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Edited by

Peter Schroeder-Heister,

Gerhard Heinzmann,

Wilfrid Hodges,

and

Pierre Edouard Bour

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Mathematics and the New Technologies Part I: Philosophical relevance of a changing culture of mathematics

Benedikt Löwe[1]

1 Mathematics and the new technologies

Mathematicians use their computers every day: they write e-mails, download papers from preprint servers, upload their own research on the same servers, log in to online communities dealing with mathematics to ask questions, they typeset their own papers with the typesetting system LaTeX, etc. But is this use of the computer and the internet relevant for questions of philosophy of mathematics about the nature of mathematics, the relationship between mathematics and the physical world, or the epistemic status of mathematical knowledge?

The traditional answer to this question is: *Not at all.* Traditionally, mathematics is seen as the paradigmatic deductive science endowed with aprioricity and a characteristic lack of spatial or temporal location of its truthmakers. One of the traditional claims is that while the mathematical discipline is a social and historical product, the underlying mathematics itself (and this is all that matters philosophically, for a traditionalist) does not depend on the way it was socially and historically produced.

The new technologies clearly have a formidable and undeniable effect on the research experience of mathematicians (such as the wide availability of papers via the internet, the communication speed and possibility of remote collaboration by the use of e-mail and visual remote connections, computer proof assistants and automated theorem provers, online crowd-sourcing of mathematical ability in order to solve open problems), but according to

[1]The author should like to thank the programme committee of CLMPS XIV for inviting him as the chair of the special symposium on *Mathematics and the new technologies*. The author acknowledges the financial support of the European Science Foundation as part of Networking Activity 359 in the EuroCoRes programme LogICCC.

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the traditionalist, this effect does not touch the philosophical aspects of mathematics.

However, in recent years, a movement called *Philosophy of Mathematical Practice* has staged a revolt against the traditionalist view. The view that mathematics should be seen as a human cultural product is not new: we find it in books like (Lakatos 1976) and (Davis & Hersh 1971), and more recently in (Hersh 1997) or (Ernest 1998). However, until ten years ago, it was seen as a *maverick* position in the philosophy of mathematics; now it represents a growing part of the philosophy of mathematics community.[1] Philosophers of mathematical practice observe that a number of philosophical statements about mathematics are either empirical statements about mathematicians or at least depend crucially on such statements. As a consequence, any philosophical position that believes in the interplay between the practice of the field studied and its philosophy, cannot ignore the fact that mathematics is a human cultural activity.

Philosophy of mathematical practice is not a homogeneous movement and does not correspond to a uniform philosophical position. For the purposes of this paper, we understand the term "philosophy of mathematical practice" to refer to the meta-philosophical stance that empirical facts of mathematics as practiced can affect philosophical questions and their answers in a philosophically relevant way.

From such a meta-philosophical position, the mentioned "formidable effect" of the new technologies on the research experience of mathematicians might also affect the philosophy of mathematics. On the other hand, even the philosopher of mathematical practice will concede that not every effect on research practice is philosophically relevant. The modern mathematician writes e-mails where Gauss wrote letters; the modern mathematician controls the typography of her papers much more than a mathematician half a century ago, but is also constrained by the rules of the universal typesetting system. Are these changes relevant for philosophy of mathematics? Or, to make the question even more extreme, if a new restaurant is built next to the mathematics department that enables researchers to have dinner and return to their offices to prove more theorems, this restaurant has an effect on their research experience. But is that new restaurant part of the story that the philosophy of mathematics needs or wants to unravel?

Clearly, not all effects of the use of new technologies are philosophically relevant, but in this tripartite paper, we are aiming to show that some of

[1] This is best witnessed by a series of proceedings volumes of related conferences (Van Kerkhove & Van Bendegem 2007; Van Kerkhove *et al.* 2010; Löwe & Müller 2010; François *et al.* 2011) and the foundation of the *Association for the Philosophy of Mathematical Practice* in 2009. An overview of the motivation behind philosophy of mathematical practice can be found in (Buldt *et al.* 2008).

them are clearly involved with some of the traditional questions of philosophy of mathematics, in particular the epistemology of mathematics. The three papers correspond to three of the four talks given in the special invited symposium *Mathematics and the New Technologies* at the *Congress for Logic, Methodology and Philosophy of Science* in Nancy on 22 July 2011. Part II (Koepke 2014) corresponds to Peter Koepke's talk entitled *Formal mathematics and mathematical practice*, and part III (Van Bendegem 2014) corresponds to Jean Paul Van Bendegem's talk entitled *Mathematics in the cloud: the web of proofs*.

The following section and Koepke's part II will highlight the effect that automated theorem provers and proof assistants have on the practice of assessing the correctness of mathematical arguments; Van Bendegem's part III will then move to the other side of mathematical epistemology, the *context of discovery* and the use of new technologies in the process of producing new mathematics.

2 A problem in the epistemology of mathematics

Philosophers of mathematics are interested in the status of mathematics as an *epistemic exception* with a type of knowledge being categorically more secure than that of other sciences (Heintz 2000; Prediger 2006). At the other end of the epistemological spectrum, we have the whimsical *knowledge by testimony*, considered *epistemologically vulnerable*.[2] And yet, mathematicians in practice often use knowledge by testimony when they use results from research papers without checking their proofs in detail. How can the epistemic exception of mathematics survive if some of the proofs rely on pointers to the literature? A simple and naïve answer to both questions would be that the deductive nature of mathematics allows referees to check correctness of the proofs of published papers with absolute certainty, and thus the written codification of mathematical knowledge is certain knowledge, relieving us from any qualms about referring to it. However this is very far from the truth; in his opinion piece published in the *Notices of the American Mathematical Society*, (Nathanson 2008) paints a dark picture of the mathematical refereeing process:

> Many (I think most) papers in most refereed journals are not refereed. There is a presumptive referee who looks at the paper, reads the introduction and the statement of the results, glances at the proofs, and, if everything seems okay, recommends publication. Some referees check proofs line-by-line, but many do not.

[2]For more details on the epistemological problem of testimony, cf. (Adler 2012).

> When I read a journal article, I often find mistakes. Whether I can fix them is irrelevant. The literature is unreliable.

The mathematical peer review process is lamentably understudied. Geist *et al.* (2010) give a description of the level of scrutiny involved in the peer review process and present two (rather preliminary) empirical studies: while, ideally, referee reports "should address Littlewood's three precepts: (1) Is it new? (2) Is it correct? (3) Is it surprising?" (Krantz 1997, 125), in practice, the level of detail of referee reports varies a lot. Among other things, the results in (Geist *et al.* 2010) show that the level of detail in which mathematical correctness is checked during the peer review process does not at all support the naïve view sketched above. In a survey of mathematical journal editors, only about half of them thought that it is the task of the referee to check the correctness of all proofs. In fact, mathematicians seem to have an almost stochastic view of the correctness checking in the peer review process:

> "Refereed proof" [is not the last word on correctness]: it just means that somebody has seen the paper, and if it is done correctly, he actually went through the proofs, and he believes that it is true, and this is very much biased by the human factor. Let's say [a famous mathematician] comes up with a paper, and I have to referee it, and then Im already preoccupied with the fact that [he] is a very well known mathematician, and so that it probably will be OK. And then there's the time pressure: you have all this stuff that you have to do, and then they ask you to review this 50 page paper, and you are sure that if you are really going to check all the details then you'll reach the conclusion "that's probably OK". You have a tendency to believe that the proofs are correct, and in addition you think "Well, he's publishing it, not I, so it's his responsibility that it is OK". Ideally, this referee has nothing else to do, he knows the subject better than the guy who wrote about it, and he will study it, and say "Yes, this is all correct". So, if the author thinks it's OK, then—let's be pessimistic—it has a probability of 95% to be OK. And so, if the referee checked it and and also thinks it's OK, then this also has a 95% chance of being correct, and so you have a very large probability that it is fine. And that is basically how it works. It's never going to be "full proof". I don't think that this exists.
>
> [If a] paper was sent to a mathematical journal of high reputation, so, say, *Acta Mathematica*; this tells us something about

> the size of the mathematical community involved. That it went to a good journal means that the journal thought of looking for good referees, so it was established more surely than that it would have been sent to the journal of a tiny mathematical society with very few members. This puts the scenario in a framework which makes it very likely that the result is correct.[3]

Weber & Mejia-Ramos (2011) investigate the techniques that mathematicians use to convince themselves that a proof is correct, and find that they are mostly heuristic techniques as is exemplified in the following quote from one of their test subjects:

> [To understand a proof] means to understand how each step followed from the previous one. I don always do this, even when I referee. I simply don always have time to look over all the details of every proof in every paper that I read. When I read the theorem, I think, is this theorem likely to be true and what does the author need to show to prove it true. And then I find the big idea of the proof and see if it will work. If the big idea works, if the key idea makes sense, probably the rest of the details of the proof are going to work too.[4]

3 The effect of the new technologies on the epistemological question

If the proof checking of human experts is considered so unreliable and just a matter of minimizing the chances that errors are missed, this opens the field for computer-checked proofs. The topic of automated theorem provers in the philosophical literature is mostly discussed as an additional epistemological issue, e.g., in the context of the computer-assisted proof of the four colour theorem (Tymoczko 1979): the elimination of the human expert seems (at least for some philosophers) to reduce our trust in the correctness of the proof. Turning this argument on its head, one could think of replacing the untrustworthy human expert (who has "all this stuff that [he has] to do [and has] a tendency of believe that the proofs are correct") by a more trustworthy machine.

In the setting of *Philosophy of Mathematical Practice*, what would we have to show in order to prove that something has an effect on the philosophy

[3]These statements are from Eva Müller-Hill's interviews with a research mathematician *Interviewpartner 6*: (Müller-Hill 2010, 342–343, 345). The statements are not actual quotations, but text based on the interview transcript and transformed into full sentences. The second paragraph is quoted from (Geist *et al.* 2010, 162–164).

[4]Test subject *M5* quoted from (Weber & Mejia-Ramos 2011).

of mathematics? If there is a believable scenario changing some features of current mathematical practice in which philosophical papers written nowadays would have to be substantially updated in order to meet the standards of philosophical discourse, this could be seen as sufficient to argue that the changes have an effect on the philosophy of mathematics.

Koepke (2014, 7) discusses the potential effect that automated proof checking has on mathematical practice, including the possibility of a new social publishing norm that requires mathematicians to submit a formalized proof with their paper (cf. also (Miller 2014) in this volume). In the following, we shall consider the following *Gedankenexperiment*: let us suppose that at some point in future, mathematicians have universally accepted that every proof has to be submitted with an attachment of a formalized proof in a regimented natural language (such as the language of the *Naproche* system, mentioned in (Koepke 2014, 6.2)) in order to be considered for publication. Correctness is then automatically checked, and the task of the referee focusses on assessing whether the paper is interesting and new. Whether we believe that this is likely to happen or not, is immaterial:[5] all that matters is that it is a possible scenario with a substantially changed culture of mathematical practice.

Let us give two examples to establish that the mentioned scenario has philosophical consequences in the above sense:

The first is the question of unreliable testimony in mathematical epistemology. As discussed in 2, the epistemologist of mathematics has to deal with a major headache: on the one hand, philosophers and mathematicians alike claim that there is an epistemic quality to mathematical knowledge that makes it more reliable than knowledge acquired by the method of induction in other sciences; on the other hand, we see a heuristic practice of checking correctness that defies the firm belief in the objectivity of mathematical knowledge. This discrepancy requires an explanation, as long as the practice of proof checking remains as it is described in (Geist *et al.* 2010; Weber & Mejia-Ramos 2011). In the described possible scenario, the discrepancy would have been resolved, or at least been replaced with a substantially different question. In the possible future in which mathematicians relegate proof checking to machines, there might be other pressing epistemological issues, but the question raised in 2 would have to be rephrased.

Our second example deals with discussions about the philosophical position of formalism: some of the critics of formalism have focused on the fact that formal derivations are far removed from typical arguments given for mathematical correctness (Rav 1999; Buldt *et al.* 2008). According to this line of argument, any version of formalism that focusses on the for-

[5]In fact, the present author does not think that this scenario is very likely.

mal derivation as the main object witnessing correctness of a mathematical statement is criticized there are hardly any formal derivations, and due to the dissimilarity between proofs and derivations, it is difficult to see the former as approximations to the latter.[6] The development of bridging tools such as *Naproche* that allow human mathematicians to use a language very similar to natural mathematical language and translate this into a formal derivation will weaken any such philosophical arguments that would then have to be reconsidered. Tanswell (2012) argues that *Naproche*, if fully developed, might reopen some of the discussions about formalism and allow philosophers to redefine formalism as a philosophical position in line with these new developments, and offer novel defenses for such a renewed position.

These two examples show that the effect of the new technologies on mathematical practice is not the equivalent of the new restaurant built next to the mathematics department, but offers genuinely new vistas in the philosophical landscape.

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[6] Also in the formal mathematics community, this has been seen as an obstacle for working towards the possible future of our *Gedankenexperiment*; cf. (Wiedijk 2007): "The other reason that there has not been much progress on the vision from the QED manifesto is that currently formalized mathematics does not resemble real mathematics at all. Formal proofs look like computer program source code."

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Benedikt Löwe
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Amsterdam
The Netherlands

Fachbereich Mathematik
Universität Hamburg
20146 Hamburg
Germany
`b.loewe@uva.nl`