CS:ST Lecture VIII

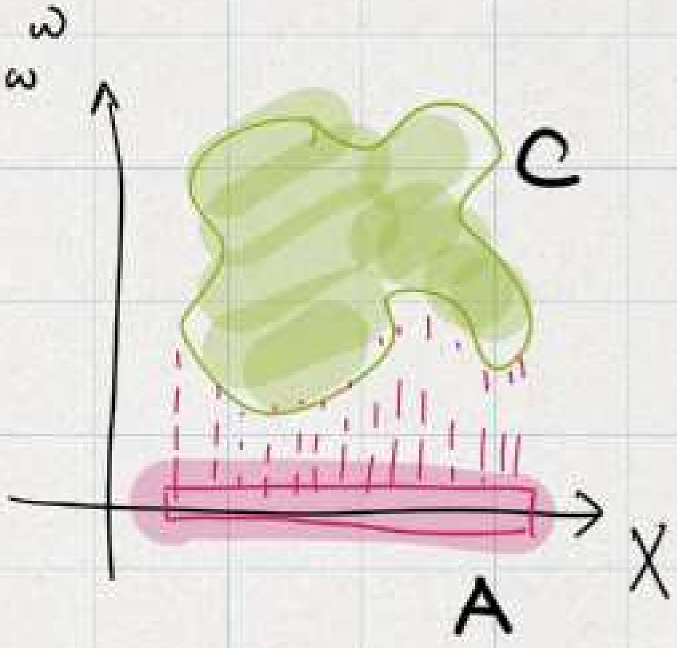
25 Sep 2020

1905

LEBESGUE'S FAMOUS MISTAKE:

He believed the Borels set are closed under projections.

→ They are not: new hierarchy on top of the Borel hierarchy with operations projection and complement.



$C \subseteq \omega^\omega \times X$ where $X = (\omega^\omega)^n$ "PRODUCT SPACE"

$$A = \mathrm{proj}_0(C) = pC = \exists^{\mathbb{R}} C$$
$$= \{x \in X ;\ \exists y \in \omega^\omega\ (x,y) \in C\}$$

PROJECTIVE HIERARCHY

If we had let

$\underset{\sim}{\Pi}^1_0$ = BOREL.

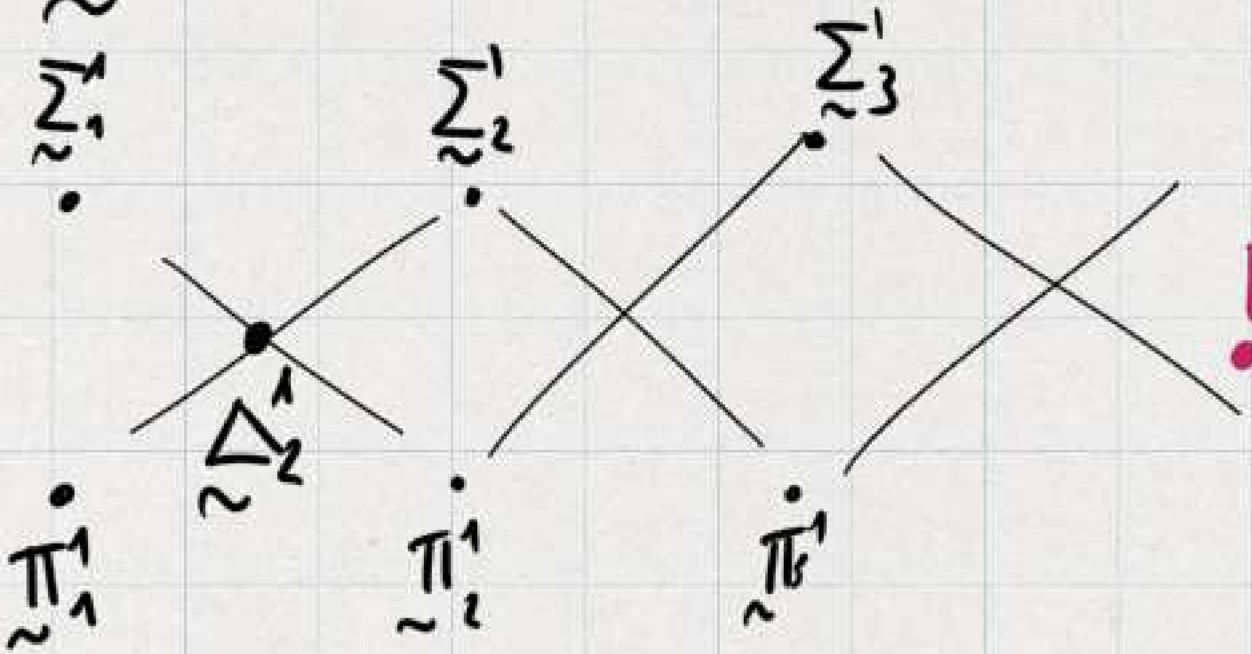
$$\underset{\sim}{\Pi}^1_0 := \underset{\sim}{\Pi}^0_1$$

$$\underset{\sim}{\Sigma}^1_{n+1} := \{pC ; C \in \underset{\sim}{\Pi}^1_n\}$$

$$\left[\underset{\sim}{\Sigma}^1_{n+1}\restriction X := \{pC ; C \in \underset{\sim}{\Pi}^1_n \restriction X \times \omega^\omega\}\right]$$

$$\underset{\sim}{\Pi}^1_n \restriction X := \{A ; X \setminus A \in \underset{\sim}{\Sigma}^1_n\}$$

$$\underset{\sim}{\Delta}^1_n := \underset{\sim}{\Sigma}^1_n \cap \underset{\sim}{\Pi}^1_n.$$



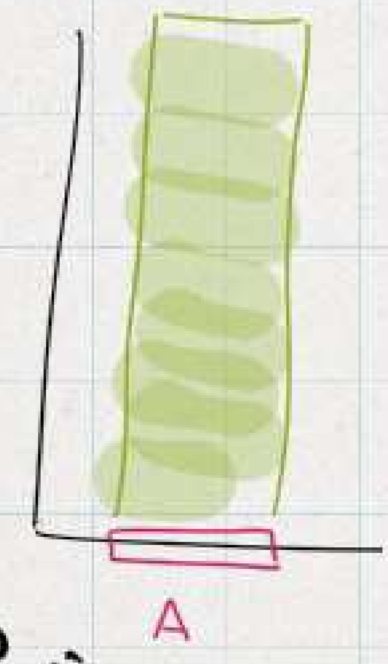
! Note that from this def. it's not obvious that

BOREL $\subseteq \underset{\sim}{\Sigma}^1_1$.

Proposition 5.2. *Assume* $\mathsf{AC}_\omega(\mathbb{R})$. *The collection of* $\mathbf{\Sigma}^1_1$ *sets contains all closed sets, is closed under countable unions and countable intersections. Therefore* $\mathrm{BOR} \subseteq \mathbf{\Sigma}^1_1$.

Proposition 5.4. $\mathbf{\Sigma}^1_1 = \{\emptyset\} \cup \{\mathrm{ran}(f) \mid f\colon \mathbb{R} \to \mathbb{R} \text{ is continuous}\}$.
Therefore $\mathbf{\Sigma}^1_1$ *is closed under continuous images:*

f cts $\iff$ $\{(x,y);\ f(x)=y\}$ is closed
$= \mathrm{graph}(f)$



If $A \subseteq X$ is closed, consider the cylindre

$$A \times \omega^\omega \subseteq X \times \omega^\omega$$

Clearly $p(A \times \omega^\omega) = A$ and $A \times \omega^\omega$ is closed.

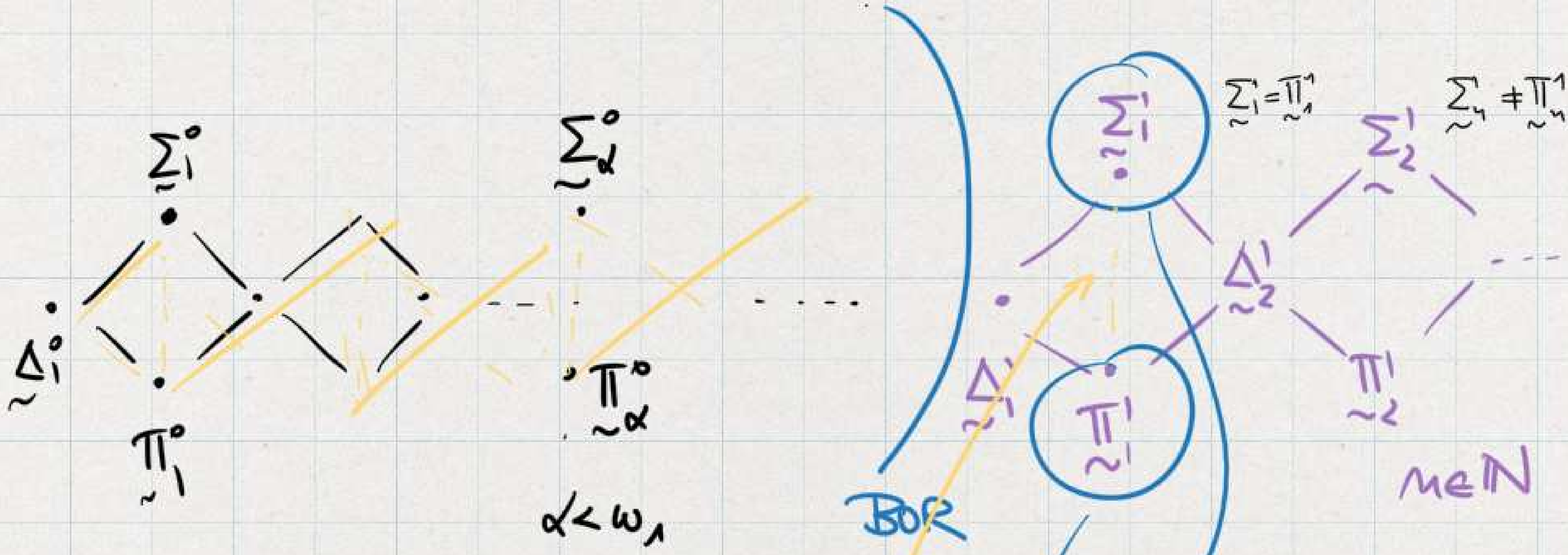
$A = \bigcup_{n \in \mathbb{N}} A_n$ where $A_n \in \mathbf{\Sigma}^1_1$, so $A_n = pC_n$ for C_n closed

$\boxed{AC_\omega(\mathbb{R})}$ to pick these

$\hat{C}_n := \{(x, ny);\ (x,y) \in C_n\}$

$C := \bigcup_{n \in \mathbb{N}} \hat{C}_n$ [Check that C is closed.]
Claim $A = pC$.

$x \in A \iff \exists n\ x \in A_n$
$\iff \exists n \exists y\ (x,y) \in C_n$
$\iff \exists n \exists y\ (x, ny) \in \hat{C}_n$
$\iff \exists z\ (x,z) \in C$
$\iff x \in pC$



Is this a proper hierarchy?

Technique of universal sets.

If Γ has univ. set, then $\check{\Gamma}$ has a univ. set.

If Γ has univ set, then $\bigcup(\omega;\Gamma)$ has a univ. set.

$$\exists^{\mathbb{R}}\Gamma = \{\underbrace{\exists^{\mathbb{R}}A}_{pA};\ A \in \Gamma\}$$

We need: If Γ has univ. set, then $\exists^{\mathbb{R}}\Gamma$ has a univ. set.

Prop. If $\underbrace{}_{\Gamma\restriction X\times\omega^\omega}$ has univ. set, then $\underbrace{}_{\exists^{\mathbb{R}}\Gamma\restriction X}$ has a univ. set.

Proof. Suppose V is universal for $\Gamma\restriction X\times\omega^\omega$. This means $V\subseteq \omega^\omega\times X\times\omega^\omega$, $V\in\Gamma\restriction\omega^\omega\times X\times\omega^\omega$, and f.e. $A\in\Gamma\restriction X\times\omega^\omega$ there is $a\in\omega^\omega$ s.t. $A=V_{(a)}$.

$$U:=\{(z,x)\in\omega^\omega\times X;\ \underline{\exists y\ (z,x,y)\in V}\}$$

Clearly U is $\exists^{\mathbb{R}}\Gamma$.

$$\exists^{\mathbb{R}}\Gamma\restriction X=\{\exists^{\mathbb{R}}A;\ A\in\Gamma\restriction X\times\omega^\omega\}$$

Suppose $A\in\exists^{\mathbb{R}}\Gamma\restriction X$ s.t. there is $C\in\Gamma\restriction X\times\omega^\omega$ s.t.

$$A=pC.$$

$$\exists^{\mathbb{R}}\Gamma=\{\exists^{\mathbb{R}}A;\ A\in\Gamma\}$$

Let $a\in\omega^\omega$ s.t. $C=V_{(a)}$.

CLAIM $A=U_{(a)}$.

$x\in A\leftrightarrow\exists y\ (x,y)\in C$
$\leftrightarrow\exists y\ (x,y)\in V_{(a)}$
$\leftrightarrow\exists y\ (a,x,y)\in V$
$\leftrightarrow(a,x)\in U$
$\leftrightarrow x\in U_{(a)}$.

This hierarchy is where the interesting set theory concerning determinacy happens:

ZFC proves Det(BOR). Martin (1975)

Martin (earlier $\sim$1970) ZFC + additional assumptions (large cardinals) proves $\mathrm{Det}(\underset{\sim}{\Sigma}^1_1)$.

It turned out that these large cardinal are necessary for these proofs:

There is a stratification that lines up determinacy in the proj. hierarchy & large cardinals:

$\mathrm{Det}(\underset{\sim}{\Pi}^1_1) \longleftrightarrow \mathrm{Det}(\underset{\sim}{\Sigma}^1_1) \leftrightsquigarrow$ a "measurable" cardinal

$\mathrm{Det}(\underset{\sim}{\Pi}^1_1) \longleftrightarrow \mathrm{Det}(\underset{\sim}{\Sigma}^1_2) \leftrightsquigarrow$ $\kappa < \lambda$ where

κ is WOODIN cardinal

λ is measurable

$\mathrm{Det}(\underset{\sim}{\Sigma}^1_{n+1}) \leftrightsquigarrow \kappa_1 < \kappa_2 < \dots < \kappa_n < \lambda$

s.t. κ_i are WOODIN

λ is measurable

not equivalence but "equivalent logical strength"

EQUICONSISTENCY.

DEFINABILITY

$\exists x^{(1)} \exists y^{(2)} \forall z^{(1)} \ldots \quad \underline{\underline{\varphi}}$

$\underline{\exists^{\mathbb{R}} C}$ If C was defined by φ

$$(x,y) \in C \longleftrightarrow \mathcal{A}^{(2)} \models \varphi(x,y)$$

$$x \in \exists^{\mathbb{R}} C \longleftrightarrow \exists y \ \ \mathcal{A}^{(2)} \models \varphi(x,y)$$

$$\longleftrightarrow \mathcal{A}^{(2)} \models \exists y^{(2)} \varphi(x, y^{(2)})$$

$\underline{\text{Ex.}}$ If A is $\underset{\sim}{\Sigma}^1_3$

C Π^1_2

D Σ^1_2

E Π^1_1

F Σ^1_1

G Π^0_1

$$\exists u^{(2)} \underbrace{\neg \exists z^{(2)} \neg}_{\forall} \exists y^{(2)} \forall x^{(1)} \psi \ldots$$

$$\underline{\exists u^{(2)} \forall z^{(2)}} \; \underbrace{\exists y^{(2)}} \forall x^{(1)} \psi \ldots$$

If you look at an arbitrary formula in the language in prenex form

$$\exists x_0^{(1)}\ \forall x_1^{(2)}\ \exists x_2^{(2)}\ \forall x_3^{(1)}\ \exists x_4^{(1)}\ \forall x_5^{(2)} \cdots$$

Andretta Lemma 4.28

each of the blocks are alternating.

then first & second order quantifiers could be mixed.

Observe

$$\exists x^{(1)}\exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

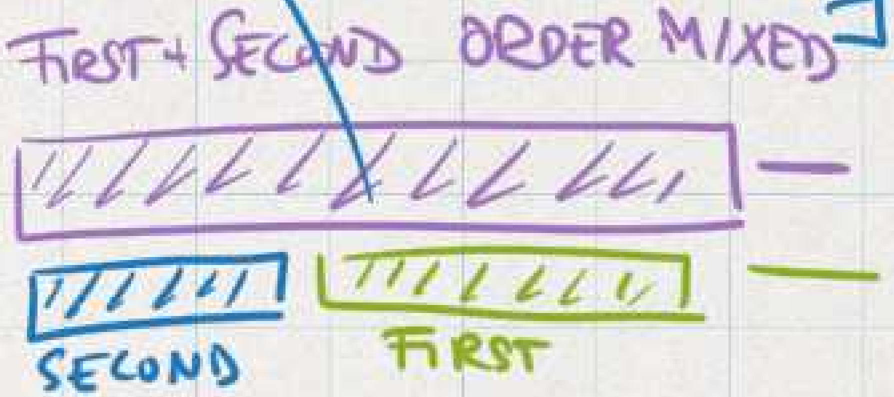
$$\forall x^{(1)}\exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

$$\exists x^{(1)}\forall y^{(2)} \underset{\not\longleftarrow}{\longrightarrow} \forall y^{(2)}\exists x^{(1)}$$

$$\rightsquigarrow$$

$$\exists x^{(2)}\exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

$\oplus$



Define Σ^1_1-formulas to be of the form $\exists x^{(2)} \varphi(x^{(2)}, \dots)$

where φ only has first-order qf.

Andretta 4E

Π^1_{n+1}-formulas $\forall x^{(2)} \varphi \dots$

φ is Σ^1_n

Σ^1_{n+1}-f. $\exists x^{(2)} \varphi \dots$

φ is Π^1_n.

Close under logical equivalence.

Theorem (Addison)

$AC_\omega(\mathbb{R})$ A is $\underset{\sim}{\Sigma}^1_n \iff$ there is $p \in \omega^\omega$ s.t.

A is $\Sigma^1_n(p)$

$[\iff$ there is φ Σ^1_n s.t.

$x \in A \iff \mathcal{A}^{(2)} \models \varphi(x, p)]$

A is $\underset{\sim}{\Pi}^1_n \iff$ there is $p \in \omega^\omega$ s.t.

A is $\Pi^1_n(p)$.

An example.

If $x \in \omega^\omega$ $n E_x m :\iff x(\langle n,m\rangle) \neq 0$.

If we have any countable structure (X,R) where X is ctble & $R \subseteq X \times X$ then we can w.l.o.g. assume that $X = \mathbb{N}$ (ω, R) and encode R in some $x \in \omega^\omega$ as follows:

$$x(\langle n,m\rangle) = \begin{cases} 0 & \text{if } n \not R m \\ 1 & \text{if } n R m. \end{cases}$$

$$LO := \{ x \,;\, (\omega, E_x) \text{ is a linear order} \}$$

$\forall n \quad n \not E_x n$

$\forall n,m,k \quad n E_x m \wedge m E_x k \rightarrow n E_x k$

$\forall n,m \quad n = m \vee n E_x m \vee m E_x n$

This is Borel.

$\exists x^{(1)}$

qf

In Lecture VI

TREE REPRESENTATION FOR CLOSED SETS

A is closed $\iff$ there is T tree s.t.

$A = [T]$.

$\boxed{\begin{array}{l} T = T_A \\ A \subseteq [T_A] \end{array}}$

$A \in \boldsymbol{\Sigma}^1_1 \iff$ there is C closed

s.t. $A = pC$

$\iff$ there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$

$A = p[T]$

Remember:

$[T] = \emptyset$

$\iff$

$(T, \supsetneq)$ is illfounded

If $T \subseteq (\omega \times \omega)^{<\omega}$ is a tree and $x \in \omega^\omega$, we define $T_x := \{ p \,;\, (x \upharpoonright lh(p), p) \in T \}$

Together $x \in A \longleftrightarrow T_x$ has an infinite branch

[is illfounded]

Theorem (Tree Representation for $\underset{\sim}{\Sigma}^1_1$ sets)

A is $\underset{\sim}{\Sigma}^1_1$ iff there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$ s.t.

$$x \in A \longleftrightarrow T_x \text{ is illfounded.}$$

Corollary ($\underset{\sim}{\Pi}^1_1$ sets)

$A \in \underset{\sim}{\Pi}^1_1$ iff there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$ s.t.

$$x \in A \longleftrightarrow T_x \text{ is wellfounded.}$$

$$\mathrm{WO} := \{x ; (\omega, E_x) \text{ is a wellorder}\}$$

$$= \mathrm{LO} \cap \{x ; \underbrace{\forall y \underbrace{\exists n \underbrace{x(y(n+1), y(n)) = 0}_{\text{qf}}}_{\Sigma^0_1}}_{\Pi^1_1}\}$$

WO is Π^1_1 (lightface).

If $x \in \mathrm{WO}$, define $\|x\| := \alpha$ iff $(\alpha, \in) \cong (\omega, E_x)$.

If $x \in \mathrm{WO}$, $\|x\| < \omega_1$. Also $\boxed{\omega \leq \|x\|.}$ This is slightly artificial.

$\omega \leq \alpha < \omega_1 \iff \exists x \in \mathrm{WO}\ \|x\| = \alpha.$

$$WO^* := \{ x \,;\, (\mathrm{fld}(x), E_x^*) \text{ is a wellorder} \}$$

$$\mathrm{fld}(x) := \{ n \in \mathbb{N} \,;\, \exists m \;\; x(\langle n,m \rangle) \neq 0 \text{ or } x(\langle m,n \rangle) \neq 0 \}$$

$$E_x^* := \{ (n,m) \,;\, n,m \in \mathrm{fld}(x) \wedge x(\langle n,m \rangle) \neq 0 \}$$

WO^* gives us all ordinal $< \omega_1$.

$$WO^* = \bigcup_{\alpha<\omega_1} WO_\alpha^*$$

Π^1_1

$$WO_\alpha := \{ x \in WO ; \|x\| = \alpha \}$$

Proposition 5.22. *For every $\alpha < \omega_1$ and every $y \in \mathrm{WO}_\alpha$, the sets*

$$\mathrm{WO}_\alpha, \quad \mathrm{WO}_{<\alpha}, \quad \mathrm{WO}_{\leq\alpha}$$

are $\Delta^1_1(y)$. In particular, they are Borel.

1. $z \in \omega^\omega$ is an isom. betw. (ω, E_x) and (ω, E_y) is arithmetical. [Being structure preserving only requires nat. number quantifiers.]
2. So $(\omega, E_x) \cong (\omega, E_y) \iff \exists z$ z is isom. ... $\longrightarrow \Sigma^1_1$.

If now $y \in WO_\alpha$, then $x \in WO_\alpha$

$\iff$

$$\underbrace{(\omega, E_x) \cong (\omega, E_y)}_{\Sigma^1_1(y)}$$

$\Pi^1_1(y)$

Still need to show $\Pi^1_1(y)$.

Π^1_1 $\Pi^1_1(y)$

$$\underbrace{x \in WO_\alpha}_{\Pi^1_1(y)} \iff \underbrace{x \in WO}_{\Pi^1_1(y)} \wedge \forall n \forall z \ulcorner z \text{ is not isom. between } (\omega, E_x) \text{ and the initial segment of } (\omega, E_y) \text{ defined by } n \urcorner$$

$\Pi^1_1(y)$

$\wedge \forall n \forall z \ulcorner z \text{ is not isom. between } (\omega, E_y) \text{ and the initial segm. of } (\omega, E_x) \text{ given by } n \urcorner$ $\Pi^1_1(y)$

Proposition 5.22. *For every $\alpha < \omega_1$ and every $y \in \mathrm{WO}_\alpha$, the sets*

$$\mathrm{WO}_\alpha, \quad \mathrm{WO}_{<\alpha}, \quad \mathrm{WO}_{\leq\alpha}$$

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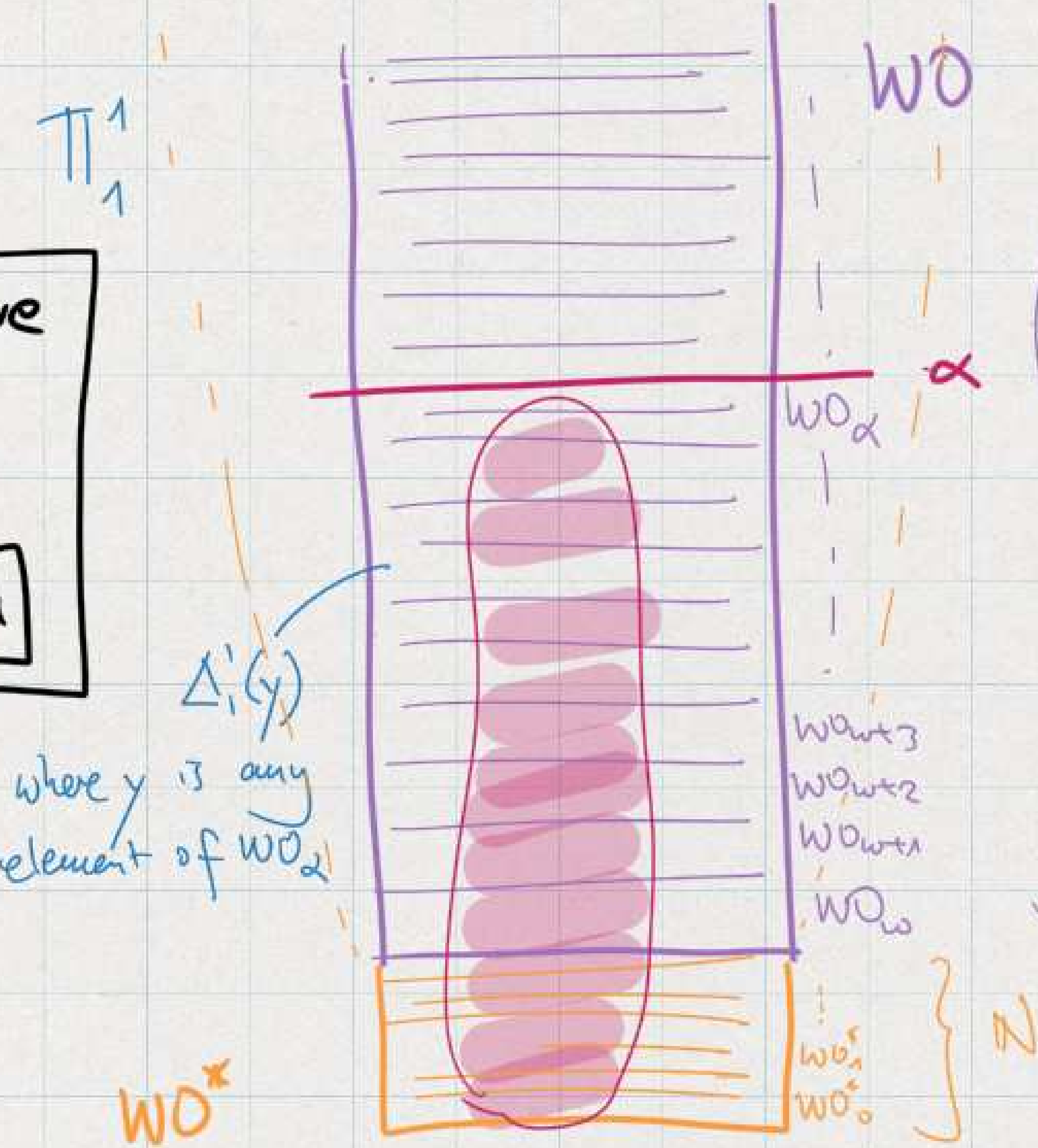
We did not prove that

$\Delta^1_1 \subseteq$ BoR!

Theorem 5.9 in

Andretta



Next time

WO is not $\underset{\sim}{\Sigma}^1_1$!

BOUNDEDNESS LEMMA

If $A \subseteq WO$ is Σ^1_1 then $\exists \alpha < \omega_1 \ A \subseteq \bigcup_{\beta<\alpha} WO_\beta$.