# Homework Sheet #4

**Capita Selecta: Set Theory**
*2020/21: 1st Semester; block 1*
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**Homework.** Homework is due on Fridays. Please submit your work as a single pdf file via Canvas. You will receive one point for each question that you attempt, independent of your performance on the question. The homework will not be formally marked, but the lecturers may give individual feedback in the case of problems.

**Deadline.** This homework set is due on **Friday, 2 October 2020, 1pm**.

11. Let $T \subseteq X^{<\omega}$ be a tree, $A \subseteq X^\omega$ be a set, and $p \in X^\omega$ be a position. Recall the definitions of *the localisation of* $T$ *at a position* $p$, denoted by $T_{\lfloor p \rfloor}$, *the localisation of* $A$ *at a position* $p$, denoted by $A_{\lfloor p \rfloor}$, and *the subtree of* $T$ *at* $p$, denoted by $T_{[p]}$, from Andretta's Definitions 1.33 & 2.12.
    (a) Observe that $[p] = [X^{<\omega}{}_{[p]}]$ and show that, in general, the games $\mathrm{G}(A \cap [X^{<\omega}{}_{[p]}])$ and $\mathrm{G}(X^{<\omega}{}_{[p]}; A)$ are different (e.g., by finding $A$ and $p$ such that different players win the games). Argue that, in contrast, the games $\mathrm{G}(X^{<\omega}{}_{[p]}; A)$ and $\mathrm{G}(A_{\lfloor p \rfloor})$ are the same game and they describe "the game $\mathrm{G}(A)$ starting from position $p$".
    (b) We call the partial function $\ell_A : X^{<\omega} \to \{\mathrm{I}, \mathrm{II}\}$ the *strategic labelling for* $A$ if $\ell(p) = \mathrm{I}$ if and only if player I has a winning strategy in $\mathrm{G}(X^{<\omega}{}_{[p]}; A)$ and $\ell(p) = \mathrm{II}$ if and only if player II has a winning strategy in $\mathrm{G}(X^{<\omega}{}_{[p]}; A)$. Using the Axiom of Choice, find a set $A$ that is determined, but the strategic labelling $\ell_A$ is not a total function.
    (c) Let $\mathbf{\Gamma}$ be a boldface pointclass. Show that the following are equivalent:
        (i) for all $A \in \mathbf{\Gamma}$, the game $\mathrm{G}(A)$ is determined and
        (ii) for all $A \in \mathbf{\Gamma}$, the strategic labelling $\ell_A$ is a total function.

12. Let $A \subseteq \omega^\omega$ be an arbitrary set and $T$ a tree such that $[T] \subseteq A$. Define $W(A, T) := \{x \in \omega^\omega\,;$ for all $n \in \omega$, we have that $x{\restriction}n \in T$ or $\ell_A(x{\restriction}n) = \mathrm{I}\}$. Prove that if player I does not have a winning strategy in $\mathrm{G}(A)$, then player II has a winning strategy in $\mathrm{G}(W(A, T))$. Check how much of the Axiom of Choice you used in your proof.

13. Prove in $\mathsf{ZFC}$ that every $\mathbf{\Sigma}^0_2$ set is determined. Check how much of the Axiom of Choice you used in your proof.

    [*Hint.* Write $A = \bigcup_{i \in \mathbb{N}} [T_i]$ and show that if player I does not have a winning strategy in $\mathrm{G}(A)$, then player II does. In order to construct this winning strategy, use the statement in question 12. iteratively.]

14. Suppose $A \subseteq \mathrm{WO}$ has the property that for each $\alpha < \omega_1$, $A \cap \mathrm{WO}_\alpha$ is countable. Show that if $A$ is Lebesgue measurable, then $A$ has measure zero.

    [*Hint.* We did not prove the *Boundedness Lemma* yet, but you are allowed to use it. You do not need to know much about Lebesgue measure. It's enough to know that each set of positive measure must contain a closed set of positive measure and that countable sets have measure zero.]