CS:ST Lecture #3

Due to a technical problem, the lecture was given on the ZOOM whiteboard without a proper pen.

RECAPITULATION of what was done in Lectures #1 & #2:

We defined games & gave necessary & sufficient conditions for winning:

NECESSARY	SUFFICIENT
I wins $\Rightarrow$ $\|A\| = 2^{\aleph_0}$	$\|A\| \leq \aleph_0 \Rightarrow$ II wins
II wins $\Rightarrow$ $\|2^\omega \setminus A\| = 2^{\aleph_0}$	$\|2^\omega \setminus A\| \leq \aleph_0 \Rightarrow$ I wins

AC $\Longrightarrow$ there is a non-determined set $A \subseteq 2^\omega$

[2^ω is wellordered]

This fragment of AC is sufficient for the proof.

This is what we're going to call a "fragment of AC".

This is relevant: AD [Axiom of Determinacy] is in conflict with AC but not with all of its fragments.

<u>Fragment</u> <u>of</u> <u>AC</u>

A conseq. of AC that does not imply AC

$ZFC \vdash \Phi$

$ZF + \Phi \nvdash AC$

DEF $\boxed{AD_M}$!!

The AXIOM OF DETERMINACY FOR GAMES ON M is "$\forall A \subseteq M^\omega$ $G(A)$ is determ."

Andretta's notation: $X \rightarrowtail Y$ & $X \twoheadrightarrow Y$ [cf. p ix]

THM ||| $\boxed{M \rightarrowtail N}$ [there is $f: M \to N$ inj.] then AD_N implies AD_M.

[Since for every M with $|M| \geq 2$ we have $\{0,1\} \rightarrowtail M$, $AD_M \Longrightarrow AD_2$.

$AC \Longrightarrow \neg AD_2$.

So, this is why it was enough to refute AD_2.

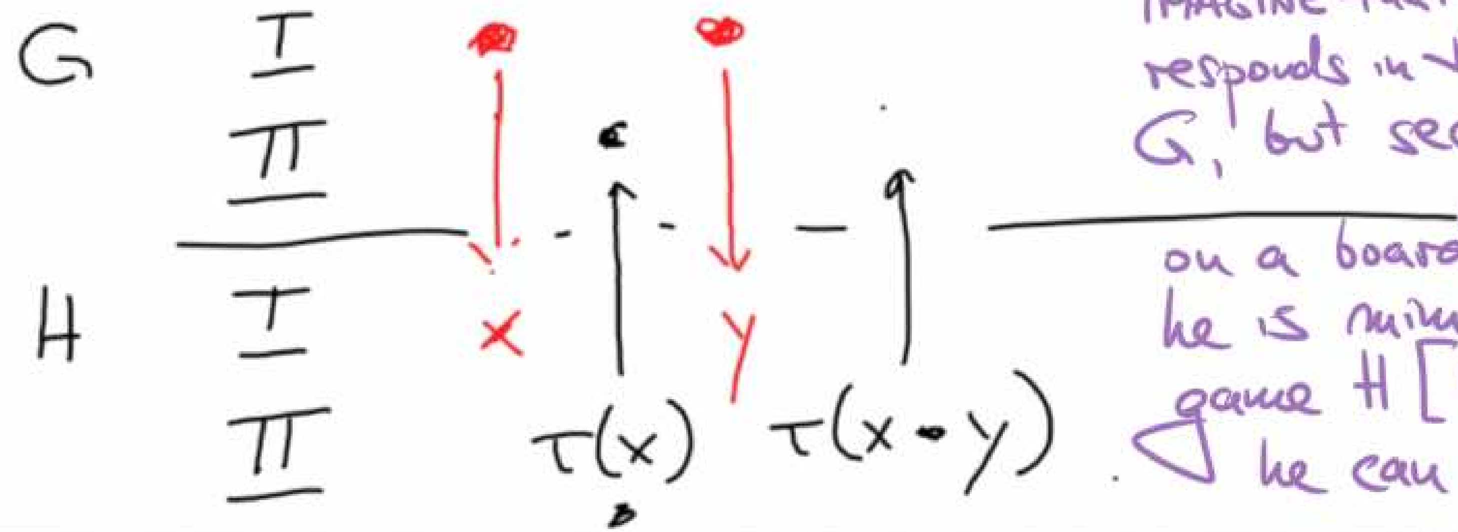
Note that AD_1 is trivially true.

This is a general method of which we're here going to see the simplest case.

AUXILIARY GAMES

G $\qquad$ H games

Supp. τ w.s. $\underline{II}$ in H



IMAGINE that player $\underline{II}$ responds in the game G, but secretly plays on a board where he is mimicking the game H [which game H he can win].

$f: M \to N$ determines
$f': M^{\omega} \to N^{\omega}$ &
$f'': M^{<\omega} \to N^{<\omega}$

But we're using the same symbol f for all three fns.

$M \overset{f}{\rightarrowtail} N$

$\cancel{\ } \longrightarrow AD_{N}$

$f: M^{\omega} \to N^{\omega}$

$f: M^{<\omega} \to N^{<\omega}$

WANT TO SHOW

$$\boxed{\begin{array}{c} \text{W.T.S. } AD_M \\ A \subseteq M^{\omega} \\ G = G(A) \end{array}}$$

$\underline{A^*} \; \underline{:=} \{f(x) ; x \in A\}$ $\quad H = G(T; A^*)$

where $T = \text{ran}(f)^{<\omega}$

$$\begin{array}{llllllll}
G & \frac{I}{II} & m_0 & f^{-1}(u_1) = m_1 & m_2 & f^{-1}(u_3) = m_3 & \\
 & & \downarrow & \uparrow & \downarrow & \uparrow & \\
H & \frac{I}{II} & f(m_0) = u_0 & \tau(u_0) & f(m_2) = u_2 & \tau(u_0 u_1 u_2) & \cdots \\
 & & & \shortparallel & & \shortparallel & \\
 & & & u_1 & & u_3 &
\end{array}$$

Then: if $m = (m_0, m_2, m_4, \ldots)$ is an arbitrary seq. for I, then let τ^* be the strategy as defined above.

Then $f(m * \tau^*) = u * \tau$ where $u_{2i} := f(m_{2i})$.

But by ass., $u * \tau \notin A^*$, so $m * \tau^* \notin A$.

REMARK Check where we're using that τ is winning in $G(T; A^*)$!

THM (GALE-STEWART 1953)

For transparency, let $M = \{0,1\} = 2$.

We call A **CLOSED** if there is a tree $T \subseteq 2^{<\omega}$ s.t. $A = [T]$.

Every closed set is determined.

[Backwards induction.]

If $A \subseteq 2^{\omega}$, then

$$T_A := \{x \upharpoonright n \,;\, n \in \omega, x \in A\}$$

$$[T_A] \supseteq A$$

In gen. $\neq$.

PROOF $\quad A = [T].$

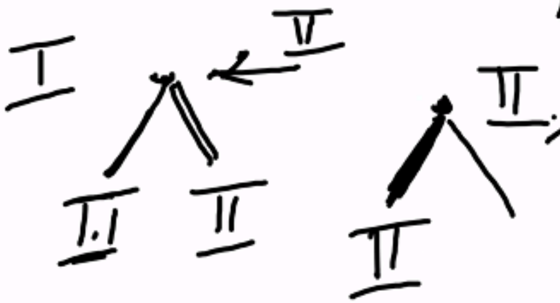
$$x \in A \longleftrightarrow \forall n \, (x \upharpoonright n \in T).$$

If x is a play of the game, then

I wins iff $\forall n \;\; x \upharpoonright n \in T.$

II wins iff $\underline{\exists n} \;\; \underline{x \upharpoonright n \notin T}.$

Partial function



$$\ell_0(p) := \begin{cases} \mathrm{II} & \text{iff } p \notin T. \\ \text{undef.} & \text{o/w.} \end{cases}$$

Suppose partial fn ℓ_n is defined.

Define ℓ_{n+1}:

$$\ell_{n+1}(p) := \ell_n(p) \quad \text{if } p \in \mathrm{dom}(\ell_n)$$

$$\ell_{n+1}(p) := \overline{\underline{\mathrm{II}}} \quad \text{if}$$

p has even length & $\ell_n(p0) = \ell_n(p1) = \overline{\underline{\mathrm{II}}}$

p has odd length & $\exists b \in \{0,1\}$ s.t. $\ell_n(pb) = \overline{\underline{\mathrm{II}}}$.

$$\ell' := \bigcup_{n\in\mathbb{N}} \ell_n$$

ℓ' part. fn. from $2^{<\omega} \longrightarrow \{\mathrm{II}\}$

$$\ell(p) := \begin{cases} \mathrm{II} & \text{if } p \in \mathrm{dom}(\ell') \\ \mathrm{I} & \text{o/w} \end{cases}$$

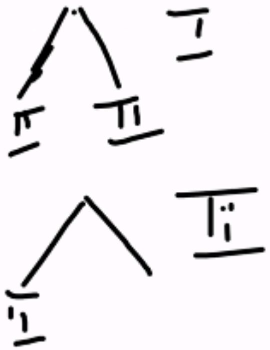
$\ell : 2^{<\omega} \longrightarrow \{\mathrm{I}, \mathrm{II}\}.$

In particular:

$$\ell(\emptyset) = \mathrm{I} \quad \text{or} \quad \ell(\emptyset) = \mathrm{II}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

I has w.s. $\qquad\qquad$ II has a w.s.



<u>Case 1</u> $\ell(\emptyset) = \mathrm{I}$.

If p has even length & $\ell(p) = \mathrm{I}$ then $\exists b\ \ell(pb) = \mathrm{I}$.

If p has odd length & $\ell(p) = \mathrm{I}$ then $\ell(p0) = \ell(p1) = \mathrm{I}$.

Define strategy σ^{ω} for I:

At p if $\ell(p) = \mathrm{II}$ give up.

if $\ell(p) = \mathrm{I}$

play 0 if $\ell(p0) = \mathrm{I}$

o/w play 1.

By properties:

$$\left.\begin{array}{l} \ell(\sigma * x \upharpoonright n) = \mathrm{I} \\ \Longrightarrow \sigma * x \upharpoonright n \in T \end{array}\right\} \Longrightarrow \sigma * x \in A \Longrightarrow \mathrm{I} \text{ wins!}$$

Case 2 $\quad \ell(\emptyset) = \overline{\underline{\text{II}}}$

$\mid \ell(p) = \overline{\underline{\text{II}}}$ & p has even length + $age(p) = 0$

$\Longrightarrow \ell(p0) = \ell(p1) = \overline{\underline{\text{II}}}$ OR $age(p0) < age(p)$ & $age(p1) < age(p)$

$\mid \ell(p) = \overline{\underline{\text{II}}}$ & p has odd length

$\Longrightarrow \exists b \ \ell(pb) = \overline{\underline{\text{II}}}$ + $age(p) = 0$ OR $age(pb) < age(p)$

Can get str. τ s.t. $\forall x \ \ell(x * \tau \upharpoonright n) = \overline{\underline{\text{II}}}$

$age(p) := n$ if $\ell(p) = \overline{\underline{\text{II}}}$ & n is least s.t. $p \in dom(\ell_n)$

Define strategy τ as:

at p if $\ell(p) = \underline{\overline{II}}$ pick b

s.t. $\ell(pb) = \underline{\overline{II}}$ and

age(pb) is minimal

[pick 0 in case of doubt]

$\ell(x * \tau \upharpoonright n) = \underline{\overline{II}}$

age$(x * \tau \upharpoonright n)$ is a decreasing seq.

hitting 0 somewhere, ~~thus~~ $\exists n$ ($x * \tau \upharpoonright n \notin T$)

q.e.d.

Relation to AC

The labelling works in ZF no matter what the set of moves M is.

The translation of the labelling into a strategy requires a method to pick from nonempty subsets of M.

This is easy if M is wellordered (e.g., $M = 2 = \{0,1\}$, $M = \omega$, $M = \underset{\text{ordinal}}{\alpha}$

Today: Subsections 1A & 1B

Section 1 (Andretta)

Fragments of AC.

Theory of cardinals / cardinalities:

$$X \sim Y \;:\Longleftrightarrow\; \exists f : X \longrightarrow Y \text{ bijection}$$

$$\underbrace{|X|}_{\text{object "the size of } X\text{"}} = \underbrace{|Y|}_{\text{"the size of } Y\text{"}}$$

[Scott's trick]

In general, the $\sim$-equivalence class of X is a proper class [except for $X = \emptyset$]. Scott's trick is used to use (some part of) the $\sim$-eq. classes as CANONICAL REPRESENTATIVE.

AC gives us canonical representatives for $\sim$-eq. classes

Thm (Zermelo).

$$\mathrm{ZFC} \vdash \forall X \,\exists \alpha \text{ ordinal } X \sim \alpha$$

$$|X| := \min\{\alpha;\ \alpha \sim X\}$$

cardinal of X

<u>Note</u>: the theory of ordinals and alephs can be developed in ZF

<u>HARTOG's THM</u> (ZF)

$\forall X \; \exists \alpha$ ordinal s.t.

$\alpha \not\rightarrowtail X$ [no inj.]

$\aleph_0 = \omega \qquad \aleph_1 := \min\{\alpha ; \alpha \not\rightarrowtail \aleph_0\}$

...

In ZF, we might have "cardinalities" that do not line up with the alephs.

X <u>finite</u> if $\exists n \in \mathbb{N}\ X \sim n$.

X <u>Dedekind-finite</u> iff $\mathbb{N} \not\rightarrowtail X$

$ZF \vdash$ every finite set is D-finite

$ZFC \vdash$ every D-finite set is finite

$ZF \nvdash$

We write $|A| \geq 2^{\aleph_0}$.
What does that mean without AC?

ZFC: $2^{\aleph_0} := \min\{\alpha ; \alpha \sim 2^{\omega}\}$

$|A| := \min\{\alpha ; \alpha \sim A\}$

In ZF, we write

$|X| = |Y|$ to mean $X \sim Y$

$|X| \leq |Y|$ to mean $X \rightarrowtail Y$

There is an inj. from 2^{ω} into A.

[In ZF, it is not necessarily the case that $Y \twoheadrightarrow X \Rightarrow X \rightarrowtail Y$!]

<u>CH</u>: $2^{\aleph_0} = \aleph_1$.

$|2^\omega|$ ORDINAL

If interpreted as: "there is a bijection between 2^ω and $\aleph_1$"

$\Longrightarrow 2^\omega$ is wellordered

<u>In ZF</u>: wCH weak CH: "$\forall A \subseteq 2^\omega$ $|A| \leq \aleph_0$ or $|A| = |2^\omega|$"

In ZFC, wCH $\longleftrightarrow$ CH. In ZF, $\not\longleftrightarrow$.

RELATIONSHIP BETWEEN POWER SETS & ORDINALS:

$$\mathcal{P}(\mathbb{N}) \sim 2^{\aleph_0}$$

IN GENERAL IN ZF

$$\mathcal{P}(\kappa) \twoheadrightarrow \kappa^+$$

$$\left[\mathcal{P}(\mathbb{N}) \twoheadrightarrow \aleph_1\right]$$

$$\cancel{\Leftrightarrow} \; 2^{\aleph_0} \geq \aleph_1$$

$$\Leftrightarrow \; \aleph_1 \rightarrowtail 2^{\aleph_0}$$

This is essentially the proof of HARTOGS'S THM.

$\kappa \sim \kappa \times \kappa$ [HESSENBERG'S THM]

$$\mathcal{P}(\kappa) \sim \mathcal{P}(\kappa \times \kappa)$$

$$\begin{aligned} &\quad\quad\quad\quad\quad\quad\quad \cup\!\!\!\!\!\mid \\ &\quad\quad\quad\quad\quad\quad\quad R \rightsquigarrow (\mathrm{field}(R), R) \end{aligned}$$

$$h : R \longmapsto \alpha \quad \text{iff} \quad (\mathrm{field}(R), R) \cong (\alpha, \in)$$

$$\alpha \rightarrowtail \kappa$$

If $\alpha < \kappa^+$, then $\exists R \in \mathcal{P}(\kappa \times \kappa)$ s.t.

$$h(R) = \alpha$$

I, X $\qquad AC_I(X)$ [FRAGMENTS]

for every family $\{A_i ; i \in I\}$ with $A_i \neq \emptyset$ & $A_i \subseteq X$

there is a choice fn $c : I \longrightarrow X$

$c(i) \in A_i$.

$$AC \iff \forall I \, \forall X \; AC_I(X)$$

<u>THM</u> If $X \xrightarrow{f} Y$, $J \xrightarrow{g} I$, then

$$AC_I(X) \Longrightarrow \cancel{AC_J(Y)}. \quad \{A_j ; j \in J\}$$

c choice fn for B_i $\quad c^*(j) := f(c(g(j)))$ $\quad B_i := f^{-1}[A_{g(j)}]$

As mentioned in class, this got a bit confusing. So, let's do it once more properly:

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<u>Theorem</u> If $X \twoheadrightarrow Y$ and $J \rightarrowtail I$, then
$$AC_I(X) \Longrightarrow AC_J(Y).$$

<u>Proof</u> Let $\{A_j ; j \in J\}$ be a family of subsets of Y s.t. $A_j \neq \emptyset \ \forall j$.
Let $f : X \twoheadrightarrow Y$ and $g : J \rightarrowtail I$ be the two functions.

For $i \in I$, let $B_i := f^{-1}\left[A_{g^{-1}(i)}\right]$.

[Note that $g^{-1}(i)$ makes sense since g is an injection.

[Note that if $A \subseteq Y$, $A \neq \emptyset$, then $f^{-1}[A] \neq \emptyset$ since f is a surjection!

So $\{B_i ; i \in I\}$ is an I-indexed family of non-empty subsets of X.

By $AC_I(X)$, find $c : I \longrightarrow X$ with $c(i) \in B_i$.

Define $c^* : J \longrightarrow Y$ by $c^*(j) := f(c(g(j)))$.

Then $c^*(j) = f(c(g(j)))$ with $c(g(j)) \in B_{g(j)}$

But then $f(c(g(j))) \in A_j$ q.e.d.