LECTURE 5 CS:St 2020

DST: INTERESTED IN SET THEO. PROPERTIES OF SUBSETS OF $\mathbb{R}$.

- Hierarchies (COMPLEXITY)

- Q: how complex should a SET THAT violates CH be?

- Q: CAN IT BE CLOSED? NO!

PRODUCT SPACES

why? X^ω IS A PRODUCT SPACE then IT IS HOMEOMORPHIC TO X^n $\forall\, 1 \leq n \leq \omega$ [SEE LEMMA 1 BELOW FOR THE MEANING OF THIS!]

LET X AND Y BE SETS WE DENOTE BY $\mathrm{Fn}(X, Y; \omega)$ THE SET OF FINITE FUNCTIONS FROM Y TO X.

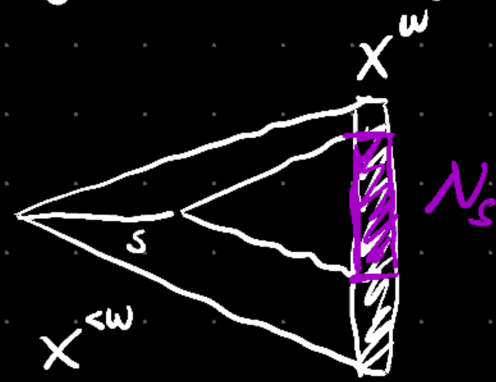
$$\mathrm{Fn}(X,Y;\omega) = \{ f \mid f \text{ IS A FUNCTION } \wedge f \subseteq Y \times X \wedge |f| < |\omega| \}$$

DEF LET X BE A NON-EMPTY SET AND Y BE A SET. THE PRODUCT TOPOLOGY X^Y IS THE TOPOLOGY GENERATED BY THE COLLECTION OF SETS OF THE FORM:

$$N_s := \{ f \in X^Y \mid s \subset f \} =: [s]$$

WHERE $s \in \mathrm{Fn}(X, Y; \omega)$.

WE WILL MOSTLY LOOK AT $Y = \omega$. X^ω



IF T IS A TREE $t, s \in T$ WE SAY THAT THEY ARE INCOMPATIBLE IFF $\neg \exists u$ st $s \subseteq u \wedge t \subseteq u$. AND WE WILL DENOTE THIS FACT BY $s \perp t$.

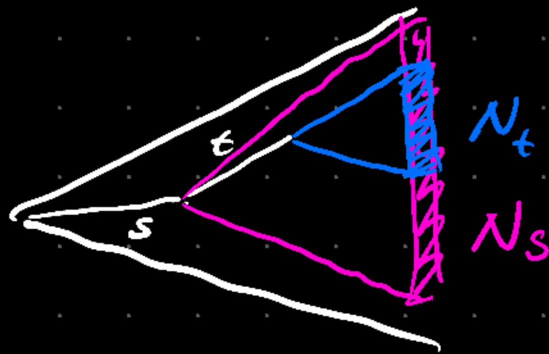
PROPOSITION (2.2) LET $X \neq \emptyset$ THEN:

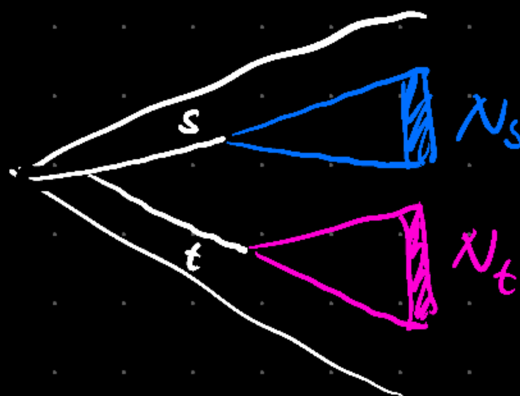
① $\forall s, t \in X^{<\omega}$ IF $s \subseteq t$ THEN $N_t \subseteq N_s$.

② $\forall s, t \in X^{<\omega}$ IF $s \perp t$ THEN $N_s \cap N_t = \emptyset$.

③ $\forall s \in X^{<\omega}$ N_s IS CLOPEN. (ZERO-DIMENSIONAL, TOTALLY DISCONNECTED)

④ X^ω IS SEPARABLE IFF X IS COUNTABLE.





LEMMA 1 For every $0 < n \leq \omega$ the space $(X^\omega)^n$ is homeomorphic to X^ω.

LEMMA Let X be a non-empty set then X^ω is completely metrizable.

Proof Let $d : X^\omega \times X^\omega \to [0,1]$

$$d(a,b) = \begin{cases} 0 & \text{if } a = b \\ \frac{1}{2^n} & \text{if } a \restriction n = b \restriction n \text{ and } a(n) \neq b(n). \end{cases}$$

QED

DEF A SPACE IS Polish IFF IT IS SEPARABLE AND COMPLETELY METRIZABLE.

COR IF $X \neq \emptyset$ AND COUNTABLE THEN IT IS Polish.

DEF LET T AND T' BE TREES AND $f: T \to T'$ BE A FUNCTION. WE SAY THAT f IS MONOTONE IFF $\forall s, t \in T \quad s \subseteq t \Rightarrow f(s) \subseteq f(t)$. MOREOVER WE SAY THAT f IS CONTINUOUS IFF IT IS MONOTONE AND

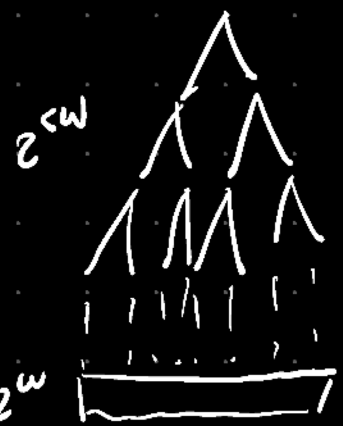
$$\forall a \in [T] \, \forall n \, \exists m \; lh(f(a \restriction m)) > n$$

LEMMA LET X AND Y BE NON-EMPTY SETS AND $f: X^\omega \to Y^\omega$ BE A FUNCTION. THEN f IS CONTINUOUS IFF THERE IS $\varphi: X^{<\omega} \to Y^{<\omega}$ which IS CONTINUOUS AND SUCH THAT

$$\forall a \in X^\omega \; f(a) = \bigcup_{n \in \omega} \varphi(a \restriction n) \, .$$

In DST they usually work on 2 spaces.

__DEF__ We call **Cantor** space the product 2^ω
We call **Baire** space the product ω^ω



not homeomorphic to $\mathbb{R}$!

- Cantor space is homeomorphic to Cantor set $E_{\frac{1}{3}}$

- Baire space is homeomorphic to the irrationals.

Q: are ω^ω and 2^ω homeomorphic? No!
(totally disconnected)

THEOREM CANTOR SPACE IS COMPACT.

ω^ω IS NOT. $\Big[$ $\{[n] \mid n\in\omega\}$ IS A COVER OF ω^ω BUT HAS NO FINITE SUBCOVER.

Q: CAN A CLOSED SET VIOLATE CH?

CAN THERE BE $C \subseteq 2^\omega$ CLOSED st

$$|\omega| < |C| < |\mathbb{R}| \text{ ?}$$

RECALL GIVEN $C \subseteq X^\omega$ THEN WE DEFINE

$$T_C = \{ s\in X^{<\omega} \mid \exists a\in C \ s\subset a\}$$

LEMMA (2.8) LET $C \subsetneq X^\omega$ BE CLOSED:

① T_C IS A PRUNED TREE AND $[T_C] = C$.

② THE BODY OF EVERY PRUNED TREE IS CLOSED.

③ UNDER $DC(X^{<\omega})$ T_C IS UNIQUE.

PROOF

① T_C IS PRUNED: LET $s \in T_C$
SO THERE IS $a \in C$ ST $s \subset a$ BUT
THEN $a \upharpoonright lh(s)+1 \in T_C$ AND $s \subset a \upharpoonright lh(s)+1$

$[T_C] = C$: $a \in C$ THEN BY DEF.
$\forall n$ $a \upharpoonright n \in T_C$ BUT THEN $a \in [T_C]$.

$a \in [T_C]$ AND $a \notin C$ THEN
$a \in X^\omega \setminus C$ SO THERE IS $n \in \omega$
ST $[a \upharpoonright n] \subseteq X^\omega \setminus C$ BUT THEN
$a \upharpoonright n \notin T_C$ ↯

② LET T BE PRUNED $[T]$ IS CLOSED.
LET $a \in X^\omega \setminus [T]$ THEN THERE IS $n \in \omega$ ST $a \upharpoonright n \notin T$ BUT THEN $[a \upharpoonright n] \subseteq X^\omega \setminus [T]$ AND $a \in [a \upharpoonright n]$
SO $X^\omega \setminus [T]$ IS OPEN.

③ ASSUME $DC(X^{<\omega})$ AND T TO BE A PRUNED TREE ST $[T] = C$.
W.T.S. $T = T_C$.

$T_C \subseteq T$ IF $t \in T_C$ THEN THERE IS $a \in C$ ST $t \subset a$. BUT THEN SINCE $a \in [T]$ THEN $t \in T$.

IF $t \in T$ THEN LOOK AT THE TREE

$$\rightarrow T_{[t]} = \{ s \in T \mid s \subseteq t \vee t \subseteq s \}$$

$$[T] = C$$
$$\cup\!|$$
$$[T_{[t]}]$$

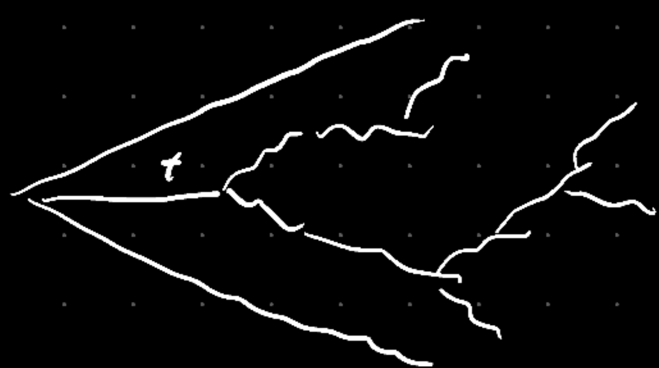
SO $T_{[t]}$ IS NON-EMPTY AND PRUNED.

By $DC(X^{<\omega})$ $[T_{[t]}] \neq \emptyset$ $a \in [T_{[t]}]$ so $a \in C$ so $t \in T_C$. $\boxtimes$

DEFINITION A TREE T IS PERFECT IFF

$\forall t \in T \, \exists s_1, s_2 \in T$ st $t \subset s_1 \wedge t \subset s_2 \wedge s_1 \perp s_2$

IF $T \neq \emptyset$



PROP (2.18) ASSUME $DC(X^{<\omega})$. FOR EVERY TREE T TFAE:

1. THERE IS A NON-EMPTY PERFECT SUBTREE OF T.

2. THERE IS A FUNCTION $f : 2^{<\omega} \to T$ which IS ORDER PRESERVING.

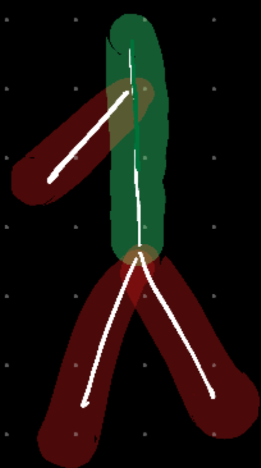
COROLLARY ASSUME $DC(X^{<\omega})$ THEN IF A TREE T HAS NON-EMPTY PERFECT SUBTREE THEN $|\mathbb{R}| \leq |[T]|$.

SO IF $T \subseteq 2^{\omega}$ THEN

$$|[T]| = |\mathbb{R}|.$$

LET T BE A TREE DEFINE THE CANTOR-BENDIXSON DERIVATIVE T' OF T AS THE TREE:

$$T' = \{ t \in T \mid \exists s_1, s_2 \quad s_1 \perp s_2 \land t \subseteq s_1 \land t \subseteq s_2 \}$$



DEFINE THE SEQUENCE:

$$T^0 = T$$

$$T^{\alpha+1} = (T^\alpha)'$$

$$T^\lambda = \bigcap_{\alpha\in\lambda} T^\alpha \quad \lambda \text{ limit.}$$

FACTS: IF $T^\alpha = T^{\alpha+1}$ THEN $\forall \beta > \alpha \; T^\alpha = T^\beta$ AND T^α IS PERFECT.

LEMMA IF T IS A TREE ON A WELL-ORDERED SET X THEN THERE IS $\bar{\alpha}$ st $T^{\bar{\alpha}} = T^{\bar{\alpha}+1}$. MOREOVER $\bar{\alpha} < (\max(\omega, |X|))^+$.

PROOF SINCE X IS WELL-ORDERABLE THEN T IS ALSO WELL-ORDERABLE. ASSUME $|T| = \kappa$ AND LET $g : \kappa \to T$ BIJECTION.

DEFINE $f : \mathrm{ORD} \to \mathrm{ORD}$

$$f(\beta) = \begin{cases} \alpha & \text{IF } \beta < \kappa \text{ AND } \\ & g(\beta) \notin T^{\alpha} \text{ BUT } \\ & g(\beta) \in T^{\gamma} \text{ FOR } \gamma < \alpha. \\ 0 & \text{O.W.} \end{cases}$$

BY REPLACEMENT $f[\kappa]$ IS A SET.

LET $\bar{\alpha}$ BE THE SMALLEST $> f[\kappa]$

THEN $T^{\bar{\alpha}} = T^{\bar{\alpha}+1}$.

FOR $\bar{\alpha} < |X^{<\omega}|^+ = (\max(\omega, |X|))^+$ ∎

Theorem Every closed subset of 2^ω is either countable or of size $|\mathbb{R}|$.

Proof

If $C \subseteq 2^\omega$ then there is T_C pruned st $C = [T_C]$.

Let $T_C^{\bar{\alpha}}$ be as before.

① If $T_C^{\bar{\alpha}}$ is non-empty by 2.18 C must have size $|\mathbb{R}|$.

② $T_C^{\bar{\alpha}} = \emptyset$:

Claim C is countable.

Proof

For all $a \in C$ let α_a be the least st

$$a \in [T_C^{\alpha_a}] \text{ but } a \notin [T_C^{\alpha_a+1}]$$

SUBCLAIM: $\exists n$ st $[T_c^{\alpha_a}] \cap N_{a\restriction n} = \{a\}$

PROOF For $n \in \omega$ LET $A_n = [T_c^{\alpha_a}] \cap N_{a\restriction n}$

LET $n \in \omega$ AND LET $b_n \in A_n$ AND $b_n \neq a$.

$\exists m \geq n$ $\quad b_n(m) \neq a(m)$ BUT THEN

$b_n \restriction m+1$ AND $a \restriction m+1$ THEY BOTH EXTEND $a \restriction n$. AND ARE INCOMPATIBLE

$$a\restriction n \in T^{\alpha_a+1}$$

BUT THEN $a \in [T^{\alpha_a+1}]$ $\lightning$ $\square$

SUBCLAIM

LET n_a BE THE SMALLEST SUCH n FOR EVERY $a \in C$.

DEFINE $f : C \to 2^{<\omega}$ AS FOLLOWS

$$f(a) = a \restriction n_a$$

f IS INJECTIVE:

IF $a \neq b$ WLOG $\alpha_a \leq \alpha_b$ THEN BOTH $a, b \in [T_C^{\alpha_a}]$ BUT $b \notin N_{a \restriction n_a}$

OTHERWISE $b \in [T_C^{\alpha_a}] \cap N_{a \restriction n_a} \neq \{a\}$

BUT THEN $a \restriction n_a \neq b \restriction n_b$.

SO $|C| \leq |2^{<\omega}| = |\omega|$ SO C IS COUNTABLE $\blacksquare$ CLAIM THEOREM.