Capita Selecta: Set Theory
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Analytic Determinacy

Notation. Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\mathrm{lh}(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

$$\begin{aligned} T_s &:= \{t \in \omega^{<\omega}\,;\, (s{\restriction}\mathrm{lh}(t), t) \in T\}, \\ T_x &:= \{t \in \omega^{<\omega}\,;\, (x{\restriction}\mathrm{lh}(t), t) \in T\} = \bigcup_{n\in\mathbb{N}} T_{x\restriction n}, \\ K_s &:= \{i \leq \mathrm{lh}(s)\,;\, s_i \in T_s\}, \text{ and} \\ K_x &:= \{i \in \omega\,;\, s_i \in T_x\} = \bigcup_{n\in\mathbb{N}} K_{x\restriction n}. \end{aligned}$$

We note that T_s is a tree of finite height (every element $t \in T_s$ has length $\leq \mathrm{lh}(s)$) and that K_s is a finite set. We observe that $T_x = \{s_i\,;\, i \in K_x\}$ (but, in general, $T_s \supsetneq \{s_i\,;\, i \in K_s\}$).

We remember that if $A \in \mathbf{\Pi}^1_1$, then there is a tree T on $\omega \times \omega$ such that

$$\begin{aligned} x \in A &\text{ if and only if } (T_x, \supsetneq) \text{ is wellfounded} \\ &\text{ if and only if } (T_x, <_{\mathrm{KB}}) \text{ is wellordered} \\ &\text{ if and only if there is an order preserving map from } (T_x, <_{\mathrm{KB}}) \text{ to } (\omega_1, <) \end{aligned}$$

where $<_{\mathrm{KB}}$ is the Kleene-Brouwer order on $\omega^{<\omega}$. For any $s \in \omega^{<\omega}$, we write $<_s$ for the order induced by the Kleene-Brouwer order on $\omega^{<\omega}$ on K_s, i.e., $i <_s j$ if and only if $s_i <_{\mathrm{KB}} s_j$. Note that since K_s is finite, $(K_s, <_s)$ is a (finite) wellorder.

Let S be any tree on ω and κ be an uncountable cardinal. A function $g : \omega \to \kappa$ is called a *KB-code for* S if for all i and j such that $s_i, s_j \in S$, we have that $s_i <_{\mathrm{KB}} s_j \leftrightarrow g(i) < g(j)$. Clearly, there is an order preserving map from $(S, <_{\mathrm{KB}})$ to $(\omega_1, <)$ if and only if there is a KB-code for S, so we can add the following equivalence to the above characterisation of $\mathbf{\Pi}^1_1$ sets:

$$x \in A \text{ if and only if there is a KB-code for } T_x$$

Shoenfield's Theorem. We first prove a tree representation theorem for $\mathbf{\Pi}^1_1$ sets.

Theorem 1 (Shoenfield)**.** If κ is uncountable, then every $\mathbf{\Pi}^1_1$ set is κ-Suslin.

P r o o f. Let $A \in \mathbf{\Pi}^1_1$ and let T be a tree on $\omega \times \omega$ such that $x \in A$ if and only if there is a KB-code for T_x. Let M be the set of all partial functions from ω into κ with finite domain. Note that $|M| = \kappa$, so it is sufficient to show that A is M-Suslin. If $s \in \omega^{<\omega}$ and $u \in M^{<\omega}$ such that $\mathrm{lh}(h) \leq \mathrm{lh}(s)$, we say that u is *coherent with* s if

(1) for all $i < \mathrm{lh}(u)$, we have that $\mathrm{dom}(u_i) = K_{s\restriction i}$,
(2) for all $i < \mathrm{lh}(u)$, $u(i)$ is an order preserving map from $(K_{s\restriction i}, <_{\mathrm{KB}})$ into $(\kappa, <)$, and
(3) for $i \leq j$, we have that $u_i \subseteq u_j$.

We now define the *Shoenfield tree* on $\omega \times M$ by $\widehat{T} := \{(s, u)\,;\, u \text{ is coherent with } s\}$ and claim that $A = \mathrm{p}[\widehat{T}]$:

"$\subseteq$": If $x \in A$, then let $g : \omega \to \kappa$ be a KB-code for T_x and define $u(i) := g{\restriction}K_{x\restriction i}$. By definition, $u{\restriction}n$ is coherent with $x{\restriction}n$ for all n, and so $(x, u) \in [\widehat{T}]$.

"$\supseteq$": If $x \in \mathrm{p}[\widehat{T}]$, find $u \in M^\omega$ such that $(x, u) \in [\widehat{T}]$; this means that for each n, $u{\restriction}n$ is coherent with $x{\restriction}n$. As noted above, we have that $T_x = \{s_i\,;\, i \in K_x\} = \{s_i\,;\, \exists n(i \in \mathrm{dom}(u(n))\}$. We define $\widehat{u} := \bigcup\{u(i)\,;\, i \in \omega\}$. By coherence, $\widehat{u}$ is a function from K_x to κ; now we define

$$g : \omega \to \kappa : n \mapsto \begin{cases} \widehat{u}(n) & \text{if } n \in K_x \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that g is a KB-code for T_x whence $x \in A$: Suppose not, then there are i and j such that $s_i, s_j \in T_x$ and $s_i <_{\mathrm{KB}} s_j \not\leftrightarrow g(i) < g(j)$. Since $i, j \in K_x$, find n large enough such that $i, j \in K_{x\upharpoonright n}$. By definition $g\upharpoonright K_{x\upharpoonright n} = u(n)$. But this means that $u(n)$ is not an order preserving map from $(K_{x\upharpoonright n}, <_{\mathrm{KB}})$ into $(\kappa, <)$, violating condition (3) of coherence. Q.E.D

Measurable Cardinals. Let X be a set. A non-empty family $U \subseteq \wp(X)$ is called a *ultrafilter over* X if for any $A, B \subseteq X$, we have that

(1) if $A, B \in U$, then $A \cap B \in U$,
(2) if $A \in U$ and $B \supseteq A$, then $B \in U$, and
(3) if $A \notin U$, then $X \backslash A \in U$.

We say that an ultrafilter is *non-trivial* if it does not contain any finite sets and if κ is any cardinal, it is called κ-*complete* if it is closed under intersections of size $< \kappa$. Note that ω-completeness follows from (1). A non-trivial κ-complete ultrafilter cannot contain any sets of size $< \kappa$.

[If $|A| = \lambda < \kappa$, then for each $a \in A$, $\{a\} \notin U$, so by (3), $X\backslash\{a\} \in U$, but then by κ-completeness, $X\backslash A = \bigcap\{X\backslash\{a\}\,;\, a \in A\} \in U$. If now $A \in U$, then $\varnothing = A \cap X\backslash A \in U$. Contradiction to non-triviality.]

An uncountable cardinal κ is called *measurable* if there is a κ-complete non-trivial ultrafilter on κ. The Axiom of Choice implies that there are non-trivial ultrafilters on ω; as mentioned, they are ω-complete, so $\aleph_0$ technically satisfies the conditions of the definition. The existence of uncountable measurable cardinals cannot be proved in ZFC and is a so-called *large cardinal axiom*. More precisely, if MC stands for "there is a measurable cardinal", then for every model $M \models \mathsf{ZFC} + \mathsf{MC}$, I can find a submodel $N \subseteq M$ such that $N \models \mathsf{ZFC} + \neg\mathsf{MC}$.

Being measurable has interesting consequences for the combinatorics on κ. We are going to use one of them in our proof of analytic determinacy. As usual, we denote by $[\kappa]^n$ the set of n-element subsets of κ. A function $f : [\kappa]^n \to \omega$ is called an n-*colouring* and a set H is called *homogeneous for* f if $f\upharpoonright[H]^n$ is constant. We call f a *finite colouring* if it is an n-colouring for some natural number $n \in \mathbb{N}$.

Theorem 2 (Rowbottom)**.** If κ is measurable, then for every countable set $\{f_s\,;\, s \in S\}$ of finite colourings, there is a set H of size κ that is homogeneous for all colourings f_s.

In our proof of analytic determinacy, we are only going to use Rowbottom's Theorem, no other properties of measurable cardinals; so, for our purposes, one could take the statement of Rowbottom's Theorem as the assumption for analytic determinacy in the next section.

Analytic Determinacy. If $\mathbf{\Gamma}$ is a boldface pointclass, then $\mathrm{Det}(\mathbf{\Gamma})$ is equivalent to $\mathrm{Det}(\breve{\mathbf{\Gamma}})$. Thus, analytic determinacy and co-analytic determinacy are equivalent.

Theorem 3 (Martin, 1969/70)**.** If there is a measurable cardinal, then every co-analytic set is determined.

Proof. Let κ be a measurable cardinal and $A \in \mathbf{\Pi}^1_1$. We aim to show that the game $\mathrm{G}(A)$ is determined. By (the proof of) Shoenfield's Theorem, we know that there is a tree $\widehat{T}$ on $\omega \times M$ such that $A = \mathrm{p}[\widehat{T}]$. (Remember that M was the set of partial functions from ω to κ with finite domain.) We are going to define a (determined) game $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ based on the Shoenfield tree and show that a winning strategy for either player in $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ can be transformed into a winning strategy for the same player in the *original game* $\mathrm{G}(A)$. This proves the theorem.

In the *auxiliary game*, player I plays elements of $\omega \times M$ and player II plays elements of ω as follows:

$$\begin{array}{c|ccccccccc} \mathrm{I} & x_0, u_0 & & x_2, u_1 & & x_4, u_2 & & x_6, u_3 & & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & & x_7 & \cdots \end{array}$$

We obtain a sequence $x \in \omega^\omega$ with $x(n) := x_n$ and a sequence $u \in M^\omega$ with $u(n) := u_n$. Player I wins $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ if $(x, u) \in [\widehat{T}]$. Note that $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ is a closed game on $\omega \times M$, thus by the Gale-Stewart Theorem, it is determined.

Let us make a number of observations about the relationship between the original game $\mathrm{G}(A)$ and the auxiliary game $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$. We call the moves u_i *auxiliary moves*. If p is a position in the auxiliary game (i.e., a finite sequence of elements of ω and elements of M in the right order), then we can define a position p^*

in the original game by forgetting about the auxiliary moves. This allows us to consider strategies τ for player II in the original game as strategies in the auxiliary game: if p is a position in the auxiliary game, we let $\tau_*(p) := \tau(p^*)$, i.e., just forget about the auxiliary moves and play as if you were playing in the original game.

Lemma 4. If player I has a winning strategy in $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$, then they have a winning strategy in $\mathrm{G}(A)$.

Proof. Suppose σ is a winning strategy in $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ and τ is any strategy for player II in the original game. As just mentioned, then τ_* is the version of that strategy in $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$. Since σ is winning, we know that $\sigma * \tau_* = (x, u) \in [\widehat{T}]$. Define a strategy σ^* in the original game as follows: while player II plays natural number moves according to τ, you produce the auxiliary play $\sigma * \tau_*$ on an auxiliary board. If that auxiliary game tells you to produce a position p by your next move, then you produce the move p^* in the original game. Then $\sigma^* * \tau = x$, and thus $x \in \mathrm{p}[\widehat{T}] = A$, so σ^* is winning. Q.E.D

Lemma 5. If player II has a winning strategy in $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$, then they have a winning strategy in $\mathrm{G}(A)$.

Proof. Let $s \in \omega^{<\omega}$. Let $k_s := |K_s|$. If $Q \in [\kappa]^{k_s}$, then there is a unique order preserving map $w : (K_s, <_s) \to (Q, <)$. Let $u_i^{s,Q} := w{\upharpoonright}K_{s\upharpoonright i}$. Then $(u_i^{s,Q}\,;\, i < \mathrm{lh}(s))$ is coherent with s. Thus, if you fix some $Q \in [\kappa]^{k_s}$, you can transform a position s in the original game into a position $s_{*,Q}$ in the auxiliary game in such a way that the auxiliary moves produce Q as the range and form a sequence coherent with the position s.

Let now τ be a strategy for player II in the auxiliary game. For each $s \in \omega^{<\omega}$, we define a k_s-colouring $f_s : [\kappa]^{k_s} \to \omega$ by $f_s(Q) := \tau(s_{*,Q})$: we colour the k_s-element subsets of κ by the answer that the strategy τ gives to the position s augmented via Q in the sense given above. By Rowbottom's theorem, there is a set $H \subseteq \kappa$ of size κ that is homogeneous for all functions f_s, i.e., if $Q, Q' \in [H]^{k_s}$, then $\tau(s_{*,Q}) = f_s(Q) = f_s(Q') = \tau(s_{*,Q'})$, so the answer of the strategy τ does not depend on the set Q as long as it is a subset of H. In particular, we can take the simplest imaginable subset of H with k_s elements: let $Q_{H,s}$ be the set consisting of the first k_s many elements of H.

Now, we define a strategy τ_H for player II in the original game by $\tau_H(s) := \tau(s_{*,Q_{H,s}})$. (Note that the precise choice of the set $Q_{H,s}$ is irrelevant in this definition by homogeneity, since $f_s(Q_{H,s}) = f_s(Q)$ for any $Q \in [H]^{k_s}$.)

We prove that if τ was winning in the auxiliary game, then τ_H is winning in the original game. Suppose not, so there is a counterstrategy σ such that $x := \sigma * \tau_H \in A$. This means (since H is uncountable) that there is an orderpreserving map from $(T_x, <_{\mathrm{KB}})$ to $(H, <)$ giving rise to a KB-code $g : \omega \to H$ for T_x. Using the KB-code g, we can now define $u_i := g{\upharpoonright}K_{x\upharpoonright i}$ and consider the play of the auxiliary game

$$\begin{array}{c|ccccccccc} \mathrm{I} & x_0, u_0 & & x_2, u_1 & & x_4, u_2 & & x_6, u_3 & & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & & x_7 & \cdots \end{array}$$

producing $(x, u) \in [\widehat{T}]$. We claim that this is a play according to τ, so we need to show that for every $i \in \mathbb{N}$, the play by player II is the τ-answer to the previous position, i.e., $x_{2i+1} = \tau(x_0, u_0, x_1, ..., x_{2i}, u_i)$. Fix $i \in \mathbb{N}$ and consider $Q := \mathrm{ran}(u_i) \subseteq H$. Then we have that $(x_0, u_0, x_1, ..., x_{2i}, u_i) = (x{\upharpoonright}2i + 1)_{*,Q}$. We see that

$$\begin{aligned} x_{2i+1} &= \tau_H(x{\upharpoonright}2i + 1) && \text{(since } x \text{ was produced by } \tau_H) \\ &= \tau((x{\upharpoonright}2n + 1)_{*,Q_{H,s}}) && \text{(by definition of } \tau_H) \\ &= \tau((x{\upharpoonright}2n + 1)_{*,Q}) && \text{(since the choice of } Q \text{ doesn't matter by homogeneity)} \\ &= \tau(x_0, u_0, x_1, ..., x_{2i}, u_i), \end{aligned}$$

so the above play is a play according to τ. But that is a contradiction, since τ was winning for player II, and so $(x, u) \notin [\widehat{T}]$. Q.E.D

Lemmas 4 & 5 together with the fact that $\mathrm{G}_{\mathrm{aux}}(\widehat{T})$ was determined (since it is a closed game) imply that $\mathrm{G}(A)$ is determined. Q.E.D