Homework Sheet #3

Capita Selecta: Set Theory
2020/21: 1st Semester; block 1
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Homework. Homework is due on Fridays. Please submit your work as a single pdf file via Canvas. You will receive one point for each question that you attempt, independent of your performance on the question. The homework will not be formally marked, but the lecturers may give individual feedback in the case of problems.

Deadline. This homework set is due on **Friday, 25 September 2020, 1pm**.

1. Let X and Y be two non-empty sets and $f : X^\omega \to Y^\omega$. We use the product topology on X^ω and Y^ω. Andretta introduces a notion called "continuous" for functions from $X^{<\omega}$ to $Y^{<\omega}$ (cf. Definition 2.4 or lecture notes of Lecture V). Show that the following are equivalent:

   (a) f is continuous;

   (b) there is a continuous function $g : X^{<\omega} \to Y^{<\omega}$ such that for all $a \in X^\omega$ we have $f(a) = \bigcup_{n\in\omega} g(a{\upharpoonright}n)$.

2. Solve Exercise 3.4 on Andretta's draft book.

3. Let (X, τ) be a topological space. We define the *strong Choquet game* $\mathrm{sCG}_{(X,\tau)}$: in each round player I plays an open subset U of X and a point $x \in U$; player II must respond with an open subset $V \subseteq U$ such that $x \in V$. In the following round player I will have to produce a new challenge with the constraint that the new open set should be a subset of V. The game continues unless one of the two players cannot move, in which case the other player wins. If the game lasts for infinitely many rounds, then player II wins if the intersection of the open sets played during the game is non-empty; otherwise, player I wins.

   Thus, a play of $\mathrm{sCG}_{(X,\tau)}$ is an ω sequence of triples (U_i, x_i, V_i). Player II wins this play if and only if

   either for all i, we have $x_i \in V_i \subseteq U_i$ and $U_{i+1} \subseteq V_i$ and $\bigcap_{i\in I} V_i \neq \varnothing$

   or there is an i such that this is not the case and for the least such i, we either have $x_i \notin U_i$ or $U_{i+1} \not\subseteq V_i$ (i.e., player I made a mistake).

   A topological space (X, τ) is *strongly Choquet* if II has a winning strategy for $\mathrm{sCG}_{(X,\tau)}$.

   Show that every Polish space is strongly Choquet.