Capita Selecta : Set Theory

2020/21 1st SEMESTER

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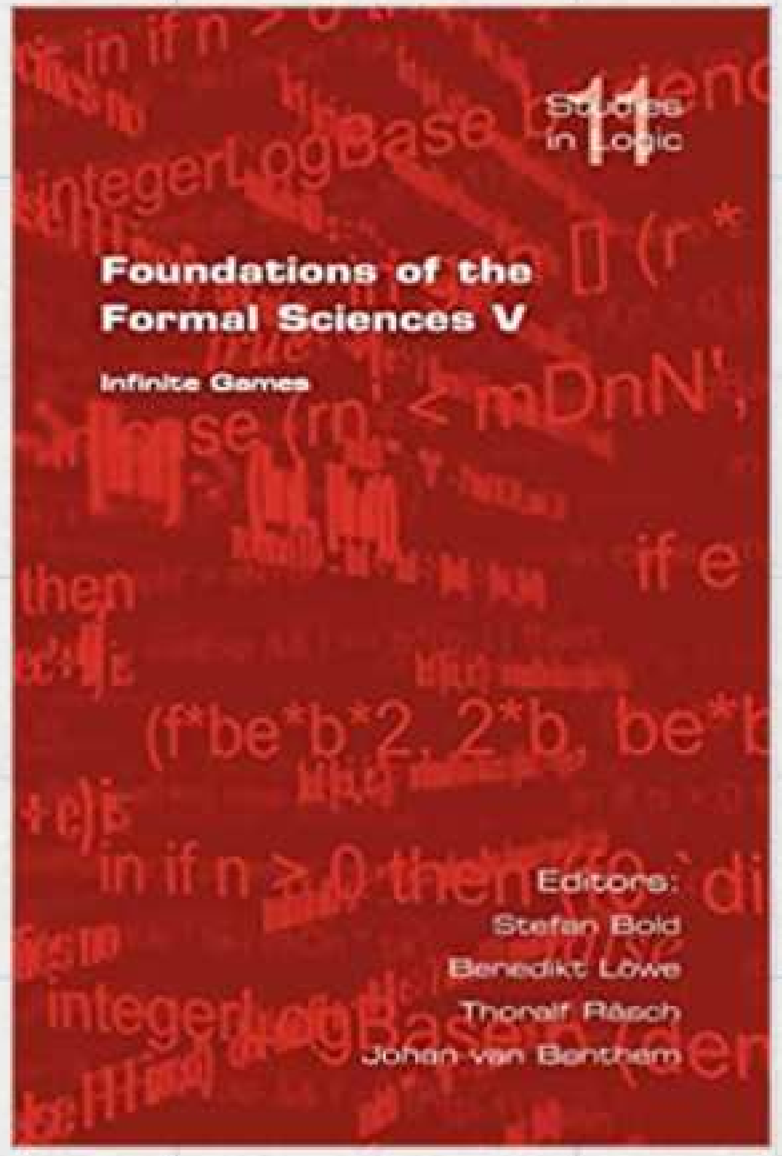
Lecture I [2 Sep 2020; 15–17]

Infinite games



What type of infinite games are we talking about?

- TWO PLAYER
- INFINITE LENGTH
- WIN-LOSE ($\sim$ ZERO SUM)
- PERFECT INFORMATION
- PERFECT RECALL

① Two players

② INFINITE LENGTH

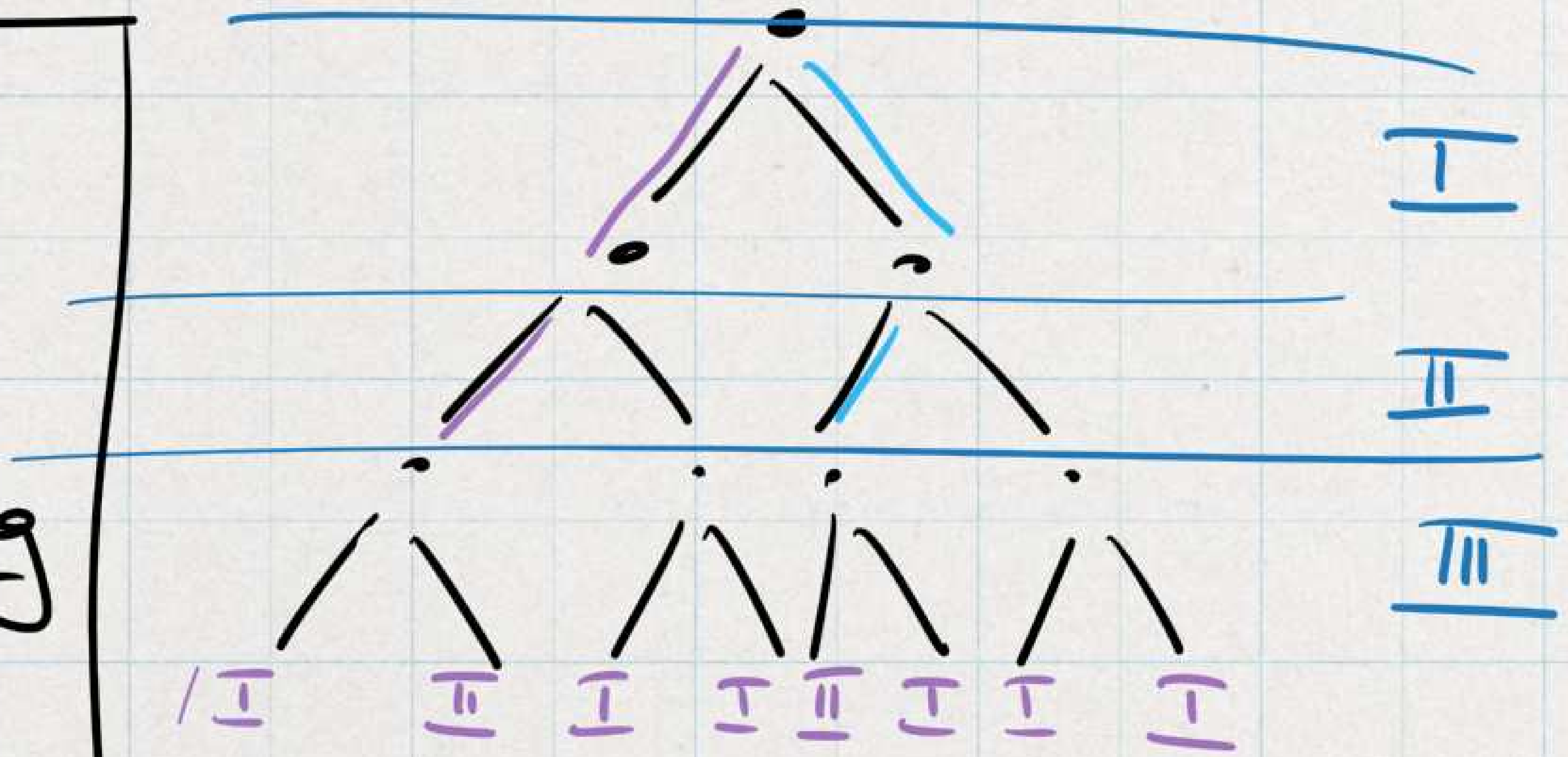
usually: Length ω

[we may say something about games of transfinite length:

$$\omega < \alpha \leq \omega_1$$

or bigger]

Three player game



No player has a strategy that enforces a win.

③ **WIN-LOSE** strengthening of ZERO SUM

We will have RUNs or PLAYs, i.e. ω-sequences of MOVES:

M move set

M^ω runs/plays

Payoff set $A \subseteq M^\omega$

Interpret $x \in A$ as I wins

$x \notin A$ as II wins.

ex. of non-zero sum

STAG HUNT

2,2	0,1
1,0	1,1

$\rightsquigarrow$ COOPERATION

zero-sum

10, −10	0,0
0,0	−1,1

④ PERFECT INFORMATION

At each step of the game, each player knows the entire state of the game.

Ex. : CHESS, CHECKERS, GO, ...

Non-Ex. : most card games ⟶ IMPERFECT INFORMATION GAME

⑤ PERFECT RECALL

At each step, each player recalls all previous moves.

HISTORY

1913

ÜBER EINE ANWENDUNG DER MENGENLEHRE AUF DIE THEORIE DES SCHACHSPIELS

VON E. ZERMELO.

Die folgenden Betrachtungen sind unabhängig von den besonderen Regeln des Schachspiels und gelten prinzipiell ebensogut für alle ähnlichen Verstandesspiele, in denen zwei Gegener unter Ausschluss des Zufalls gegeneinander spielen; es soll aber der Bestimmtheit wegen hier jeweilig auf das Schach als das bekannteste aller derartigen Spiele exemplifiziert werden. Auch handelt es sich nicht um irgend eine Methode des praktischen Spiels, sondern lediglich um die Beantwortung der Frage: kann der Wert einer beliebigen während des Spiels möglichen Position für eine der spielenden Parteien sowie der bestmögliche Zug mathematisch-objektiv bestimmt oder wenigstens definiert werden, ohne dass auf solche mehr subjektiv-psychologischen wie die des "vollkommenen Spielers" und dergleichen Bezug genommen zu werden brauchte? Dass dies wenigstens in einzelnen besonderen Fällen möglich ist, beweisen die sogenannten "Schachprobleme," d. h. Beispiele von Positionen, in denen der Anziehende *nachweislich* in einer vorgeschriebenen Anzahl von Zügen das Matt erzwingen kann. Ob aber eine solche Beurteilung der Position auch in anderen
chführung der Analyse in der unübersehbaren Kompli-
tzungen ein praktisch unüberwindliches Hindernis findet,
kbar ist und überhaupt einen Sinn hat, scheint mir doch
sein, und erst diese Feststellung dürfte für die praktische
d der "Eröffnungen," wie wir sie in den Lehrbüchern des
ere Grundlage bilden. Die im folgenden zur Lösung des
hode ist der "Mengenlehre" und dem "logischen Kalkül"
ie Fruchtbarkeit dieser mathematischen Disziplinen in
ausschliesslich um *endliche* Gesamtheiten handelt.

Ernst Zermelo

German logician

Ernst Friedrich Ferdinand Zermelo was a German logician and mathematician, whose work has major implications for the foundations of mathematics. He is known for his role in developing Zermelo–Fraenkel axiomatic set theory and his proof of the well-ordering theorem. Wikipedia

Born: July 27, 1871, Berlin

Died: May 21, 1953, Freiburg im Breisgau

Zermelo's Theorem

Every two-player, finite, win-lose, perf. inf., perfect recall game is DETERMINED.

One of the two players has a w.S.

Application for chess:

One of the following is true:

A) WHITE has w.s.

B) BLACK has w.s.

C) both players have drawing str.

FINITE

1913 Zermelo
DET. of finite games

1944 von Neumann-Morgenstern

MATHEMATICAL GAME THEORY

INFINITE

1920/30 POLAND
a lot of unpublished work on infinite games

BANACH-MAZUR 1935
inf. game characterising Baire property

1953 Gale-Stewart

1960 Blackwell
DESCRIPTIVE SET THY ⟷ INF. GAMES
game proof Π^1_1-uniformisation theorem
⇝ Solovay

Mycielski
AD
AXIOM OF DETERMINACY

DEEP CONNECTION BETWEEN AD & LARGE CARDINALS

1985 MARTIN-STEEL THM

Let M be the set of moves.

$$\begin{array}{c|cccccccc|cc} \mathrm{I} & m_0 & & m_2 & & m_4 & & m_6 & & & \cdots \\ \mathrm{II} & & m_1 & & m_3 & & m_5 & & m_7 & & \cdots \end{array}$$

Players play in ALTERNATION

Player I starts (note that this is an asymmetry)

Together, they produce a function

$$x : \mathbb{N} \longrightarrow M \quad \left[\text{i.e., } x \in M^\omega \right]$$

$\{x ; x : \mathbb{N} \to M\}$

We say $p \in M^{<\omega}$ is a POSITION.

$x \in M^\omega$ is a RUNS or PLAYS.

$A \subseteq M^\omega$ is a PAYOFF SET.

$G(A)$

I wins if the play $x \in A$

II wins if $x \notin A$

Our definition seems somewhat restrictive:

1. Both players have the same moves.
2. You might want to have non-alternating moves.

We'll see that these seemingly more general games are special cases of our games.

from Andretta's book

1.C Trees

Definition 1.25. Let X be a non-empty set. A **tree on** X is a $T \subseteq {}^{<\omega}X$ closed under initial segments, that is

$$\forall t \in T \, \forall s \subseteq t \, (s \in T).$$

The elements of T are called **nodes**. If $s \subset t$ and $s, t \in T$, then t is an **extension** of s, and if $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$ then t is an **immediate extension** of s. An $s \in T$ is a **terminal node** if it has no extensions, and the set of all terminal nodes is denoted by $\mathbf{tn}(T)$. A tree T is **pruned** if it has no terminal nodes, i.e., $\mathbf{tn}(T) = \emptyset$. A **branch** of a tree T on X is a sequence $f \in {}^{\omega}X$ such that

$$\forall n \in \omega \, (f \restriction n \in T).$$

The **body** of T is the set of all of its branches

$$[T] = \{f \in {}^{\omega}X \mid \forall n (f \restriction n \in T)\}.$$

A **sub-tree** of T is an $S \subseteq T$ which is closed under initial segments.

1.C Trees

from Andretta's book

$X^{<\omega}$ =

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POSITIONS

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= X^{ω}

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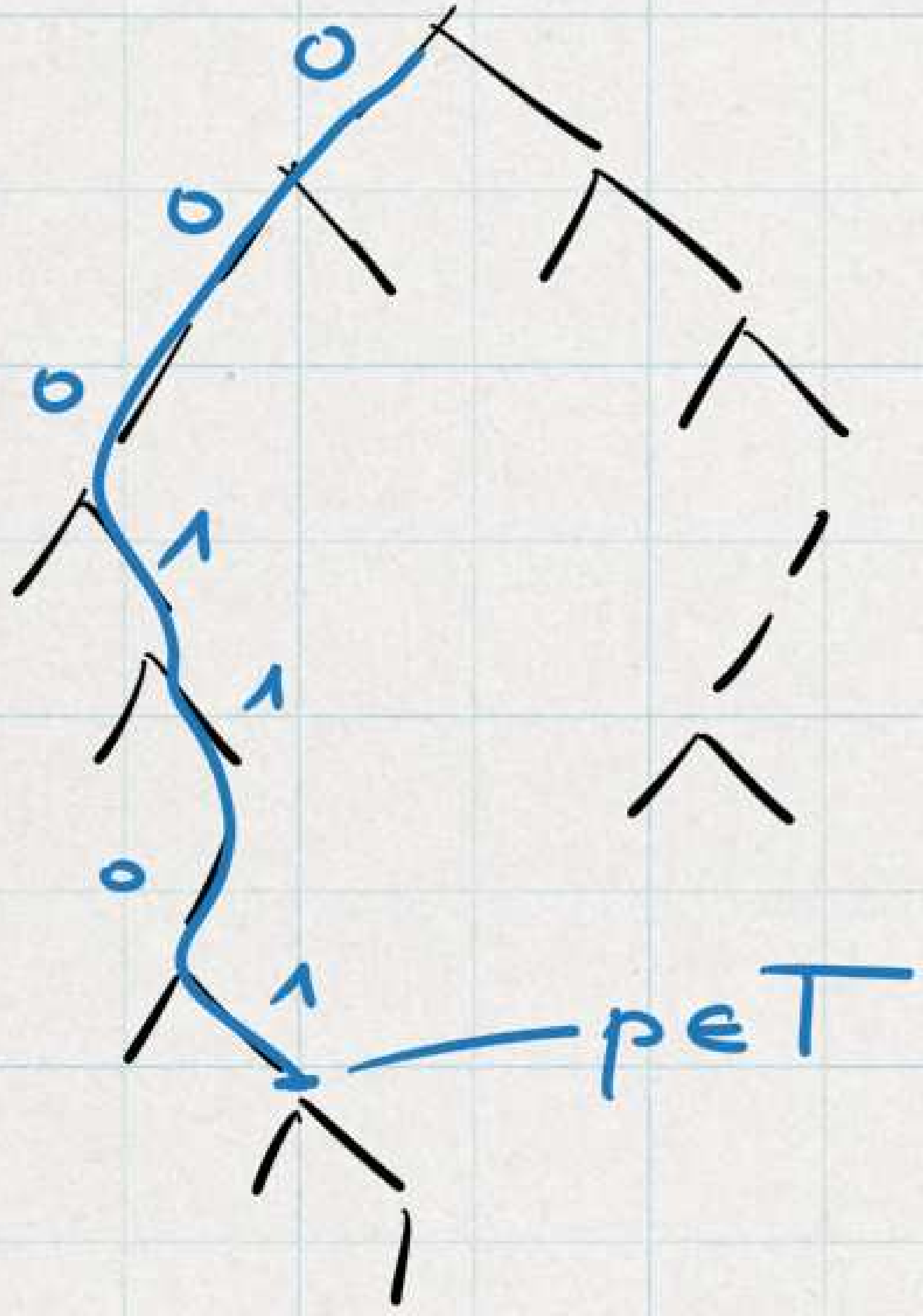
$T \subseteq \{0,1\}^{<\omega}$



$p = \langle 0001101 \rangle$.

$G(A_T) \approx$

Define a modified game $G(T; A)$

$T \subseteq M^{<\omega}$ tree

$A \subseteq M^\omega$ payoff set

$A_T := A \cap [T] \cup \{x ; x \notin [T]$ & the least n s.t. $x \upharpoonright n \notin T$ is even$\}$

I	m_0		m_2		m_4		...
II		m_1		m_3		m_5	...

$x(n) := m_n$

$x \in M^\omega$

We say that x is a win for I if

$x \in A$ and $x \in [T]$ or

$x \notin [T]$ and the least n s.t. $x \upharpoonright n \notin T$ is even

Called a game with rules: T as a finitary rule system

Whoever violates a rule loses; if no one violates a rule, then A det. the win.

So the games with rules look more general, but $G(T;A)$ is recaptured by $G(A_T)$.

- Now we can do different move sets:

  E.g., M_{I} move set for $\underline{\mathrm{I}}$

  M_{II} move set for $\underline{\mathrm{II}}$

  $M := M_{\mathrm{I}} \cup M_{\mathrm{II}}$ Rule tree T:

  $T := \{p \,;\, \forall n \;\; p(n) \in M_{\mathrm{I}}$ if n is even & $p(n) \in M_{\mathrm{II}}$ if n is odd$\}$

- Non-alternating moves

  E.g., $M_{\mathrm{I}} := M$

  $M_{\mathrm{II}} := M^2$

  $M^* := M_{\mathrm{I}} \cup M_{\mathrm{II}}$ and apply the idea of the rule tree above.

Strategies

(Andretta 19.8)

A strategy is just a function

$$\sigma : M^{<\omega} \longrightarrow M$$

[One could think of strategies for player I as

$$\sigma : M^{\text{even}} \longrightarrow M$$

& strategies for player II as

$$\tau : M^{\text{odd}} \longrightarrow M.$$

INFORMATIONAL OVERKILL !

f two strategies σ, τ, we define by recursion

$\sigma * \tau \in M^{\omega}$

$$\sigma * \tau(2n) := \sigma\big((\sigma*\tau)\restriction 2n\big)$$

$$\sigma * \tau(2n+1) := \tau\big((\sigma*\tau)\restriction 2n+1\big)$$

σ is a WINNING STR. FOR I / II IN $G(A)$ if $\forall \tau \ \sigma * \tau \in A$

$\forall \tau \ \tau * \sigma \notin A$