := F(p,\lambda_0)$ has the properties stated in the theorem. □

Remarks 4. We collect a few more observations in the context of Theorem 6.

- The proof shows that the map g can in fact be choosen arbitrarily close to f.
- One can always choose U to be some open neighborhood of $f^{-1}(Z)$. Another standard situation for applications is $U = M$.
- If $f|_{\partial M} \pitchfork Z$, we can choose $U = M \setminus \partial M$, and get g transverse to Z which is homotopic to f relative to ∂M.
- It is interesting to compare the content of the theorem in the case when $Z = \{q\}$ is a point in N with the content of Sard's theorem. Theorem 6 asserts that we can make q a regular value by perturbing f, whereas Sard's theorem asserts that one can find regular values of the original f which are arbitrarily close to q.

References

[1] M. Hirsch, *Differential topology*, Springer GTM 33, 1976.