UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2005/2006; 1st Semester
dr Benedikt Löwe

Homework Set # 10 *Deadline:* November 22nd, 2005

Exercise 32 (total of seven points).

Let $\mathcal{L} := \{+, \cdot, 0, 1, -\}$ be the language of Boolean algebras and Φ_{BA} be the axioms of Boolean algebras. Let

$$\varphi :\Leftrightarrow \forall x \forall y \Big(\big((x \neq x \cdot y) \wedge (y \neq x \cdot y)\big) \to (x \cdot y = 0) \Big),$$
$$\psi :\Leftrightarrow \exists x \big((x \neq 0) \wedge (x \neq 1)\big).$$

Let Φ_0, Φ_1, Φ_2, and Φ_3 be the deductive closures of Φ_{BA}, $\Phi_{\mathrm{BA}} \cup \{\neg\psi\}$, $\Phi_{\mathrm{BA}} \cup \{\varphi\}$, and $\Phi_{\mathrm{BA}} \cup \{\varphi, \psi\}$, respectively. Investigate whether Φ_i is a complete theory. If it isn't, give a formula σ such that $\sigma \notin \Phi_i$ and $\neg\sigma \notin \Phi_i$. If it is complete, give a brief argument why. (1 point each for Φ_0 and Φ_1, 2 points for Φ_2, 3 points for Φ_3.)

Exercise 33 (total of three points).

Give the names of the following logicians and mathematicians (1 point each):

- X was one of the students of David Hilbert who was a teacher at the *Gymnasium Arnoldinum* from 1929 to 1948.
- Y was an important figure in the history of the *Deutsche Mathematiker-Vereinigung*. He was married to the granddaughter of Hegel, and is popularly known for the "Y bottle", a two-dimensional manifold not embeddable into $\mathbb{R}^3$.
- Z received his PhD degree in 1924 at the UvA for a thesis entitled *Intuitionistische axiomatiek der projectieve meetkunde* and was the PhD supervisor of a (retired) ILLC member.

(*One extra point:* What is the canonical webpage for finding information about supervisor-student relations in mathematics?)

Exercise 34 (total of five points).

Let $\mathbf{P} := \langle P, \leq \rangle$ be a **partial preorder** (*i.e.*, $\leq$ is a reflexive and transitive relation). For $x, y \in P$, define $x \equiv y$ by $x \leq y$ & $y \leq x$. Show that $\equiv$ is an equivalence relation (1½ points). Let $D := P/\!\equiv$ be the set of $\equiv$-equivalence classes. For $\mathbf{d}, \mathbf{e} \in D$, define $\mathbf{d} \leq \mathbf{e}$ if and only if there are $x \in \mathbf{d}$ and $y \in \mathbf{e}$ such that $x \leq y$. Show that this is well-defined (2 points) and that $\langle D, \leq \rangle$ is a partial order (1½ points).

Exercise 35 (total of seven points).

(1) Find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \oplus \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \oplus \mathbf{W}$ and explain why (2 points).
(2) Similarly, find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \otimes \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \otimes \mathbf{W}$ and explain why (2 points).
(3) In the first two tasks, you can choose one wellorder to be finite. Why can't both wellorders be finite in such an example (1 point)?
(4) Consider $\mathbf{L} := \langle \mathbb{Q}, \leq \rangle$ to be the rational numbers with the usual ordering. Find out whether $\mathbf{L} \oplus \mathbf{L}$ is isomorphic to $\mathbf{L}$ and give an argument (2 points).

    Hint. The Cantor Isomorphism Theorem (sometimes called "back-and-forth theorem") for countable linear orders may help. If you use it, you don't have to prove it, but please state it clearly with a proper reference to the literature and make sure that you apply it properly.