LECTURE X   CS&T 2020

we will prove $AD \Rightarrow PSP$.

actually we will prove

$$\mathrm{DET}(\underset{\sim}{\Gamma}) \Rightarrow \mathrm{PSP}(\underset{\sim}{\Gamma})$$

$\underset{\sim}{\Gamma}$ boldface.

$$\underset{\sim}{\Sigma}^0_1,\ \Lambda^\omega_2$$

DEF A boldface pointclass $\underset{\sim}{\Gamma}$ is class function:

1) For every top. space X $\quad \underset{\sim}{\Gamma}(X) \subseteq P(X)$
2) For every top. space X $\quad X, \emptyset \in \underset{\sim}{\Gamma}(X)$
3) For every pair X, Y of top. spaces and continuous function $f : X \to Y$ we have

$$A \in \underset{\sim}{\Gamma}(Y) \Rightarrow f^{-1}[A] \in \underset{\sim}{\Gamma}(X)$$

$\underset{\sim}{\Gamma}(\omega^\omega) \leftarrow$ what we worked with so far!

$$\underset{\sim}{\Sigma}^0_1 \upharpoonright 2^\omega = \underset{\sim}{\Sigma}^0_1{}_{\mathbb{R}}(2^\omega)$$

Theorem For every boldface pointclass $\underset{\sim}{\Gamma}$:

$$DET(\underset{\sim}{\Gamma}(\omega^\omega)) \Rightarrow PSP(\underset{\sim}{\Gamma}(\omega^\omega))$$

Proof Let $A \subseteq \omega^\omega$ we define the $*$-game $G^*(A)$ the following game:

$$\begin{array}{llll} & \in \omega^{<\omega} \times \omega^{<\omega} & \supsetneq s_{0,i_0} & \emptyset \\ \mathrm{I} & (s_{0,0}, s_{0,1}) & (s_{1,0}, s_{1,1}) & \text{in } \omega^{<\omega} \\ & & & [s] = N_s \\ \mathrm{II} & & i_0 & \end{array}$$

In each turn I plays $s_{n,0}, s_{n,1} \in \omega^{<\omega} \times \omega^{<\omega}$ st $[s_{n,0}] \neq \emptyset$ $[s_{n,1}] \neq \emptyset$, $s_{n,0} \perp s_{n,1}$

If $n>0$ then $s_{n-1,i_{n-1}} \subset s_{n,0}$, $s_{n-1,i_{n-1}} \subset s_{n,1}$.

Then player I wins iff $x \in A$ where x is the unique element in $\bigcap_{n \in \mathbb{N}} [s_{n,i_n}]$.

This is one of those games in which the set of moves $\mathcal{M} = \underset{\omega^{<\omega} \times \omega^{<\omega}}{\underset{\parallel}{\mathcal{M}_{\mathrm{I}}}} \uplus \underset{2}{\underset{\parallel}{\mathcal{M}_{\mathrm{II}}}}$ [Lecture 2]

A PLAY OF THE GAME CAN BE SEEN AS A SEQUENCE

$$z = \{ (s_{n,0}, s_{n,1}, i_n) \mid n \in \mathbb{N} \}$$

IF WE ARE AT ROUND $n+1$ OF A PLAY SAY

$$P = \{ (s_{k,0}, s_{k,1}, i_k) \mid k \leq n \}$$

WE CALL A PAIR $(s,t) \in \omega^{<\omega} \times \omega^{<\omega}$ LEGAL FOR P IFF:

① $lh(s) \geq n$, $lh(t) \geq n$

② $s_{n,i_n} \subset s$, $s_{n,i_n} \subset t$

③ $s \perp t$

LET T BE THE TREE OF LEGAL POSITIONS OF THE GAME ON $\mathcal{M}$.

I WINS A PLAY $z \in \mathcal{M}^{\omega}$ OF $G^{*}(A)$ IFF $z \in [T]$ AND $x \in A$ WHERE x IS THE UNIQUE ELEMENT OF $\bigcap_{n \in \mathbb{N}} [s_{n,i_n}]$. IN THIS CASE I DENOTE x AS $\lim(z)$.

NOTE THAT THIS IS NOT CONFORM TO OUR GAMES ON $\mathcal{M}$.

WE DEFINE A CONTINUOUS MAP FROM $[T]$ TO ω^ω

$$\mathcal{X}(z) = \lim(z)$$

THIS IS CONTINUOUS!

LOOK AT $G(\mathcal{X}^{-1}(A))$ ON $\mathcal{M}$ THIS IS EQUIVALENT TO $G^{*}(A)$. [[CHECK!]]

$$\mathrm{DET}(\underset{\sim}{\Gamma}(\omega^\omega))$$

IN LECTURE 3:

Proposition 19.13. $Y \rightarrowtail X \wedge \mathsf{AD}_X \Rightarrow \mathsf{AD}_Y$. *Hence if X and Y are in bijection*

$$\mathsf{AD}_X(\Gamma(x^\omega)) \Leftrightarrow \mathsf{AD}_Y(\Gamma(y^\omega))$$

$g: X \to Y$ BIJ.

$\downarrow$ $f: \omega^\omega \to \mathcal{M}^\omega$

$$f(n_0 n_1 \ldots) = g(n_0) g(n_1) \ldots$$

$\mathcal{M} = \mathcal{M}_I \uplus \mathcal{M}_{II}$ IS IN BIJ WITH ω.

$\mathcal{M}_I = \omega^{<\omega} \times \omega^{<\omega}$, $\mathcal{M}_{II} = 2$

IF $A \in \underset{\sim}{\Gamma}(\omega^\omega) \Rightarrow \underbrace{\chi^{-1}[A]}_{A^* =} \in \underset{\sim}{\Gamma}([T]) \Rightarrow \varphi^{-1}[A^*] \in \underset{\sim}{\Gamma}(\omega^\omega)$

USING NEW DEF

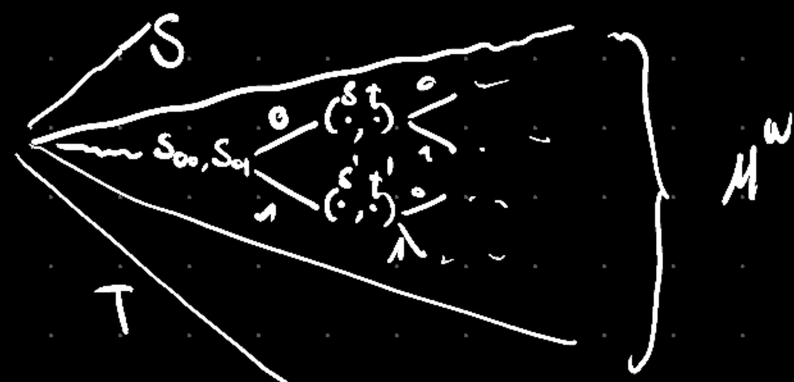
NOW WE CAN USE $\mathrm{DET}(\underset{\sim}{\Gamma}(\omega^\omega))$ TO CONCLUDE THAT $A \in \underset{\sim}{\Gamma}(\omega^\omega) \Rightarrow G^*(A)$ IS DETERMINED.

LEMMA ① IF I HAS A WIN. S. FOR $G^*(A)$ THEN A HAS A PERFECT SUBSET

② IF II HAS A WIN. S. FOR $G^*(A)$ THEN A IS COUNTABLE.

PROOF

① ASSUME THAT I HAS A W.S. σ FOR $G^*(A)$
LOOK AT THE STRATEGIC TREE S OF σ



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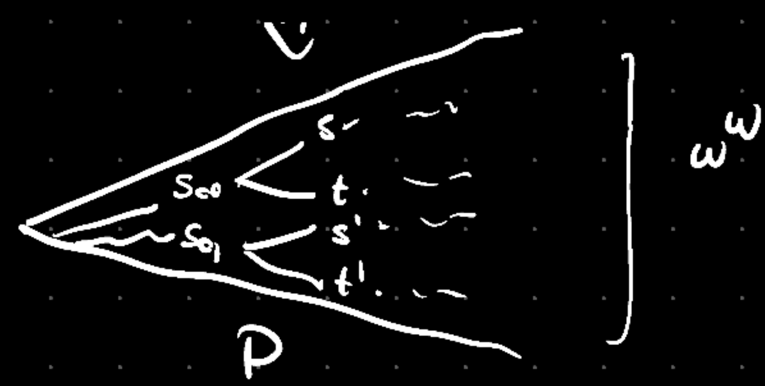
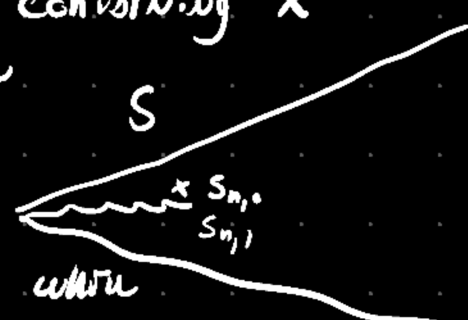


$$P = \{ a \in \omega^{<\omega} \mid \exists z \in [S]\ a \subset \lim(z) \}$$

CLAIM $[P]$ IS A PERFECT SUBSET OF A.

- P IS NON-EMPTY AND PRUNED SINCE S IS.
- $[P] \subseteq A$: $\quad x \in [P]$

START PLAY G AGAINST A STRATEGY IN WHICH $\underline{\text{II}}$ SELECTS THE OPEN BALL CONTAINING x
IF AT SOME POINT THERE ARE $S_{n,0}$, $S_{n,1}$ st $x \notin [S_{n,0}]$, $x \notin [S_{n,1}]$ THEN



IF YOU LOOK AT $x \restriction m$ WHERE $m \geq \max\{lh(s_{n,0}), lh(s_{n,1})\}$ THEN $x \restriction m \notin P$.
BUT THIS MEANS THAT $x \in [P]$ ↯

- P IS PERFECT: IF $a \in P$ THEN $\exists z \in [S]$

$$z = \{(S_{n,0}, S_{n,1}, i_n) \mid n \in \mathbb{N}\}$$

s.t. $a \subset \lim(z)$.

LOOKS AT n BIG ENOUGH SUCH THAT
$lh(S_{n,0}) > lh(a)$ AND $lh(S_{n,1}) > lh(a)$ SO
$a \subset S_{n,0}$ AND $a \subset S_{n,1}$ AND
$S_{n,0} \perp S_{n,1}$ SO $S_{n,0}$ AND $S_{n,1}$ ARE
IN P SINCE σ IS WINNING FOR I.

SO $[P]$ IS PERFECT IN A.

a) ASSUME II HAS W.S. ON $G^*(A)$ σ.

FOR $x \in A$ THE POSITION

$$p = \{(S_{k,0}, S_{k,1}, i_k \mid k \leq n\}$$

IS GOOD FOR x IF IT IS IN THE STRATEGIC TREE S FOR σ AND $x \in S_{n,i_n}$

LET

$$G = \{p \mid p \text{ IS A GOOD POS. FOR SOME } x \in A\}$$

For $p \in G$ as before

$$A_p = \{ y \in [S_{n,i_n}] \mid \forall \langle s_{n+1,0}, s_{n+1,1} \rangle \text{ legal for } P \text{ if } \sigma(P^\frown \langle s_{n+1,0}\, s_{n+1,1} \rangle) = i \text{ then } y \notin [S_{n+1,i}] \}$$

we will show that $A \subseteq \bigcup_{p \in G} A_p$

- if $x \in A$ then there is a maximal good pos. for x o.w. player I would win. call it P_x

$$x \in A_{P_x}.$$

- $|A_p| \leq 1$: if $x, y \in A_p$ and $x \neq y$

look at minimal m st $x(m) \neq y(m)$

$$S_{n,i_n} \subseteq x \restriction m+1 = S_{n+1,0}$$

$$S_{n,i_n} \subseteq y \restriction m+1 = S_{n+1,1}$$

so $\langle S_{n+1,0}, S_{n+1,1} \rangle$ is legal for P and it does not matter what II does either $x \in S_{n+1,i_{n+1}}$ or $y \in S_{n+1,i_{n+1}}$ $\}$ in

SO $|A_p| = 1$ AND A IS COUNTABLE. ∎

<u>COR</u> AD $\Rightarrow$ PSP.

COR (FROM MARTIN DET($\underset{\sim}{\Delta}^1_1$)) PSP($\underset{\sim}{\Delta}^1_1$)

<u>THEOREM</u> (Q-16) PSP($\underset{\sim}{\Sigma}^1_1$)

PROOF A MODIFIED VERSION OF $*$-GAME GIVES YOU A WAY OF USING LOWER LEVELS OF DET TO SHOW PSP($\underset{\sim}{\Sigma}^1_1$).

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