CS:ST Lecture XI

7 October 2020

$Det(\utilde{\Sigma}^1_1)$

Goal: Proving analytic determinacy! From extra assumptions!!

Lectures XI & XII:

Wed
1. "extra assumptions"
2. Prove tree representation theorem for $\utilde{\Pi}^1_1$ sets

Fri
3. Define auxiliary game that is determined and allows to "translate" the w.s.

→ ZFC is not able to prove more than $Det(\utilde{\Delta}^1_1)$!

<u>Theorem</u> In ZFC, $\mathrm{Det}(\underset{\sim}{\Sigma}^1_1) \Longleftrightarrow \mathrm{Det}(\underset{\sim}{\Pi}^1_1)$.

[It is not true that "A is determined $\Longleftrightarrow$ $\omega^\omega \setminus A$ is determined".]

<u>Proof.</u> Suppose $P \in \underset{\sim}{\Pi}^1_1$ and $A := \omega^\omega \setminus P$.

Then A is $\underset{\sim}{\Sigma}^1_1$. Let's assume $\mathrm{Det}(\underset{\sim}{\Sigma}^1_1)$ and show that $G(P)$ is determined.

Define $A^* := \bigcup_{n \in \mathbb{N}} nA$

$nA := \{nx \,;\, x \in A\}$

Shift function $x^{\leftarrow}(n) := x(n+1)$ $\langle n \rangle ^\frown x$

$sh : x \longmapsto x^{\leftarrow}$ is continuous

$sh^{-1}[A] = A^*$ So A^* is a cts preimage of A, so $A^* \in \underset{\sim}{\Sigma}^1_1$.

$\Longrightarrow G(A^*)$ is determined

$$A^* = \bigcup_{n \in \mathbb{N}} \underline{n}A$$

determined

$G(P)$		$G(A^*)$
II has w.s. in $G(P)$	$\longleftarrow$	I has w.s.
I has w.s. in $G(P)$	$\longleftarrow$	II has w.s.

$\longrightarrow$ P is determined. q.e.d.

[So: if we want to prove analytic determinacy $Det(\underset{\sim}{\Sigma}^1_1)$ we can also just prove $Det(\Pi^1_1)$]

(1) "Additional assumptions"

LARGE CARDINAL AXIOMS

We are not going to see any details: just a broad overview and the statement that we need later.

"LARGE CARDINAL AXIOM"

⟶ a property of cardinals Φ is a "large cardinal property" if it makes cardinals κ with $\Phi(\kappa)$ so large that ZFC cannot prove their existence.

Example A cardinal κ is called WORLDLY if

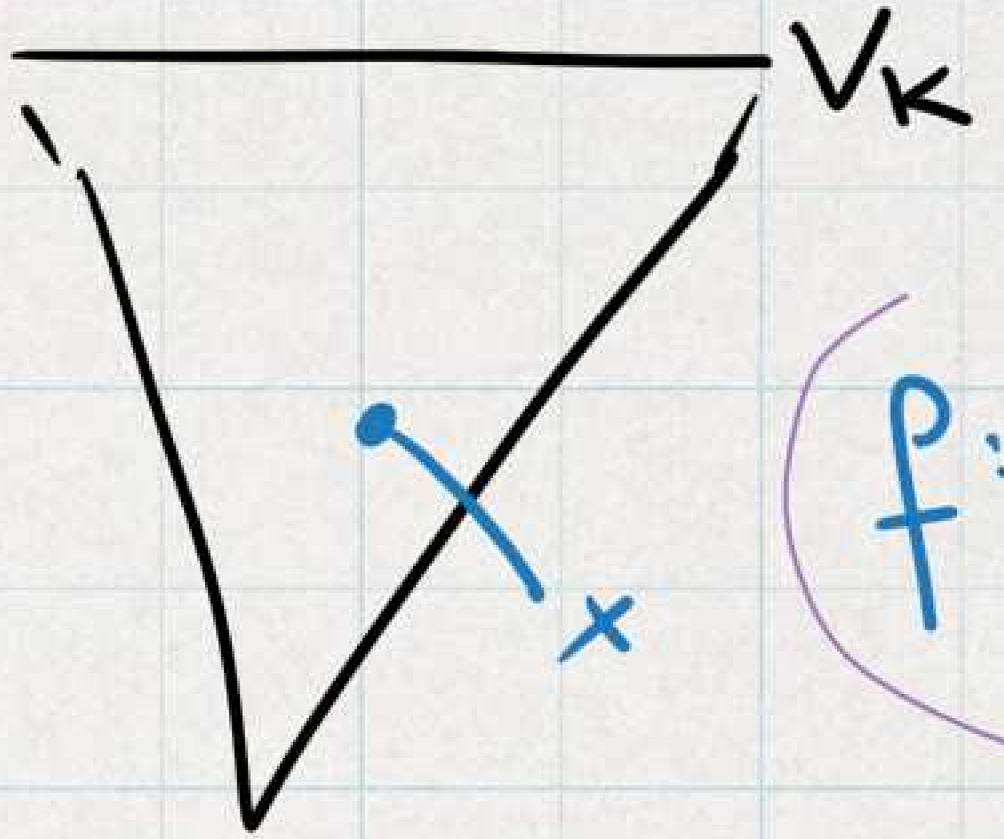
$$V_\kappa \models \underline{ZFC}.$$

[κ is closed under all set-theoretic constructions]

— Gödel's Incompleteness Thm implies immediately that $ZFC \nvdash$ "there are worldly cardinals".

— The smallest worldly cardinal* is not $\underline{\text{regular}}$

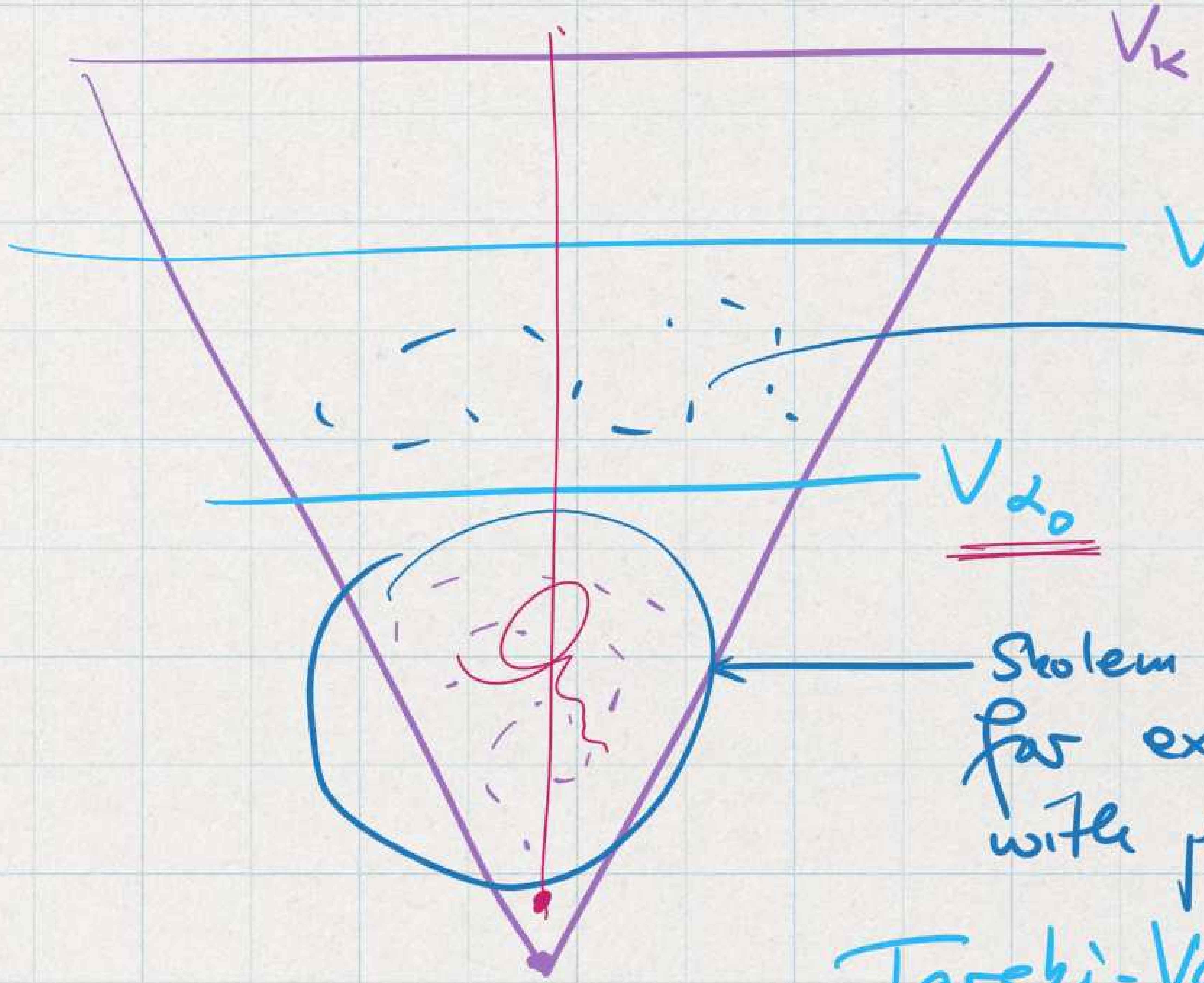
$$cf\,\kappa = \kappa.$$



$f : x \longrightarrow \kappa$ unbounded

not definable in V_κ

Proof Sketch of this:
Build the Skolem hull inside V_κ.



Skolem witnesses for ex. formulas with par. in V_{α_0}

Skolem witnesses for ex. formulas with par. in $\emptyset$

Tarski-Vaught

$$\boxed{\text{Therefore } \mathrm{cf}\,\kappa = \aleph_0.}$$

$$\Uparrow$$

ZFC $\vdash$ $\bigcup_{n\in\mathbb{N}} V_{\alpha_n} \preccurlyeq V_\kappa$.

$\Longrightarrow$ min. of κ: $\bigcup V_{\alpha_n} = V_\kappa$.

MEASURABLE CARDINALS

0.A.6 Filters and ideals

A **filter** on a set $X \neq \emptyset$ is a non-empty collection of subsets of X closed under intersections and supersets, i.e.,

- $A, B \in F \Rightarrow A \cap B \in F$,
- $A \in F \wedge A \subseteq B \subseteq X \Rightarrow B \in F$.

A filter is **proper** if $\emptyset \notin F$, and it is **principal** if it is the collection of all supersets of some $B \subseteq X$, i.e. it is of the form $\{A \subseteq X \mid B \subseteq A\}$.

A proper filter on X which is maximal under inclusion is called an **ultrafilter** on X.

A filter F is **κ-complete** if it is closed under intersections of $< \kappa$ elements, i.e., if for any $\gamma < \kappa$ and any choice of $A_\alpha \in F$, then $\bigcap_{\alpha<\gamma} A_\alpha \in F$. Note that this definition makes sense for all ordinals $\kappa > 2$, although it is most useful when κ is a cardinal. Whenever F is an ultrafilter, this can be restated as follows: if $\gamma < \kappa$ and $\bigcup_{\alpha<\gamma} A_\alpha \in F$, then $A_\alpha \in F$ for some $\alpha < \gamma$. Dually, an ideal is κ-complete if it is closed under $< \kappa$-unions or, equivalently, if its dual filter is κ-complete. Thus every filter and ideal is ω-complete, and a filter (ideal) is ω_1-complete just in case it is closed under countable intersections (unions). Often ω_1-completeness is called σ-completeness.

Definition 14.1. An ordinal $\kappa > \omega$ is **measurable** if there is a κ-complete, non-principal ultrafilter on κ.

If D is a κ-complete, non-principal ultrafilter on a set X, its characteristic function $\mu \colon \mathscr{P}(X) \to 2$ satisfies:

(88a) $\mu(\emptyset) = 0,$

(88b) $\mu(\{x\}) = 0$ for all $x \in X$,

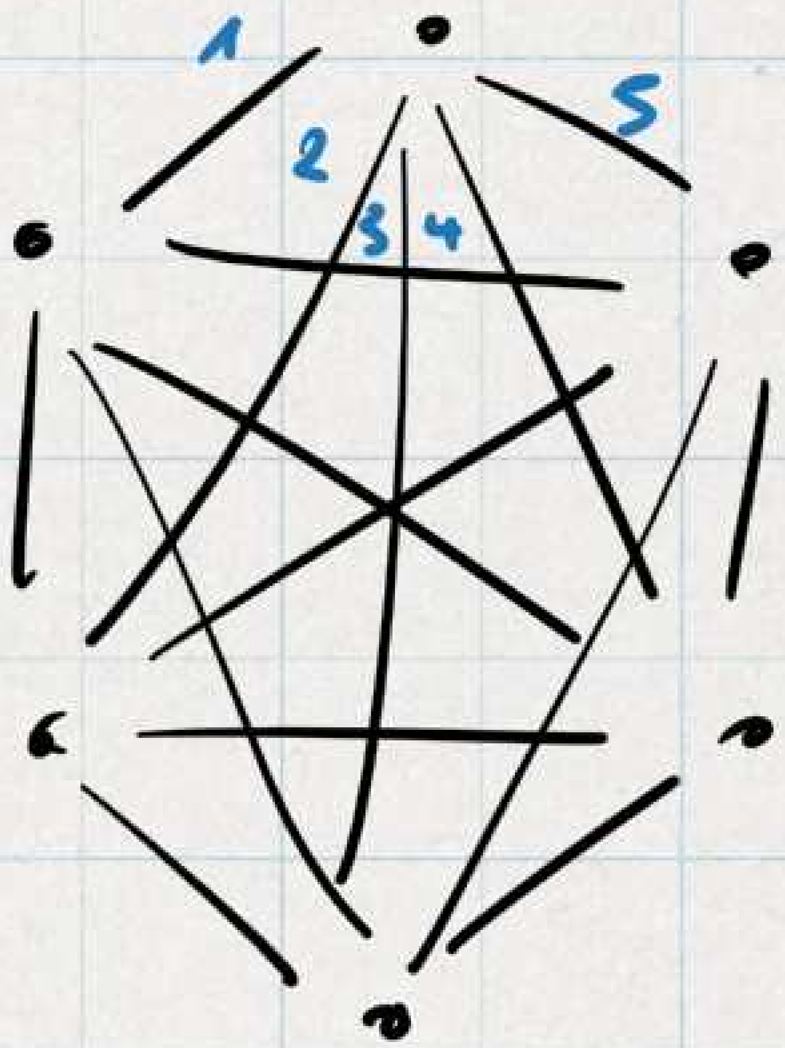
(88c) $\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha)$ for all $\gamma < \kappa$ and pairwise disjoint $X_\alpha \subseteq X$ and $\alpha < \gamma$,

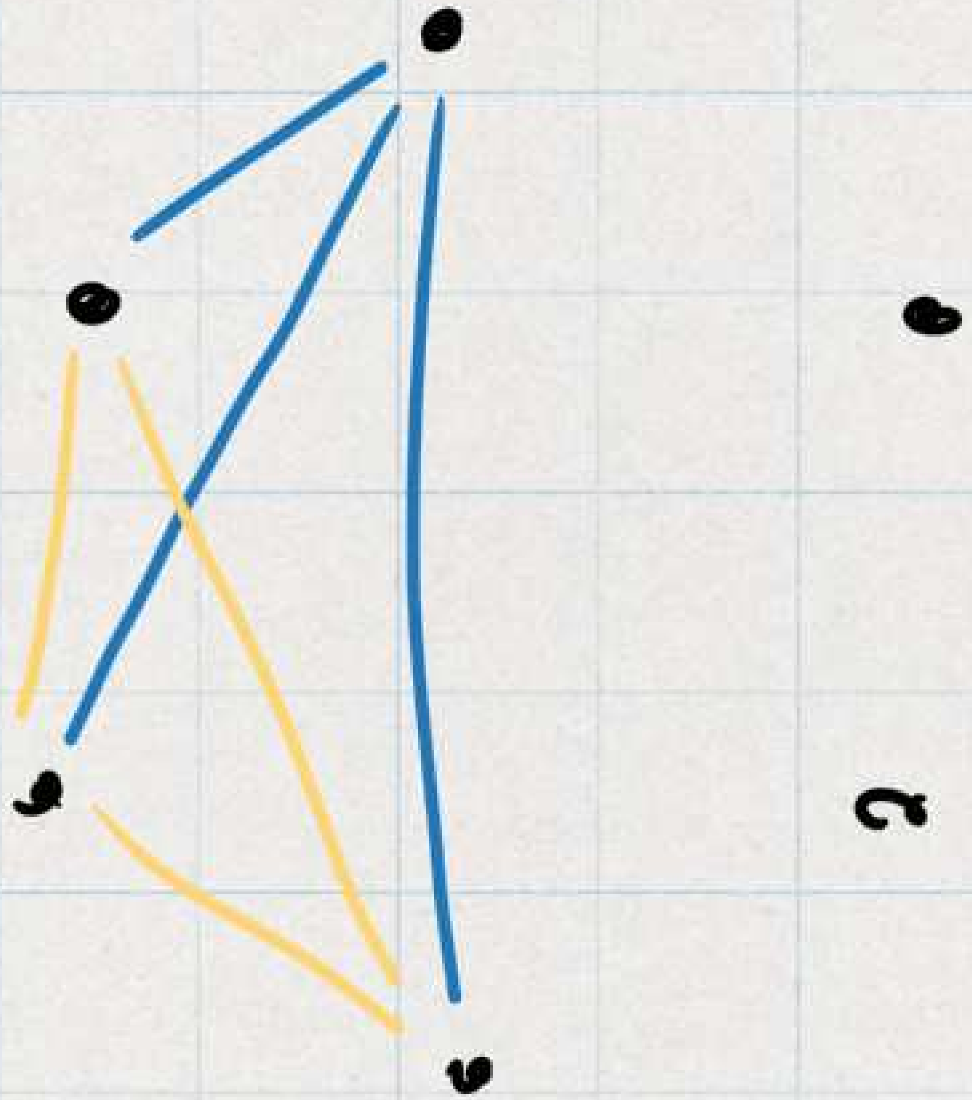
and, conversely, any μ satisfying (88a), (88b), and (88c) is the characteristic function of a κ-complete, non-principal ultrafilter on X. A function μ as above is a probability measure on X in the sense of Section 8.A: (88c) is a strengthening of σ-additivity and it is called κ-additivity, while (88b) is dubbed in this context non-triviality rather than continuity. Therefore κ is measurable just in case there is a κ-complete, non-trivial measure $\mu \colon \mathscr{P}(\kappa) \to \{0, 1\}$.

In Lectures XI & XII, we are not going to use this definition, but only a combinatorial consequence (see next page).

COLOURINGS (Ramsey Theory)

If you colour the edges in a complete hexagon in BLUE and GREEN then there is a monochromatic triangle.





W.L.O.G. three edges from top vertex BLUE.

If all yellow lines are coloured GREEN, then they form a GREEN triangle.
Otherwise, one of them is BLUE and then that one with the edges to the top vertex are a BLUE triangle.

$[6]^2$

As usual, we denote by $[\kappa]^n$ the set of n-element subsets of κ. A function $f : [\kappa]^n \to \omega$ is called an *n-colouring* and a set H is called *homogeneous for f* if $f \upharpoonright [H]^n$ is constant. We call f a *finite colouring* if it is an n-colouring for some natural number $n \in \mathbb{N}$.

MONOCHROMATIC

Theorem 2 (Rowbottom)**.** If κ is measurable, then for every countable set $\{f_s\,;\, s \in S\}$ of finite colourings, there is a set H of size κ that is homogeneous for all colourings f_s.

$f : [6]^2 \longrightarrow 2$ Found H of size 3 that is homogeneous for f.

Theorem (Ramsey). For every finite colouring of ω there is an infinite homogeneous subset.

We'll show $\boxed{\text{If there is a cardinal } \kappa \text{ satisfying Rowbottom's Thm, then } \mathrm{Det}(\underset{\sim}{\Sigma}^1_1).}$

$\underset{\sim}{\Pi}^1_1$

[There will be a pdf-file with notes of the proof in Lectures XI & XII.]

② Tree Representation for $\underset{\sim}{\Pi}^1_1$ sets.

[Shoenfield's Thm
Suslin property]

13 The Shoenfield Absoluteness Theorem

14 A Primer of Large Cardinals

26 Suslin Sets and Cardinals

WORK IN PROGRESS.

Exercise 26.1 (The Kunen–Martin Theorem). Let R be a κ-Suslin well-founded relation on $\mathbb{R}$, $R = \mathrm{p}[T]$. Show that $\mathbf{Coll}(\omega, \kappa) \Vdash$ "$\mathrm{p}[\check{T}]$ is a well-founded relation." Conclude that $\|R\| < \kappa^+$.

<u>Def.</u> Let X be an arbitrary set and $A \subseteq (\omega^\omega)^n$. We say A is X-Suslin if there is a tree T on $\omega^n \times X$ s.t. $A = p[T]$.

$$\forall \vec{x} \in (\omega^\omega)^n \quad \vec{x} \in A \longleftrightarrow \exists y \in X^\omega \ (\vec{x}, y) \in [T].$$

<u>REMEMBER</u>

$A \in \underset{\sim}{\Sigma}^1_1$ iff there is a tree T on $\omega \times \omega$ s.t. $A = p[T]$.

iff A is $\mathbb{N}$-Suslin / ω-Suslin / $\aleph_0$-Suslin

Corollary Not every $\underset{\sim}{\Pi}^1_1$-set can be ω-Suslin.

Lemma If there is an injection from X to Y and A is X-Suslin, then A is Y-Suslin.

[Obvious: take $T' \subseteq (\omega^n \times Y)^{<\omega}$ as the image of $T \subseteq (\omega^n \times X)^{<\omega}$.]

Remark Every set $A \subseteq \omega^\omega$ is ω^ω-Suslin. [Think about it.]

So the notions of ω-Suslin, $\aleph_1$-Suslin, $\aleph_2$-Suslin, ... $2^{\aleph_0}$-Suslin form a new "tree-based" complexity hierarchy.

Theorem (Shoenfield's Thm)

Every $\boldsymbol{\Pi}^1_1$-set is $\aleph_1$-Suslin.

[κ-Suslin for $\kappa \geq \aleph_1$].

Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\mathrm{lh}(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

$$\boxed{T_s := \{t \in \omega^{<\omega}\,;\,(s\restriction\mathrm{lh}(t), t) \in T\}},$$

$$\boxed{T_x := \{t \in \omega^{<\omega}\,;\,(x\restriction\mathrm{lh}(t), t) \in T\}} = \bigcup_{n\in\mathbb{N}} T_{x\restriction n},$$

$$K_s := \{i \leq \mathrm{lh}(s)\,;\,s_i \in T_s\}, \text{ and}$$

$$K_x := \{i \in \omega\,;\,s_i \in T_x\} = \bigcup_{n\in\mathbb{N}} K_{x\restriction n}.$$

$T_{x\restriction n} \subseteq T_x$

C closed

$A = p(C)$

T

$A = p[T]$

$x \in A \iff T_x$ is illfounded

We note that T_s is a tree of finite height (every element $t \in T_s$ has length $\leq \mathrm{lh}(s)$) and that K_s is a finite set. We observe that $T_x = \{s_i\,;\,i \in K_x\}$ (but, in general, $T_s \supsetneq \{s_i\,;\,i \in K_s\}$).

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$$\boxed{K_s := \{i \leq \mathrm{lh}(s)\,;\, s_i \in T_s\}} \text{ and}$$

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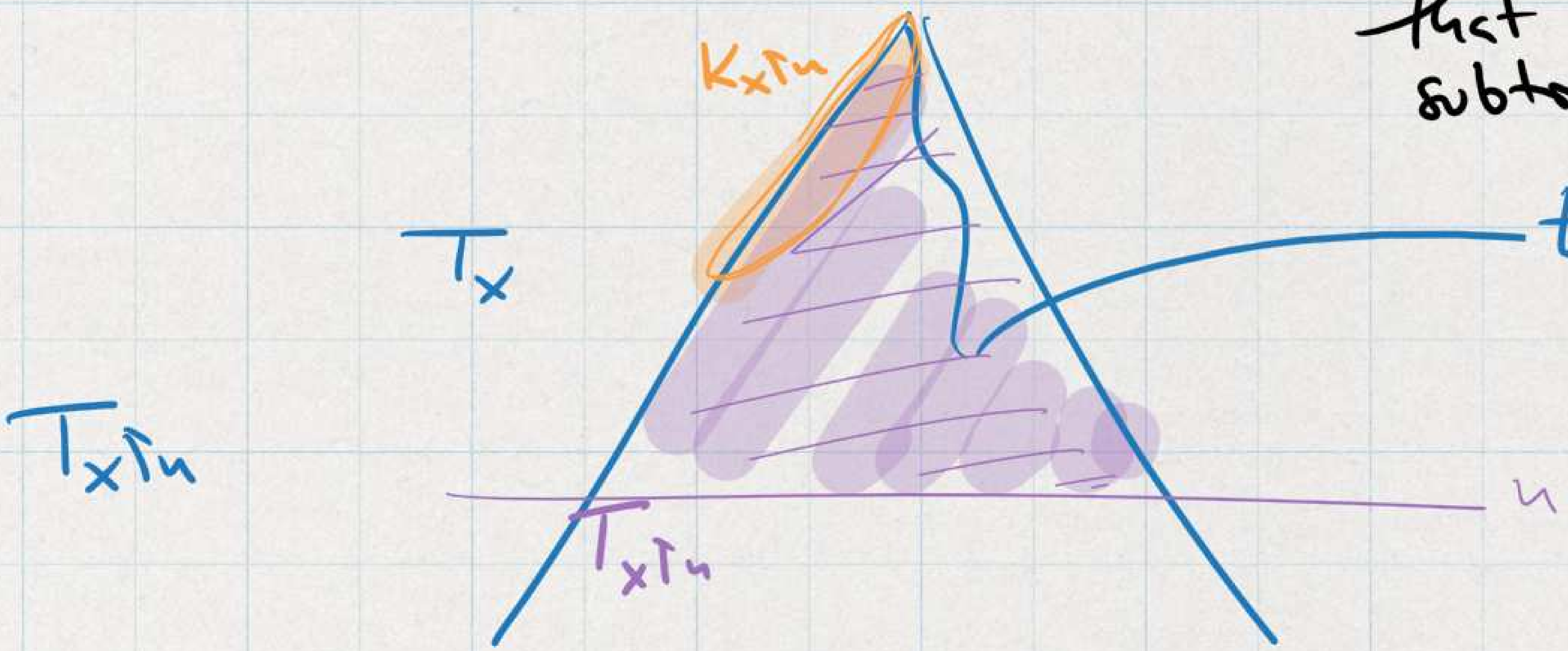
Fix $i \longmapsto s_i$

$\omega \longrightarrow \omega^{<\omega}$

$K_s := \{i \leq \mathrm{lh}(s)\,;\, s_i \in T_s\}$

K_s is a set of natural numbers that corresponds to a finite subtree of T_s.



$t \in T_x \longleftrightarrow (x{\upharpoonright}\mathrm{lh}(t), t) \in T$

$T_{x\upharpoonright n}$ may be infinite since it can contain ∞ly many sequences of a given length.

Remember the Representation Theorem for $\underset{\sim}{\Pi}^1_1$-sets:

A is $\underset{\sim}{\Pi}^1_1$ iff there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$ with

$$x \in A \iff T_x \text{ is wellfounded}$$

$$\iff \underline{(T_x, \supsetneqq)} \text{ is wellfounded}$$

We now linearise $(T_x, \supsetneqq)$ as follows:

$<_{BK}$ is linear on $\omega^{<\omega}$

Define $<_{BK}$ on $\omega^{<\omega}$ as follows:

$$s <_{BK} t \iff \underline{t \subsetneqq s} \text{ OR } \Big[\exists n < lh(s), lh(t) \;\; s{\restriction}n = t{\restriction}n \text{ AND } s(n) \underline{< t(n)}\Big]$$

We can check:

$(T, \supsetneq)$ wellfdd. $\Longleftrightarrow$ $(T, <_{BK})$ is a wellorder

$\boxed{A \in \underset{\sim}{\Pi}^1_1}$

$x \in A$ $\Longleftrightarrow$ $(T_x, \supsetneq)$ is wellfdd

$\Longleftrightarrow$ $(T_x, <_{BK})$ is a wellorder

[A function $g: \omega \longrightarrow \omega_1$ is called a BK-code for a tree T iff

$\forall i,j$ such $s_i, s_j \in T$ we have $s_i <_{BK} s_j \longleftrightarrow g(i) < g(j)$]

$\Longleftrightarrow$ there is a BK-code for T_x.

Need to prove $(T_x, <_{BK})$ is wellorder, $T_x \subseteq \omega^{<\omega}$

iff there is BK-code for T_x.

$[g : \omega \longrightarrow \aleph_1$ s.t. $\forall i, j$: $s_i, s_j \in T_x$ then $s_i <_{BK} s_j \iff g(i) < g(j)]$

" $\Longrightarrow$ " $(T_x, <_{BK})$ is a ctble wellorder

$\Longrightarrow$ there is an o.p. injection $f : T_x \longrightarrow \aleph_1$

$s <_{BK} t \Longrightarrow f(s) < f(t)$

g is a BK-code.

$$g : \omega \longrightarrow \aleph_1$$
$$i \longmapsto f(s_i) \quad \text{if } s_i \in T_x$$
$$i \longmapsto 0 \quad \text{if } s_i \notin T_x$$

" $\Longleftarrow$ " $f(s_i) := g(i)$ for $s_i \in T_x$

By ass. $f : (T_x, <_{BK}) \longrightarrow (\aleph_1, \in)$ order preserving

q.e.d.