Interactions between Large Cardinals and Strong Logics

Dissertation

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0. Introduction

0.1. Large cardinals from model creation tools

Large cardinals are one of the focal points of set-theoretic research. On the one hand, they directly advance our mathematical theory of the infinite. As Akihiro Kanamori puts it in his monograph *The Higher Infinite*, "large cardinals are the trustees of older traditions in direct line from Cantor's original investigation of [...] the transfinite numbers" ([Kan03, p. XV]). In this vein, for many they provide "natural" additions to Zermelo-Fraenkel set theory with the axiom of choice (ZFC), which inform understanding of the infinite. On the other hand, since the initialisation of their modern presentation in the 1960s, large cardinals have turned out to provide the backbone to the project of gauging the strength of our mathematical axioms. As a consequence of Gödel's Incompleteness Theorems, we cannot hope to ever prove the consistency of foundational mathematical theories. The best we can do is to establish relative consistency: that under the assumption of consistency of one theory, we can prove the consistency of another. In that light, one of the major tasks of set theory is to give a description how the strengths of different theories relate to each other. Carrying out this project, large cardinals turned out to be the most useful additions to ZFC. Today, we virtually know of no sensible extension of ZFC, whose consistency does not follow from the existence of some large cardinal.

Large cardinals are intimately connected to model theory of extensions of first-order logic, today often dubbed *strong logics*. In fact, some of the earliest large cardinal axioms considered posited the existence of so called *weakly* and *strongly compact cardinals*. They can be motivated along the following lines. The probably most important tool in first-order model theory is the Compactness Theorem. It says that to check satisfiability of a first-order theory, it is sufficient to check satisfiability of all its finite subtheories. In 1962, Alfred Tarski reported on a problem he had given to his student William Hanf about how the Compactness Theorem carries over to infinitary logics (cf. [Tar62]). More specifically, for an infinite, regular cardinal κ, Tarski and Hanf were interested in the logic $\mathcal{L}_{\kappa\kappa}$, which allows to form conjunctions and disjunctions over sets of formulas and quantification over strings of variables of length smaller than κ, respectively. Note that in these terms first-order logic can be denoted as $\mathcal{L}_{\omega\omega}$. The following concepts were raised:

(i) A theory $T \subseteq \mathcal{L}_{\kappa\kappa}$ is called *$<\kappa$-satisfiable* if all of its $<\kappa$-sized subsets have models.

(ii) κ is called *strongly compact* if it is uncountable and every $<\kappa$-satisfiable theory $T \subseteq \mathcal{L}_{\kappa\kappa}$ has a model.

(iii) κ is called *weakly compact* if it is inaccessible and every $<\kappa$-satisfiable theory $T \subseteq \mathcal{L}_{\kappa\kappa}$ with size $|T| = \kappa$ has a model.

Notice that the Compactness Theorem for first-order logic precisely says that ω is strongly compact, modulo the countability of ω. Soon it was understood that strongly compact cardinals are *measurable*, while measurable cardinals are weakly compact. Further, Hanf obtained the result that below a weakly compact cardinal are many inaccessible cardinals. This hence answered the question affirmatively, whether the first measurable cardinal is strictly larger than the first inaccessible, a problem that had been open since Stanisław Ulam's initial treatment of measurable cardinals in [Ula30] (cf. [Kan03, Section 4] for an overview of the history and proofs of the above reported results). Model theory of strong logics thus proved to provide deep insights into long standing open problems in set theory.

In the subsequent years after this initial breakthrough, large cardinal axioms started to be stated as statements about certain elementary embeddings, formalised via the existence of ultrafilters – and later of extenders – and by the 1980s, these axioms were established as the yardstick of measuring consistency strengths of strong set-theoretic assumptions. Nevertheless, model theory of strong logics turned out to be quite flexible, and many of these axioms were proven to be equivalent to statements about logics possessing certain model-theoretic properties. This thesis stands in the tradition of this latter type of research and adds to the theory of relations between large cardinals and properties of strong logics. We will approach these questions from both angles: providing model-theoretic characterisations of large cardinal axioms yet unknown to have such equivalent formulations; and providing the large cardinal strength of properties of logics where this strength is yet unknown.

Before we step into the technical treatment of our topic in Chapter 1, we give a short overview of the early development of the field in the 1970s and 1980s. We by no means claim to give a complete report of the history of relations between large cardinals and strong logics, but only outline the results most important for this thesis. We assume here that the reader is familiar with standard large cardinal notions. Should that not be the case, the relevant definitions can be found in Section 1.3 below. The large cardinal notions we will briefly consider here are:

(1) Supercompact cardinals.

(2) Extendible and $C^{(n)}$-extendible cardinals.

(3) Vopěnka's Principle.

Until recently, properties of logics which were known to be related to large cardinals, were often of two kinds. On the one hand, *compactness properties*, like the defining properties of strongly compact cardinals, generalise the Compactness Theorem to stronger logics. More specifically, the *compactness number* of a logic $\mathcal{L}$ is the smallest cardinal κ such that any $<\kappa$-satisfiable theory of $\mathcal{L}$ is satisfiable. On the other hand, *Löwenheim-Skolem properties* generalise another of the most prominent theorems of first-order model theory: the Löwenheim-Skolem Theorem says (in one version) that any structure in a countable vocabulary has a countable elementary substructure. Similarly, the *Löwenheim-Skolem-Tarski* (LST) number of a logic $\mathcal{L}$ is (in one version) the smallest cardinal κ such that any structure in a vocabulary of size $< \kappa$ has a substructure of size $< \kappa$, which is an elementary substructure regarding the logic $\mathcal{L}$. In 1971, Menachem Magidor showed that the smallest supercompact cardinal can be characterised as the LST number of second-order logic $\mathcal{L}^2$, and further, that the smallest extendible cardinal can be characterised as the compactness number of $\mathcal{L}^2$ (for both results cf. [Mag71]).

Miroslav Benda considered more complicated forms of compactness properties. *Compactness for type omission* provides models of a large theory omitting a large type, by models of small parts of the theory omitting small parts of the type. In 1976, Benda showed that κ being a supercompact cardinal is equivalent to $\mathcal{L}_{\kappa\kappa}$ having certain omitting-types-compactness properties (cf. [Ben78]).

A parallel development that related model theory of strong logics with set theory was the comparison of definability in set theory and definability by strong logics carried out by Jouko Väänänen. He showed that for some logics, the classes of structures which are definable by them correspond precisely to the classes of structures definable by some level of the Lévy hierarchy, and gave this situation the name *symbiosis*. Importantly, he introduced *sort logics*. These are extensions of second-order logic, graded by the natural numbers, and the n-th level $\mathcal{L}^{\mathrm{s},n}$ is able to define the classes which are $\Sigma_n \cup \Pi_n$ definable in the Lévy hierarchy.

This early history of relating large cardinals and model theory culminated in two results by Jonathan Stavi and Johann Makowsky. Stavi showed that the large cardinal axiom schema known as *Vopěnka's Principle* (VP) is equivalent to the axiom schema:

"Every logic has an LST number."

Stavi never published his result, but it can be found in [MV11, Theorem 6]. Similarly, Makowsky showed in 1985 (cf. [Mak85]) that VP is equivalent to the axiom schema:

"Every logic has a compactness number."

In particular, VP can be seen as a bound on the possible large cardinal strengths of these types of properties. For a long time it therefore looked like large cardinal axioms with higher strength than VP, like, for example, the existence of huge cardinals, were out of reach of model theoretic means. We will see below that this is not the case, as recent results showed that model theory can indeed provide such strong assumptions. Nevertheless, it took around thirty years until advancements related to questions of this thesis picked up pace again.

One major infusion that allowed to systematise the study of model theory of strong logics came from the outside of it. In 2012, Joan Bagaria introduced the so called *$C^{(n)}$-cardinals* (cf. [Bag12]), for a natural number n. The $C^{(n)}$-version of some large cardinal property P often provides natural strengthenings of P. Bagaria showed that considering *$C^{(n)}$-extendible cardinals* provides major insights into the structure of Vopěnka's Principle.

Considering VP's equivalences to statements about properties of logics, one might expect that one could undergo an analysis similar to Bagaria's, providing insights into the structure of the existence of compactness numbers for every logic. And indeed, Will Boney showed in [Bon20] that the smallest $C^{(n)}$-extendible cardinal gives us precisely the compactness number for Väänänen's sort logic $\mathcal{L}^{s,n}$. We will see that this is a pattern. Several properties of logics stratify themselves along the lines of the $C^{(n)}$-extendible cardinals. Moreover, other large cardinals that can be varied via relations to the class $C^{(n)}$ provide other such stratifications of model-theoretic properties in other regions of the large cardinal hierarchy.

What is the reason that properties of strong logics are systematically related to large cardinals? Results from first-order model theory (for example, the Compactness Theorem and the Löwenheim-Skolem Theorems) can be seen as model creation tools. Given some first-order theory, they provide new models of this theory, often with certain desired properties. The higher ranks of the consistency strength hierarchy usually state the existence of large cardinals which are witnessed by some elementary embedding $j : N \to M$ between transitive and, in particular, well-founded structures. The model creation tools from first-order model theory *can* provide elementary embeddings, namely when the theory we are building a new model of is some elementary diagram. However, as first-order logic cannot define well-foundedness, the elementary embeddings we get from first-order model theory usually do not witness the existence of any large cardinal. Here strong logics come into play. In many extensions of first-order logic, well-foundedness is expressible. And hence, model creation tools that provide models of theories of stronger logics do imply the existence of the correct type of elementary embeddings between transitive structures. Moreover, many strong large cardinal assumptions require more and more properties of the target M of some witnessing elementary embedding as above. And often, the required properties are expressible in some strong logic. In many cases, models of theories of strong logics are thus exactly the type of objects we require for the existence of some large cardinal.

0.2. Overview of the dissertation

This dissertation operates at the intersection of two areas of mathematical logic, namely the theory of *large cardinals*, and *abstract model theory*. While the former one is a subdiscipline of set theory, the latter one could be viewed as studying model-theoretic notions, in particular for extensions of first-order logic also known as *strong logics*, by set-theoretic means. Accordingly, our perspective is mostly that of a set theorist, and we assume familiarity with standard set-theoretic notions and techniques, like the most common large cardinal notions (inaccessibility, measurability, supercompactness, etc.) or forcing, as can be found in textbooks such as [Jec03, Kan03].

Chapter 1 will review many notions and results from the literature necessary for the other chapters of the thesis. In Section 1.1, we will review some very basic concepts from set theory which are crucial throughout. While we imagine that most readers will have seen systems such as second-order logic, we do not assume familiarity with abstract model theory (the treatment of logics as abstract objects of study). We will therefore consider its basic setup in Section 1.2. We also review most of the concrete logics we will deal with (Section 1.2.2), including a thorough treatment of *sort logic* (Section 1.2.4). Section 1.3 reviews the most important large cardinal notions we will consider. It can also be read as a reference to classical results about connections between large cardinals and properties of strong logics, which we will point out going along. The subsequent chapters all make reference to the concepts and notation introduced in Chapter 1. Besides that, while they sometimes make reference to notions and results treated elsewhere, by and large they can be read independently of each other. As an exception to this rule, before reading Section 4.8 the reader should consult Section 3.2, and possibly Sections 3.3 and 3.5. For Section 5.4 the reader should further be aware of some of the material on Π_n-strong cardinals discussed in Sections 2.3.1 and 2.3.2.

Chapter 2 studies compactness properties involving Henkin models of an abstract logic $\mathcal{L}$. We introduce the concept of a *strong Henkin model*, building on a weaker notion we will call *weak Henkin models*, which was considered in [BDGM24]. Compactness properties involving the latter notion are known to characterise *strong* and *Woodin* cardinals. In Section 2.2, we show that compactness properties for our stronger notion can characterise stronger large cardinals, namely supercompact cardinals when considering second-order logic (Theorem 2.2.4), $C^{(n)}$-extendible cardinals when considering sort logic (Theorem 2.2.8), and Vopěnka's Principle (VP) (Corollary 2.2.12). In Section 2.3, we study further applications of weak Henkin models. We consider a weakening of VP known as *weak Vopěnka's Principle* (WVP), introduced in [ART88]. While VP was known to have model-theoretic characterisations since the 1980's, whether the same is true for WVP has been open. Recently, WVP was shown to be connected to so-called *Π_n-strong cardinals* (cf. [BW23]). We show that compactness properties for weak Henkin models of sort logic can characterise Π_n-strong cardinals (Theorem 2.3.6), and as a result WVP (Corollary 2.3.11). We further give some other applications of compactness properties for weak Henkin models to give characterisations of cardinals which are *jointly Π_n-strong and strongly compact* (Theorem 2.3.25), and of *superstrong* cardinals (Theorem 2.3.28).

Chapter 3 introduces the new notion of *cardinal correctly extendible cardinals* and some of its variants. They are motivated by relations to compactness cardinals of the *equicardinality logic* $\mathcal{L}(\mathsf{I})$ we prove to hold in Section 3.5, and to *upward Löwenheim-Skolem-Tarski* (ULST) numbers of $\mathcal{L}(\mathsf{I})$ considered in Chapter 4, Section 4.8. Sections 3.3 and 3.4 analyse how cardinal correctly extendible cardinals relate to strongly compact, supercompact, and extendible cardinals. In particular, we observe some interesting interaction between extendible cardinals and the inner model HOD, by showing that under certain assumptions on the relation between the universe and HOD, the smallest extendible cardinal may consistently cease to be extendible in HOD while preserving its cardinal correct extendibility (Theorem 3.4.10 and Corollary 3.4.14).

Chapter 4 considers ULST numbers, which codify how variants of the upward Löwenheim-Skolem Theorem hold for strong logics. They were introduced in [GKV20]. The authors showed first results that the ULST number of second-order logic is partially extendible. We confirm their conjecture that the ULST number of second-order logic is precisely the first extendible cardinal (Theorem 4.5.1). We introduce the stronger notion of *strong* ULST (SULST) numbers and analyse the connections between ULST and SULST numbers of several logics and large cardinals. The logics and large cardinals we relate include the well-foundedness logic and measurable cardinals (Corollary 4.4.3), sort logics and $C^{(n)}$-extendible cardinals (Corollary 4.6.1), and as a result VP (Corollary 4.6.4). We relate ULST and SULST numbers of infinitary logics and variations of tall cardinals (Corollaries 4.7.9, 4.7.20, and 4.7.24), and of $\mathcal{L}(\mathsf{I})$ and variations of cardinal correctly extendible cardinals (Corollary 4.8.3). Our results imply that for some logics the notions of ULST, SULST, and compactness numbers coincide, while for others they can be separated. Finally, we introduce the notion of *$\mathcal{L}$-extendible cardinals* and show that for a large class of logics, their existence is equivalent to the existence of ULST numbers (Section 4.9).

Chapter 5 analyses model-theoretic properties of *class logics*, which are logics that have, for some fixed vocabulary, a proper class of sentences. While most classically considered class logics do not allow for much interesting model theory, we build on results by Trevor Wilson to show how restricted class logics can exhibit compactness properties and (upward) Löwenheim-Skolem properties. In particular, we provide a second characterisation of Π_n-strong cardinals and WVP by model-theoretic means through Löwenheim-Skolem properties of class extensions of sort logics (Theorem 5.4.3 and Corollary 5.4.6). In Section 5.3, we analyse classical compactness and upward Löwenheim-Skolem properties of class versions of first- and second-order logic, and of sort logics. Finally, we show how to characterise *Shelah cardinals* by a newly introduced compactness property of second-order class logics (Theorem 5.5.1).

Chapter 6 considers the relationship of properties of strong logics $\mathcal{L}$ and certain reflection principles involving some set theoretic predicate R, mediated through what is known as *symbiosis* between $\mathcal{L}$ and R. Bagaria and Väänänen in [BV16] and Galeotti, Khomskii, and Väänänen in [GKV20] showed how under assumption of symbiosis between $\mathcal{L}$ and R, reflection principles involving R are equivalent to downward and upward Löwenheim-Skolem properties of $\mathcal{L}$, respectively. Building on results from the author's Master's thesis [Osi21], we show that the same is true for transfer between compactness properties of $\mathcal{L}$ and certain reflection principles in classes of partial orders defined via R (Theorem 6.3.12). We further give a proof of a statement by Bagaria how weak downward reflection principles transfer to weak downward Löwenheim-Skolem properties (Theorem 6.4.13).

Finally, in Chapter 7 we consider *compactness for type omission properties.* We show that properties of finitary logics give rise to large cardinals stronger than VP by showing that certain compactness for type omission properties of the well-foundedness logic are equivalent to the existence of huge cardinals (Theorem 7.3.2). That this was possible was known for infinitary logics previously, but was open for finitary ones. Further, we show how compactness for type omission properties for infinitary equicardinality logics give rise to a large cardinal notion in between supercompactness and extendibility (Theorem 7.4.1).

1. Preliminaries

1.1. Basic set theory

Let us start by reviewing some basic set-theoretic notation and results, all of which are standard. All definitions and results of this section not accompanied by references can be found in any comprehensive introduction to axiomatic set theory, such as [Jec03]. We work in ZFC throughout, formalised in a first-order meta-language with a single binary relation symbol $\in$.

Throughout the thesis we will be assessing the interplay between models of set theory and how they interpret whether some structure $\mathcal{A}$ satisfies a sentence φ of some given logic. In particular, we are interested in whether some model of set theory is correct about whether $\mathcal{A}$ satisfies φ. To assess this, it will often be important to consider the precise complexity of the set-theoretic formulas defining some logic we operate with. We therefore review the *Lévy hierarchy* of formulas in the language of set theory, and we shall be precise about the concrete coding of many of the objects we introduce. A formula is $\Delta_0 = \Sigma_0 = \Pi_0$ if it does not contain any unbounded quantifiers. Recursively, a formula Φ is Σ_{n+1} if it is of the form $\Phi = \exists x \Psi$ for some Π_n formula Ψ, and Φ is Π_{n+1} if it is of the form $\Phi = \forall x \Psi$ for a Σ_n formula Ψ. It is Δ_n if there are Σ_n and Π_n formulas Φ_0 and Φ_1, respectively, and $\vdash (\Phi \leftrightarrow \Phi_0) \wedge (\Phi \leftrightarrow \Phi_1)$, i.e., that Φ is equivalent to both a Σ_n and a Π_n formula is a theorem of first-order logic.

A *class* is simply a formula $\Phi(x, p_1, \ldots, p_n)$ in connection with some (optional) set parameters $p_1, \ldots, p_n$. We identify Φ with the collection of sets which Φ is true of and write $\Phi = \{a \colon \Phi(a, p_1, \ldots, p_n)\}$. If this collection Φ is *not* a set, we call Φ a *proper* class. Note that any set M is a class, as $M = \{a \colon a \in M\}$. Given these conventions, for formulas like $\mathrm{Ord}(x)$, expressing that x is an ordinal, or $\mathrm{Card}(x)$, expressing that x is an infinite cardinal, we write Ord and Card to denote the classes of all ordinals and cardinals, respectively.

For any formula Φ, and some transitive class M, we write Φ^M for the relativisation of Φ to M, i.e., all quantifiers in Φ are relativised to M. If a set M is transitive then the Δ_0 formulas are *absolute* between M and V, i.e., for any $a_1, \ldots, a_n \in M$ and any Δ_0 formula $\Phi(x_1, \ldots, x_n)$,

$$\Phi^M(a_1, \ldots, a_n) \text{ iff } \Phi(a_1, \ldots, a_n).$$

If $\Phi(x_1, \ldots, x_n)$ is Σ_1, then Φ is *upwards absolute* from M to V, i.e.,

$$\Phi^M(a_1, \ldots, a_n) \text{ implies } \Phi(a_1, \ldots, a_n).$$

If $\Phi(x_1, \ldots, x_n)$ is Π_1, then Φ is *downwards absolute* from V to M, i.e.,

$$\Phi(a_1, \ldots, a_n) \text{ implies } \Phi^M(a_1, \ldots, a_n).$$

This implies that Δ_1 formulas are absolute between transitive M and V.

If for some formula Φ and a Σ_n formula Ψ, we have $\mathrm{ZFC} \vdash \Phi \leftrightarrow \Psi$, we say Φ is Σ_n^{ZFC}. Analogously, we define Π_n^{ZFC}. If a formula is both Σ_n^{ZFC} and Π_n^{ZFC}, it is Δ_n^{ZFC}. Similar to the above remarks, Σ_1^{ZFC}, Π_1^{ZFC}, and Δ_1^{ZFC} formulas are upwards absolute, downwards absolute, and absolute, respectively, for transitive models of ZFC. Notice that only a finite amount of ZFC is needed to witness that a formula is, for instance, Σ_n^{ZFC}. Then the upwards absoluteness holds for transitive models of this finite fragment of ZFC, and analogously for Π_n^{ZFC} and Δ_n^{ZFC}. As we will most often work with models of some amount of ZFC, and we might usually adjoin the "right" fragment of ZFC to show that some formula is, for example, Σ_n^{ZFC} by simply adding it as a conjunct to the sentences already satisfied by the model, we will drop the superscripts "ZFC" for these classes of formulas throughout and simply write Σ_n, Π_n and Δ_n for Σ_n^{ZFC}, Π_n^{ZFC}, and Δ_n^{ZFC}. If $\mathcal{K}$ is some class and for $\Gamma \in \{\Sigma_n, \Pi_n, \Delta_n\}$ there is some Γ formula $\Phi(x_1, \ldots, x_n)$ such that $\mathcal{K} = \{(a_1, \ldots, a_n) \colon \Phi(a_1, \ldots, a_n)\}$, then we say that $\mathcal{K}$ is Γ *definable*, or that $\mathcal{K}$ *can be defined in a* Γ *way*, or that being in $\mathcal{K}$ is a Γ *property*. When this is well-known, we will often use that some property is Γ definable for $\Gamma \in \{\Sigma_n, \Pi_n, \Delta_n\}$ without extra notice.

We will fix a formalisation $\mathcal{L}_{\omega\omega}$ of first-order logic inside of set theory. For this purpose, we need to talk about vocabularies τ and corresponding τ-structures $\mathcal{A}$. These definitions are the standard ones from first-order model theory. Nevertheless, let us be precise for the concrete set-theoretic coding we use, as this will be used for our treatment of general logics, and, as remarked earlier, evaluating these objects inside models of set theory will be crucial. Still, that we use this specific coding is not essential.

We consider many-sorted structures, i.e., structures that possibly come with multiple domains. In this framework, a vocabulary τ, next to possible relation, function, and constant symbols, contains a finite number of *sort symbols*. The precise definitions are given below (cf. Definitions 1.1.1 and 1.1.2), but let us give some intuition. A τ-structure in the many-sorted sense possibly has multiple domains A_s, one for each sort symbol $s \in \tau$. Further, each relation, constant, and function symbol comes with a *configuration* of sorts, specifying which domains it shall be defined on. So a vocabulary $\tau = \{s_1, s_2, R\}$ might consist of two sort symbols s_1 and s_2 and a relation symbol R with configuration $\mathrm{conf}(R) = (s_1, s_2)$. A τ-structure $\mathcal{A} = (A_1, A_2, R^{\mathcal{A}})$ then consists of two sets A_1 and A_2, the *domains* in sort s_1 and s_2, respectively, and a relation $R^{\mathcal{A}} \subseteq A_1 \times A_2$. We do *not* assume that the different domains have to be disjoint. Using many-sorted structures is not strictly necessary, as multiple domains can be coded into a single one by using additional unary predicates. It will however be convenient, as we will often work with *sort logic*, whose semantics is naturally developed in the context of many-sorted structures.

Definition 1.1.1. We define the following notions.

(i) For each $s \in \omega$, we call the pair $(0, s)$ a *sort symbol*, and say that s *is a sort*.

(ii) For each set a and $s \in \omega$, we call $(1, (s, a))$ a *constant symbol of sort* s.

(iii) For each set a and $s_1, \ldots, s_k \in \omega$, we call $(2, (s_1, \ldots, s_k, a))$ a *relation symbol of arity* k *between the sorts* $s_1, \ldots, s_k$.

(iv) For each set a and $s_1, \ldots, s_{k+1}$, we call $(3, (s_1, \ldots, s_{k+1}, a))$ a *function symbol of arity k between the sorts $s_1, \ldots, s_k$ to the sort s_{k+1}*.

(v) We let conf be the function that returns the *configuration of sorts* of a symbol, i.e.,

  (a) if $c = (1, (s, a))$ is a constant symbol, then $\text{conf(c)} = s$.

  (b) if $R = (2, (s_1, \ldots, s_k, a))$ is a relation symbol then $\text{conf}(R) = (s_1, \ldots, s_k)$.

  (c) if $f = (3, (s_1, \ldots, s_{k+1}, a))$ is a function symbol, then $\text{conf}(f) = (s_1, \ldots, s_{k+1})$.

A *vocabulary* τ is a set of sort symbols, relation symbols, function symbols and constant symbols. If τ is a vocabulary, we write $s(\tau)$ for the set of sort symbols appearing in τ. We assume that $s(\tau)$ is always non-empty and finite. Further, for any relation, function or constant symbol $x \in \tau$, all sorts involved in x must appear in $s(\tau)$, i.e., if $\text{conf}(x) = (s_1, \ldots, s_k)$, then $\{s_1, \ldots, s_k\} \subseteq s(\tau)$.

The sets a appearing in part (ii) to (iv) of the definition above are simply there to generate a proper class of relation, function and constant symbols. Note that we can read off of a given set whether it is a sort, relation, function or constant symbol, as well as its arity and configuration, given its coding as a simple finite tuple. These concepts are therefore easily seen to be Δ_0 definable. Being a vocabulary is a Δ_1 property, as it involves restriction to finitely many sort symbols and being finite is Δ_1. In particular, all the notions of the above definition are absolute between transitive models and V.

As usual, a τ-structure provides interpretations for the symbols appearing in τ:

Definition 1.1.2. Let τ be a vocabulary with $s(\tau) = \{s_1, \ldots, s_n\}$. A tuple $\mathcal{A} = (A_1, \ldots, A_n, F)$ is called a *τ-structure* iff $A_1, \ldots, A_n$ are non-empty sets called the *domains for sorts* $s_1, \ldots, s_n$, respectively, and F is a function with domain $\tau \setminus s(\tau)$ such that

(i) if $c = (1, (s_i, a)) \in \tau$ is a constant symbol, then $c^{\mathcal{A}} = F(c) \in A_i$.

(ii) if $R \in \tau$ is a relation symbol and $\text{conf}(R) = (s_{i_1}, \ldots, s_{i_k})$, then $R^{\mathcal{A}} = F(R)$ is a relation $R^{\mathcal{A}} \subseteq \Pi_{j=1}^{k} A_{s_{i_j}}$.

(iii) if $f \in \tau$ is a function symbol and $\text{conf}(f) = (s_{i_1}, \ldots, s_{i_{k+1}})$, then $f^{\mathcal{A}} = F(f)$ is a function $f^{\mathcal{A}} : \prod_{j=1}^{k} A_{s_{i_j}} \to A_{s_{i_{k+1}}}$.

We also write $A = \bigcup_{1 \leq i \leq n} A_i$.

That $\mathcal{A}$ is a τ-structure is a Δ_1 property of $\mathcal{A}$ and τ. If $\tau = \{s_0, R, f, c\}$ for a sort symbol s_0, a relation symbol R, a function symbol f, and a constant symbol c, then if $\mathcal{A} = (A_0, F)$ is a τ-structure, we will most often write the more customary expression $\mathcal{A} = (A, R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}})$ to denote $\mathcal{A}$. We then mean that $F(R) = R^{\mathcal{A}}$, $F(f) = f^{\mathcal{A}}$ and $F(c) = c^{\mathcal{A}}$, and similar for other vocabularies.

Coming back to first-order logic $\mathcal{L}_{\omega\omega}$, we fix a coding of formulas of first-order logic over some vocabulary τ. We assume that all logical symbols, i.e., variables, equality, boolean connectives and quantifiers, are coded as finite tuples of natural numbers, similar to the coding of the non-logical symbols above. Further, we assume that formulas over τ are finite tuples of logical symbols and members of τ. For a precise definition of our syntax, consider Appendix A. We write $\mathcal{L}_{\omega\omega}[\tau]$ for the collection of first-order formulas over τ. Being a first-order formula is Δ_1 definable, i.e., there is a Δ_1 formula $\Phi(x, y)$ such that $\Phi(\varphi, \tau)$ holds of some sets φ and τ iff τ is a vocabulary and $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$. Let us assume throughout that the vocabulary of the language of set theory $\{\in\}$ is given by the concrete binary predicate $\in = (2, (0, 0))$. The satisfaction relation is defined in the usual recursive way on the complexity of φ. Keep in mind that we also denote the meta-language relation symbol as $\in$. Let us write $\mathcal{A} \models \varphi[f]$ to express that $\mathcal{A}$ is a τ-structure for some vocabulary τ, $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$, and f is a variable assignment on the free variables of φ, taking values in A, such that φ is satisfied by $\mathcal{A}$ under the assignment f. We also write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are among $x_1, \ldots, x_n$, and $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in A$ to denote that $\mathcal{A} \models \varphi[f]$ with $f(x_i) = a_i$. First-order satisfaction can also be defined by a Δ_1 formula $\Psi(x, y, z)$ such that for any sets $\mathcal{A}$, φ, f:

$$\mathcal{A} \models \varphi[f] \text{ iff } \Psi(\mathcal{A}, \varphi, f).$$

In particular, both syntax and semantics of first-order logic are absolute for transitive models of set theory. That these constructions are possible is to some extent folklore. For proofs of how to carry them out, including the Δ_1 definitions of the concepts involved, consider, for example, [Bar75, Section III.1]. Note that there the infinitary logic $\mathcal{L}_{\infty\omega}$ is considered instead. The first-order case is essentially the same, only replacing arbitrary conjunctions and disjunctions by finite ones.

Note that all formulas Φ of our meta-language have a formal analogue, i.e., a set $\varphi \in \mathcal{L}_{\omega\omega}[\{\in\}]$, which mirrors the structure of the formula Φ. Usually it should be clear from context whether we are talking about a formula of the meta-language or of a formula in $\mathcal{L}_{\omega\omega}$. But for the moment, let us write $\ulcorner\Phi\urcorner$ to denote the formal analogue of the meta-language formula Φ. For any formula $\Phi(x_1, \ldots, x_n)$, given some transitive set M and considering the structure $(M, \in)$ with the membership relation restricted to M and any $a_1, \ldots, a_n \in M$, we can prove:

$$\Phi^M(a_1, \ldots, a_n) \text{ iff } (M, \in) \models \ulcorner\Phi(a_1, \ldots, a_n)\urcorner.$$

While formally, for $\varphi \in \mathcal{L}_{\omega\omega}[\{\in\}]$, the assertion $(M, \in) \models \varphi(a_1, \ldots, a_n)$ is only defined if M is a set, abusing notation we also write $V \models \varphi(a_1, \ldots, a_n)$ to denote that $\Phi(a_1, \ldots, a_n)$ holds, where $\varphi = \ulcorner\Phi\urcorner$, and similarly for other classes. Given the above remarks, the two perspectives are essentially the same for sets M.

Recall Tarski's undefinability of truth (cf., e.g., [Jec03, Theorem 12.7]), i.e., there is no formula $T(x)$ such that for all formulas $\Phi(x_1, \ldots, x_n)$ of the meta-language:

$$\text{ZFC} \vdash \forall x_1, \ldots, x_n(\Phi(x_1, \ldots, x_n) \leftrightarrow T((\ulcorner\Phi\urcorner, x_1, \ldots, x_n)).$$

On the other hand, the global satisfaction relation T_n restricted to Σ_n formulas is uniformly definable in a Σ_n way (cf., e.g., [Kan03, Section 0]), i.e., there is a Σ_n formula $T_n(x)$ in the meta-language such that for all Σ_n formulas $\Phi(x_1, \ldots, x_n)$:

$$\text{ZFC} \vdash \forall x_1, \ldots, x_n(\Phi(x_1, \ldots, x_n) \leftrightarrow T_n((\ulcorner\Phi\urcorner, x_1, \ldots, x_n))).$$

In particular, if M is some transitive set, we can formalise that $(M, \in)$ is a Σ_n-elementary substructure of the universe by writing down the following as a formula of set theory:

> For all x, if $x = (\varphi, x_1, \ldots, x_n)$ is a finite tuple such that $\varphi \in \mathcal{L}_{\omega\omega}[\{\in\}]$ is a Σ_n formula, and $x_1, \ldots, x_n \in M$, then
>
> $$(M, \in) \models \varphi(x_1, \ldots, x_n) \text{ iff } T_n((\varphi, x_1, \ldots, x_n)).$$

If the above holds, then for any Σ_n formula $\varphi \in \mathcal{L}_{\omega\omega}[\tau]$ and any $a_1, \ldots, a_n \in M$, $(M, \in) \models \varphi(a_1, \ldots, a_n)$ iff $(V, \in) \models \varphi(a_1, \ldots, a_n)$ (where the latter assertion abuses notation as described above). We denote this situation as $M \prec_{\Sigma_n} V$.

Following Bagaria [Bag12], we write $C^{(n)} = \{\alpha \colon V_\alpha \prec_{\Sigma_n} V\}$ for the class of all ordinals for which V_α is a Σ_n-elementary substructure of the universe. Using the above remark and the fact that "x is an ordinal and $y = V_x$" is Π_1, Bagaria shows that $C^{(n)}$ is definable by a Π_n formula. Using the Reflection Theorem for the formula T_n shows that for every n, $C^{(n)}$ forms a club class. Note that all V_α are transitive, so in particular absolute with respect to the Δ_0 formulas, so $C^{(0)}$ is simply the class of all ordinals. Further, the class $C^{(1)}$ consists of precisely the uncountable fixed points of the $\beth$-function. For proofs of all of these remarks, cf. [Bag12, Section 1].

1.2. Abstract model theory

We assume that the reader is familiar with model theory of first-order logic, and that they have seen other logics, like expansions of first-order logic by additional quantifiers, or second-order logic. Such extensions of first-order logic, also known as *strong logics*, are our main objects of study. Many of our results concern some specific logic. For example, we will consider large cardinal notions that arise from certain model-theoretic properties of second-order logic. On the other hand, many large cardinal axiom schemas are equivalent to schemas making assertions that *all* logics exhibit a certain property. To make sense of such schemas, we have to give a formal definition of what we mean by "logic".

The point of view we will adopt is the one taken by the field of *abstract model theory*. Its basic ideas go back to Per Lindström [Lin69] and Jon Barwise [Bar74] and were codified in the collective monograph [BF85]. A *logic* in this sense has, like first-order logic, a set of *sentences* for every vocabulary τ. Abstract model theory then generalises from first-order logic what it means to sensibly ascribe truth values to these sentences as interpreted in some τ-structure.

1.2.1. Abstract logics

In this section we will define what we mean by an *abstract logic* throughout this thesis (Definition 1.2.1). Most notions are standard and can, for instance, be found in [BF85]. For technical details, we will follow [Osi21].

Before we can give our definition, we need some other standard notions. If $\mathcal{A}$ and $\mathcal{B}$ are τ-structures, an isomorphism $f : \mathcal{A} \to \mathcal{B}$ is a bijection $f : \bigcup_{s \in s(\tau)} A_s \to \bigcup_{s \in s(\tau)} B_s$ which restricts to bijections $A_s \to B_s$ for every $s \in s(\tau)$ and preserves the constants, relations, and functions defined on $\mathcal{A}$ and $\mathcal{B}$ in the obvious way. We write $\mathcal{A} \cong \mathcal{B}$ if there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$. Further, if τ and σ are vocabularies, a map $f : \tau \to \sigma$ is called a *renaming* if it is a bijection between the respective sets of sort symbols, relation symbols, function symbols, and constant symbols, respecting arity and configuration. If $f : \tau \to \sigma$ is a renaming and $\mathcal{A}$ is a τ-structure, we can use f to turn $\mathcal{A}$ into an $f(\sigma)$-structure by interpreting, for example, some relation symbol $f(r) \in \sigma$ by $r^{\mathcal{A}}$. We will denote this $f(\sigma)$-structure as $f(\mathcal{A})$ and call it as well a *renaming* of $\mathcal{A}$.

Definition 1.2.1. An *abstract logic* $\mathcal{L}$ is a pair consisting of a definable class function that maps every vocabulary τ to a class $\mathcal{L}[\tau]$, called the *class of $\mathcal{L}$-sentences over τ*, and a definable class relation $\models_{\mathcal{L}}$, called the *satisfaction relation of $\mathcal{L}$*, such that:

(i) If $\mathcal{A} \models_{\mathcal{L}} \varphi$, then $\varphi \in \mathcal{L}[\tau]$ for some vocabulary τ and $\mathcal{A}$ is a τ-structure. In this case we say that $\mathcal{A}$ *is a model of* φ.

(ii) If $\sigma \subseteq \tau$ for vocabularies τ and σ, then $\mathcal{L}[\sigma] \subseteq \mathcal{L}[\tau]$.

(iii) If $\mathcal{A} \cong \mathcal{B}$ for τ-structures $\mathcal{A}$ and $\mathcal{B}$, then for every $\varphi \in \mathcal{L}[\tau]$:

$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } \mathcal{B} \models_{\mathcal{L}} \varphi.$$

(iv) If $\varphi \in \mathcal{L}[\sigma]$ and $\sigma \subseteq \tau$ for vocabularies τ and σ, then for every τ-structure $\mathcal{A}$:

$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } (\mathcal{A} \upharpoonright \sigma) \models_{\mathcal{L}} \varphi.$$

(v) If $f : \tau \to \sigma$ is a renaming, then there is a bijection $f : \mathcal{L}[\tau] \to \mathcal{L}[\sigma]$ such that:

$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ iff } f(\mathcal{A}) \models_{\mathcal{L}} f(\varphi).$$

We call (iv) the *reduct property* and (v) the *renaming property.* If $f : \mathcal{L}[\tau] \to \mathcal{L}[\sigma]$ is the bijection from (v), we call for $T \subseteq \mathcal{L}[\tau]$ the image $f``T$ a *copy* of T, and for $\varphi \in T$, we call $f(\varphi)$ a *renaming* of φ. If clear from context which abstract logic we are talking about, we will most often drop the subscript and simply write $\models$ for $\models_{\mathcal{L}}$. If the abstract logic $\mathcal{L}$ further satisfies the following condition, we simply say that $\mathcal{L}$ is a *logic.*

(vi) There is a cardinal κ such that for any vocabulary τ and any $\varphi \in \mathcal{L}[\tau]$ there is $\tau_0 \in \mathcal{P}_\kappa \tau$ such that $\varphi \in \mathcal{L}[\tau_0]$, and such that if $\tau \in H_\kappa$, then $\mathcal{L}[\tau] \subseteq H_\kappa$.

We call the smallest κ as above the *strong dependence number* $\text{dep}^*(\mathcal{L})$ of $\mathcal{L}$.

If $\text{dep}^*(\mathcal{L}) = \kappa$, note that the renaming property implies that φ is equivalent up to renaming to a sentence in $\mathcal{L} \cap H_\kappa$. When analysing models of $\mathcal{L}$-sentences, we can therefore restrict attention to sentences in H_κ. We will use this observation tacitly below. We further get that there are not more than $\max(2^{<\kappa}, |\mathcal{P}_\kappa \tau|)$ many sentences in $\mathcal{L}[\tau]$.

Note that there is in general no requirement on what the sentences of our abstract logics look like. They are simply some abstractly given sets. In particular, for some set-sized vocabulary τ, the collection $\mathcal{L}[\tau]$ may form a proper class. We call such abstract logics (*proper*) *class logics*. We will see that sensible model theory of proper class logics is very limited. For this reason, it has become somewhat customary to restrict attention to abstract logics for which this cannot happen, so for which $\mathcal{L}[\tau]$ is always a set. By our comments above, assuming the existence of a strong dependence number is one way to achieve this goal. To make sure for a convenient formulation of our theorems, we reserve the term *logic* to abstract logics with a strong dependence number, as we will most often want to exclude proper class logics from the discussion.[1]

First-order logic can be considered as a logic, letting $\mathcal{L}_{\omega\omega}[\tau]$ be the first-order sentences over τ, i.e., formulas without any free variables, and $\models_{\mathcal{L}_{\omega\omega}}$ be the usual satisfaction relation. Note that we also used $\mathcal{L}_{\omega\omega}[\tau]$ to denote the set of first-order formulas with possible free variables over τ. It should be clear from context which of the perspectives we are taking

Abstract model theory identifies the meaning of a sentence with the class of structures satisfying it. This allows to compare abstract logics via their expressive strengths. For this purpose, if τ is a vocabulary and $\varphi \in \mathcal{L}[\tau]$ we write

$$\mathrm{Mod}^{\tau}_{\mathcal{L}}(\varphi) = \{\mathcal{A} \colon \mathcal{A} \text{ is a } \tau\text{-structure such that } \mathcal{A} \models_{\mathcal{L}} \varphi\},$$

for the class of models of φ. Again, we drop the superscript τ and subscript $\mathcal{L}$ if they are clear from context. If $\mathcal{K}$ is some class of τ-structures for some vocabulary τ, we also call $\mathcal{K}$ a *model class*. If there is a sentence $\varphi \in \mathcal{L}[\tau]$ such that $\mathcal{K} = \mathrm{Mod}^{\tau}_{\mathcal{L}}(\varphi)$, then we say that $\mathcal{K}$ is *definable* or *axiomatisable* in $\mathcal{L}$, or simply $\mathcal{L}$-*definable* or $\mathcal{L}$-*axiomatisable*.

Definition 1.2.2. Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be abstract logics. We say that $\mathcal{L}_1$ *is an extension of* $\mathcal{L}_0$, in symbols $\mathcal{L}_0 \leq \mathcal{L}_1$, iff for every vocabulary τ and every $\varphi \in \mathcal{L}_0[\tau]$, there is some $\psi \in \mathcal{L}_1[\tau]$ such that $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$, i.e., if every $\mathcal{L}_0$-definable model class is also $\mathcal{L}_1$-definable. We also say that $\mathcal{L}_0$ is *bounded* by $\mathcal{L}_1$.

If $\mathcal{L}_0 \leq \mathcal{L}_1$ and further there is $\psi \in \mathcal{L}_1$ such that for no $\varphi \in \mathcal{L}_0$, $\mathrm{Mod}(\varphi) = \mathrm{Mod}(\psi)$, then we say that $\mathcal{L}_1$ is a *proper* extension of $\mathcal{L}_0$, and write $\mathcal{L}_0 < \mathcal{L}_1$.

Definition 1.2.3. A logic $\mathcal{L}$ is called *strong* iff $\mathcal{L}_{\omega\omega} < \mathcal{L}$.

We call any subset $T \subseteq \mathcal{L}[\tau]$ an $\mathcal{L}$-*theory over* τ or simply a *theory*. Given some τ-structure $\mathcal{A}$ we write

$$\mathrm{Th}_{\mathcal{L}}(\mathcal{A}) = \{\varphi \in \mathcal{L}[\tau] \colon \mathcal{A} \models_{\mathcal{L}} \varphi\},$$

for the $\mathcal{L}$-*theory of* $\mathcal{A}$. Also here, if clear from context, we drop the subscript and simply write $\mathrm{Th}(\mathcal{A})$.

[1] The difference of $\mathrm{dep}^*(\mathcal{L})$ to the more often considered *dependence number* $\mathrm{dep}(\mathcal{L})$ is the requirement that $\mathcal{L}[\tau] \subseteq H_\kappa$ for $\tau \in H_\kappa$. The technical advantage that $\mathrm{dep}^*(\mathcal{L})$ has over $\mathrm{dep}(\mathcal{L})$ is that it excludes some pathological abstract logics which have a large set of sentences over small vocabularies.

1.2.2. Examples of strong logics

Let us mention some of the most important examples of strong logics we will deal with. We will point out the semantic features of each logic. Their syntax can be understood to be defined in the obvious way. Note that when interacting with set theory, in particular with elementary embeddings of the universe, also the precise coding of the syntax as set-theoretic objects will be relevant. For such a precise definition, the reader is referred to Appendix A.

(a) The logic $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ expands first-order logic by adding the *well-foundedness quantifier* Q^{WF} with the semantics:

$$\mathcal{A} \models \mathsf{Q}^{\mathrm{WF}} xy\varphi(x,y) \text{ iff } \{(a,b) \in A^2 \colon \mathcal{A} \models \varphi(a,b)\} \text{ is a well-founded relation.}$$

(b) The logic $\mathcal{L}(\mathsf{I})$ expands first-order logic by the *Härtig* or *equicardinality* quantifier I with the semantics:

$$\mathcal{A} \models_{\mathcal{L}(\mathsf{I})} \mathsf{I}xy\varphi(x)\psi(y) \text{ iff } |\{a \in A \colon \mathcal{A} \models \varphi(a)\}| = |\{a \in A \colon \mathcal{A} \models \psi(a)\}|.$$

(c) Second-order logic $\mathcal{L}^2$ expands first-order logic by introducing n-ary variables $X^{(n)}$ for every natural number and thus allowing to quantify over n-ary relations over the structure in question, i.e.,

$$\mathcal{A} \models \exists X^{(n)}\varphi(X) \text{ iff there is some } B \subseteq A^n \text{ such that } \mathcal{A} \models \varphi(B),$$

and similarly for second-order universal quantification. If X is a unary relation variable, we simply write $\exists X\varphi(X)$.

(d) Let us consider the abstract logic $\mathcal{L}_{\infty\infty}$, with the intention to define infinitary logics $\mathcal{L}_{\kappa\lambda}$ as a subsystem. For $\mathcal{L}_{\infty\infty}$, we allow consideration of conjunctions and disjunctions over arbitrary sets of formulas, and quantification over arbitrary sequences of variables. i.e., if $T \subseteq \mathcal{L}_{\infty\infty}$ is a set of formulas, we allow the expressions $\bigwedge T$ and $\bigvee T$, whose truth is evaluated in the following way:

$$\mathcal{A} \models \bigwedge T \text{ iff } \mathcal{A} \models \varphi \text{ for all } \varphi \in T, \text{ and}$$
$$\mathcal{A} \models \bigvee T \text{ iff } \mathcal{A} \models \varphi \text{ for some } \varphi \in T.$$

Further, if Z is some set, $(x_z \colon z \in Z)$ is a sequence of variables, and φ is some formula of $\mathcal{L}_{\infty\infty}$, then $\exists(x_z \colon z \in Z)\varphi, \forall(x_z \colon z \in Z)\varphi$ are formulas; then $\mathcal{A} \models \exists(x_z \colon z \in Z)\varphi$ iff

$$\text{there is some sequence } (a_z \colon z \in Z) \text{ such that } \mathcal{A} \models \varphi(a_z \colon z \in Z),$$

and dually for the universal quantifier. Omitting the part about infinite sequences of variables, we get the abstract logic $\mathcal{L}_{\infty\omega}$, which allows for arbitrary conjunctions and disjunctions but has no infinite quantifiers. Note that both $\mathcal{L}_{\infty\infty}$ and $\mathcal{L}_{\infty\omega}$ are proper class logics.

(e) For regular cardinals $\kappa \geq \lambda$, the formulas of the logic $\mathcal{L}_{\kappa\lambda}$ are formed exactly as $\mathcal{L}_{\infty\infty}$ but restricting to conjunctions and disjunctions of sets of sentences T of size $|T| < \kappa$ and strings of quantifiers indexed by sets $Z \in H_\lambda$. For the satisfaction relation, we simply let $\mathcal{A} \models_{\mathcal{L}_{\kappa\lambda}} \varphi$ iff $\varphi \in \mathcal{L}_{\kappa\lambda}$ and $\mathcal{A} \models_{\mathcal{L}_{\infty\infty}} \varphi$.

(f) Further, we allow combinations of these logics. In particular, we consider the infinite versions $\mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}_{\kappa\lambda}(\mathsf{I})$ and $\mathcal{L}^2_{\kappa\lambda}$. For $\mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\mathrm{WF}})$ and $\mathcal{L}_{\kappa\lambda}(\mathsf{I})$, we simply add infinite boolean connectives and infinite first-order quantifiers. For $\mathcal{L}^2_{\kappa\lambda}$, we also allow quantification over infinite sequences of second-order variables, with the semantics defined in the obvious way. Similarly, we consider proper class logics like $\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$, etc.

Note that any sentence of the logics $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}(\mathsf{I})$, and $\mathcal{L}^2$ can be considered as a finite object and, as for first-order logic, given the coding we fix in Appendix A their syntax is Δ_1 definable. In particular, if τ is a vocabulary and $(M, \in)$ is some transitive model of set theory such that $\tau \in M$, then $(\mathcal{L}[\tau])^M = \mathcal{L}[\tau] \in M$ for these logics. For the infinitary logics $\mathcal{L}_{\kappa\lambda}$ (and analogously for $\mathcal{L}^2_{\kappa\lambda}$ etc.), also their syntax is Δ_1 definable, but using κ and λ as parameters. So to compute $\mathcal{L}_{\kappa\lambda}[\tau]$ in M, this only really makes sense if M contains κ and λ. Further, even if $\kappa, \lambda \in M$, it might happen that, for example, $T \subseteq \mathcal{L}_{\kappa\omega}[\tau]$ such that $|T| < \kappa$ for $\tau \in M$ but $T \notin M$, and therefore also $\bigwedge T \notin M$. On the other hand, if $T, \kappa \in M$, then $M \models |T| < \kappa$ (as κ is a cardinal) and so $\bigwedge T \in \mathcal{L}^M_{\kappa\omega}$. We can reason similarly for sequences of variables. Thus, $\mathcal{L}^M_{\kappa\lambda} = \mathcal{L}_{\kappa\lambda} \cap M \subseteq \mathcal{L}_{\kappa\lambda}$.

We say that some transitive model M of set theory *is correct about $\mathcal{L}$-satisfaction* or simply *correct about* $\mathcal{L}$ if for every $\varphi \in \mathcal{L}[\tau] \cap M$ and τ-structure $\mathcal{A} \in M$:

$$\mathcal{A} \models_{\mathcal{L}} \varphi \text{ if and only if } (M, \in) \models \text{“}\mathcal{A} \models_{\mathcal{L}} \varphi\text{”}.$$

It will be crucial throughout our discussion to consider whether some model of set theory is correct about some given logic. Recall that any transitive model is correct about first-order satisfaction. The following observations are well-known and easy to show by induction on formulas of the considered logics, using the extra assumptions we specify below in those inductive steps which distinguish the logic in question from first-order logic. All the models are considered to be transitive.

(i) Any transitive model is correct about $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$-satisfaction.

(ii) Any transitive model is correct about $\mathcal{L}_{\kappa\omega}$-satisfaction.

(iii) If $M^\lambda \subseteq M$, then M is correct about $\mathcal{L}_{\kappa\lambda}$-satisfaction.

(iv) If M is correct about cardinals, i.e., $\mathrm{Card}^M = \mathrm{Card} \cap M$, then M is correct about $\mathcal{L}(I)$-satisfaction.

(v) If for any $A \in M$, $\bigcup_{n\in\omega} \mathcal{P}(A^n) \subseteq M$, then M is correct about $\mathcal{L}^2$-satisfaction.

The usefulness of the strong logics above for large cardinal theory stems from the fact that they can define important properties of set-theoretic models. For example, the well-foundedness logic $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ may express that some model of set theory is well-founded.

Thus, in particular, $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ allows us to consider transitive models of set theory by considering the transitive collapse of a well-founded model. The other logics considered are able to define additional properties of set-theoretic interest. For each of our logics, we will collect the sentences most important for large cardinal theory and the properties they are defining in Lemma 1.2.4 below. We will make reference to these sentences throughout the later chapters.

Before we do this, let us make some notational remarks to state formulas in a more convenient way. We denote quantification over ordinals by $\forall\alpha\varphi$, i.e., as a shorthand for $\forall x(\mathrm{Ord}(\alpha) \to \psi)$, and similar for existential quantification and also utilising other lower case Greek letters. When writing down formulas in the language of set theory, we will often use natural language to denote formal assertions. For example, we might write something like

$$\varphi(x, y) = \exists y(y \text{ is a vocabulary}),$$

for the formal expression, written down in the language of set theory, expressing that y fulfils the properties of Definition 1.1.1. For better readability, we may also use quotation marks, for example, $\varphi(x, y) = \exists y(\text{“}y \text{ is a vocabulary”})$. Similarly, when working with some vocabulary τ and we want to express something in some logic $\mathcal{L}$ and there is an obvious way for this, we may simply state the desired property in ordinary English. For example, if τ contains some binary relation symbol $<$, we might write:

> “Fix a formula $\varphi \in \mathcal{L}(\mathsf{Q}^{\mathrm{WF}})[\tau]$ saying that $<$ is a well-order.”

Then we mean that φ is the obvious formula expressing this, namely, the conjunction of the (first-order) axioms of linear orders, together with the sentence $\mathsf{Q}^{\mathrm{WF}}xy(x < y)$. Finally, to state things more concisely, let us further fix the following abbreviations for first-order formulas, expressing the often used properties indicated.

$\mathrm{func}(x)$	“x is a function”
$\mathrm{dom}(f) = y$	“the domain of f is y”
$x = (x_1, \ldots, x_n)$	“x is the tuple $(x_1, \ldots, x_n)$”
$\mathrm{limit}(\gamma)$	“γ is a limit ordinal”
$\mathrm{succ}(\gamma)$	“γ is a successor ordinal”
Ext	$\forall x\forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$

Lemma 1.2.4. Each sentence below is in the language $\{\in\}$ of set theory.

(a) There is $\varphi_{\mathrm{WF}} \in \mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ such that $(M, E) \models \varphi_{\mathrm{WF}}$ iff E is well-founded.

(b) There is $\varphi_{\mathrm{Card}} \in \mathcal{L}(\mathsf{I})$ such that any transitive model $(M, \in)$ satisfying φ_{Card} computes cardinals correctly, i.e., $(M, \in) \models \varphi_{\mathrm{Card}}$ iff

$$\text{for every } a \in M, \mathrm{Card}(a) \text{ iff } (M, \in) \models \mathrm{Card}(a).$$

(c) (Magidor [Mag71]) There is $\Phi \in \mathcal{L}^2$, known as *Magidor's* Φ, such that $(M, E) \models \Phi$ iff there exists a limit ordinal α such that $(M, E) \cong (V_\alpha, \in)$.

(d) There is a sentence $\Phi^* \in \mathcal{L}^2$ such that $(M, E) \models \Phi^*$ iff there exists an ordinal α such that $(M, E) \cong (V_\alpha, \in)$. We will call Φ^* *Magidor's* Φ^*.

(e) If κ is regular, for every $a \in H_\kappa$ there is $\sigma_a(x) \in \mathcal{L}_{\kappa\omega}$ such that for any transitive set M and any $b \in M$:

$$(M, \in) \models \sigma_a(b) \text{ iff } a = b.$$

(f) There is $\varphi_{\mathrm{WF}} \in \mathcal{L}_{\omega_1\omega_1}$ such that $(M, E) \models \varphi_{\mathrm{WF}}$ iff E is well-founded.

(g) For every regular κ and $\lambda < \kappa$ there is a sentence $\psi_\lambda \in \mathcal{L}_{\kappa\kappa}$ such that for any transitive M, $(M, \in) \models \psi_\lambda$ iff $M^\lambda \subseteq M$.

Proof. For (a), let $\varphi_{\mathrm{WF}} = \mathsf{Q}^{\mathrm{WF}} xy(x \in y)$. For (b), let

$$\varphi_{\mathrm{Card}} = \forall x[\mathrm{Card}(x) \leftrightarrow (\mathrm{Ord}(x) \wedge \forall y(y \in x \to \neg \mathsf{I} zz(z \in y, z \in x)))].$$

For (c), first note that if $(M, \in)$ is transitive and $a, b \in M$, then there is a formula $\psi(x, y) \in \mathcal{L}^2$ such that $M \models \psi(a, b)$ iff $b = \mathcal{P}(a)$. For instance, the following formula does the job:

$$\begin{aligned}\psi(x,y) =& \forall v[v \in y \to \exists w(w \subseteq x \wedge v = w)] \\ & \wedge \forall X(\forall v(X(v) \to v \in x) \to \exists w(w \in y \wedge \forall z(X(z) \leftrightarrow z \in w))).\end{aligned}$$

In the following, let us write $\mathcal{P}_r(x) = y$ for $\psi(x, y)$. Note that $M \models \exists y(\mathcal{P}_r(a) = y)$ iff $\mathcal{P}(a) \in M$. If $\beta \in M$ is any ordinal and $f \in M$, then consider the following formula $\chi(\beta, f)$. It has the property, that $M \models \chi(\beta, f)$ iff f is a function with domain β such that for any $\gamma < \beta$, $f(\gamma) = V_\gamma$:

$$\begin{aligned}\chi(\beta, f) = \mathrm{func}(f) \wedge \mathrm{dom}(f) = \beta \wedge \forall \gamma < \beta[&(\gamma = 0 \to f(\gamma) = \emptyset) \\ &\wedge (\mathrm{succ}(\gamma) \wedge \gamma = \delta + 1 \to f(\gamma) = \mathcal{P}_r(f(\delta))) \\ &\wedge (\mathrm{limit}(\gamma) \to f(\gamma) = \bigcup_{\delta<\gamma} f(\delta))].\end{aligned}$$

Let us write $\exists^2 y(y = V_\beta)$ for the second-order assertion $\exists f(\psi(\beta + 1, f) \wedge f(\beta) = y)$. Note that if $M \cap \mathrm{Ord}$ is a limit ordinal, then $M \models \exists^2 y(y = V_\beta)$ iff $V_\beta \in M$. Now let $\Phi \in \mathcal{L}^2$ be the conjunction of the following sentences:

(i) Extensionality: $\forall x, y(x = y \leftrightarrow \forall z(z \in y \leftrightarrow z \in y))$.

(ii) Well-foundedness: $\forall X \exists x(X(x) \wedge \forall y \neg(y \in x \wedge X(y)))$.

(iii) There is no largest ordinal: $\forall \beta \exists \gamma(\beta \in \gamma)$.

(iv) For every ordinal β, V_β exists: $\forall \beta \exists^2 y(y = V_\beta)$.

(v) Every set is contained in some V_β: $\forall x \exists \beta \exists^2 y(y = V_\beta \wedge x \in y)$.

We claim that $(M, E) \models \Phi$ iff $(M, E) \cong (V_\alpha, \in)$ for some limit ordinal α. For the substantial direction, assume that $(M, E) \models \Phi$. By (i) and (ii), (M, E) is isomorphic to some transitive model, so let us assume that $E = \in$ and M is transitive. By (iii), $\alpha = M \cap \mathrm{Ord}$ is a limit ordinal. We claim that $M = V_\alpha$. Note that by (v), M thinks that every set is contained in some V_β for a $\beta < \alpha$. By usage of second-order logic, this V_β is the real V_β. This implies that $M \subseteq V_\alpha$. To argue that $V_\alpha \subseteq M$, it is sufficient to show that $V_\beta \in M$ for all $\beta < \alpha$. But this simply follows from (iv).

For (d), we argue similarly, but we have to be more careful: if $M = V_{\beta+1}$, due to the standard encoding of ordered pairs and rank reasons, M does not have a function f such that $f(\beta) = V_\beta$. The following sentence encodes that $M = V_{\lambda+1}$ for some limit ordinal λ.

(i) Extensionality: $\forall x, y(x = y \leftrightarrow \forall z(z \in y \leftrightarrow z \in y))$.

(ii) Well-foundedness: $\forall X \exists x(X(x) \wedge \forall y \neg(y \in x \wedge X(y)))$.

(iii) There is a largest ordinal λ and λ is a limit: $\exists \lambda(\forall \beta(\beta \leq \lambda) \wedge \mathrm{limit}(\lambda))$.

(iv) For every $\beta < \lambda$, V_β exists: $\forall \beta(\beta < \lambda \rightarrow \exists^2 y(y = V_\beta))$.

(v) V_λ exists: $\exists x(\forall z(z \in x \leftrightarrow \exists \beta < \lambda(\exists^2 y(y = V_\beta \wedge z \subseteq y))))$.

(vi) "I am the real power set of V_λ":

$$\forall y(y \subseteq V_\lambda) \wedge \forall X(\forall v(X(v) \rightarrow v \in V_\lambda)) \rightarrow \exists z(\forall v(X(v) \leftrightarrow v \in z))).$$

Let us call this sentence Φ_λ^{+1}. Similarly, we can construct a sentence Φ_{s}^{+1} truthfully saying "I am $V_{\alpha+1}$ for some successor α", by minor technical adaptations. Then the desired sentence Φ^* encoding that the model is isomorphic to some V_α for some ordinal α is the disjunction of Magidor's Φ with Φ_λ^{+1} and Φ_{s}^{+1}.

For (e), we proceed by $\in$-induction on $a \in H_\kappa$. If $a = \emptyset$, let $\sigma_a(x) = \neg\exists y(y \in x)$. If $a \in H_\kappa$ and $\sigma_c(x)$ is defined for all $c \in a$, define:

$$\sigma_a(x) = \forall y(y \in x \leftrightarrow \bigvee_{c \in a} \sigma_c(y)).$$

Note that $\sigma_a(x) \in \mathcal{L}_{\kappa\omega}$ because $a \in H_\kappa$. For (f), the axiom of choice implies that $\varphi_{\mathrm{WF}} = \neg\exists(x_i \colon i < \omega) \bigwedge_{i<\omega} x_{i+1} \in x_i$ is as desired. For (g), let ψ_λ be:

$$\forall(x_i \colon i < \lambda)\exists f(\mathrm{func}(f) \wedge \exists y(\sigma_\lambda(y) \wedge \mathrm{dom}(f) = y \wedge \bigwedge_{i<\lambda} \exists z(\sigma_i(z) \wedge f(z) = x_i))).$$

□

1.2.3. Generalising model theory to abstract logics

Abstract Logics enable us to formulate a lot of model-theoretic notions from first-order logic in a general way for any logic $\mathcal{L}$. Consider the Compactness Theorem for first-order logic:

Theorem 1.2.5. Let $T \subseteq \mathcal{L}_{\omega\omega}$ be a set of sentences of first-order logic. Then T has a model iff every finite subset of T has a model.

Substituting $\mathcal{L}_{\omega\omega}$ by a stronger logic usually leads to a failure of this theorem. On the other hand, it is possible that similar compactness properties hold at higher cardinals. We can formulate these properties in a natural way:

Definition 1.2.6. Let $\mathcal{L}$ be a logic and κ be cardinal.

(1) Let $T \subseteq \mathcal{L}$ be an $\mathcal{L}$-theory. We say that T is $<\kappa$-satisfiable iff every subset $T_0 \subseteq T$ such that $|T_0| < \kappa$ has a model.

(2) We say that $\mathcal{L}$ is *κ-compact*, or that κ is a *compactness cardinal* for $\mathcal{L}$, iff every $<\kappa$-satisfiable $\mathcal{L}$-theory has a model.

(3) If there is a cardinal γ such that $\mathcal{L}$ is γ-compact, we call the smallest such cardinal δ the *compactness number of* $\mathcal{L}$ and write $\mathrm{comp}(\mathcal{L}) = \delta$.

We will see that the existence of compactness numbers for strong logics is often equivalent to the existence of large cardinals.

Another important theorem for first-order logic is the downward Löwenheim-Skolem Theorem. It can be formulated in different strengths.

Theorem 1.2.7 (Downward Löwenheim-Skolem Theorem, Version 1)**.** If φ is a first-order sentence with an infinite model, then φ has a countable model.

Theorem 1.2.8 (Downward Löwenheim-Skolem Theorem, Version 2)**.** If φ is a first-order sentence over a countable vocabulary τ and $\mathcal{A}$ is an infinite τ-structure such that $\mathcal{A} \models \varphi$, then there is a countable substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \models \varphi$.

Again, we will see that both theorems may fail for stronger logics and therefore we are interested in analogues of these properties at higher cardinals.

Definition 1.2.9. Let $\mathcal{L}$ be a logic and κ a cardinal.

(1) If any satisfiable sentence of $\mathcal{L}$ has a model of size $< \kappa$, we call κ a *Löwenheim-Skolem (LS) number* of $\mathcal{L}$. If such a cardinal exists, we call the smallest such δ *the* Löwenheim-Skolem number of $\mathcal{L}$ and write $\mathrm{LS}(\mathcal{L}) = \delta$.

(2) If for any $\varphi \in \mathcal{L}[\tau]$ with $|\tau| < \kappa$, it is the case that for any τ-structure $\mathcal{A} \models \varphi$, there is some substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| < \kappa$ and $\mathcal{B} \models \varphi$, we call κ a *Löwenheim-Skolem-Tarski (LST) number* of $\mathcal{L}$. If such a cardinal exists, we call the smallest such δ *the* Löwenheim-Skolem-Tarski number of $\mathcal{L}$ and write $\mathrm{LST}(\mathcal{L}) = \delta$.[2]

[2] In the literature, sometimes one also finds the LST number being defined as providing a small $\mathcal{L}$-elementary substructure (cf. Definition 1.2.11), as opposed to a small substructure that satisfies a single designated sentence (and in fact, this is how we presented the LST number in Section 0.1). It is folklore that for many logics these two definitions are equivalent.

It is provable in ZFC that every logic has an LS number.

Proposition 1.2.10 (Folklore)**.** Let $\mathcal{L}$ be a logic. Then $\mathcal{L}$ has an LS number.

Proof. Let $\kappa = \mathrm{dep}^*(\mathcal{L})$. As any sentence of φ is up to renaming equivalent to a sentence in H_κ, we might restrict to $\varphi \in \mathcal{L} \cap H_\kappa$ to analyse possible sizes of models of sentences of $\mathcal{L}$. For any satisfiable $\varphi \in \mathcal{L} \cap H_\kappa$, let

$$\delta_\varphi = \min\{|\mathcal{A}| \colon \mathcal{A} \models \varphi\}.$$

Then $X = \{\delta_\varphi \colon \varphi \in \mathcal{L} \cap H_\kappa \text{ is satisfiable}\}$ is a set of cardinals. In particular, $\delta = \sup(X)$ is a cardinal. Clearly, δ^+ is an LS number of $\mathcal{L}$. □

On the other hand, we will later see that the existence of LST numbers of strong logics often has large cardinal strength.

In first-order model theory, we are often interested in elementary embeddings between structures, i.e., maps that preserve satisfaction of first-order formulas. Recall that if $\mathcal{A}$ and $\mathcal{B}$ are τ-structures, then an embedding $e : A \to B$ is called an *elementary embedding* if for every first-order formula $\varphi(x_1, \ldots, x_n)$ over τ with free variables among $\{x_1, \ldots, x_n\}$, and every $a_1, \ldots, a_n$:

$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \text{ iff } \mathcal{B} \models \varphi(e(a_1), \ldots, e(a_n)).$$

Note that this makes reference to free variables, a concept we do not have available for general abstract logics. However, free variables can be coded by considering additional constant symbols. This observation is also what lies behind the idea for the usage of *elementary diagrams* in first-order model theory. If $\mathcal{A}$ is a τ-structure, we can take a set of fresh, distinct constant symbols $\{c_a \colon a \in A\}$ which are not in τ, consisting of one constant for every $a \in A$. Interpret every constant c_a by a itself, letting $c_a^{\mathcal{A}} = a$, and consider the $(\tau \cup \{c_a \colon a \in A\})$-structure $(\mathcal{A}, c_a^{\mathcal{A}})_{a \in A}$. Then we call

$$\mathrm{ElDiag}(\mathcal{A}) = \mathrm{Th}_{\mathcal{L}_{\omega\omega}}((\mathcal{A}, c_a^{\mathcal{A}})_{a \in A}),$$

the *elementary diagram* of $\mathcal{A}$. It is easy to check that for any τ-structure $\mathcal{B}$, the existence of an elementary embedding $e : \mathcal{A} \to \mathcal{B}$ is equivalent to $(\mathcal{B}, e_a^{\mathcal{B}})_{a \in A} \models \mathrm{ElDiag}(\mathcal{A})$ with $e_a^{\mathcal{B}} = e(a)$. Analogously, we therefore consider the following notion:

Definition 1.2.11. Let $\mathcal{L}$ be a logic, τ a vocabulary and consider τ-structures $\mathcal{A}$ and $\mathcal{B}$ with an embedding $e : \mathcal{A} \to \mathcal{B}$. Further let $\{c_a \colon a \in A\}$ be a set of distinct constant symbols which are not in τ and let $c_a^{\mathcal{A}} = a$.

(1) $\mathrm{ElDiag}_{\mathcal{L}}(\mathcal{A}) = \mathrm{Th}_{\mathcal{L}}((\mathcal{A}, c_a^{\mathcal{A}})_{a \in A})$ is called the $\mathcal{L}$-*elementary diagram* of $\mathcal{A}$.

(2) The map e is an $\mathcal{L}$-*elementary embedding* iff $(\mathcal{B}, e(c_a^{\mathcal{A}}))_{a \in A} \models \mathrm{ElDiag}_{\mathcal{L}}(\mathcal{A})$.

If $A \subseteq B$ and the identity map $\mathrm{id} : A \to B$ is an $\mathcal{L}$-elementary embedding, we also write $\mathcal{A} \prec_{\mathcal{L}} \mathcal{B}$ and say that $\mathcal{A}$ *is an* $\mathcal{L}$-*elementary substructure of* $\mathcal{B}$.

Even though the word "elementary" seems to suggest preservation of first-order formulas, the name $\mathcal{L}$-elementary embedding (diagram, substructure) has become customary. For the designated logics $\mathcal{L}$ we considered before, like $\mathcal{L}_{\kappa\kappa}$, $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}(\mathsf{I})$ and $\mathcal{L}^2$, whose semantics are also defined with the help of variables, the respective notion of $\mathcal{L}$-elementary embedding is equivalent to preservation of satisfaction of formulas with free variables.[3]

1.2.4. Sort logics

Sort logics were introduced by Jouko Väänänen in [Vää79]. Their usage can be motivated by the following observations. Remember that we associate the expressive power of a logic by the model classes definable in it. From our ZFC standpoint, any model class is definable by some formula in the language of set theory. On the other hand, no logic can define *all* model classes. For let $\mathcal{L}$ be an arbitrary logic. By definition, $\mathcal{L}$ is definable by some formula in the language of set theory of complexity, say, Σ_n, possibly with parameters from a set X. Let $\operatorname{dep}^*(\mathcal{L}) = \kappa$. Then to analyse model classes definable by $\mathcal{L}$, we can restrict to classes definable by some $\varphi \in \mathcal{L} \cap H_\kappa$. If $\mathcal{K}$ is such a class,

$$\mathcal{A} \in \mathcal{K} \text{ iff } \mathcal{A} \models_{\mathcal{L}} \varphi,$$

is a Σ_n definition of $\mathcal{K}$ with parameters in $X \cup H_\kappa$. But surely there are model classes which are not definable in a Σ_n way with parameters in $X \cup H_\kappa$. As a result, $\mathcal{L}$ cannot define all model classes.

Väänänen's *sort logic* is graded by the natural numbers. The n-th level $\mathcal{L}^{\mathrm{s},n}$ of sort logic, for a natural number n, can define all model classes closed under isomorphism that are Σ_n or Π_n definable in the Lévy hierarchy. In particular, we will see that all model classes closed under isomorphism are definable by some level of sort logic. In light of our above remarks, this is the best we can do.

Let us first give some intuition about the way sort logic operates. Its main feature are *sort quantifiers*, written as $\widetilde{\exists}$ and $\widetilde{\forall}$. A formula $\widetilde{\exists} X \varphi(X)$ involving a sort quantifier over some relation variable X of arity n is true in a structure $\mathcal{A}$ if $\mathcal{A}$ can be expanded by an additional domain B such that there is a subset $Y \subseteq B^n$ and the expanded structure satisfies the formula $\varphi(Y)$, i.e., the sort quantifiers search outside the structure $\mathcal{A}$ itself for some relation on additional sorts satisfying the formula φ. The n-th level $\mathcal{L}^{\mathrm{s},n}$ is then allowed to ask about alternations of n-sort quantifiers. We will also consider fragments $\mathcal{L}^{\mathrm{s},\Sigma_n}$ and $\mathcal{L}^{\mathrm{s},\Pi_n}$ of $\mathcal{L}^{\mathrm{s},n}$, which are the formulas of $\mathcal{L}^{\mathrm{s},n}$ with a leading existential and universal sort quantifier, respectively. For regular cardinals κ, we further consider infinitary versions $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$, $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\kappa\omega}$, and $\mathcal{L}^{\mathrm{s},\Pi_n}_{\kappa\omega}$, which each expand $\mathcal{L}_{\kappa\omega}(\mathsf{Q}^{\mathrm{WF}})$.

Väänänen gave a modern presentation of the syntax and semantics of sort logic in [Vää14]. On the other hand, there is no modern presentation giving detailed proofs of the main features of sort logic. As sort logic turned out to be central for the intersection of large cardinal theory and abstract model theory, we believe that it is worthwhile to provide here such an account of sort logics most important features.

[3]Though note that for $\mathcal{L}^2$, this only means preservation of satisfaction with regards to formulas with first-order free variables, as the $\mathcal{L}^2$-elementary diagram considers individual constants which are interpreted as elements of the model, as opposed to subsets.

For reasons of comprehensibility, we first state the main properties of sort logic, before continuing with the technical details and their proofs. In later chapters, the properties we list below are often sufficient to follow our arguments, while the technical details are mostly relevant for the proofs that these properties hold. With each of the properties listed, we state where to find them. We assume κ to be a regular cardinal.

(a) If $\mathcal{K}$ is a model class of τ-structures for some $\tau \in H_\kappa$, closed under isomorphism, then:

   (i) $\mathcal{K}$ is Σ_n definable with parameters in H_κ iff $\mathcal{K}$ is axiomatisable by some $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_n}_{\kappa\omega}[\tau]$.

   (ii) $\mathcal{K}$ is Π_n definable with parameters in H_κ iff $\mathcal{K}$ is axiomatisable by some $\varphi \in \mathcal{L}^{\mathrm{s},\Pi_n}_{\kappa\omega}[\tau]$.

   For both statements, cf. Corollary 1.2.23.

(b) There is a sentence $\Phi^{\mathrm{s}} \in \mathcal{L}^{\mathrm{s},1}$ such that $(M, E) \models \Phi^{\mathrm{s}}$ iff $(M, E) \cong (V_\alpha, \in)$ for some limit ordinal α (cf. Lemma 1.2.15).

(c) There is a sentence $\Phi^{\mathrm{s},*} \in \mathcal{L}^{\mathrm{s},2}$ such that $(M, E) \models \Phi$ iff $(M, E) \cong (V_\alpha, \in)$ for some ordinal α (cf. Corollary 1.2.18).

(d) For $n \geq 1$, there is a sentence $\Phi^{(n)} \in \mathcal{L}^{\mathrm{s},n}$ such that $(M, E) \models \Phi^{(n)}$ iff $(M, E) \cong (V_\alpha, \in)$ for some $\alpha \in C^{(n)}$ (cf. Corollary 1.2.17).

(e) If $\alpha \in C^{(n)}$, then V_α is correct about $\mathcal{L}^{\mathrm{s},n}$-satisfaction (cf. Corollary 1.2.21).

(f) The satisfaction relation $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$ is Δ_{n+1} definable using κ as a parameter (cf. Corollary 1.2.22).

(g) Every logic is bounded by some level of sort logic, i.e., for every logic $\mathcal{L}$ there is some natural number n and some cardinal γ such that $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\gamma\omega}$ (cf. Corollary 1.2.24).

Let us now proceed with the precise definition of sort logic and proofs of the above statements. Väänänen introduces sort logic as an expansion of second-order logic. For technical reasons we will divert from this, and present it as an extension of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$. Our outline thus contains some novelty, as we show that basing sort logic on $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ is sufficient to define it and still end up with its main features.

Recall that a *sort* is simply a natural number (cf. Definition 1.1.1). We assume we have a proper class of individual variables, denoted by lowercase letters $x, y, z, u, v, w, \ldots$ Additionally, for every arity we fix a proper class of relation variables, denoted by capitalised letters, $X, Y, Z, R, S, T, \ldots$ We assume that every individual variable has a sort $s(x) \in \omega$ and every relation variable has, similarly to relation symbols, a configuration $\mathrm{conf}(X) \in \omega^{<\omega}$, simultaneously determining its arity. The variables are coded as sets in a convenient way so that conf is Δ_1 definable, as carried out in Appendix A. Recall that for a vocabulary τ, we write $s(\tau)$ for the collection of sort symbols appearing τ.

We will refer to the collection of sorts used by X as $s(X)$, i.e., if X has configuration $(n_1, \dots, n_k)$, then $s(X) = \{n \in \omega \colon \exists i(1 \leq i \leq k \text{ and } n = n_i)\}$.

We will need an infinitary version of sort logic, for which we first define its syntax by expanding on the class of formulas of $\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$. We will then later restrict to an extension of $\mathcal{L}_{\kappa\omega}(\mathsf{Q}^{\mathrm{WF}})$ for some regular cardinal κ. So let us recursively define a class of formulas $\mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ for some vocabulary τ by the following clauses. For the concrete sets we assume are used when coding formulas of sort logic, cf. Definition A.2.

(i) If $r \in \tau$ with $\mathrm{conf}(r) = (n_1, \dots, n_k)$ and for $1 \leq i \leq k$, x_i is an individual variable, or a constant symbol in τ, such that $s(x_i) = n_i$, then $r(x_1, \dots, x_k) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(ii) If X is a relation variable with $\mathrm{conf}(X) = (n_1, \dots, n_k)$ and for $1 \leq i \leq k$, x_i is an individual variable, or a constant symbol in τ, such that $s(x_i) = n_i$, then $X(x_1, \dots, x_k) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(iii) If f is a function symbol with $\mathrm{conf}(f) = (n_1, \dots, n_{k+1})$ and for $1 \leq i \leq k+1$, x_i is an individual variable, or a constant symbol in τ, such that $s(x_i) = n_i$, then $f(x_1, \dots, x_k) = x_{k+1} \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(iv) If x and y each are an individual variable or a constant symbol in τ, then $x = y \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(v) If $\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$, then $\neg\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(vi) If $T \subseteq \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ is a set, then $\bigwedge T, \bigvee T \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$.

(vii) If $\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ and x and y are individual variables, then $\mathsf{Q}^{\mathrm{WF}} xy\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$, *provided* φ does not contain any sort quantifiers.

(viii) If $\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ and x is an individual variable, then $\exists x\varphi, \forall x\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$, *provided* φ does not contain any sort quantifiers.

(ix) If $\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ and X is a relation variable, then $\widetilde{\exists} X\varphi, \widetilde{\forall} X\varphi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}$, *provided* $s(X) \cap s(\tau) = \emptyset$ and no free variable of φ involves a sort $n \in s(X)$.

The technical condition on the sorts of X in the last item is to prevent additional sorts added from interfering badly with sorts already introduced. Further, we do *not* allow sort quantifiers to appear in the scope of the well-foundedness quantifier Q^{WF} and of first-order quantification $\exists$ and $\forall$. These two restrictions are mostly in place for technical reasons, as they do not limit what we will want to express with sort logic, but make some of the proofs somewhat easier.

Recall from Section 1.1 that domains of different sort are allowed to be non-disjoint. Note that we *do* allow formulas of the form $x = y$ for individual variables of different sorts. In particular, sort logic is allowed to enquire whether two objects in two different domains are equal.

We say that a formula has complexity $\Delta_0 = \Sigma_0 = \Pi_0$ in the sort quantifiers if it is free of sort quantifiers. Recursively, it has complexity Σ_{n+1} if it has the form $\widetilde{\exists} X\psi$ for some

Π_n formula ψ. And it has complexity Π_{n+1} if it has the form $\widetilde{\exists} X\psi$ for some Σ_n formula ψ. Let us write $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega}$ for the collection of formulas of complexity up to Σ_n. Analogously collect the formulas which have complexity up to Π_n by $\mathcal{L}^{\mathrm{s},\Pi_n}_{\infty\omega}$. Now $\mathcal{L}^{\mathrm{s},n}_{\infty\omega}$ has as sentences all infinitary boolean combinations of sentences in $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega} \cup \mathcal{L}^{\mathrm{s},\Pi_n}_{\infty\omega}$.

If we restrict in clause (vi) to sets to of size $|T| < \kappa$ for some regular cardinal κ, we write $\mathcal{L}^{\mathrm{s}}_{\kappa\omega}$ for the resulting collection of sentences, and similarly we restrict to $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\kappa\omega}$, $\mathcal{L}^{\mathrm{s},\Pi_n}_{\kappa\omega}$ and $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$. We assume that if the vocabulary $\tau \in H_\kappa$, then a formula of $\mathcal{L}^{\mathrm{s}}_{\kappa\omega}$ can only use variables in H_κ. Note that all other non-logical symbols are coded as elements of H_ω and formulas are coded as finite tuples (cf. Appendix A). This makes sure that $\mathcal{L}^{\mathrm{s}}_{\kappa\omega}[\tau] \subseteq H_\kappa$ for $\tau \in H_\kappa$ and so $\mathrm{dep}^*(\mathcal{L}^{\mathrm{s}}_{\kappa\omega}) = \kappa$. If $\kappa = \omega$, we simply write $\mathcal{L}^{\mathrm{s}}$, $\mathcal{L}^{\mathrm{s},n}$, $\mathcal{L}^{\mathrm{s},\Sigma_n}$, and $\mathcal{L}^{\mathrm{s},\Pi_n}$.

We will define the semantics of sort logic making reference to variable assignments. A *variable assignment* for a τ-structure M is a function f that has as its domain a set of variables in sorts contained in $s(\tau)$ such that for any individual variable x, $f(x) \in M_{s(x)}$ and for any relation variable X of configuration $(n_1, \ldots, n_k)$, $f(X) \subseteq M_{n_1} \times \cdots \times M_{n_k}$.

If f is a variable assignment for a τ-structure M and S is a set of variables, we say that a variable assignment g for a τ-structure M is a an *S-variant of f* iff $S \subseteq \mathrm{dom}(g)$ and $f \setminus \{(x, f(x)) \colon x \in S\} = g \setminus \{(x, g(x)) \colon x \in S\}$. If $S = \{x\}$ is a singleton, we simply call g an x-variant.

We now give the semantics of sort logic. To make for a simpler statement of our definition, if f is a variable assignment for a τ-structure M and $c \in \tau$ is a constant symbol, we let $f(c) = c^M$. Let us state the full recursive definition, even though only the points involving sort quantifiers are novel.

Definition 1.2.12. Let M be a τ-structure, f a variable assignment for M, and n a natural number. We define recursively:

(i) For every $r \in \tau$, if $s(r) = (n_1, \ldots, n_k)$, and for $1 \leq i \leq n$, x_i is a variable symbol, or a constant symbol in τ, of sort $s(x_i) = n_i$, then

$$M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} r(x_1, \ldots, x_k)[f] \text{ iff } (f(x_1), \ldots, f(x_k)) \in r^M.$$

(ii) For every relation variable X, if $s(X) = (n_1, \ldots, n_k)$, and for $1 \leq i \leq n$, x_i is a variable symbol, or a constant symbol in τ, of sort $s(x_i) = n_i$, then

$$M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} X(x_1, \ldots, x_k)[f] \text{ iff } (f(x_1), \ldots, f(x_k)) \in f(X).$$

(iii) If x, y are variables or constant symbols, respectively, and $s(x), s(y) \in s(\tau)$, then $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} x = y[f]$ iff $f(x) = f(y)$.

(iv) $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \neg\varphi[f]$ iff $M \not\models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$.

(v) $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \bigvee T[f]$ iff $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$ for some $\varphi \in T$.

(vi) $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \bigwedge T[f]$ iff $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$ for all $\varphi \in T$.

(vii) If x is a variable, then $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \exists x\varphi[f]$ iff there is an x-variant g of f such that $M \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[g]$.

(viii) If x and y are variables, then $M \models_{\mathcal{L}^{s,n}_{\infty\omega}} \mathsf{Q}^{\mathrm{WF}} xy\varphi[g]$ iff

$$\{(g(x), g(y)) \colon g \text{ is an } \{x, y\}\text{-variant of } f \text{ such that } M \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi[g]\}$$

is well-founded.

(ix) If $\varphi \in \mathcal{L}^{s,\Pi_{n-1}}_{\infty\omega}$, then $M \models_{\mathcal{L}^{s,n}_{\infty\omega}} \widetilde{\exists} X\varphi[f]$ iff $s(X) = (n_1, \dots, n_k)$ and there is an expansion of M to a $\tau \cup \{n_1, \dots, n_k\}$-structure M' and a variable assignment g for M' such that $f \subseteq g$ and $M' \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi[g]$.

(x) If $\varphi \in \mathcal{L}^{s,\Sigma_{n-1}}_{\infty\omega}$, then $M \models_{\mathcal{L}^{s,n}_{\infty\omega}} \widetilde{\forall} X\varphi[f]$ iff $s(X) = (n_1, \dots, n_k)$ and for all expansions of M to a $\tau \cup \{n_1, \dots, n_k\}$-structure M' and any variable assignment g for M' such that $f \subseteq g$, $M' \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi[g]$.

If κ is regular, and $\varphi \in \mathcal{L}^{s,n}_{\kappa\omega}$, we define the satisfaction relation of $\mathcal{L}^{s,n}_{\kappa\omega}$ by letting $\mathcal{A} \models_{\mathcal{L}^{s,n}_{\kappa\omega}} \varphi$ iff $\varphi \in \mathcal{L}^{s,n}_{\kappa\omega} \wedge \mathcal{A} \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi$.

Let us make a few notational remarks. As we deal with quantification over different sorts, when introducing a variable in a sentence of sort logic, we will denote its configuration of sorts by a superscript, which we will leave out afterwards. For example, we will write things like

$$\varphi = \exists x^{s_0} y^{s_1} (x = y) \text{ and } \psi = \widetilde{\exists} X^{(s_1, s_1)} (\forall x^{s_1} (X(x, x))).$$

Our main use for sort quantifiers is to add a model of set theory, witnessing that some structure satisfies some desired property. To add a model of set theory, we have to use a sort quantifier, quantifying over a binary relation variable X. We then specify some property the model of set theory shall satisfy, for example, that it is extensional. For this we have to write out the extensionality axiom using the variable X. Concretely, this would look like this:

$$\chi = \widetilde{\exists} X^{(s_1, s_1)} (\forall x^{s_1} \forall y^{s_1} (X(x, y) \leftrightarrow \forall z^{s_1} (X(z, x) \leftrightarrow X(z, y)))).$$

This sentence is true in some structure $\mathcal{A}$, if we can add an additional universe B with a binary relation E, interpreting X, so that (B, E) satisfies the extensionality axiom (in particular, this concrete sentence χ is trivially true in any structure $\mathcal{A}$). In the following, when we say that ψ is some set-theoretic statement written down using a relation variable $X^{(s_1, s_1)}$, we mean that we replace all instances of $\in$ occurring in ψ by X and use quantification over variables in the sort s_1, as done above when spelling out the extensionality axiom above.

Notice that if $\varphi_0 = (\widetilde{\forall} X^{(s_1, \dots, s_n)} \chi) \wedge \psi \in \mathcal{L}^{s,n}_{\kappa\omega}$ and $\varphi_1 = (\widetilde{\forall} X^{(s_1, \dots, s_n)} \chi) \vee \psi$ and none of the sorts $s_1, \dots, s_n$ appears in ψ, then φ_0 is equivalent to $\widetilde{\forall} X^{(s_1, \dots, s_n)} (\chi \wedge \psi)$, and φ_1 is equivalent to $\widetilde{\forall} X^{(s_1, \dots, s_n)} (\chi \vee \psi)$. The same is true if we substitute $\widetilde{\forall}$ by $\widetilde{\exists}$. Thus, we may absorb conjuncts and disjuncts into the scope of sort quantifiers without altering the meaning of a sentence, as long as they do not interfere with the sorts quantified over. In particular, this shows that if $\varphi \in \mathcal{L}^{s,\Pi_n}_{\infty\omega}$ and ψ is free of sort quantifiers, then $\varphi \wedge \psi$ is equivalent to a formula in $\mathcal{L}^{s,\Pi_n}_{\infty\omega}$, i.e., with a leading universal sort quantifier. The same holds for $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega}$. In the following, for better readability we will often write $\varphi \wedge \psi$ for φ and ψ as above, but formally with the intention to denote the equivalent formula in $\mathcal{L}^{\mathrm{s},\Pi_n}_{\infty\omega}$ or in $\mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega}$.

Our first aim is to show that if $\mathcal{K}$ is a Σ_n or Π_n definable model class closed under isomorphism, then it is $\mathcal{L}^{\mathrm{s},n}$ axiomatisable. The following lemma provides the first step.

Lemma 1.2.13. Let $p, \tau \in H_\kappa$ and $\Phi(x,p)$ be a Σ_1 formula defining with the parameter p a class $\mathcal{K}$ of τ-structures closed under isomorphism. Then there is a formula $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_1}_{\kappa\omega}[\tau]$ such that $\mathcal{K} = \mathrm{Mod}(\varphi)$.

Proof. Conceptually, the proof is simple. We write down a sentence of $\mathcal{L}^{\mathrm{s},\Sigma_1}_{\kappa\omega}$, that we can add a (by usage of Q^{WF}) transitive set M as an additional sort to the model which contains the model itself, and which believes that the model is in $\mathcal{K}$. If the model is in $\mathcal{K}$ then surely this can be witnessed by some transitive set. On the other hand, if the model is not in $\mathcal{K}$, no such transitive set could be added: this would imply $\mathcal{A} \in \mathcal{K}$ by Σ_1 definability and therefore upward absoluteness of $\mathcal{K}$ from transitive sets. We use $\mathcal{L}_{\kappa\omega}$ to define the parameter p. Because this is our first time dealing with sort logic, we spell out all the details to make sure everything works. Subsequently, we will be less detailed.

Since Φ is a Σ_1 formula, we have that $\Phi(x,p) = \exists y \Psi(x,y,p)$ for some Δ_0 formula Ψ. As $\tau, p \in H_\kappa$, we have available the formulas σ_τ and σ_p. For simplicity, assume that τ uses only one sort s_0 and contains only unary relation symbols, and further assume that it is given by $\tau = \{R_i \colon i < \gamma\}$ for some $\gamma < \kappa$. Note that formally, if z is a τ-structure, then $z = (a,f)$, where a is the domain and f is a function assigning interpretations to the R_i. Take a new sort symbol s_1 and consider the following formula $\chi^{\mathrm{s}}(a,f)$, in which all set-theoretic formulas are written down using a binary relation variable $X^{(s_1,s_1)}$, and a and f are individual variables in sort s_1:

$$
\begin{aligned}
\chi^{\mathrm{s}}(a,f) = \exists y^{s_1}(\sigma_\tau(y)
&\wedge \text{“}(a,f) \text{ is a } y\text{-structure”} \\
&\wedge \forall x^{s_0} \exists z^{s_1}(X(z,a) \wedge z = x) \\
&\wedge \forall x^{s_1} \exists z^{s_0}(X(x,a) \wedge z = x) \\
&\wedge \bigwedge_{R \in \tau} \forall x^{s_0} \forall z^{s_1} v^{s_1}((\sigma_R(v) \wedge x = z) \to (X(z,f(v)) \leftrightarrow R(x))).
\end{aligned}
$$

The purpose of this formula is the following: If $(M, \in)$ is transitive and there is a pair (a,f) in M such that the structure $(\mathcal{A}, M, \in)$ satisfies $\chi^{\mathrm{s}}(a,f)$ – where M is taken to be the interpretation of the sort symbol s_1 and $\in$ interprets the relation variable X – then note that (a,f) is the actual structure $\mathcal{A}$. With χ^{s} at hand consider the following sentence, where Ψ is written down using the relational variable X:

$$
\begin{aligned}
\varphi = \widetilde{\exists} X^{(s_1,s_1)}[&\forall x^{s_1} y^{s_1}(x = y \leftrightarrow \forall z^{s_1}(X(z,x) \leftrightarrow X(z,y))) \\
&\wedge \mathsf{Q}^{\mathrm{WF}} x^{s_1} y^{s_1} X(x,y) \\
&\wedge \exists a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v) \wedge \exists y^{s_1} \Psi((a,f),y,v))].
\end{aligned}
$$

We show that φ is the formula of $\mathcal{L}^{\mathrm{s},\Sigma_1}_{\kappa\omega}$ we were looking for. Note that the outer quantifier is the only sort quantifier, so $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_1}_{\kappa\omega}$.

If $\mathcal{A} \models \varphi$, then this is witnessed by a new sort with domain M and some relation $X \subseteq M^2$ such that (M, X) is extensional and well-founded. So we can assume, by collapsing, that M is transitive and $X = \in$. We then have that $(M, \in)$ contains some pair (a, f) which by satisfying the formula χ^{s} is the actual structure $\mathcal{A}$. It contains p by the usage of σ_p. Furthermore, M has some $y \in M$ such that $(M, \in) \models \Psi(\mathcal{A}, y, p)$. Because Ψ is Δ_0 and M is transitive, really $\Psi(\mathcal{A}, y, p)$ holds and thus $\Phi(\mathcal{A}, p)$.

And if $\Phi(\mathcal{A}, p)$ holds, this is verified by some transitive set containing $\mathcal{A}$ and p. But then expanding $\mathcal{A}$ to $(\mathcal{A}, M, \in)$ witnesses that $\mathcal{A}$ satisfies φ. □

Lemma 1.2.14. Let $p, \tau \in H_\kappa$ and $\Phi(x, p)$ be a Π_1 formula defining with the parameter p a class $\mathcal{K}$ of τ-structures closed under isomorphism. Then there is a formula $\varphi \in \mathcal{L}^{\mathrm{s},\Pi_1}_{\kappa\omega}[\tau]$ such that $\mathcal{K} = \mathrm{Mod}(\varphi)$.

Proof. Let $\Phi(x, p) = \forall y \Psi(x, y, p)$ where Ψ is a Δ_0 formula. Using χ^{s} as in the proof above, let

$$\begin{aligned}\varphi = \widetilde{\forall} X^{(s_1,s_1)}([&\forall x^{s_1} y^{s_1}(x = y \leftrightarrow \forall z^{s_1}(X(z,x) \leftrightarrow X(z,y))) \wedge \\ &\mathsf{Q}^{\mathrm{WF}} x^{s_1} y^{s_1}(x \in y) \wedge \exists a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v))] \\ &\to \forall a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v) \to \forall y^{s_1} \Psi((a,f), y, p)))\end{aligned}$$

If $\mathcal{A} \models \varphi$, then any expansion of $\mathcal{A}$ by a transitive set M which contains $\mathcal{A}$ and p will have a $y \in M$ such that $M \models \Psi(\mathcal{A}, y, p)$. Then clearly, $\Phi(\mathcal{A})$ holds. On the other hand, if $\Phi(\mathcal{A}, p)$ holds, because it is Π_1, by downward absoluteness, any transitive M that contains $\mathcal{A}$ and p will believe $\Phi(\mathcal{A}, p)$. Thus $\mathcal{A} \models \varphi$. □

By now we showed that the Σ_1 and the Π_1 definable model classes are $\mathcal{L}^{\mathrm{s},1}$-axiomatisable, respectively. We will argue inductively that Σ_n and Π_n definable classes are $\mathcal{L}^{\mathrm{s},n}$-axiomatisable. For this, we will need a few facts about $C^{(n)}$-classes for $n \geq 1$. Recall that $C^{(n)}$ is definable by a Π_n formula. Now consider the class

$$\mathcal{K}^{(n)} = \{\mathcal{A} \colon \exists \alpha \in C^{(n)} \text{ such that } \mathcal{A} \cong (V_\alpha, \in)\}.$$

If $n \geq 2$, then

$$\mathcal{A} \in \mathcal{K}^{(n)} \text{ iff } \forall \beta(\beta \in C^{(n-1)} \wedge \mathcal{A} \in V_\beta \to V_\beta \models \text{“}\exists \alpha \in C^{(n)} \colon \mathcal{A} \cong V_\alpha\text{”})$$

provides a Π_n definition of $\mathcal{K}^{(n)}$. To see that the above equivalence holds, let $\mathcal{A} \in \mathcal{K}^{(n)}$ with $\mathcal{A} \cong V_\alpha$ and $\alpha \in C^{(n)}$. Take any $\beta \in C^{(n-1)}$ with $\mathcal{A} \in V_\beta$. We can compute the transitive collapse of $\mathcal{A}$ in V_β and so $V_\beta \models \mathcal{A} \cong V_\alpha$. Now because $\beta \in C^{(n-1)}$ and $C^{(n)}$ is Π_n definable, the fact that $\alpha \in C^{(n)}$ is downward absolute to V_β, so $V_\beta \models \alpha \in C^{(n)}$. And clearly, if every V_β with $\beta \in C^{(n-1)}$ believes $\mathcal{A}$ to be in $\mathcal{K}^{(n)}$, then this is true (just take a V_β that is correct about $\mathcal{K}^{(n)}$).

In our induction for $n \geq 2$, we will use that $\mathcal{K}^{(n-1)}$ is definable by a $\mathcal{L}^{s,\Pi_{n-1}}$-formula. Note that for $n = 1$, the above equivalence is still true, but the right hand side is not a Π_1 statement because "$\mathcal{A} \in V_\beta$" is already Π_1. We can therefore not use our already obtained results to derive a formula axiomatising $\mathcal{K}^{(1)}$. We thus need the following lemma, which shows that we can construct analogues to Magidor's Φ in $\mathcal{L}^{s,\Pi_1}$. That this is possible is due to an ability of sort quantifiers to emulate second-order quantification.

Lemma 1.2.15. The following hold:

(i) There is a sentence $\Phi^s \in \mathcal{L}^{s,\Pi_1}[\{\in\}]$ such that $(M, E) \models \Phi$ iff $(M, E) \cong (V_\alpha, \in)$ for some limit ordinal α.

(ii) There is a sentence $\Phi^{(1)} \in \mathcal{L}^{s,\Pi_1}[\{\in\}]$ such that $(M, E) \models \Phi^{(1)}$ iff $(M, E) \cong (V_\alpha, \in)$ for some $\alpha \in C^{(1)}$.

Proof. For (i), take φ to be the (first-order) statement in the language of set theory that says "There is no largest ordinal, for every ordinal α, V_α exists, every set is in some V_β, and the power set axiom holds". We fix a sentence $\varphi_{\text{sub}} \in \mathcal{L}^{s,\Pi_1}$ which is true in some transitive model $(M, \in)$ iff for every $a \in M$, $\mathcal{P}(a) \subseteq M$. The sentence φ_{sub} can be constructed as follows:

$$\begin{gathered}\widetilde{\forall} X^{s_1}[\exists y^{s_0} \forall x^{s_1}(X(x) \to \exists z^{s_0}(x = z \wedge z \in y)) \to \\ \exists v^{s_0}[\forall z^{s_0}(z \in v \to \exists x^{s_1}(z = x \wedge X(x))) \\ \wedge \forall x^{s_1}(X(x) \to \exists z^{s_0}(z = x \wedge z \in v))]]\end{gathered}$$

The sentence φ_{sub} is true in some model $(M, \in)$ if for every possible expansion of $(M, \in)$ by a set N and a unary relation $X \subseteq N$, if X is a subset of some element y of M, then X is an actual element of M. Thus $\mathcal{P}(y) \subseteq M$. Now define Φ^s as

$$\Phi = \text{Ext} \wedge \mathsf{Q}^{\text{WF}} xy(x \in y) \wedge \varphi \wedge \varphi_{\text{sub}}.$$

This sentence expresses that its model (M, E) is extensional, well-founded, satisfies φ, and contains any subset of any of its members. Now clearly, if α is a limit ordinal, then $V_\alpha \models \Phi^s$. And if $(M, E) \models \Phi^s$, by well-foundedness and extensionality, we can without loss of generality assume that $(M, \in)$ is given by its transitive collapse. We claim that with $\alpha = M \cap \text{Ord}$, we have that α is a limit ordinal and $M = V_\alpha$. Clearly α has to be a limit because M does not have a largest ordinal. Because M believes that each of its elements has a rank we have to have $M \subseteq V_\alpha$. It is thus sufficient to show $V_\beta \in M$ for all $\beta < \alpha$. Note $\emptyset = V_0 \in M$. Assume that $\beta = \gamma + 1$ and $V_\gamma \in M$. Then by virtue of φ_{sub}, we have $\mathcal{P}(V_\gamma) = V_\beta \subseteq M$. Furthermore, by φ, M has a set x that it believes to be V_β. Together this implies that $x = V_\beta$. Finally, let β be a limit and assume $V_\gamma \in M$ for all $\gamma < \beta$. Then $V_\beta = \bigcup_{\gamma<\beta} V_\gamma \subseteq M$. But again, by φ, M has a set x that it believes to be V_β. This together implies $V_\beta \in M$.

For (ii), recall that $C^{(1)}$ consists precisely of the uncountable $\beth$-fixed points and let

$$\Phi^{(1)} = \Phi^s \wedge \forall \beta(\text{Ord}(\beta) \to \exists x \exists y \exists f[y = V_\beta \wedge \text{Card}(x) \wedge \text{"}f : y \to x \text{ is a bijection"}]).$$

By (i), any model of $\Phi^{(1)}$ can be assumed to be some V_α with α a limit ordinal. But then by the second part of $\Phi^{(1)}$, whenever β is some ordinal below α, V_α contains a bijection between V_β and some cardinal $\kappa < \alpha$. As V_α contains the real V_β, this really gives rise to a bijection $V_\beta \to \kappa$. Thus, α is a $\beth$-fixed point. Note that both Φ^{s} and $\Phi^{(1)}$ are members of $\mathcal{L}^{\mathrm{s},\Pi_1}$. □

Now we are ready to show our desired theorem.

Theorem 1.2.16. The following hold:

(i) For $n \geq 1$, if $\tau, p \in H_\kappa$ and $\Phi(x,p)$ is a Σ_n formula defining a class $\mathcal{K}$ of τ-structures closed under isomorphism, then there is a formula $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_n}_{\kappa\omega}[\tau]$ such that $\mathcal{K} = \mathrm{Mod}(\varphi)$.

(ii) For $n \geq 1$, if $\tau, p \in H_\kappa$ and $\Phi(x,p)$ is a Π_n formula defining a class $\mathcal{K}$ of τ-structures closed under isomorphism, then there is a formula $\varphi \in \mathcal{L}^{\mathrm{s},\Pi_n}_{\kappa\omega}[\tau]$ such that $\mathcal{K} = \mathrm{Mod}(\varphi)$.

Proof. We show the two statements simultaneously by induction on n. The base case $n = 1$ was shown in Lemmas 1.2.13 and 1.2.14. We have to treat the case $n = 2$ and $n > 2$ separately. First let $n = 2$, and let us show (i). We assume that $\Phi(x,p) = \exists y \Psi(x,y,p)$ for some Π_1 formula Ψ. By Lemma 1.2.15, the sentence $\Phi^{(1)} \in \mathcal{L}^{\mathrm{s},\Pi_1}$ axiomatises the class $\mathcal{K}^{(1)}$. Consider

$$\varphi = \widetilde{\exists} X^{(s_1,s_1)}(\Phi^{(1)} \wedge \exists a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v) \wedge \exists y^{s_1} \Psi((a,f),y,v))).$$

We assume that s_1 does not appear in τ, and that χ^{s} is as in the proof of Lemma 1.2.13. We take $\Phi^{(1)}$ to be written out using X as the binary relation. Then any set M expanding a structure $\mathcal{A}$ to witness that $\mathcal{A} \models \varphi$ via $X \subseteq M^2$ satisfies $\Phi^{(1)}$, so has the property that $(M,X) \cong (V_\alpha, \in)$ for some $\alpha \in C^{(1)}$. Then if $\mathcal{A} \models \varphi$, it thus can be expanded by a model $(V_\alpha, \in)$ containing $\mathcal{A}, p$ and with $\alpha \in C^{(1)}$ and such that there is some $y \in V_\alpha$ such that $V_\alpha \models \Psi(\mathcal{A}, y, p)$. Because $\alpha \in C^{(1)}$, we get that $\Psi(\mathcal{A}, y, p)$ really holds and thus $\mathcal{A} \in \mathcal{K}$. On the other hand, if $\mathcal{A} \in \mathcal{K}$, then this surely is witnessed by some V_α where $\alpha \in C^{(1)}$, and we can expand $\mathcal{A}$ accordingly by V_α to verify that $\mathcal{A} \models \varphi$. Finally, since $\Phi^{(1)}$ has complexity Π_1 in the sort quantifiers, $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_2}_{\kappa\omega}$.

Proceeding to (ii) assume that $\Phi(x,p)$ is Π_2 and so $\Phi(x,p) = \forall y \Psi(x,y,p)$ for some Σ_1 formula Ψ. Consider

$$\begin{aligned}\varphi = \widetilde{\forall} X^{(s_1,s_1)}(&[\Phi^{(1)} \wedge \exists a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v))] \\ &\to \forall a^{s_1} f^{s_1} v^{s_1}(\chi^{\mathrm{s}}(a,f) \wedge \sigma_p(v) \to \forall y^{s_1} \Psi((a,f),y,v))),\end{aligned}$$

with the analogous assumptions on X and $\Phi^{(1)}$. Then if $\mathcal{A} \models \varphi$, take any V_α such that $\alpha \in C^{(1)}$, $\mathcal{A}, p \in V_\alpha$ and V_α is correct about $\Phi(x)$. Because $(\mathcal{A}, V_\alpha, \in)$ satisfies the antecedent of φ, we get that $V_\alpha \models \forall y \Psi(\mathcal{A}, y, p)$, i.e., $V_\alpha \models \Phi(\mathcal{A})$. As we took V_α such that it is correct about Φ, we get $\mathcal{A} \in \mathcal{K}$. And if $\mathcal{A} \in \mathcal{K}$, and (M,X) is an expansion of $\mathcal{A}$ satisfying the antecedent of φ, Then (M, X) is without loss of generality given by $(V_\alpha, \in)$ for some $\alpha \in C^{(1)}$. Then because Φ is Π_2, it is downward absolute to V_α and thus $V_\alpha \models \forall y \Psi(\mathcal{A}, y, p)$, verifying that $\mathcal{A} \models \varphi$. Because the implication turns $\Phi^{(1)}$ into a Σ_1 sentence of sort logic, φ has complexity Π_2.

Now to the case in which $n > 2$: Recall that by our earlier remarks, the class $\mathcal{K}^{(n-1)}$ is Π_{n-1} definable. By induction hypothesis, there is thus a formula $\Phi^{(n-1)} \in \mathcal{L}^{s,\Pi_{n-1}}$ axiomatising it. Then we can use the same proof as above, substituting occurrences of $\Phi^{(1)}$ by $\Phi^{(n-1)}$, occurrences of Σ_1, Σ_2, Π_1, Π_2 by Σ_{n-1}, Σ_n, Π_{n-1}, Π_n, and occurrences of $C^{(1)}$ by $C^{(n-1)}$, respectively. □

Because we will utilise the sentences $\Phi^{(n)}$ from the above proof ubiquitously, let us fix them in a separate statement.

Corollary 1.2.17. For every $n \geq 1$, there is a sentence $\Phi^{(n)} \in \mathcal{L}^{s,n}[\{\in\}]$ such that:

$$(M, E) \models \Phi^{(n)} \text{ iff there is an ordinal } \alpha \in C^{(n)} \text{ such that } (M, E) \cong (V_\alpha, \in).$$

Because the class $\{(M, E) \colon \exists\alpha((V_\alpha, \in) \cong (M, E))\}$ is Π_2-definable, Theorem 1.2.16 further implies:

Corollary 1.2.18. There is a sentence $\Phi^{s,*} \in \mathcal{L}^{s,2}[\{\in\}]$ such that:

$$(M, E) \models \Phi^{s,*} \text{ iff there is an ordinal } \alpha \text{ such that } (M, E) \cong (V_\alpha, \in).$$

Our next goal is to show the opposite direction, and to turn axiomatisations in $\mathcal{L}^{s,n}$ into definitions in the Lévy hierarchy. We need the following lemma.

Lemma 1.2.19. Satisfaction for $\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$ is Δ_1 definable.

Proof. For $i \in \{0, 1\}$, we will give formulas $\mathrm{Sat}_i(x, y, z, v)$ such that for any sets $\mathcal{A}$, φ, τ, and f,

$$\begin{aligned} \mathrm{Sat}_i(\mathcal{A}, \varphi, \tau, f) \text{ iff } & \tau \text{ is a vocabulary, } \varphi \in \mathcal{L}_{\infty\omega(\mathsf{Q}^{\mathrm{WF}})}[\tau] \text{ is a formula,} \\ & \mathcal{A} \text{ is a } \tau\text{-structure, } f \text{ is a variable assignment for } \mathcal{A} \\ & \text{with } \mathrm{dom}(f) \text{ containing all free variables of } \varphi, \text{ and} \\ & \mathcal{A} \models_{\mathcal{L}_{\infty\omega(\mathsf{Q}^{\mathrm{WF}})}} \varphi[f]. \end{aligned}$$

We already know that being a vocabulary τ and a τ-structure are Δ_1. That f is a variable assignment for $\mathcal{A}$ with a domain containing any free variables of φ is Δ_1 as well. Furthermore, given some vocabulary τ, $\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})[\tau]$ is Δ_1 definable. This can be shown, for example, exactly as in [Bar75, Section III.1] (here the argument is carried out for $\mathcal{L}_{\infty\omega}$ but as $\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$ simply adds a finitary quantifier, this can easily be modified to our case). Because well-foundedness is absolute for transitive models (of some large enough finite fragment ZFC* of ZFC), one can show that $\mathcal{A} \models_{\mathcal{L}_{\infty\omega(\mathsf{Q}^{\mathrm{WF}})}} \varphi[f]$ is absolute for transitive models. Therefore, the following provide Σ_1 and Π_1 formulas, respectively:

$$\begin{aligned}\mathrm{Sat}_0(x,y,z,v) = \exists M(&\text{“}M \text{ is transitive”} \wedge (M,\in) \models \mathrm{ZFC}^* \\ &\wedge \text{“}z \text{ is a vocabulary”} \wedge y \in \mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})[z] \\ &\wedge \text{“}x \text{ is a } \tau\text{-structure”} \wedge \text{“}v \text{ is a variable assignment} \\ &\text{for } x \text{ containing the free variables of } y\text{”} \\ &\wedge x,y,z,v \in M \wedge (M,\in) \models \text{“}x \models_{\mathcal{L}_{\infty\omega(\mathsf{Q}^{\mathrm{WF}})}} y[v]\text{”}).\end{aligned}$$

$$\begin{aligned}\mathrm{Sat}_0(x,y,z,v) = \forall M([&\text{“}M \text{ is transitive”} \wedge (M,\in) \models \mathrm{ZFC}^* \\ &\wedge \text{“}z \text{ is a vocabulary”} \wedge y \in \mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})[z] \\ &\wedge \text{“}x \text{ is a } \tau\text{-structure”} \wedge \text{“}v \text{ is a variable assignment} \\ &\text{for } x \text{ containing the free variables of } y\text{”} \\ &\wedge x,y,z,v \in M] \to (M,\in) \models \text{“}x \models_{\mathcal{L}_{\infty\omega(\mathsf{Q}^{\mathrm{WF}})}} y[v]\text{”}).\end{aligned}$$

□

Now we can show our desired theorem.

Theorem 1.2.20. Let $n \geq 1$.

(a) Let τ be a vocabulary and $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega}[\tau]$ be a formula. Then there is a Σ_n formula $\mathrm{Sat}(x,y)$ using φ and τ as parameters such that for any sets $\mathcal{A}$ and f:

$$\begin{aligned}\mathrm{Sat}(\mathcal{A},f) \text{ iff } &\mathcal{A} \text{ is a } \tau\text{-structure and } f \text{ is a variable assignment for } \mathcal{A} \\ &\text{such that } \mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f].\end{aligned}$$

In particular, if $\alpha \in C^{(n)}$ such that $\varphi, \tau, \mathcal{A} \in V_\alpha$, then $\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$ iff $V_\alpha \models$ “$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$”.

(b) Let τ be a vocabulary and $\varphi \in \mathcal{L}^{\mathrm{s},\Pi_n}_{\infty\omega}[\tau]$ be a formula. Then there is a Π_n formula $\mathrm{Sat}(x,y)$ using φ and τ as parameters such that for any sets $\mathcal{A}$ and f:

$$\begin{aligned}\mathrm{Sat}(\mathcal{A},f) \text{ iff } &\mathcal{A} \text{ is a } \tau\text{-structure and } f \text{ is a variable assignment for } \mathcal{A} \\ &\text{such that } \mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f].\end{aligned}$$

In particular, if $\alpha \in C^{(n)}$ such that $\varphi, \tau, \mathcal{A} \in V_\alpha$, then $\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$ iff $V_\alpha \models$ “$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi[f]$”.

Proof. Let us show the two parts (a) and (b) simultaneously by induction on n. First consider $n = 1$ and let $\varphi \in \mathcal{L}^{\mathrm{s},\Sigma_n}_{\infty\omega}[\tau]$ be a formula with free variables given by the members of a set S. We argue for our claim by induction on φ. If $\varphi \in \mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$, we may simply take the Σ_1 formula from the previous Lemma 1.2.19. Thus, the base case is covered, and we can further assume that φ contains a sort quantifier. Then $\varphi = \widetilde{\exists} X \psi(X, S)$ for some formula $\psi \in \mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})$ free of sort quantifiers, with free variables given by a set S. Let $\mathrm{Sat}(x, y)$ be given by:

$$\begin{aligned}\exists M \exists g(&\text{“}x \text{ is a } \tau\text{-structure”}\\ &\wedge \text{“}y \text{ is a variable assignment for } x\\ &\text{with } \mathrm{dom}(y) \text{ containing the free variables of } \varphi\text{”}\\ &\wedge \text{“}M = (\mathcal{A}, B_1) \text{ is an expansion of } x\text{”}\\ &\wedge \text{“}g \supseteq y \text{ is a variable assignment for } (\mathcal{A}, B_1)\text{”}\\ &\wedge (\mathcal{A}, B_1) \models_{\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})} \psi[g]).\end{aligned}$$

Note that $(\mathcal{A}, B_1) \models_{\mathcal{L}_{\infty\omega}(\mathsf{Q}^{\mathrm{WF}})} \psi[g]$ is a Δ_1 statement by Lemma 1.2.19 and therefore the above is a Σ_1 formula. We further use that being an expansion is expressible in a Δ_1 way.

Let us argue for the “in particular” part. We argued above that $\mathrm{Sat}(\mathcal{A}, f)$ holds true if and only if $\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},\Sigma_1}_{\infty\omega}} \varphi[f]$. Note that these computations can be carried out in V_α for V_α containing φ, τ, and $\mathcal{A}$. Furthermore, V_α recognizes that $\varphi \in \mathcal{L}^{\mathrm{s},1}_{\infty\omega}$ is actually in $\mathcal{L}^{\mathrm{s},\Sigma_1}_{\infty\omega}$. Thus $V_\alpha \models$ “$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},1}_{\infty\omega}} \varphi[f]$” iff $V_\alpha \models$ “$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},\Sigma_1}_{\infty\omega}} \varphi[f]$” iff $V_\alpha \models \mathrm{Sat}(\mathcal{A}, f)$. If $\alpha \in C^{(1)}$, it is correct about Σ_1 formulas, and in particular about $\mathrm{Sat}(x, y)$. Therefore $V_\alpha \models$ “$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},1}_{\infty\omega}} \varphi[f]$” iff $\mathrm{Sat}(\mathcal{A}, f)$ iff $\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},1}_{\infty\omega}} \varphi[f]$.

If $\varphi = \widetilde{\forall} X^{s_1} \psi(X, S)$ is Π_1 in sort logic, replacing, in $\mathrm{Sat}(x, y)$ as above, the explicitly written down existential quantifiers with universal ones and exchanging the last conjunction by an implication, results in the desired Π_1 formula. The “in particular” part follows analogously by using that V_α for $\alpha \in C^{(1)}$ is also correct about Π_1 formulas.

If $n > 1$, and $\varphi = \widetilde{\exists} X \psi(X, S)$ is Σ_n, then ψ is an $\mathcal{L}^{\mathrm{s},\Pi_{n-1}}_{\infty\omega}$ formula and by induction hypothesis there is a Π_{n-1} formula $\mathrm{Sat}_p(v, w)$ such that $\mathcal{B} \models_{\mathcal{L}^{\mathrm{s},\Pi_{n-1}}_{\infty\omega}} \psi[g]$ iff $\mathrm{Sat}_p(\mathcal{B}, g)$. Therefore the following formula is a Σ_n formula and works as desired:

$$\begin{aligned}\exists M \exists g(&\text{“}x \text{ is a } \tau\text{-structure”}\\ &\wedge \text{“}y \text{ is a variable assignment for } x\\ &\text{with } \mathrm{dom}(y) \text{ containing the free variables of } \varphi\text{”}\\ &\wedge \text{“}M = (\mathcal{A}, B_1) \text{ is an expansion of } x\text{”}\\ &\wedge \text{“}g \supseteq y \text{ is a variable assignment for } (\mathcal{A}, B_1)\text{”}\\ &\wedge \mathrm{Sat}_p((\mathcal{A}, B_1), g))\end{aligned}$$

The Π_n case is again analogous, replacing existential quantifiers by universal ones appropriately. In both cases, the “in particular” part works as above. □

Our results have a number of useful consequences.

Corollary 1.2.21. Let $\alpha \in C^{(n)}$. For any $\varphi \in \mathcal{L}^{\mathrm{s},n}_{\infty\omega}[\tau]$ and τ-structure $\mathcal{A}$, if $\varphi, \mathcal{A} \in V_\alpha$, then

$$\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi \text{ iff } V_\alpha \models \text{“}\mathcal{A} \models_{\mathcal{L}^{\mathrm{s},n}_{\infty\omega}} \varphi\text{”}.$$

Proof. For $\psi \in \mathcal{L}^{s,\Sigma_n}_{\infty\omega} \cup \mathcal{L}^{s,\Pi_n}_{\infty\omega}$, this follows from Theorem 1.2.20. Recall that any $\varphi \in \mathcal{L}^{s,n}_{\infty\omega}$ is simply a boolean combination of members of $\mathcal{L}^{s,\Sigma_n}_{\infty\omega} \cup \mathcal{L}^{s,\Pi_n}_{\infty\omega}$. But then clearly, correctness of V_α about satisfaction of members of $\mathcal{L}^{s,\Sigma_n}_{\infty\omega} \cup \mathcal{L}^{s,\Pi_n}_{\infty\omega}$ implies correctness about satisfaction towards $\psi \in \mathcal{L}^{s,n}_{\infty\omega}$. □

Corollary 1.2.22. For any $n \geq 1$ and any regular cardinal κ:

(a) The satisfaction relation $\models_{\mathcal{L}^{s,n}_{\infty\omega}}$ is Δ_{n+1} definable.

(b) The satisfaction relation $\models_{\mathcal{L}^{s,n}_{\kappa\omega}}$ is Δ_{n+1} definable using κ as a parameter.

Proof. For part (a), for $i \in \{0, 1\}$, we will give formulas $\mathrm{Sat}_i(x, y, z, v)$ such that for any sets $\mathcal{A}, \varphi, \tau, f$,

$$\begin{aligned}\mathrm{Sat}_i(\mathcal{A}, \varphi, \tau, f) \text{ iff } & \tau \text{ is a vocabulary, } \varphi \in \mathcal{L}^{s,n}_{\infty\omega}[\tau] \text{ is a formula,}\\ & \mathcal{A} \text{ is a } \tau\text{-structure, } f \text{ is a variable assignment for } \mathcal{A}\\ & \text{with } \mathrm{dom}(f) \text{ containing all free variables of } \varphi, \text{ and}\\ & \mathcal{A} \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi[f].\end{aligned}$$

Again, checking the syntactical parts of the statement can be carried out in a Δ_1 way. By Corollary 1.2.21, $\mathcal{A} \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi[f]$ is absolute between V_α and V for $\alpha \in C^{(n)}$. This implies that the following give equivalent formulas as desired:

$$\begin{aligned}\mathrm{Sat}_0(x, y, z, v) = \exists\alpha\exists X(& \alpha \in C^{(n)} \wedge X = V_\alpha \wedge z \text{ is a vocabulary } \wedge\\ & y \in \mathcal{L}^{s,n}_{\infty\omega}[z] \wedge x \text{ is a } \tau\text{-structure } \wedge v \text{ is a variable}\\ & \text{assignment for } x \text{ containing the free variables of } y\\ & \wedge x, y, z, v \in X \wedge X \models \text{“}x \models_{\mathcal{L}^{s,n}_{\infty\omega}} y[v]\text{”}).\end{aligned}$$

$$\begin{aligned}\mathrm{Sat}_0(x, y, z, v) = \forall\alpha\forall X([& \alpha \in C^{(n)} \wedge X = V_\alpha \wedge z \text{ is a vocabulary } \wedge\\ & y \in \mathcal{L}^{s,n}_{\infty\omega}[z] \wedge x \text{ is a } \tau\text{-structure } \wedge v \text{ is a variable}\\ & \text{assignment for } x \text{ containing the free variables of } y\\ & \wedge x, y, z, v \in X] \rightarrow X \models \text{“}x \models_{\mathcal{L}^{s,n}_{\infty\omega}} y[v]\text{”}).\end{aligned}$$

Since "$\alpha \in C^{(n)}$" is a Π_n statement and "$X = V_\alpha$" is a Π_1 statement, these formulas are Σ_{n+1} and Π_{n+1}, respectively.

For part (b), recall that $\mathcal{A} \models_{\mathcal{L}^{s,n}_{\kappa\omega}} \varphi$ iff $\varphi \in \mathcal{L}^{s,n}_{\kappa\omega}$ and $\mathcal{A} \models_{\mathcal{L}^{s,n}_{\infty\omega}} \varphi$. Thus, that $\models_{\mathcal{L}^{s,n}_{\kappa\omega}}$ is Δ_{n+1} definable follows from part (a), using that we can use the parameter κ to check that $\varphi \in \mathcal{L}^{s,n}_{\kappa\omega}$. □

Corollary 1.2.23. Let $\tau \in H_\kappa$ be a vocabulary and $\mathcal{K}$ a class of τ-structures closed under isomorphism. Then:

(a) $\mathcal{K}$ is Σ_n definable with parameters in H_κ iff $\mathcal{K}$ is axiomatisable by some $\varphi \in \mathcal{L}^{s,\Sigma_n}_{\kappa\omega}[\tau]$.

(b) $\mathcal{K}$ is Π_n definable with parameters in H_κ iff $\mathcal{K}$ is axiomatisable by some $\varphi \in \mathcal{L}^{s,\Pi_n}_{\kappa\omega}[\tau]$.

Proof. This follows immediately from Theorems 1.2.16 and 1.2.20. □

Sort logics can be seen as the most powerful logics available, as every logic is bounded by some level of sort logic:

Corollary 1.2.24. For every logic $\mathcal{L}$, there is some cardinal κ and some natural number n such that $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$.

Proof. Let $dep^*(\mathcal{L}) = \gamma$. Then to analyse classes definable by sentences of $\mathcal{L}$ it is sufficient to restrict to sentences $\psi \in \mathcal{L} \cap H_\gamma$. Take a natural number n and a cardinal $\kappa \geq \gamma$ large enough such that $\mathcal{L}$ is definable by a Σ_n formula with parameters in H_κ. We claim that $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$: For any $\psi \in \mathcal{L} \cap H_\gamma$,

$$\mathcal{A} \models_{\mathcal{L}} \psi,$$

is a Σ_n definition of $\mathrm{Mod}(\psi)$ with parameters in H_κ. As $\mathrm{Mod}(\psi)$ is closed under isomorphism, Corollary 1.2.23 implies that $\mathrm{Mod}(\psi)$ is axiomatisable by some $\varphi \in \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$. □

1.3. Large cardinals

We collect the most important classical large cardinal notions that will accompany us throughout, fixing some notation going along. If not mentioned otherwise, all these notions can be found in comprehensive set theory textbooks like [Jec03] or [Kan03]. Going along, we point out some of the classical results that connected large cardinals with model theory.

1.3.1. Measurable cardinals, elementary embeddings and ultrafilters

Most large cardinals useful in our considerations about model theory of strong logics appear as critical points of elementary embeddings. For this, if $j : (N, \in) \to (M, \in)$ is an elementary embedding between two transitive models of (some finite fragment of) ZFC, it is standard to see that as long as j is not the identity, it moves some ordinal, i.e., there is $\delta \in M$ such that $j(\delta) > \delta$ (cf. [Kan03, Section 5]). The smallest such ordinal is called the *critical point* $\mathrm{crit}(j)$ of j. Often, elementary embeddings j are of interest that have V as their domain, and in particular the target of j is then also a proper class $M \subseteq V$. The most basic large cardinal axiom of this type then states the existence of what is known as *measurable* cardinals:

Definition 1.3.1 (Measurable cardinals; Definition 1)**.** A cardinal κ is *measurable* iff κ is the critical point of an elementary embedding $j : V \to M$ where M is some transitive class.

It is a theorem of Kunen that a non-trivial elementary embedding $j : V \to V$ is inconsistent with ZFC (cf., e.g., [Kan03, Section 23]). Much of the theory of large cardinals plays out between these two extremes. While the assumptions on the target M

of an embedding witnessing measurability of a cardinal are quite minimal (M is only needed to be transitive), demanding more properties of M that bring it closer and closer to V, i.e., existence of elementary embeddings more and more resembling the inconsistent embedding $j : V \to V$, typically leads to stronger and stronger assumptions.

Note that according to the above definition, stating the *existence* of a measurable cardinal would demand of us stating the existence of an elementary embedding $j : V \to M$. As such an embedding is a proper class, this cannot be formulated as a statement in the language of set theory. This is a typical situation in large cardinal theory and it requires us to formalise the notion of measurability in a different way. This can be achieved by requiring the existence of certain *ultrafilters*.

Definition 1.3.2. Let S be a set.

(1) A *filter over* S is a set $F \subseteq \mathcal{P}(S)$ such that
  (i) $\emptyset \notin S$.
  (ii) If $X, Y \in F$, then $X \cap Y \in F$.
  (iii) If $X \in F$ and $Y \subseteq S$ such that $X \subseteq Y$, then $Y \in F$.

(2) A filter F over S is called *non-principal* iff there is no $X \subseteq S$ such that $F = \{Y \subseteq S \colon X \subseteq Y\}$.

(3) A filter U over S is called an *ultrafilter* iff U is non-principal and for every $X \subseteq S$ either $X \in U$ or $S \setminus X \in U$.

(4) For a cardinal κ, a filter F is called κ*-complete* iff it is closed under intersections of length $< \kappa$, i.e., for any $\delta < \kappa$ and any $\{X_i \colon i < \delta\} \subseteq U$, also $\bigcap_{i<\delta} X_i \in U$.

Definition 1.3.3 (Measurable cardinals; Definition 2)**.** Let κ be an uncountable cardinal. Then κ is called *measurable* iff there is a κ-complete ultrafilter over κ.

Let us point out how the two definitions of measurable cardinals are connected. Recall the *ultraproduct* construction (cf., e.g., [CK73, Chapter 4]). If U is an ultrafilter over a set S, τ is a vocabulary, and for every $s \in S$ we are given some τ-structure $\mathcal{A}_s$, we write $\prod_{s\in S} \mathcal{A}_s/U$ for the ultraproduct of the $\mathcal{A}_s$ modulo U. Recall that $\prod_{s\in S} \mathcal{A}_s/U$ is made up out of equivalence classes $[f]_U$ of functions $f \in \prod_{s\in S} \mathcal{A}_s$. If all the $\mathcal{A}_s$ are the same structure $\mathcal{A}$, we call $\prod_{s\in S} \mathcal{A}_s/U$ the *ultrapower of* $\mathcal{A}$, and denote it as $\mathrm{Ult}(\mathcal{A}, U)$. The usefulness of ultraproducts in model theory stems from the following well-known result:

Theorem 1.3.4 (Łos' Theorem)**.** Let $\varphi(x_1, \dots, x_n)$ be a first-order formula over τ and $[f_1]_U, \dots, [f_n]_U \in \prod_{s\in S} \mathcal{A}_s/U$. Then

$$\prod_{s\in S} \mathcal{A}_s/U \models \varphi([f_1]_U, \dots, [f_n]_U) \text{ iff } \{s \in S \colon \mathcal{A}_s \models \varphi(f_1(s), \dots, f_n(s))\} \in U.$$

Recall that we may, using what is known as *Scott's Trick*, also construct an ultrapower $(\mathrm{Ult}(V,U), \in^{\mathrm{Ult}(V,U)})$ of $(V, \in)$ modulo U (cf., e.g., [Kan03, Section 5]), which we will denote simply by $\mathrm{Ult}(V,U)$. If $f : S \to V$ is some function, we write $[f]_U \in \mathrm{Ult}(V,U)$ for the Scott equivalence class of f. When clear out of context, we will usually drop the subscript and write $[f]$ for $[f]_U$.

We get a version of Łos' Theorem:

Theorem 1.3.5 ([Kan03, Theorem 5.2])**.** For any formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory and $f_1, \ldots, f_n$ functions $S \to V$:

$$\mathrm{Ult}(V,U) \models \varphi([f_1]_U, \ldots, [f_n]_U) \text{ iff } \{s \in S \colon \varphi^V(f_1(s), \ldots, f_n(s))\} \in U.$$

The importance of completeness of ultrafilters comes with the following result.

Theorem 1.3.6 ([Kan03, Proposition 5.3])**.** Let U be an ultrafilter over a set S. Then $\mathrm{Ult}(V,U)$ is well-founded iff U is ω_1-complete.

In particular, for an ω_1-complete ultrafilter, we can consider the transitive collapse M_U of $\mathrm{Ult}(V,U)$ (with checking before that $\in^{\mathrm{Ult}(V,U)}$ is set-like). In the following, we will therefore identify $\mathrm{Ult}(V,U)$ with its transitive collapse M_U and hence $\in^{\mathrm{Ult}(V,U)}$ with $\in$.

For a set x, let $c_x : S \to V$ be the function with constant value x, i.e., $c(s) = x$ for all $s \in S$. Using Łos' Theorem, one can check that $x \mapsto [c_x]_U$ defines an elementary embedding $j_U : V \to M_U$. We thus have a device that allows us to build elementary embeddings of V into transitive classes, provided we have an ω_1-complete ultrafilter.

The connection between ultrafilters of a certain completeness and measurable cardinals is now the following. If $j : V \to M$ is an elementary embedding with a critical point $\mathrm{crit}(j) = \kappa$, then κ is a regular uncountable cardinal and for $X \subseteq \kappa$:

$$X \in U \text{ iff } \kappa \in j(X)$$

defines a κ-complete ultrafilter U over κ (cf. [Kan03, Theorem 5.6]). On the other hand, if for an uncountable cardinal κ there is a κ-complete ultrafilter over κ, our remarks above show that $j_U : V \to \mathrm{Ult}(V,U)$ is an elementary embedding. Moreover, the κ-completeness of U implies that j_U has a critical point $\mathrm{crit}(j_U) = \kappa$ (see [Kan03, Proposition 5.4]). Putting everything together, we have the equivalence of our two definitions 1.3.1 and 1.3.3:

Theorem 1.3.7. Let κ be an uncountable cardinal. Then there is a κ-complete ultrafilter U over κ iff there is an elementary embedding $j : V \to M$ with critical point $\mathrm{crit}(j) = \kappa$.

1.3.2. Interactions between elementary embeddings and syntax of logics

Before we continue with our treatment of large cardinals, let us first make a few remarks about the interaction of logics with elementary embeddings. A typical use case of elementary embeddings in model theory of strong logics is the following. Suppose

$T \subseteq \mathcal{L}[\tau]$ for some logic $\mathcal{L}$ and vocabulary τ, and we want to argue that T is satisfiable. We might know that T is $<\kappa$-satisfiable for some cardinal κ. Suppose we have an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > |T|$, $j``T \in M$ and such that $M$ is correct about $\mathcal{L}$-satisfaction. By elementarity, in $M$, $j(T)$ is a $<j(\kappa)$-satisfiable $j(\tau)$-theory. In particular, as $|j``T| = |T| < j(\kappa)$, and $M \models j``T \subseteq j(T)$, then $M$ has a $j(\tau)$-structure $\mathcal{A}$ that it believes to be a model of $j``T$. Because M is correct about $\mathcal{L}$-satisfaction, this is really a model of $j``T$. Now if we would know that $j``T$ is a copy of T, we could rename $\mathcal{A} \restriction j``\tau$ to a τ-structure satisfying T.

We can now appreciate the importance of the precise definitions of the syntaxes of the logics considered in Appendix A: they make sure that $j``T$ in many cases is indeed a copy of T. Let us argue for this. In arguments of this kind, it is important to consider how sentences of our logics look as material sets, to make sure that when applying j to some $\varphi \in \mathcal{L}$, then $j(\varphi)$ is really a renaming of φ. First, note that our vocabularies are coded in a way such that any elementary embedding provides a renaming: Suppose that $j : V \to M$ is an elementary embedding and τ is a vocabulary. Recall that non-logical symbols are finite tuples from which we can read off all relevant information about the symbol, like whether it is a function or relation symbol, or what its arity or configuration is (cf. Definition 1.1.1). Therefore, j preservers all of these information. For instance, if $r = (2, (n_1, \ldots, n_k, a))$ is a relation symbol, then by elementarity $j(r) = (2, (n_1, \ldots, n_k, j(a)))$ is a relation symbol with the same configuration. Using these observations, it is simple to see that $j``\tau$ is a renaming of τ.

Similarly, consider the official syntax of, for example, $\mathcal{L}_{\kappa\kappa}$ for some regular cardinal κ, fixed in Appendix A, Definition A.2. Note that if $j : V \to M$ is elementary and $\mathrm{crit}(j) \geq \kappa$, then, if $T = \{\varphi_i \colon i < \gamma\} \subseteq \mathcal{L}_{\kappa\kappa}$ for $\gamma < \kappa$ and $\varphi = (10, T) = \bigwedge T \in \mathcal{L}_{\kappa\kappa}$, then $j(\varphi) = j((10, T)) = (10, j(T))$. But $j(T) = j(\{\varphi_i \colon i < \gamma\}) = \{j(\varphi_i) \colon i < j(\gamma)\} = j``T$. Thus $j(\varphi) = \bigwedge j``T$. Or similarly, $j(\exists (x_i \colon i < \gamma)\psi) = \exists (j(x_i) \colon i < \gamma) j(\psi)$. Using observations like this, it is easy to show by induction on φ, that $j(\varphi)$ is simply a renaming of φ. We summarise this in the following proposition.

Proposition 1.3.8. Let $j : V \to M$ be elementary such that $\mathrm{crit}(j) \geq \kappa$ and τ a vocabulary. Then $j``\mathcal{L}_{\kappa\kappa}[\tau]$ is a copy of $\mathcal{L}_{\kappa\kappa}[\tau]$, i.e., for every $\varphi \in \mathcal{L}_{\kappa\kappa}[\tau]$ and for any $\tau$-structure $\mathcal{A}$, when renaming $\mathcal{A}$ to a $j``\tau$-structure $\mathcal{A}^*$ along the renaming $j : \tau \to j``\tau$, we get:

$$\mathcal{A} \models_{\mathcal{L}_{\kappa\kappa}} \varphi \text{ iff } \mathcal{A}^* \models_{\mathcal{L}_{\kappa\kappa}} j(\varphi).$$

Using their official syntaxes' codings from Appendix A, analogous assertions are true for the logics $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}(\mathsf{I})$, $\mathcal{L}^2$, and $\mathcal{L}^{\mathrm{s},n}$, introduced in Section 1.2, as well as their infinite versions, $\mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}_{\kappa\lambda}(\mathsf{I})$, $\mathcal{L}^2_{\kappa\lambda}$, and $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$. Note that because $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}(\mathsf{I})$, $\mathcal{L}^2$, and $\mathcal{L}^{\mathrm{s},n}$ are finitary logics, *any* elementary embedding $j : V \to M$ restricts to renamings of any of their theories, independent of their critical point (while, for example, for $\mathcal{L}_{\kappa\kappa}$ this is only true for elementary embeddings with critical point at least κ).

Let us return to our consideration of large cardinals. Measurable cardinals have traditionally not received the most prominent attention in model theory of strong logics. Nevertheless, the following folklore characterisation has been known for a long time, at least since Chang and Keisler's *Model Theory*, in which it appears as an exercise.

Theorem 1.3.9 ([CK73, Exercise 4.2.6]). The following are equivalent for a cardinal κ:

(1) κ is measurable.

(2) κ is a chain compactness cardinal for the logic $\mathcal{L}_{\kappa\kappa}$, i.e., if $T \subseteq \mathcal{L}_{\kappa\kappa}$ is a theory which can be written as an increasing union $\bigcup_{\alpha<\kappa} T_\alpha$ such that every T_α has a model, then T has a model.

The following can be proved with similar arguments.

Proposition 1.3.10. The following are equivalent for a cardinal κ:

(1) κ is the smallest measurable cardinal.

(2) κ is the smallest chain compactness cardinal for the logic $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, i.e., the smallest cardinal κ such that if $T \subseteq \mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ is a theory which can be written as an increasing union $\bigcup_{\alpha<\kappa} T_\alpha$ such that every T_α has a model, then T has a model.

1.3.3. Supercompact cardinals

As in many parts of large cardinal theory, supercompact cardinals play a major role in model theory of strong logics. Also they are characterised by the existence of certain ultrafilters. We will present this perspective first, before stating to which kind of elementary embeddings they give rise to. Instead of a single ultrafilter living on κ, supercompact cardinals come with many ultrafilters, more specifically, one for every ordinal $\lambda \geq \kappa$ living on $\mathcal{P}_\kappa\lambda$. These further need additional properties.

Definition 1.3.11. Let κ be a cardinal, $\lambda \geq \kappa$ an ordinal and F a filter over $\mathcal{P}_\kappa\lambda$.

(1) The filter F is called *fine* iff for every $\alpha < \lambda$, $\{s \in \mathcal{P}_\kappa\lambda \colon \alpha \in s\} \in F$.

(2) A fine ultrafilter U is called *normal* iff every function f which is regressive on a set in U is constant on a set in U, i.e., if $f : \mathcal{P}_\kappa\lambda \to \lambda$ and there is $X \in U$ such that $f(s) \in s$ for all $s \in X$, then there is $\gamma < \lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda \colon f(s) = \gamma\} \in U$.

Definition 1.3.12. Let κ be a cardinal.

(1) For $\lambda \geq \kappa$, κ is called λ-*supercompact* iff there is a fine, normal, κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$.

(2) We call κ *supercompact* iff κ is λ-supercompact for every $\lambda \geq \kappa$.

Taking the ultrapower $j_U : V \to M_U$ of V by a fine, normal, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$, we call j_U a (λ-) *supercompactness embedding.* Again, κ-completeness amounts to $\mathrm{crit}(j_U) = \kappa$, while normality and fineness can be used to show that $j_U(\kappa) > \lambda$ and $j_U``\lambda \in M_U$. One can further use the fact that $j_U``\lambda \in M_U$ to show that $M_U^\lambda \subseteq M_U$. On the other hand, if $j : V \to M$ is elementary with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j``\lambda \in M$, one can show that letting for $X \subseteq \mathcal{P}_\kappa\lambda$:

$$X \in U \text{ iff } j``\lambda \in j(X),$$

defines a fine, normal, κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$ (see [Jec03, Lemma 20.13] for a proof of these statements). Thus we get:

Theorem 1.3.13. The following are equivalent:

(1) κ is supercompact.

(2) For every $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.

(3) For every $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j``\lambda \in M$.

It can be shown that if κ is 2^κ supercompact, κ is a limit of measurable cardinals (cf. [Kan03, Proposition 22.1]). In particular, then $V_\kappa \models \exists\alpha(\alpha \text{ is measurable})$ and so supercompact cardinals exceed measurable cardinals in consistency strength.

Supercompact cardinals are of major importance in model theory of strong logics, because, as we will later see, they correspond to the first level of several hierarchies that amount to stronger and stronger assumptions about model-theoretic properties of logics. Contrary to what their name suggests, they are better known as cardinals giving rise to Löwenheim-Skolem properties (in contrast to compactness). The following is a classic result of Magidor:

Theorem 1.3.14 (Magidor [Mag71])**.** Let κ be a cardinal. Then κ is the first supercompact cardinal iff $\kappa = \mathrm{LST}(\mathcal{L}^2)$.

Nevertheless, supercompactness *has* a known characterisation in terms of compactness principles for strong logics. We will be able to appreciate this result better, after we introduced strongly compact cardinals.

1.3.4. Strongly compact cardinals

Probably the archetype of cardinals connected to model theory are strongly compact cardinals, as their original motivation stems from the question whether infinitary logics are compact. Nevertheless, in modern research they are usually treated as a weakening of supercompact cardinals. Let us take this perspective for a moment.

Definition 1.3.15. Let κ be a cardinal.

(1) For $\lambda \geq \kappa$, κ is called λ-*compact* iff there is a fine, κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$.

(2) We call κ *strongly compact* iff κ is λ-compact for every $\lambda \geq \kappa$.

Note that the difference to supercompactness is only the missing normality condition. Again, we consider the ultrapower $j_U : V \to M_U$ of V, this time by a fine, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$. Again, κ-completeness amounts to $\mathrm{crit}(j) = \kappa$. On the other hand, we are missing normality. While this is needed in a proof to show that $j_U``\lambda \in M_U$, fineness is sufficient to show that there is a set $d \in M_U$ such that $M_U \models |d| < j(\kappa)$ and $j``\lambda \subseteq d$. I.e., we can cover $j``\lambda$ by a set which is "small" from $M$'s perspective (note that if $j``\lambda \in M_U$, such a cover in M_U is given by $j``\lambda$ itself). See [Kan03, Theorem 22.17] for a proof of these statements. We remark that it can further be shown that $M^\kappa \subseteq M$ (cf. [Ham09, Theorem 2.11]). On the other hand, if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and there is a $d \in M$ such that $j``\lambda \subseteq d$ and $M \models |d| < j(\kappa)$, then one can check that letting for $X \subseteq \mathcal{P}_\kappa\lambda$,

$$X \in U \text{ iff } d \in j(X),$$

defines a fine, κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$. Thus we get:

Theorem 1.3.16. The following are equivalent:

(1) κ is strongly compact.

(2) For every $\lambda \geq \kappa$, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\kappa \subseteq M$ and there is $d \in M$ such that $j``\lambda \subseteq d$ and $M \models |d| < j(\kappa)$.

Moreover, one can show that condition (2) is actually sufficient to get the covering property for the pointwise image of *any* set, not just for ordinals (see again [Kan03, Theorem 22.17]). Thus, the above conditions (1) and (2) are equivalent to:

(3) For every set x, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $M^\kappa \subseteq M$ and there is $d \in M$ such that $j``x \subseteq d$ and $M \models |d| < j(\kappa)$.

This covering of pointwise images by a "small" set in M is precisely what is needed in arguments that give us compactness properties of infinitary logics $\mathcal{L}_{\kappa\kappa}$. In published proofs of this fact (cf. [Kan03, Proposition 4.1] or [Jec03, Lemma 20.2]), this is often obscured by deriving an ultrafilter, stating that a Łos like theorem can be proven for $\mathcal{L}_{\kappa\kappa}$, and combining these two facts to show that a model of a theory can be constructed by an ultraproduct. However, this is not strictly needed, as the compactness property of $\mathcal{L}_{\kappa\kappa}$ can be derived directly from (3), without going through the ultraproduct construction. As the argument for this is instructive, we will show here how to do this (see the forward direction of the proof):

Theorem 1.3.17. Let κ be a cardinal. Then κ is strongly compact iff $\kappa = \mathrm{comp}(\mathcal{L}_{\kappa\kappa})$.

Proof. For the forward direction, suppose that κ is strongly compact. Because a compactness cardinal of $\mathcal{L}_{\kappa\kappa}$ cannot be smaller than κ, it is sufficient to show that $\mathcal{L}_{\kappa\kappa}$ is κ-compact. To this end, let $T \subseteq \mathcal{L}_{\kappa\kappa}$ be a $< \kappa$-satisfiable theory. By (3) above, take an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $M^\kappa \subseteq M$ and with a $d \in M$ such that $j``T \subseteq d$ and $M \models |d| < j(\kappa)$. We can without loss of generality assume (by considering $d \cap j(T) \in M$) that $d \subseteq j(T)$. By elementarity, M believes that $j(T)$ is $<j(\kappa)$-satisfiable. Therefore M believes that there is a model $\mathcal{A} \models d$. In particular, for every $\varphi \in T$, we have:

$$M \models \text{``}\mathcal{A} \models j(\varphi)\text{''}.$$

Because M is closed under κ-sequences, it is correct about $\mathcal{L}_{\kappa\kappa}$-satisfaction and so from the outside we therefore see that really $\mathcal{A} \models j``T$. Now since $\mathrm{crit}(j) = \kappa$, by Proposition 1.3.8, $j``T$ is a copy of T. Hence, $\mathcal{A}$ can be renamed to our desired model of T.

And now let $\kappa = \mathrm{comp}(\mathcal{L}_{\kappa\kappa})$. We show that with $\lambda \geq \kappa$ there is a fine and κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$. To this purpose, take a fresh constant symbol d and consider the $\mathcal{L}_{\kappa\kappa}$-theory

$$T = \mathrm{ElDiag}_{\mathcal{L}_{\kappa\kappa}}(V_{\lambda+2}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\},$$

where the constants c_i in the second part of T refer to the constants used in the elementary diagram. Let $T_0 \subseteq T$ such that $|T_0| < \kappa$. Then there is some $X \subseteq \lambda$ such that $|X| < \kappa$ and

$$T_0 \subseteq \mathrm{ElDiag}_{\mathcal{L}_{\kappa\kappa}}(V_{\lambda+2}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i \in X\}.$$

Letting $d^M = X$, clearly $M = (V_{\lambda+2}, \in, d^M) \models T_0$. Thus, by compactness, we get that there is a model $N \models T$. Notice that the $\mathcal{L}_{\kappa\kappa}$-diagram contains the $\mathcal{L}_{\kappa\kappa}$-sentences expressing well-foundedness and extensionality. We can therefore, by collapsing, without loss of generality assume that $N = (N, \in, c_x^N, d^N)_{x \in V_{\lambda+2}}$ is transitive. The elementary diagram gives us an elementary embedding $j : V_{\lambda+2} \to N$ by $x \mapsto c_x^N$. Note that $c_i^N \in d^N$ for all $i < \lambda$ and so $|d^N| \geq \lambda$. Further $N \models |d| < c_\kappa = j(\kappa)$ and so $j(\kappa) > \lambda$. In particular, j has a critical point $\mathrm{crit}(j) \leq \kappa$. Because $\mathcal{L}_{\kappa\kappa}$ can define all ordinals $< \kappa$, these have to be fixed by j and so $\mathrm{crit}(j) = \kappa$. Further, the theory T makes sure that $j``\lambda \subseteq d^N$. Then as before, we can use j to derive our desired ultrafilter. Note that $\mathcal{P}(\mathcal{P}_\kappa\lambda) \subseteq V_{\lambda+2}$, so it makes sense to define for $X \subseteq \mathcal{P}_\kappa\lambda$:

$$X \in U \text{ iff } d^N \in j(X).$$

It is standard to check that U is a fine and κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$. □

In the above proof, the compactness property of $\mathcal{L}_{\kappa\kappa}$ is not sufficient to derive an embedding $j : V_{\lambda+2} \to N$ such that $j``\lambda \in N$, as we cannot make sure by an $\mathcal{L}_{\kappa\kappa}$-theory alone that a set covering $j``\lambda$ does not contain any additional elements. Benda realised that strengthening compactness by an omitting types property gets us around this problem, and hence to supercompactness.

For this, if $\mathcal{L}$ is a logic, we say that a *type* p of $\mathcal{L}$ is simply a set of formulas of $\mathcal{L}$ in a single free variable (we could strengthen this to types in multiple free variables, but as in set theory we can code finite sequences by single elements, this would not amount to any differences in our applications). If $\mathcal{A}$ is a structure in the language of p, we say that $\mathcal{A}$ *realises* p, if there is an $a \in A$ such that $\mathcal{A} \models \varphi(a)$ for every $\varphi(x) \in p$. We say that $\mathcal{A}$ *omits* p if $\mathcal{A}$ does not realise p. Note that an abstractly given logic $\mathcal{L}$ might not have free variables available, but we can code variables as additional constants. In this case, a type of $\mathcal{L}$ over a vocabulary τ is a collection p of sentences $\varphi \in \mathcal{L}[\tau \cup \{c\}]$ for a constant symbol $c \notin \tau$, and a τ-structure $\mathcal{A}$ realises p if there is an expansion $(\mathcal{A}, c^{\mathcal{A}})$ satisfying every $\varphi \in p$. Again, $\mathcal{A}$ omits p if it does not realise it.

We state Benda's result in its modern presentation due to Boney.

Theorem 1.3.18 (Benda, Boney [Ben78, Bon20])**.** The following are equivalent for a cardinal κ:

(1) κ is supercompact.

(2) For any $\lambda \geq \kappa$, if $T \subseteq \mathcal{L}_{\kappa\kappa}$ is a theory that can be written as an increasing union $T = \bigcup_{s \in \mathcal{P}_\kappa \lambda} T_s$ and $p(x) = \{\varphi_i(x) \colon i < \lambda\} \subseteq \mathcal{L}_{\kappa\kappa}$ is a type such that with $p_s = \{\varphi_i(x) \colon i \in s\}$ there is a club subset X of $\mathcal{P}_\kappa \lambda$ such that for $s \in X$, T_s has a model omitting p_s, then T has a model omitting p.

We will not give the full proof here, but give an idea on how omitting a type can allow us to get normality. Consider again the theory

$$T = \mathrm{ElDiag}_{\mathcal{L}_{\kappa\kappa}}(V_{\lambda+2}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\},$$

but now accompanied by a type

$$p(x) = \{x \in d \wedge x \neq c_i \colon i < \lambda\}.$$

If N is a (transitive) model of T, we get an elementary embedding $j : V_{\lambda+2} \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $j``\lambda \subseteq d^N$ and $N \models |d| < j(\kappa)$. Now suppose $N$ omits $p$. Then any $x \in d^N$ cannot satisfy all formulas in $p$. But then in particular, there has to be $i < \lambda$ such that $x = c_i^N = j(i)$. Thus, $x \in j``\lambda$ and we get $j``\lambda = d^N$. But then deriving an ultrafilter using $j``\lambda$, gives us a normal ultrafilter along the lines of Theorem 1.3.13. Type omission can thus be used to rule out cases in which we have some undesired object like $d \supsetneq j``\lambda$.

Obviously every supercompact cardinal is strongly compact, while it is consistent that the smallest strongly compact cardinal is the smallest measurable cardinal, and hence not supercompact (cf. [Mag76]). Nevertheless, strongly compact cardinals exceed measurables significantly in consistency strength. In fact, whether the existence of supercompact and strongly compact cardinals is equiconsistent is a longstanding open problem.

Recall that a cardinal κ is strongly compact iff every κ-complete filter over any set can be extended to a κ-complete ultrafilter (cf., e.g., [Kan03, Proposition 4.1]). We will sometimes consider the following variation of strongly compact cardinals introduced by Bagaria and Magidor.

Definition 1.3.19 ([BM14a, Definition 4.6])**.** If $\kappa \leq \delta$ are cardinals, we say that δ is *κ-strongly compact* iff every δ-complete filter over any set can be extended to a κ-complete ultrafilter.

Like strongly compact cardinals, they have a characterisation via elementary embeddings and via the existence of specific ultrafilters.

Theorem 1.3.20 (Bagaria & Magidor [BM14b, Theorem 1.2])**.** The following are equivalent for uncountable $\kappa \leq \delta$:

(1) δ is κ-strongly compact.

(2) For every $\alpha \geq \delta$ there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) \geq \kappa$ and there is $d \in M$ with $j``\alpha \subseteq d$ and $M \models |d| < j(\delta)$.

(3) For every set x there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) \geq \kappa$ and there is $d \in M$ with $j``x \subseteq M$ and $M \models |d| < j(\delta)$.

(4) For every set x, there is a κ-complete, fine ultrafilter over $\mathcal{P}_\delta(x)$.

We can utilise ω_1-strongly compact cardinals to characterise $\mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

Theorem 1.3.21 (Magidor)**.** Let δ be a cardinal. A cardinal δ is a strong compactness cardinal for $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ iff δ is ω_1-strongly compact. In particular, $\delta = \mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ iff δ is the smallest ω_1-strongly compact cardinal.

We are not aware of any published proof of this fact, besides the following one given by Gitman and the author in [GO24].

Proof. Suppose that δ is ω_1-strongly compact. Fix a vocabulary τ. Let T be a $<\delta$-satisfiable theory of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})[\tau]$. We have to find a model of T. By Theorem 1.3.20, ω_1-strong compactness of δ gives us an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa \geq \omega_1$ and $d \in M$ such that $j``T \subseteq d$ and $M \models |d| < j(\delta)$. In $M$, let $S = j(T) \cap d$. Observe that $M \models |S| < j(\delta)$ and $j``T \subseteq S \subseteq j(T)$. By elementarity, M believes that every subset of $j(T)$, a theory in $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})[j(\tau)]$ of size $<j(\delta)$, is satisfiable. It follows that M has a $j(\tau)$-structure $\mathcal{A} \models S$. As M is transitive, it is correct about the well-foundedness quantifier, so $\mathcal{A}$ is really a model of S and, in particular, of $j``T \subseteq S$. Using the renaming which takes $\tau$ to $j``\tau$, we get that $\mathcal{A}$ can be renamed to a model of T.

Next, suppose that δ is a strong compactness cardinal for $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$. To show that δ is ω_1-strongly compact, by the proof of [BM14a, Theorem 4.7] it is sufficient to produce for every $\alpha > \delta$ a fine ω_1-complete ultrafilter over $\mathcal{P}_\delta(\alpha)$. If γ is an ordinal with $\mathcal{P}(\alpha) \in V_\gamma$ and we have an elementary embedding $j : V_\gamma \to M$ with M transitive, $\mathrm{crit}(j) \geq \omega_1$, $d \in M$ with $j``\alpha \subseteq d \subseteq j(\alpha)$ and $M \models |d| < j(\delta)$, then it is routine to check that U defined by

$$X \in U \text{ iff } X \subseteq \mathcal{P}_\delta(\alpha) \text{ and } d \in j(X)$$

is a fine ω_1-complete ultrafilter over $\mathcal{P}_\delta(\alpha)$. Let T be the following $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$-theory:

$$\mathrm{ElDiag}_{\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})}(V_\gamma, \in) \cup \{c_i \in d \wedge |d| < c_\delta \colon i < \alpha\},$$

where d is a new constant symbol not used in the elementary diagram.

The theory T is $<\delta$-satisfiable by V_γ itself, interpreting d by those $i < \alpha$ such that $c_i \in d \wedge |d| < c_\delta$ appears as a sentence of $T_0 \in \mathcal{P}_\delta T$. So T has a well-founded model M, and by collapsing we can assume without loss of generality that M is transitive. Thus, we get an elementary embedding $j : V_\gamma \to M$. Clearly, $|d^M| \geq \alpha$ by T and so $j(\delta) = c_\delta^M > |d^M|^M \geq |d^M| \geq \alpha$, so j has a critical point $\mathrm{crit}(j) \leq \delta$. It is easy to check along the lines of [Kan03, Theorem 5.6] that $\mathrm{crit}(j)$ is a measurable cardinal, so in particular, $\mathrm{crit}(j) \geq \omega_1$. Finally, $j``\alpha \subseteq d^M$, so letting $d_0 = d^M \cap j(\alpha)$ we get $j``\alpha \subseteq d_0 \subseteq j(\alpha)$ and $M \models |d_0| \leq |d^M|^M < j(\delta)$. □

1.3.5. $C^{(n)}$-extendible cardinals

Recall that $C^{(n)}$, for a natural number n, is the club class of ordinals α such that V_α is a Σ_n-elementary substructure of V. The $C^{(n)}$-extendible cardinals introduced by Joan Bagaria in [Bag12] are, in a sense we will understand later, a natural strengthening of supercompact cardinals. Their original definition diverts us from large cardinal properties witnessed by the existence of ultrafilters and elementary embeddings of the universe.

Definition 1.3.22. Let κ be a cardinal and n a natural number.

(1) For $\alpha > \kappa$, κ is called *α-$C^{(n)}$-extendible* iff there is some ordinal β and an elementary embedding $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $j(\kappa) \in C^{(n)}$.

(2) We say that κ is *$C^{(n)}$-extendible* iff it is α-$C^{(n)}$-extendible for every $\alpha > \kappa$.

Bagaria introduced $C^{(n)}$-extendibles as a generalisation of the classical notion of *extendible* cardinals (cf., e.g., [Kan03, Section 23]). Extendible cardinals are defined in precisely the same way as the above notion, but missing the condition that $j(\kappa) \in C^{(n)}$. Bagaria's $C^{(1)}$-extendible cardinals are precisely the classical extendible cardinals (cf. [Bag12, Section 3]).

Bagaria also considered an apriori stronger notion, in which, for $\alpha \in C^{(n)}$, one can find an emedding $j : V_\alpha \to V_\beta$ witnessing (1) for a $\beta \in C^{(n)}$. Tsaprounis [Tsa12] showed that these notions coincide. Hamkins and Gitman [GH19] further showed that using this formulation one can drop the assumption that $j(\kappa) \in C^{(n)}$ and still arrive at the same notion. It is further a classical result for extendible cardinals, i.e., the case $n = 1$, that the condition $j(\kappa) > \alpha$ is superfluous. The proof carries over to the general case. Let us collect these remarks.

Theorem 1.3.23. The following are equivalent for a cardinal κ:

(1) κ is $C^{(n)}$-extendible.

(2) For every $\alpha > \kappa$ with $\alpha \in C^{(n)}$, there is $\beta \in C^{(n)}$ and an elementary embedding $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

(3) For every $\alpha > \kappa$ with $\alpha \in C^{(n)}$, there is $\beta \in C^{(n)}$ and an elementary embedding $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$.

Proof. For the equivalence of (1) and (2), consider [GH19, Theorem 15]. For the equivalence of (2) and (3) in the case $n = 1$, see [Kan03, Proposition 23.15]. The general case is an easy variation of this proof. □

It is the formulations (2) and (3) that we will most often be interested in, in the context of model theory of strong logics.

With increasing n, the existence of $C^{(n)}$-extendible cardinals becomes a stronger and stronger assumption. Already the classical notion exceeds supercompact cardinals:

Proposition 1.3.24. If κ is extendible, it is a supercompact limit of supercompact cardinals.

Proof. Combine [Kan03, Proposition 23.6] and [Kan03, Proposition 23.7]. □

Bagaria generalised this to $C^{(n)}$-extendible cardinals.

Theorem 1.3.25 (Bagaria [Bag12, Proposition 3.5])**.** If κ is $C^{(n+1)}$-extendible, then it is a limit of $C^{(n)}$-extendible cardinals.

Note that "κ is $C^{(n)}$-extendible" is a Π_{n+2} statement:

$$\forall\alpha\exists\beta\exists j(\alpha > \kappa \to (j : V_\alpha \to V_\beta \text{ is elementary} \land \text{crit}(j) = \kappa \land j(\kappa) \in C^{(n)})).$$

It is again a classical result that extendible cardinals are in $C^{(3)}$ (see [Kan03, Proposition 23.15]). Therefore extendible cardinals are correct about extendibility. Bagaria showed that also this generalises to larger n.

Theorem 1.3.26 (Bagaria [Bag12, Proposition 3.4])**.** If κ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$.

In particular, if κ is $C^{(n+1)}$-extendible, and therefore a limit of $C^{(n)}$-extendibles, by virtue of being in $C^{(n+2)}$, V_κ will satisfy that there is a $C^{(n)}$-extendible cardinal. Hence, the existence of $C^{(n+1)}$-extendible cardinals exceeds the existence of $C^{(n)}$-extendibles in consistency strength.

We further get a strong reflection result about $C^{(n)}$-extendible cardinals.

Corollary 1.3.27 (Bagaria [Bag12, Proposition 3.6])**.** If κ is $C^{(n+2)}$-extendible, then there is a proper class of $C^{(n)}$-extendible cardinals.

Proof. If κ is $C^{(n+2)}$-extendible it is in particular $C^{(n+1)}$-extendible, and thus a limit of $C^{(n)}$-extendible cardinals. Thus V_κ satisfies that there is a proper class of $C^{(n)}$-extendible cardinals. The latter is a Π_{n+4}-statement:

$$\forall\gamma\exists\delta(\delta > \gamma \land \delta \text{ is } C^{(n)}\text{-extendible}).$$

Because $\kappa \in C^{(n+4)}$, it reflects this statement to V. □

One of the first usages of extendible cardinals was Magidor's theorem about compactness cardinals of second-order logics:

Theorem 1.3.28 (Magidor [Mag71])**.** The following are equivalent for a cardinal κ:

(1) κ is extendible.

(2) $\kappa = \text{comp}(\mathcal{L}^2_{\kappa\kappa})$.

(3) $\kappa = \text{comp}(\mathcal{L}^2_{\kappa\omega})$.

One may further show that $\text{comp}(\mathcal{L}^2)$ is the smallest extendible cardinal. We will see below that these results generalise to $C^{(n)}$-extendible cardinals.

1.3.6. Vopěnka's Principle

Vopěnka's Principle is an axiom schema that can be understood as a limit point for many model-theoretic properties. While this fact about Vopěnka's Principle has been known for a long time, developments of recent years enable us to understand this relation in a much more structured way. In particular, its connections to $C^{(n)}$-extendible cardinals allow us to find precise connections between model theory and large cardinals.

There are many equivalent formulations of Vopěnka's Principle. The one we will adopt is the following:

Definition 1.3.29. *Vopěnka's Principle* (VP) is the axiom:

> For every proper class $\mathcal{K}$ of structures in a joint vocabulary τ, there are distinct $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ such that there is an elementary embedding $e : \mathcal{A} \to \mathcal{B}$.

In our context of ZFC, we understand this as an axiom schema which states for every meta-theoretic formula $\Phi(x, y_1, \ldots, y_n)$ that, if $\Phi(x, p_1, \ldots, p_n)$ defines a class of structures in a joint vocabulary via possible parameters $p_1, \ldots, p_n$, then we find $\mathcal{A} \neq \mathcal{B}$ such that $\Phi(\mathcal{A}, p_1, \ldots, p_n)$ and $\Phi(\mathcal{B}, p_1, \ldots, p_n)$ and an elementary embedding $e : \mathcal{A} \to \mathcal{B}$.

We will also want to consider local versions of VP. For this, we write VP(Π_n) for the statement of VP restricted to classes definable by Π_n formulas without any parameters. We further write VP($\mathbf{\Pi}_n$) for the statement of VP restricted to classes definable by Π_n formulas with parameters. Recall that the global satisfaction relation, restricted to a fixed level of the Lévy hierarchy is definable by a single formula in the language of set theory. Thus we can state VP(Π_n) as a single axiom, and do not need to treat it as a schema.

Let us further write VP($\kappa, \mathbf{\Sigma}_n$) for the following statement:

> For every proper class $\mathcal{K}$ of structures in a joint vocabulary $\tau \in H_\kappa$ such that $\mathcal{K}$ is Σ_n definable with parameters in H_κ, for every $\mathcal{B} \in \mathcal{K}$ there exists $\mathcal{A} \in \mathcal{K} \cap H_\kappa$ such that there is an elementary embedding $e : \mathcal{A} \to \mathcal{B}$.

This is one of the *structural reflection principles*, introduced by Bagaria and his co-authors in [BCMR15]. Note that if for every n, there is a proper class of κ such that VP($\kappa, \mathbf{\Sigma}_n$) holds, then VP holds as well.

VP is an unusual large cardinal assumption in that it does not talk about the existence of cardinals at all. Nevertheless, relatively early several connections to large cardinal assumptions were discovered. For example, in [SRK78] it was shown that if VP holds, there exists a proper class of extendible cardinals, and if κ is a *huge* cardinal (and less), then V_κ satisfies VP.

Modern results relate VP closely to $C^{(n)}$-extendible cardinals. In fact, they form what could be called the backbone of VP in that existence of a $C^{(n)}$-extendible cardinal corresponds precisely to restrictions of VP to lower levels of the Lévy hierarchy. Bagaria's following theorem makes this precise.

Theorem 1.3.30 (Bagaria [Bag12, Corollary 4.13])**.** The following are equivalent for $n \geq 1$:

(1) $\mathrm{VP}(\Pi_{n+1})$.

(2) $\mathrm{VP}(\kappa, \mathbf{\Sigma}_{n+2})$ holds for some cardinal κ.

(3) There exists a $C^{(n)}$-extendible cardinal.

The proof further shows that if κ is $C^{(n)}$-extendible, then κ is one of the cardinals witnessing (2). Recall that the existence of a $C^{(n+2)}$-extendible cardinal implies that there is a proper class of $C^{(n)}$-extendible cardinals. Therefore, Bagaria's theorem above implies:

Theorem 1.3.31 (Bagaria [Bag12, Corollary 4.15])**.** The following are equivalent:

(1) VP.

(2) For every n, $\mathrm{VP}(\Pi_n)$ holds.

(3) For every n, there is a cardinal κ such that $\mathrm{VP}(\kappa, \mathbf{\Sigma}_n)$ holds.

(4) For every n, there is a proper class of cardinals κ such that $\mathrm{VP}(\kappa, \mathbf{\Sigma}_n)$ holds.

(5) For every n, there is a $C^{(n)}$-extendible cardinal.

(6) For every n, there is a proper class of $C^{(n)}$-extendible cardinals.

The following theorem is a first glimpse in which way ($C^{(n)}$)-extendible cardinals are direct strengthenings of supercompact cardinals.

Theorem 1.3.32 (Bagaria [Bag12, Corollary 4.6])**.** The following are equivalent:

(1) $\mathrm{VP}(\Pi_1)$.

(2) $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ holds for some cardinal κ.

(3) There exists a supercompact cardinal.

And again, the proof shows that if κ is supercompact, then κ is one of the cardinals witnessing (2). Summarising Bagaria's results, we have that VP is stratified into a hierarchy along restrictions of its statements to the Π_n formulas,. And these restrictions correspond precisely to the existence of large cardinals, starting with supercompact cardinals on the level $n = 1$, and proceeding with $C^{(n)}$-extendible cardinals, leading to stronger and stronger statements with rising n. Bagaria and Lücke called this a "pattern" in the large cardinal hierarchy [BL24]. It is this and similar hierarchies that we will pay close attention to throughout this thesis.

Similar to the equivalence between VP and the existence of $C^{(n)}$-extendible cardinals for every n, VP is equivalent to global statements involving properties of logics. The equivalence of VP to the following statement (2) was proven by Stavi, but never published (see [MV11]). The equivalence to (3) is due to Makowsky [Mak85].

Theorem 1.3.33 (Makowsky, Stavi)**.** The following are equivalent:

(1) VP.

(2) Every logic has an LST number.

(3) Every logic has a compactness number.

We will see throughout the thesis that there are more statements of this type, relating VP to statements about *all* logics exhibiting some property.

Consider Magidor's theorems 1.3.14 and 1.3.28 on the one hand, and the equivalence of VP and the existence of $C^{(n)}$-extendible cardinals for every n on the other hand. In the light of Makowsky's and Stavi's results, the question is then natural whether we can also have local forms of Theorem 1.3.33 above. Indeed, results by Boney and Poveda show that this is the case.

Theorem 1.3.34 (Boney [Bon20, Theorem 4.1])**.** Let κ be a cardinal and n a natural number.

(a) κ is $C^{(n)}$-extendible iff $\kappa = \text{comp}(\mathcal{L}^{\text{s},n}_{\kappa\omega})$.

(b) $\kappa = \text{comp}(\mathcal{L}^{\text{s},n})$ iff κ is the smallest $C^{(n)}$-extendible cardinal.

Theorem 1.3.35. The following are equivalent for a natural number n and a cardinal κ:

(1) κ is $C^{(n)}$-extendible.

(2) $\kappa = \text{LST}(\mathcal{L}^{\text{s},n+1}_{\kappa\omega})$.

(3) For all $\beta > \kappa$ such that $\beta \in C^{(n+1)}$ and for all $x \in V_\beta$ there is an $\alpha < \kappa$ such that $\alpha \in C^{(n+1)}$ and an elementary embedding $j : V_\alpha \to V_\beta$ such that $x \in \text{ran}(j)$ and $j(\text{crit}(j)) = \kappa$.

Proof. Boney [Bon20, Theorem 4.13] shows a slightly weaker statement than the equivalence of (2) and (3). Poveda [Pov20, Theorem 5.2.3] shows the equivalence of (3) and (1). The latter result was also already implicit in [BGS17, Theorem 3.8]. □

One can adopt this result to also give a characterisation of the smallest $C^{(n)}$-extendible cardinal as $\text{LST}(\mathcal{L}^{\text{s},n+1})$.

Notice that Makowsky's and Stavi's theorems can be derived as corollaries: For instance, for the compactness case, if VP holds, then for every n, there exists a proper class of $C^{(n)}$-extendible cardinals. Now if $\mathcal{L}$ is any logic, $\mathcal{L} \leq \mathcal{L}^{\text{s},n}_{\gamma\omega}$ for some γ and n. Then taking any $C^{(n)}$-extendible cardinal $\kappa > \gamma$, we get $\text{comp}(\mathcal{L}) \leq \text{comp}(\mathcal{L}^{\text{s},n}_{\gamma\omega}) \leq \text{comp}(\mathcal{L}^{\text{s},n}_{\kappa\omega})$. On the other hand, if there is a compactness cardinal for any logic, then in particular $\text{comp}(\mathcal{L}^{\text{s},n})$ exists for every n. Hence, there exists a $C^{(n)}$-extendible cardinal for every n.

The pattern leading up via the $C^{(n)}$-extendible cardinals to VP is therefore precisely mirrored by compactness and Löwenheim-Skolem properties of stronger and stronger logics. We summarise these relations in Figure 1.1. We will see that this hierarchy is repeated when considering other properties of logics, namely *upward LST numbers* and *Henkin compactness properties*.

It it such hierarchies which let us consider VP as a limit point of model-theoretic properties. For instance, for any fragment of VP, such as VP(Π_n), we find some logic $\mathcal{L}$ such that the existence of a compactness number of $\mathcal{L}$ is stronger than VP(Π_n). More precisely, such a logic is given by $\mathcal{L}^{\mathrm{s},m}$ for any $m \geq n$. And further, assuming the existence of compactness numbers of $\mathcal{L}^{\mathrm{s},m}$ for all natural numbers gives us the full Vopěnka's Principle. Bagaria and his co-authors found other similar patterns in the

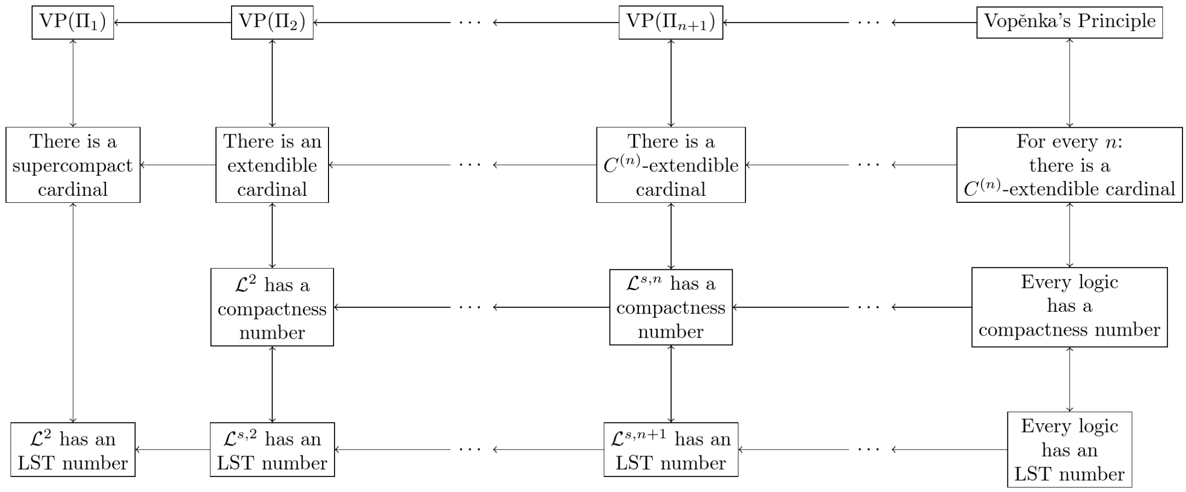

Figure 1.1.: Relations between VP, $C^{(n)}$-extendible cardinals, compactness numbers, and LST numbers.

large cardinal hierarchy (cf. [BW23, BL24] and the survey paper [Bag23]). In particular, Bagaria and Wilson showed that an analogous stratification is formed by what is now known as Π_n-*strong-cardinals* below *weak Vopěnka's Principle* (cf. [BW23]). The latter is an assumption arising from a weakening of a category-theoretic formulation of VP. We will show that also Π_n-strong cardinals, and hence weak Vopěnka's Principle, relate to patterns of certain properties being exhibited by stronger and stronger logics.

1.3.7. Extenders

We saw that many elementary embeddings $j : V \to M$ we are concerned with arrive as the canonical embeddings provided by the ultrapower construction. As it turns out, not all elementary embeddings we will need can be derived like this. Instead, we will have to deal with so called *extenders* and corresponding *extender powers*. They provide a much more flexible way of approximating elementary embeddings $j : V \to M$. Informally, an extender E is a collection of ultrafilters which fit together in some suitable way such that computing the ultrapowers of the universe by them results in a directed system of models with elementary embeddings between them. The directed limit M_E of these models then comes with an elementary embedding $j_E : V \to M_E$. If the ultrafilters in E are derived from some ambient embedding $j : V \to M$, the embedding j_E can embody many of the relevant properties of j. In this section we will summarise those parts of the basic theory of extenders we will need throughout this thesis. We will first consider extender powers of the universe V, and then lay out how to adapt the construction for set-sized models. In particular we will consider how the extender power of some V_α relates to the extender power of V.

Extender powers of the universe

The following procedure is standard and our exposition is based on [Kan03, Chapter 26]. Recall the direct limit construction from model theory, which we will not review (cf., e.g., [Kan03, Section 0]). We will omit proofs. Suppose we have an elementary embedding $j : V \to M$ with a critical point κ and suppose there is some ordinal $\lambda > \kappa$ and $\kappa \leq \zeta < \lambda$ such that ζ is the smallest ordinal with the property that $j(\zeta) \geq \lambda$. Most often we will consider the case $\zeta = \kappa$. We write $[\alpha]^{<\omega}$ for the set of finite subsets of α, and for a natural number n, we write $[\alpha]^n$ for the n-sized subsets of α. The ultrafilters we are interested in are indexed by $a \in [\lambda]^{<\omega}$ and in live on $[\zeta]^{|a|}$. Note that if $X \subseteq [\zeta]^{|a|}$, then $j(X) \subseteq [j(\zeta)]^{|a|}$ and $\lambda \subseteq j(\zeta)$ by assumption. It thus makes sense to ask whether $a \in j(X)$, and indeed, it is standard to check that

$$X \in E_a \text{ iff } X \subseteq [\zeta]^{|a|} \text{ and } a \in j(X),$$

defines a κ-complete ultrafilter over $[\zeta]^{<\omega}$. We write $M_a = \mathrm{Ult}(V, E_a)$ for the corresponding ultrapower and $j_a : V \to M_a$ for the standard elementary embedding $x \mapsto [c_x]_{E_a}$. This comes with an elementary embedding $k_a : M_a \to M$ by letting $[f]_{E_a} \mapsto j(f)(a)$. And further, the ultrapowers M_a indexed by $a \in [\zeta]^{<\omega}$ form a directed system with elementary embeddings $i_{ab} : M_a \to M_b$ whenever $a \subseteq b$ (note that if $a, b \in [\zeta]^{<\omega}$, then $c = a \cup b \in [\zeta]^{<\omega}$ and so this gives directedness). To construct i_{ab}, let for $b = \{\alpha_1, \dots, \alpha_n\}$ with $\alpha_1 < \dots < \alpha_n$ and $a = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$ the map $\pi_{ba} : [\zeta]^n \to [\zeta]^m$ be defined by

$$\{\beta_1, \dots, \beta_n\} \mapsto \{\beta_{i_1}, \dots, \beta_{i_m}\}.$$

Then defining $i_{ab} : M_a \to M_b$ by $[f]_{E_a} \mapsto [f \circ \pi_{ba}]_{E_b}$ results in the desired properties that $((M_a \colon a \in [\lambda]^{<\omega}), (i_{ab} \colon a \subseteq b))$ forms a directed system such that $j_b = i_{ab} \circ j_a$ and $k_a = k_b \circ i_{ab}$. In particular, we can consider the direct limit M_E of the M_a. As a direct limit, M_E consists of equivalence classes of elements of the M_a that are eventually identified via the i_{ab}: if $x \in M_E$, then x is the equivalence class (via Scott's trick) of some $f \in M_a$ for some a, i.e., x consists of all $[g]_{E_b}$ such that there is some c with $i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b})$. We write $[a, [f_{E_a}]]_E \in M_E$ for such an equivalence class and often drop the indices for better readability. As a direct limit, M_E comes with elementary embeddings

$$\begin{aligned} &j_E : V \to M_E, && x \mapsto [a, [c_x]] \text{ (for any } a \in [\lambda]^{<\omega}), \\ &k_{aE} : M_a \to M_E, && [f]_{E_a} \mapsto [a, [f]], \\ &k_E : M_E \to M, && [a, [f]] \mapsto j(f)(a). \end{aligned}$$

such that $j = k_E \circ j_E$, $k_{aE} = k_{bE} \circ i_{ab}$ and $k_a = k_E \circ k_{aE}$. Here the specific shapes of the maps j_E and k_{aE} come from the direct limit construction, while that k_E takes the specified form can be checked along the lines of the proof of [Kan03, Theorem 26.1]. Note that in general, taking a direct limit may lead to ill-founded models. In this case though, consider the relation $\in^{M_E}$ on M_E which comes from the direct limit construction. As we have an elementary embedding $k_E : M_E \to M$, an ill-founded $\in^{M_E}$ would lead to M being ill-founded. Therefore, M_E is well-founded, and so we can, by collapsing, identify $\in^{M_E}$ with $\in$ and assume that M_E is transitive.

Fixing notation, we call $E = (E_a \colon a \in [\lambda]^{<\omega})$ the (κ, λ)-*extender derived from* j. We call κ the *critical point* and λ the *length* of E. If $\zeta > \kappa$, we further call E a *long* extender. The following theorem summarises the most important results we will need about the resulting embeddings and models.

Theorem 1.3.36 ([Kan03, Theorem 26.1])**.** Given the above construction, the following hold:

(1) $M_E = \{j_E(f)(a) \colon a \in [\lambda]^{<\omega}, f : [\zeta]^{|a|} \to V\}$ and $j_E(f)(a) = [a, [f]]$.

(2) $\mathrm{crit}(j_E) = \kappa$ and $j_E(\zeta) \geq \lambda$.

(3) $\mathrm{crit}(k_E) \geq \lambda$.

(4) k_E is the inverse collapsing isomorphism.

(5) For any γ such that $|V_\gamma|^M \leq \lambda$: $V_\gamma^M \subseteq \mathrm{ran}(k_E)$, $V_\gamma^{M_E} = V_\gamma^M$ and $k_E(x) = x$ for $x \in V_\gamma^{M_E}$.

The reader is also referred to [Tsa12, Proposition A.3.] for a thorough proof of this result. We made explicit that $j_E(f)(a) = [a, [f]]$ which is implicity proven there.

For ease of presentation, we considered embeddings $j : V \to M$. Rarely, we will be interested in performing an extender construction for other inner models. In this case, if $j : N \to M$ is an elementary embedding between inner models with $\mathrm{crit}(j) = \kappa$ and ζ some smallest ordinal such that $j(\zeta) \geq \lambda > \kappa$, a similar construction can be carried out, taking ultrapowers of N and a direct limit of these ultrapowers. The changes to be made are that one considers for $a \in [\lambda]^{<\omega}$ ultrafilters E_a on $\mathcal{P}^N([\zeta]^{<\omega})$ instead of on $\mathcal{P}([\zeta]^{<\omega})$, and that the ultrapowers of N consist of equivalence classes of functions $f \in^{[\zeta]^{|a|}} N \cap N$. Consider [Kan03, Section 26] for details. We call $(E_a \colon a \in [\lambda]^{<\omega})$ the N-(κ, λ)-*extender derived from* j. An analogue of Theorem 1.3.36 can then be stated (cf. [Kan03, Theorem 26.1]).

For completeness, let us mention that there is a also an axiomatic definition of what it means to be an N-(κ, λ)-*extender* E, which is not dependent on an ambient elementary embedding, specifying combinatorial properties of the objects involved in $E = (E_a \colon a \in [\lambda]^{<\omega})$. Among these properties is that $(N, \in, E_a)$ computes E_a to be an ultrafilter and so one can proceed to build ultrapowers of N and consider their direct limit M_E. The combinatorial properties involved imply that M_E is well-founded, and so the direct limit gives an elementary embedding $j_E : N \to M_E$ between transitive models. They further imply that $\mathrm{crit}(j_E) = \kappa$ and there is some smallest ordinal $\zeta \geq \kappa$ such that $j_E(\zeta) \geq \lambda$. Therefore, one can use j_E to derive an extender E^* in the way outlined above. One can show that this N-(κ, λ)-extender E^* derived from j_E is again the N-(κ, λ)-extender E and therefore the two ways of considering extenders are essentially equivalent. Again, consider [Kan03, Section 26] for details about all the mentioned properties. We will mostly be concerned with extenders derived from an embedding, so we will omit the exposition here. Note however, that to give a formalisation of some large cardinal notion in terms of the existence of extenders, the latter notion, which is formulated without reference to an elementary embedding, is the appropriate one.

Ultrapowers and extender powers of set models

As model-theoretic properties of logics often interact with set models, rather than with proper classes, we will repeatedly want to derive extenders from embeddings between set-sized structures, which are provided by our model-theoretic tools. It is folklore that the extender power of the universe interacts nicely with building extender powers of set-sized models. For instance, building an extender power of, say, some V_α, we have that the resulting model m_E is an initial segment of the extender power M_E. While this is quite clear, to argue that specific properties of m_E carry over to M_E is sometimes delicate, as some of the constructions for V only carry over to a V_α modulo technical difficulties. As we will often need to make sure for a precise consideration of these relations, and many of the technical details are rarely (if ever?) spelled out, we therefore want to dedicate this section to present some of the required adaptations to treat this case, and to fix the notation with which we will refer to it. We will in particular be interested in how the extender power of some V_α and the associated embeddings of this construction relate to the extender power of V. The main properties we will need in later chapters are summarised in Theorems 1.3.39 and 1.3.40.

We start by considering how ultrapowers of some V_α relate to ultrapowers of V, as the main technicalities appear already in this case. Let α be some ordinal, $S \in V_\alpha$ and U an ultrafilter over S. We make some extra assumptions: Let us assume that either, α is some successor ordinal $\alpha = \beta + 1$, or that $\operatorname{cof}(\alpha) > |S|$. We consider $m = \operatorname{Ult}(V_\alpha, U)$. We have the standard map $j_m : V_\alpha \to m$, $x \mapsto [c_x]_U$. Using Łos' Theorem, this provides an elementary embedding also in this set-sized case. First we note that building the ultrapower $M = \operatorname{Ult}(V, U)$ and the standard map $j_U : V \to M$, $x \mapsto [c_x]_U$, we have

$$j_U \restriction V_\alpha = j_m$$

and that

$$V^M_{j_U(\alpha)} = m.$$

This follows from the following chain of equivalences.

$$\begin{aligned} [f]_U \in V^M_{j_U(\alpha)} &\text{ iff } M \models \operatorname{rk}([f]_U) < j_U(\alpha) = [c_\alpha]_U \\ &\text{ iff } \{s \in S \colon \operatorname{rk}(f(s)) \in c_\alpha(s)\} \in U \\ &\text{ iff } \{s \in S \colon \operatorname{rk}(f(s)) \in \alpha\} \in U \\ &\text{ iff } [f]_U = [g]_U \text{ for some } g : S \to V_\alpha \\ &\text{ iff } [f]_U \in \operatorname{Ult}(V_\alpha, U) = m. \end{aligned}$$

Now suppose $j : V_\alpha \to N$ is an elementary embedding and $x \in j(S)$ is some set such that for $X \subseteq S \in V_\alpha$,

$$X \in U \text{ iff } x \in j(X),$$

defines an ultrafilter on U. Again build the ultrapower $m = \mathrm{Ult}(V_\alpha, U)$ and the standard map $j_m : V_\alpha \to m$ as above. We would like to build a factor map $k : m \to N$ such that $j = k \circ j_m$. Dealing with V, this is no problem, but note that this is different in our case. We would like to define for $f : S \to V_\alpha$, the map k as $[f]_U \mapsto j(f)(x)$. But note that in general $f \notin V_\alpha$ and so $j(f)$ might not be defined. This is where our assumptions on α come into play. If $\mathrm{cof}(\alpha) > |S|$, then any map $f : S \to V_\alpha$ is in V_α and so we can define k as intended. If $\alpha = \beta + 1$ is a successor cardinal, we can code f by an element of V_α as follows. Recall that a *flat pairing function* constructs a pair of two sets that does not increase ranks, i.e., if $a, b \in V_\eta$ for some η, we can build an object $(a, b)^* \in V_\eta$ such that $(a, b)^* = (c, d)^*$ iff $a = c$ and $b = d$. Fix such a flat pairing function $(a, b) \mapsto (a, b)^*$.[4] Now if $f : S \to V_{\beta+1}$, let $f^* = \{(s, b)^* \colon s \in S, b \in f(s)\}$. Note that $f^* \in V_{\beta+1}$. Letting $f^*(s) = \{b : (s, b)^* \in f^*\} \in V_{\beta+1}$, we have $f^*(s) = f(s)$ for all $s \in S$.

For notational homogeneity, in the following, if $\alpha = \beta + 1$ and $f : S \to V_\alpha$ is a function write f^* for the coded function as constructed above. And if α is a limit ordinal of cofinality $\mathrm{cof}(\alpha) > |S|$, simply let $f^* = f$. Thus, in both cases we can define

$$k_m : m \to N, [f]_U \mapsto j(f^*)(k).$$

Then k_m behaves in the desired way:

Claim 1.3.37. The map k_m is elementary and $j = k_m \circ j_m$.

Proof. The following chain of equivalences shows elementarity.

$$\begin{aligned} m \models \varphi([f]_U) &\text{ iff } \{s \in S \colon \varphi^{V_{\alpha+1}}(f(s))\} \in U \\ &\text{ iff } x \in j(\{s \in S \colon \varphi^{V_{\alpha+1}}(f(s))\}) \\ &\text{ iff } x \in j(\{s \in S \colon \varphi^{V_{\alpha+1}}(f^*(s))\}) \\ &\text{ iff } x \in \{s \in j(S) \colon N \models \varphi(j(f^*)(s))\} \\ &\text{ iff } N \models \varphi(j(f^*)(x)) \\ &\text{ iff } N \models \varphi(k_m([f]_U)). \end{aligned}$$

Further, because $c_y^*(s) = c_y(s) = y$ for all $s \in S$ and $y \in V_\alpha$, by elementarity of j we have $j(c_y^*)(x) = j(y)$. Thus we get $k_m(j_m(y)) = k_m([c_y]_U) = j(c_y^*)(x) = j(y)$. □

Now we want to consider extender powers of V_α. The construction is essentially analogous to the class case, but at some points we have to use coded functions as above. Let $j : V_\alpha \to N$ be elementary and suppose j has a critical point $\mathrm{crit}(j) = \kappa$ and that we have some ordinal $\lambda > \alpha$ such that for some smallest ordinal $\zeta \geq \kappa$, $j(\zeta) \geq \lambda$. Further,

[4] The most well-known such function is sometimes called the *Quine-Rosser definition of an ordered pair*. For a definition, cf. [Ros53, p. 281].

assume that either $\alpha = \beta + 1$ is a successor ordinal, or that $\mathrm{cof}(\alpha) > |\zeta|$. Then for $a \in [\lambda]^{<\omega}$, the same arguments as in the case of class embeddings give that

$$X \in E_a \text{ iff } X \subseteq [\zeta]^{|a|} \text{ and } a \in j(X),$$

defines a κ-complete ultrafilter over $[\zeta]^{|a|}$, which comes with an ultrapower $m_a = \mathrm{Ult}(V_\alpha, E_a)$ and an elementary map $j_{a,m} : V_\alpha \to m_a$, $x \mapsto [c_x]_{E_a}$. By our construction above, using coded functions f^*, we get factor maps

$$k_{a,m} : m_a \to N, [f]_{E_a} \mapsto j(f^*)(a).$$

Define, analogously to the class case, for $a \subseteq b$ elementary maps

$$i_{ab,m} : m_a \to m_b, [f]_{E_a} \mapsto [f \circ \pi_{ba}]_{E_b}.$$

Also analogously, one checks that $j_{b,m} = i_{ab,m} \circ j_{a,m}$, and, using the coding, that $k_{a,m} = k_{b,m} \circ \pi_{ba}$.

We can therefore consider the direct limit m_E of the m_a, writing $[a, [f]_{E_a}]_E \in m_E$ for the equivalence classes making up m_E, but mostly dropping the indices. We get elementary maps

$$\begin{aligned} j_{E,m} : V_\alpha \to m_E, &\quad x \mapsto [a, [c_x]] \text{ (for any } a\text{)}, \\ k_{a,E,m} : m_a \to m_E, &\quad [f]_{E_a} \mapsto [a, [f]], \\ k_{E,m} : m_E \to N, &\quad [a, [f]] \mapsto j(f^*)(a), \end{aligned}$$

where the definitions of $j_{E,m}$ and $k_{a,E,m}$ work as in the class case and come from the direct limit construction, and we have $j_{E,m} = k_{a,E,m} \circ k_{a,m}$. The definition of $k_{E,m}$ requires going through f^* again, so let us check that the desired properties carry over.

Claim 1.3.38. The map $k_{E,m}$ is elementary, $j = k_{E,m} \circ j_{E,m}$ and for any a, $k_{a,m} = k_{m,E} \circ k_{a,m,E}$.

Proof. We have $k_{E,m} \circ k_{a,E,m}([f]) = k_{E,m}([a, [f]]) = j(f^*)(a) = k_{a,m}([f])$. Further, $k_{E,m} \circ j_{E,m}(x) = k_{E,m}([a, [c_x]]) = j(c_x^*)(a) = j(x)$, where the latter equality holds as shown in Claim 1.3.37. So left to show is elementarity. This follows easily from the commutativity and elementarity of the other maps involved: $m_E \models \varphi([a, [f]])$ iff $m_a \models \varphi([f])$ iff $N \models \varphi(k_{a,m}([f]))$ iff $N \models \varphi(j(f^*)(a))$ iff $N \models \varphi(k_{E,m}([a, [f]]))$. □

The main results about correspondence of j with $j_{E,m}$ are exactly as in the class case, summarised by Theorem 1.3.36 above. The proof goes completely analogous, (cf., e.g., [Kan03, Theorem 26.1]), substituting having to deal with coded functions f^* at the appropriate places.

Theorem 1.3.39. The following hold:

(1) $m_E = \{j_{E,m}(f^*)(a) \colon a \in [\lambda]^{<\omega} \text{ and } f : [\zeta]^{|a|} \to V_\alpha\}$.

(2) $\mathrm{crit}(j_{E,m}) = \kappa$ and $j_{E,m}(\zeta) \geq \lambda$.

(3) $k_{E,m} \restriction \lambda = \mathrm{id}$.

(4) If $j(\zeta) = \lambda$, then $j_{E,m}(\zeta) = \lambda$ and $k_{E,m}(\lambda) = \lambda$.

(5) $k_{E,m}$ is the inverse collapsing isomorphism.

(6) For any γ such that $|V_\gamma^N|^N \leq \lambda$: $V_\gamma^N \subseteq \mathrm{ran}(k_{E,m})$, $V_\gamma^{m_E} = V_\gamma^N$ and $k_{E,m}(x) = x$ for $x \in V_\gamma^{m_E}$.

Notice the unusual statement of assertion (3), which is not talking about critical points. The reason is that in the set-sized case, $k_{E,m}$ might simply not move any ordinal at all. Let, for example, $\zeta = \alpha$ and $j : V_{\alpha+1} \to V_{\beta+1}$ be such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha + 1$, i.e., j is an $(\alpha+1)$-extendibility embedding. Derive the (κ, β)-extender $(E_a \colon a \in [\beta]^{<\omega})$ consisting of ultrafilters E_a over $[\alpha]^{|a|}$. Then considering the extender power m_E of $V_{\alpha+1}$, we get $j_{E,m}(\alpha) = j(\alpha) = \beta$. And therefore with $k_{E,m} : m_E \to V_{\beta+1}$ we have $k_{E,m} \restriction \beta + 1 = \mathrm{id}$ (and, clearly, m_E cannot contain any ordinal $> \beta$).

Now we want to connect the embedding $j_{E,m}$ with the extender power of the universe $j_E : V \to M_E$. Let us first note that for $a \in [\lambda]^{<\omega}$ and $X \subseteq [\zeta]^{|a|}$, we have that $a \in j_{E,m}(X)$ iff $k_{E,m}(a) \in k_{E,m} \circ j_{E,m}(X)$ iff $a \in j(X)$. In particular, deriving an extender from $j_{E,m}$ again results in $(E_a \colon a \in [\lambda]^{<\omega})$. We already showed that with $M_a = \mathrm{Ult}(V, E_a)$ and $j_a : V \to M_a$ the usual map, we have that $V_{j_a(\alpha)}^{M_a} = m_a$ and that $j_a \restriction V_\alpha = j_{a,m}$. Note further that $V_{j_E(\alpha)}^{M_E}$ is the direct limit of the $V_{j_a(\alpha)}^{M_a}$. This holds because

$$\begin{aligned} M_E \models [a,[f]] \in V_{j_E(\alpha)} &\text{ iff } M_E \models k_{aE}([f]) \in V_{k_{aE} \circ j_a(\alpha)} \\ &\text{ iff } M_a \models [f] \in V_{j_a(\alpha)}. \end{aligned}$$

As $V_{j_a(\alpha)}^{M_a} = m_a$ and m_E is the direct limit of the m_a, this means that $V_{j_E(\alpha)}^{M_E} = m_E$. Hence, $m_E \subseteq M_E$. Finally, if $x \in V_\alpha$, then $j_E(x) = [a, [c_x]] = j_{E,m}(x)$ and so $j_E \restriction V_\alpha = j_{E,m}$.

We summarise the situation in the following theorem.

Theorem 1.3.40. Suppose there is an elementary embedding $j : V_\alpha \to M$ with a critical point $\mathrm{crit}(j) = \kappa$, for some ordinal λ there is a smallest cardinal $\zeta \geq \kappa$ such that $j(\zeta) \geq \lambda$ and α is a successor ordinal or of cofinality $\mathrm{cof}(\alpha) > \zeta$. Let E be the (κ, λ)-extender derived from j. Consider the canonical extender power embeddings $j_{E,m} : V_\alpha \to m_E$ and $j_E : V \to M_E$. Then:

(1) $m_E \subseteq M_E$ and $j_E \restriction V_\alpha = j_{E,m}$.

(2) $\mathrm{crit}(j_{E,m}) = \mathrm{crit}(j_E) = \kappa$.

(3) ζ is the smallest ordinal such that $j_{E,m}(\zeta) = j_E(\zeta) \geq \lambda$.

(4) If $j(\zeta) = \lambda$, then $j_{E,m}(\zeta) = j_E(\zeta) = \lambda$.

(5) The (κ, λ)-extenders derived from $j_{E,m}$ and from j_E are both again E.

2. Henkin-Compactness Properties

Remarks on co-authorship. The results of Sections 2.2 and 2.3.4 are joint with Alejandro Poveda and appear in [OP24]. The results of Section 2.3.2 are joint with Will Boney.

2.1. Introduction

In [Bon20], Will Boney characterised strong cardinals by a compactness property involving Henkin models of second-order theories. In [BDGM24], this served as motivation to introduce the notion of a Henkin model for an arbitrarily given strong logic $\mathcal{L}$ and the authors used a compactness property involving this concept to characterise Woodin cardinals. The Henkin models considered in this result exhibit some specificities, which lead to a natural strengthening of the concept of a Henkin model we will introduce. To distinguish between the two, we will call the two notions *weak* and *strong* Henkin models, respectively (cf. Definitions 2.2.1 and 2.2.3). The chapter is structured as follows. In Section 2.2, we consider compactness properties involving strong Henkin models, and in Section 2.3, we consider compactness properties involving weak Henkin models.

The relevant background and definitions for Henkin models are discussed in Section 2.2.1. We proceed to consider how compactness properties involving strong Henkin models of $\mathcal{L}^2$ can be used to characterise supercompact cardinals (Section 2.2.2). We then introduce a compactness property involving strong Henkin models called *n-strong-Henkin-compactness number* (n-SHC *number*), for some natural number n (Section 2.2.3). We show that the existence of n-SHC numbers of the sort logics $\mathcal{L}^{s,n}$ provides another stratification of Vopěnka's Principle corresponding to the hierarchy of $C^{(n)}$-extendible cardinals. In particular, Vopěnka's Principle is equivalent to the existence of n-SHC numbers for all logics.

In Section 2.3, we continue the study of compactness properties involving weak Henkin models. We study what is known as *weak Vopěnka's Principle*, a weakening of Vopěnka's Principle motivated by a category-theoretic formulation of the latter. Bagaria and Wilson in [BW23] provided an analysis of weak Vopěnka's Principle, showing that it has a stratification by so-called Π_n*-strong cardinals*, which is completely analogous to the analysis of the usual Vopěnka's Principle in terms of $C^{(n)}$-extendible cardinals. The main result of Section 2.3 is that compactness properties for weak Henkin models are able to characterise Π_n-strong cardinals and thus give rise to a stratification of weak Vopěnka's Principle in model-theoretic terms. The compactness properties we will employ are further a direct weakening of the ones used to characterise $C^{(n)}$-extendibles in Section 2.2.

The relevant definitions of weak Vopěnka's Principle and Π_n-strong cardinals are presented in Section 2.3.1. The main results on characterisations of Π_n-strong cardinals and weak Vopěnka's Principle can be found in Section 2.3.2. Finally, we present some further applications of compactness properties for weak Henkin models, by characterising cardinals which are jointly Π_n-strong and strongly compact (Section 2.3.3), and superstrong cardinals (Section 2.3.4).

2.2. Strong Henkin models

2.2.1. Motivation and definitions

Recall the *Henkin semantics* of a second-order sentence $\varphi \in \mathcal{L}^2[\tau]$. Given some τ-structure $\mathcal{A}$ and a subset $P \subseteq \mathcal{P}(A)$, we say that the pair $(\mathcal{A}, P)$ is a *Henkin model* of φ, if when restricting the second-order quantifiers of φ to range over P (in contrast to over the full power set of A), $\mathcal{A}$ computes to be a model of φ. Now note that if M is some transitive set such that $\mathcal{A} \in M$ and

$$M \models \text{“}\mathcal{A} \models_{\mathcal{L}^2} \varphi\text{”},$$

then $(\mathcal{A}, \mathcal{P}^M(A))$ is a Henkin model of φ. Thus, having a Henkin model of some second-order sentence is similar to evaluating the truth of "$\mathcal{A} \models_{\mathcal{L}^2} \varphi$" in some transitive set that might not contain the full power set of A.

This observation served as motivation in [BDGM24] to generalise the notion of Henkin model to arbitrary strong logics $\mathcal{L}$. We present a simplified version of the notion considered there. Recall from Definition 1.2.1 that a copy T^* of some $\mathcal{L}$-theory T over a vocabulary τ is the image of T under the renaming of $\mathcal{L}$-sentences induced by some renaming $f : \tau \to \tau^*$.

Definition 2.2.1. Let $\mathcal{L}$ be a logic, τ a vocabulary, $T \subseteq \mathcal{L}[\tau]$ an $\mathcal{L}$-theory, M a transitive set and $\mathcal{A} \in M$. Then $(M, \mathcal{A})$ is called a *weak $\mathcal{L}$-Henkin model of T* iff there is a copy T^* of T such that for any $\varphi \in T^*$:

$$M \models \text{“}\mathcal{A} \models_{\mathcal{L}} \varphi\text{”}.$$

We call these models *weak*, to distinguish them from the stronger ones we will consider below.

Recall the notion of a strong cardinal. For $\kappa < \lambda$, a cardinal κ is called λ-*strong* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. It is called *strong* if it is λ-strong for every $\lambda > \kappa$.

Boney in [Bon20] used a compactness property involving the classical Henkin models for second-order logic to characterise strong cardinals. To restate his theorem in terms of weak Henkin models, let us make a few remarks. Notice that our notion of weak $\mathcal{L}$-Henkin model imposes basically no restrictions on the transitive set M involved. In practice, we always want M to satisfy some additional properties. In particular, we need that M is a model of enough of set theory that it interacts somewhat nicely with the logic $\mathcal{L}$ in question. For this purpose, let us fix a finite fragment ZFC^* of (consequences of) ZFC, which in particular includes the statements that V_α exists for every ordinal α, that every set is contained in some V_α, and that Magidor's Φ^* (cf. Lemma 1.2.4) is true precisely in the structures (M, E) isomorphic to some $(V_\alpha, \in)$, and such that ZFC proves that any V_β with β a limit ordinal satisfies ZFC*. Boney's result (originally phrased with reference to the classical Henkin semantics) can now be cast in the following way:

Theorem 2.2.2 (Boney [Bon20, Theorem 4.7])**.** The following are equivalent for a cardinal κ and $\beth_\lambda = \lambda > \kappa$:

(1) κ is λ-strong.

(2) For any theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ that can be written as an increasing union $T = \bigcup_{\alpha\in\kappa} T_\alpha$ of theories T_α which each have a model of size $\geq \kappa$, there is a weak $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models$ ZFC*, $V_\lambda \subseteq M$ and $|A| \geq \lambda$.

Note that in the notion of a weak Henkin model, it is from the outside that we see that $M \models$ "$\mathcal{A} \models_{\mathcal{L}} \varphi$" for every φ in (a copy of) some theory T, while M itself might not have access to T provided $T \notin M$. It is thus natural to ask about strengthening Henkin models in the following way.

Definition 2.2.3. Let $\mathcal{L}$ be a logic, τ a vocabulary, $T \subseteq \mathcal{L}[\tau]$ an $\mathcal{L}$-theory, M a transitive set and $\mathcal{A} \in M$. Then the pair $(M, \mathcal{A})$ is called a *strong $\mathcal{L}$-Henkin model of* T iff $T \in M$ and

$$M \models \text{“}\mathcal{A} \models_{\mathcal{L}} T\text{”}.$$

We will study this stronger notion and show that it indeed can be used to provide characterisations of much stronger cardinals.

2.2.2. Supercompact cardinals

The typical use case of Henkin models is that T is some elementary diagram of some structure $\mathcal{B}$, and that a Henkin model $(M, \mathcal{A})$ of T gives rise to an elementary embedding $\mathcal{B} \to \mathcal{A}$. In the case of weak Henkin models, it is only in V that we can compute this embedding. But in the case of strong Henkin models, M has access to T, and so the elementary embedding lives already in M. This, combined with considering fully $<\kappa$-satisfiable theories, comes along with a jump up in large cardinal strength.

Theorem 2.2.4. The following are equivalent:

(1) κ is supercompact.

(2) For every λ, if $T \subseteq \mathcal{L}^2_{\kappa\omega}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models$ ZFC* and $V_\lambda \subseteq M$.

(3) For every λ, if $T \subseteq \mathcal{L}^2_{\kappa\kappa}$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2_{\kappa\kappa}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models$ ZFC*, $V_\lambda \subseteq M$ and $M^\lambda \subseteq M$.

Proof. Clearly (3) implies (2). We first show that (1) implies (3). So let T be a $<\kappa$-satisfiable $\mathcal{L}^2_{\kappa\kappa}$-theory. Take a $\beth$-fixed point λ of cofinality at least κ such that $V_\lambda \models \mathrm{ZFC}^*$ and large enough such that $T \in V_\lambda$ and V_λ has a model for every $< \kappa$-sized subset of T. By supercompactness, let $j : V \to N$ be elementary with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $N^\lambda \subseteq M$. Notice that the restriction $i = j \restriction V_\lambda : V_\lambda \to V^N_{j(\lambda)}$ is an elementary embedding; this implies $V^N_{j(\lambda)} \models \mathrm{ZFC}^*$. Because V_λ believes that T is $<\kappa$-satisfiable, by elementarity, $V^N_{j(\lambda)} \models$ "$i(T)$ is $<i(\kappa)$-satisfiable". Because $\lambda = \beth_\lambda$ and $T \in V_\lambda$, $|T| < \lambda$. Thus, by N's closure, $i``T \in N$ and $|i``T|^N < \lambda < i(\kappa)$. Furthermore, $i(T) \in V^N_{j(\lambda)}$ and so also $i``T \in V^N_{j(\lambda)}$ and $V^N_{j(\lambda)} \models |i``T| < \lambda$. Therefore $V^N_{j(\lambda)}$ believes that there is a model $\mathcal{B} \models i``T$. Because $\mathrm{crit}(i) = \kappa$ and $T \subseteq \mathcal{L}^2_{\kappa\kappa}$, with the renaming $i : \tau \to i``\tau$, we have that $i``T$ is a copy of $T$. Note that by closure under $\lambda$-sequences, $N$ and hence $V^N_{j(\lambda)}$ knows about the renamings $i : \tau \to i``\tau$ and $i : T \to i``T$. We can therefore rename $\mathcal{B}$ in $V^N_{j(\lambda)}$ to a τ-structure $\mathcal{A}$, which $V^N_{j(\lambda)}$ believes to satisfy T. Notice that $V_\lambda \subseteq V^N_{j(\lambda)}$. Further, $\mathrm{cof}(j(\lambda))^N \geq j(\kappa) > \lambda > \kappa$. By closure of N, this implies that $V^N_{j(\lambda)}$ is λ-closed. Summarising, $(V^N_{j(\lambda)}, \mathcal{A})$ is a strong Henkin model as desired.

Now assume (2) and let us show (1). Take a cardinal $\lambda > \kappa$ of $\mathrm{cof}(\lambda) \geq \kappa$. Consider the theory

$$T = \mathrm{ElDiag}_{\mathcal{L}^2_{\kappa\omega}}(V_{\lambda+1}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\},$$

where d is a new constant and the c_i are the constants used in the elementary diagram. If $T_0 \subseteq T$ is of size $< \kappa$, there is $X \subseteq \lambda$ such that $|X| < \kappa$ and the sentence "$c_i \in d \wedge |d| < c_\kappa$" is contained in T_0 iff $i \in X$. Then letting $d = X$, we get that $(V_{\lambda+1}, \in, d)$ witnesses that T_0 is satisfiable. So by (2), we get a transitive set M satisfying ZFC^* such that $V_\alpha \subseteq M$ for some large $\alpha > \lambda$ and $\mathcal{A} \in M$ such that $M \models$ "$\mathcal{A} \models T$". We may take α large enough such that $T \in V_\alpha$. Note that T is a theory in a language $\tau = \{\in, c_x, d \colon x \in V_{\lambda+1}\}$. Because with T, also $\tau \in V_\alpha$, and thus also the structure $N = (V_{\lambda+1}, \in, c^N_x)_{x \in V_{\lambda+1}}$ in which every c_x is interpreted by x itself, and which witnesses that $(V_{\lambda+1}, \in)$ satisfies its own elementary diagram, is in V_α. Hence, this structure is also in M. Because first-order satisfaction is absolute between M and V, M understands that T contains the elementary diagram of $(V_{\lambda+1}, \in)$ and therefore believes that there is an elementary embedding $j : V_{\lambda+1} \to \mathcal{A}$. Again, by absoluteness of first-order satisfaction, this is really an elementary embedding. Note that T contains Magidor's Φ^*, as clearly $V_{\lambda+1}$ is isomorphic to some V_α. Together with $M \models \mathrm{ZFC}^*$, this implies that M believes $\mathcal{A}$ to be some rank-initial segment and so we have to have $A = V^M_{\beta+1}$ for some β. Because $c_i^{\mathcal{A}} \in d^{\mathcal{A}}$ for every $i < \lambda$, we get that $j(\kappa) > |d|^{\mathcal{A}} \geq \lambda$. In particular, $\mathrm{crit}(j) \leq \kappa$. Because also $\mathcal{L}_{\kappa\omega}$-satisfaction is absolute for transitive models and $\mathcal{L}_{\kappa\omega}$ can define all ordinals $< \kappa$, those have to be fixed by j. Thus $\mathrm{crit}(j) = \kappa$. Note that $j``\lambda$ is definable from $j$ and $\lambda$ and so $j``\lambda \in M$ and therefore in $V^M_{\beta+1}$. Summarising, we have an elementary embedding $j : V_{\lambda+1} \to V^M_{\beta+1}$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j``\lambda \in V^M_{\beta+1}$. We can therefore let, for $X \subseteq \mathcal{P}_\kappa\lambda$:

$$X \in U \text{ iff } j``\lambda \in j(X).$$

It is standard to check that this defines a fine, normal and κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$. To check normality, for example, if f is a regressive function on $\mathcal{P}_\kappa\lambda$ and so $\{s \in \mathcal{P}_\kappa\lambda \colon f(s) \in s\} \in U$. Then $j``\lambda \in \{s \in \mathcal{P}_{j(\kappa)}j(\lambda) \colon j(f)(s) \in s\}$ and hence $j(f)(j``\lambda) = j(\gamma)$ for some $\gamma < \lambda$. Therefore $\{s \in \mathcal{P}_\kappa\lambda \colon f(s) = \gamma\} \in U$. Hence κ is λ-supercompact for arbitrarily large λ. □

We may also characterise the smallest supercompact cardinal, by a slight adaptation to the above proof.

Theorem 2.2.5. The following are equivalent for any cardinal κ:

(1) κ is the smallest supercompact cardinal.

(2) κ is the smallest cardinal such that for every λ, if $T \subseteq \mathcal{L}^2$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$ and $V_\lambda \subseteq M$.

(3) κ is the smallest cardinal such that for every λ, if $T \subseteq \mathcal{L}^2$ is a $<\kappa$-satisfiable theory, then there is a strong $\mathcal{L}^2$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$, $V_\lambda \subseteq M$ and $M^\lambda \subseteq M$.

Proof. If κ is supercompact, by Theorem 2.2.4 we know that property (3) holds with respect to $T \subseteq \mathcal{L}^2_{\kappa\omega}$ and thus in particular for $T \subseteq \mathcal{L}^2$. Thus it is sufficient to show that if κ is as in (2), then there is a supercompact γ such that $\gamma \leq \kappa$. Let $\lambda > \kappa$ be of $\mathrm{cof}(\lambda) \geq \kappa$. Consider

$$T = \mathrm{ElDiag}_{\mathcal{L}^2}(V_{\lambda+1}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\}.$$

By the same argument as in the proof before before, we get an M such that for some α much larger than λ, we have $T \in V_\alpha$, $V_\alpha \subseteq M$, and $M \models$ "$\mathcal{A} \models T$". And further, we get an elementary embedding $j_\lambda : V_{\lambda+1} \to \mathcal{A}$ such that $A = V^M_{\beta+1}$. Again, our theory implies that $j_\lambda(\kappa) > \lambda$ and $j``\lambda \in V^M_{\beta+1}$. But this time, we do not have $\mathcal{L}_{\kappa\omega}$ at our disposal and so we only get $\mathrm{crit}(j_\lambda) \leq \kappa$. Nevertheless, this argument shows that for a proper class of λ, we have an elementary embedding as above with $\mathrm{crit}(j_\lambda) \leq \kappa$. As there are only κ-many possible values for $\mathrm{crit}(j_\lambda)$, there is a fixed γ which is the critical point of j_λ for a proper class of λ. Then let for $X \subseteq \mathcal{P}_\gamma\lambda$, $X \in U$ iff $j``\lambda \in j(X)$. This U can be shown to be a fine, normal, γ-complete ultrafilter U over $\mathcal{P}_\gamma\lambda$ by standard arguments. Then $\gamma \leq \kappa$ is supercompact. □

2.2.3. $C^{(n)}$-extendible cardinals and Vopěnka's Principle

In this section we will show that an analogue of Theorem 2.2.4 for sort logics provides the hierarchy of $C^{(n)}$-extendible cardinals, and thus another stratification of VP. To show this, we will use the following characterisation of $C^{(n)}$-extendibility provided by Bagaria and Goldberg.

Theorem 2.2.6 (Bagaria & Goldberg [BG24, Theorem 2.6])**.** The following are equivalent for any cardinal κ and any natural number $n \geq 1$:

(1) κ is $C^{(n)}$-extendible.

(2) For every $\lambda > \kappa$, $\lambda \in C^{(n+1)}$, there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $M \models$ "$\lambda \in C^{(n+1)}$".

(3) For every $\lambda > \kappa$, $\lambda \in C^{(n+1)}$, there is a fine, normal and κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda\colon \mathrm{ot}(s) \in C^{(n+1)}\} \in U$.

Recall that $\mathrm{VP}(\Pi_1)$ is equivalent to the existence of a supercompact cardinal, and that $\mathrm{VP}(\Pi_{n+1})$ is equivalent to the existence of a $C^{(n)}$-extendible cardinal (cf. Section 1.3.6). In this vein, $C^{(n)}$-extendible cardinals can be seen as a generalisation of supercompact cardinals. Bagaria's and Goldberg's theorem explains this observation in terms of large cardinals, without the need to go through the equivalence to fragments of VP. Extendibility and $C^{(n)}$-extendibility can hence be understood as direct strengthenings of supercompactness by simply adding correctness assumptions about the target of supercompact embeddings.

Similarly, the characterisation of supercompactness by compactness properties with $\mathcal{L}^2$-Henkin models as in Theorem 2.2.4 leads to $C^{(n)}$-extendibility when considering $\mathcal{L}^{\mathrm{s},n}$-Henkin models (with some natural extra assumptions) instead. For this purpose, for each natural number $n > 1$, fix a finite fragment ZFC^*_n of (consequences of) ZFC containing the sentence that $\Phi^{(n)} \in \mathcal{L}^{\mathrm{s},n}$ is true in precisely those models which are isomorphic to some V_α for $\alpha \in C^{(n)}$ (cf. Corollary 1.2.17), and the sentence $\Phi^{\mathrm{s},*} \in \mathcal{L}^{\mathrm{s},n}$, which is true in precisely those models which are isomorphic to some V_α for α any ordinal (cf. Corollary 1.2.18). For uniformity of notation, let $\mathrm{ZFC}^*_1 = \mathrm{ZFC}^*$ be the fragment introduced in Section 2.3.1.

We can now define the property we consider and prove our theorem.

Definition 2.2.7. Let $\mathcal{L}$ be a logic, $n \geq 1$ a natural number, and κ a cardinal. We say that κ is an *n-strong-Henkin-compactness* (n-SHC) *number* of $\mathcal{L}$ iff for any $\lambda \in C^{(n)}$ and any $<\kappa$-satisfiable theory $T \subseteq \mathcal{L}$ there is a strong $\mathcal{L}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*_n$ and $V_\lambda \prec_{\Sigma_n} M$.

Theorem 2.2.8. The following are equivalent for any cardinal κ and any natural number $n \geq 1$:

(1) κ is $C^{(n)}$-extendible.

(2) κ is an $(n+1)$-SHC number of $\mathcal{L}^{\mathrm{s},n+1}_{\kappa\omega}$.

Proof. The proof proceeds similar to the supercompactness case. Assume (1) and let us show (2). Let T be $<\kappa$-satisfiable. Take $\lambda = \beth_\lambda \in C^{(n+1)}$ of $\mathrm{cof}(\lambda) \geq \kappa$, such that $V_\lambda \models \mathrm{ZFC}^*_{n+1}$ and large enough such that V_λ verifies that T is $<\kappa$-satisfiable. By Theorem 2.2.6, take $j : V \to N$ with $\mathrm{crit}(j) = \kappa$, $N^\lambda \subseteq N$ and $N \models \lambda \in C^{(n+1)}$. Again, $i = j \restriction V_\lambda : V_\lambda \to V^N_{j(\lambda)}$ is elementary. In particular, $V^N_{j(\lambda)} \models \mathrm{ZFC}^*_{n+1}$; further

$V^N_{j(\lambda)} \models$"$i(T)$ is $<i(\kappa)$ satisfiable" and so $V^N_{j(\lambda)}$ has a model $\mathcal{B}$ for the copy i"T. As earlier, by closure of N, this can be renamed to a τ-structure $\mathcal{A} \in V^N_{j(\lambda)}$ which $V^N_{j(\lambda)}$ believes to satisfy T. Again, $V_\lambda \subseteq V^N_{j(\lambda)}$. Finally, if $a \in V_\lambda$ and $\Phi(x)$ is a Σ_{n+1} formula:

$$\begin{aligned} V_\lambda \models \Phi(a) &\text{ iff } N \models \Phi(a) \\ &\text{ iff } V^N_{j(\lambda)} \models \Phi(a). \end{aligned}$$

Here the first "iff" holds as $N \models \lambda \in C^{(n+1)}$, and the second one because really $\lambda \in C^{(n+1)}$ and so by elementarity, $N \models j(\lambda) \in C^{(n+1)}$. The equivalence shows that $V_\lambda \prec_{\Sigma_{n+1}} V^N_{j(\lambda)}$ and so, summarising, $(V^N_{j(\lambda)}, \mathcal{A})$ is a strong Henkin model as desired.

Now assume (2), let $\lambda > \kappa$ be in $C^{(n+1)}$, and consider

$$T = \mathrm{ElDiag}_{\mathcal{L}^{s,n+1}_{\kappa\omega}}(V_{\lambda+1}, \in) \cup \{c_i \in d \wedge |d| < c_\kappa \colon i < \lambda\}.$$

Again, for $<\kappa$-satisfiable theories of T, we can get a model by considering $V_{\lambda+1}$ itself. So for some $\alpha \in C^{(n+1)}$ much greater than λ and such that $T \in V_\alpha$, by assumption we get an $M \models \mathrm{ZFC}^*_{n+1}$ such that $V_\alpha \prec_{\Sigma_{n+1}} M$ and there is $\mathcal{A} \in M$ which M believes to be a model of T. As before, M has a first-order elementary embedding $j : V_{\lambda+1} \to \mathcal{A}$. Note that $V_\lambda \models \Phi^{(n+1)}$ and so $V_{\lambda+1}$ satisfies the relativisation of $\Phi^{(n+1)}$ to V_λ, i.e., a sentence coding that $\Phi^{(n+1)}$ holds in the rank initial segment cut off at the largest ordinal λ of $V_{\lambda+1}$. Then M believes that this sentence holds in $V^M_{\beta+1}$ and so $M \models \beta \in C^{(n+1)}$. Again, our theory implies that $j(\kappa) > \lambda$ and because j"λ is definable in M, we have j"$\lambda \in V^M_{\beta+1}$. Summarising, we have an elementary embedding $j : V_{\lambda+1} \to V^M_{\beta+1}$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and j"$\lambda \in V^M_{\beta+1}$. Define a fine, normal and κ-complete ultrafilter on $\mathcal{P}_\kappa\lambda$ as usual, by letting $X \in U$ iff j"$\lambda \in j(X)$. By Theorem 2.2.6, it suffices to verify that $X = \{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in C^{(n+1)}\} \in U$. Note that because $\lambda \in C^{(n+1)}$ and $\mathrm{cof}(\lambda) \geq \kappa$, for $s \in \mathcal{P}_\kappa\lambda$ we have

$$V_{\lambda+1} \models \forall s \in \mathcal{P}_\kappa\lambda(s \in X \leftrightarrow V_\lambda \models \text{"}\mathrm{ot}(s) \in C^{(n+1)}\text{"}).$$

By elementarity,

$$V^M_{\beta+1} \models \forall s \in \mathcal{P}_{j(\kappa)}j(\lambda)(s \in j(X) \leftrightarrow V^M_\beta \models \text{"}\mathrm{ot}(s) \in C^{(n+1)}\text{"}).$$

So we have to show that $V^M_\beta \models$ "$\lambda = \mathrm{ot}(j$"$\lambda) \in C^{(n+1)}$". Because $M \models$"$\beta \in C^{(n+1)}$", this is equivalent to $M \models$"$\lambda \in C^{(n+1)}$". Recall that $C^{(n+1)}$ is Π_{n+1} definable. As really $\alpha \in C^{(n+1)}$, and $\alpha > \lambda \in C^{(n+1)}$, hence $V_\alpha \models$ "$\lambda \in C^{(n+1)}$". Because $V_\alpha \prec_{\Sigma_{n+1}} M$ by assumption, this implies $M \models$ "$\lambda \in C^{(n+1)}$", verifying $X \in U$. □

We would like to remark that extra assumptions on a Henkin model $(M, \mathcal{A})$, like $V_\lambda \subseteq M$ in Theorem 2.2.4 and $V_\lambda \prec_{\Sigma_{n+1}} M$ in Theorem 2.2.8, are crucial, if $(M, \mathcal{A})$ is supposed to be useful. The general definition of a (weak or strong) $\mathcal{L}$-Henkin model only requires there to be some transitive set M that believes in a structure satisfying some $\mathcal{L}$-sentences. However, M might be very incorrect about the logic $\mathcal{L}$. The assumptions on the Henkin models as above are natural to provide some partial correctness of M.

As we did with Theorem 2.2.5, it is easy to adapt the argument for Theorem 2.2.8 to characterise the smallest $C^{(n)}$-extendible cardinal.

Theorem 2.2.9. The following are equivalent for every cardinal κ and every natural number $n \geq 1$:

(1) κ is the smallest $C^{(n)}$-extendible cardinal.

(2) κ is the smallest n-SHC number of $\mathcal{L}^{\mathrm{s},n+1}$.

We get the following results about VP and its local forms.

Corollary 2.2.10. The following are equivalent for every $n \geq 2$.

(1) $\mathrm{VP}(\Pi_n)$

(2) $\mathcal{L}^{\mathrm{s},n}$ has an n-SHC number.

Proof. This follows immediately from Theorems 2.2.9 and 1.3.30. □

Note that in our characterisation of supercompactness in Theorem 2.2.5, we might add in conditions (2) and (3), that if $(M, \mathcal{A})$ is the strong Henkin model provided, then for $\lambda \in C^{(1)}$ also $V_\lambda \prec_{\Sigma_1} M$. The reason for this is that if $\lambda \in C^{(1)}$ and $j : V \to N$ is a λ-supercompactness embedding, then $\lambda \in (C^{(1)})^N$. Hence, combined with Theorem 1.3.32 our argument shows:

Corollary 2.2.11. The following are equivalent:

(1) $\mathrm{VP}(\Pi_1)$.

(2) $\mathcal{L}^2$ has a 1-SHC number.

Finally, we can characterise the global notion.

Corollary 2.2.12. The following are equivalent:

(1) VP.

(2) For every logic $\mathcal{L}$ and every natural number n, there is an n-SHC number of $\mathcal{L}$.

Proof. Assume that VP holds and let $\mathcal{L}$ be any logic. Then $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$ for some κ and some $n > 1$ (cf. Corollary 1.2.24). Because VP holds, there is a proper class of $C^{(n-1)}$-extendible cardinals, so there is some $C^{(n-1)}$-extendible cardinal $\delta \geq \kappa$. By Theorem 2.2.8, δ is an n-SHC number for $\mathcal{L}^{\mathrm{s},n}_{\delta\omega}$ and thus in particular for $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega} \leq \mathcal{L}^{\mathrm{s},n}_{\delta\omega}$.

And if (2) holds, then in particular, every $\mathcal{L}^{\mathrm{s},n}$ has an n-SHC number. By Theorem 2.2.9, then for every n there is a $C^{(n)}$-extendible cardinal. Hence, VP holds. □

Figure 2.1 updates Figure 1.1 from Section 1.3.6 by the new properties we considered.

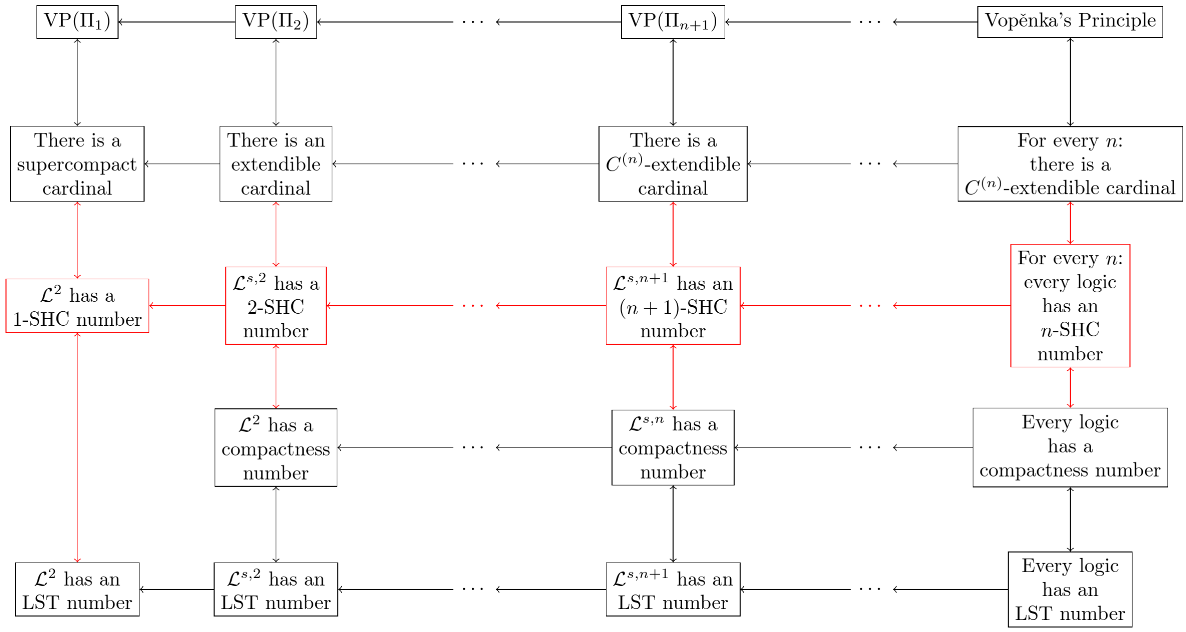

Figure 2.1.: Relations between VP, $C^{(n)}$-extendible cardinals, SHC numbers, compactness numbers, and LST numbers.

2.3. Weak Henkin models

2.3.1. Motivation and definitions

Vopěnka's Principle is equivalent to the existence of large cardinals and to several model-theoretic statements, in particular to the existence of strong-Henkin-compactness cardinals (cf. Theorem 2.2.12). And, as a finely grained analysis, we saw that the stratification of VP along the Lévy hierarchy corresponds precisely to stratifications of its large cardinal formulation in form of the $C^{(n)}$-extendible cardinals (Theorem 1.3.30), and to stratifications of its model-theoretic formulation in form of compactness cardinals (Theorem 1.3.34), and n-SHC numbers for sort logics (Corollary 2.2.10), respectively. On the other hand, it has been known for a long time that Vopěnka's Principle has consequences for category theory. In [AR94, Chapter 6], Adámek and Rosický show for a long list of statements important in category theory that they are equivalent to Vopěnka's Principle. For instance, it is equivalent to the statement "The locally presentable categories are precisely the complete and bounded categories" and to many more. As they write, "assuming [...] Vopěnka's Principle, the structure of locally presentable categories becomes much more transparent" ([AR94, p. 241]).

The category-theoretic formulations of Vopěnka's Principle motivated consideration of an axiom called *weak Vopěnka's Principle*. Its precise consistency strength was unknown for a long time until this question recently got solved by Trevor Wilson (cf. [Wil22b] and Theorem 2.3.5 below). As we will see, Bagaria and Wilson showed that similar to the situation with Vopěnka's Principle, also weak Vopěnka's Principle can be stratified along the existence of certain large cardinals called Π_n-*strong* cardinals, for natural numbers n. This left open whether we can push this analogy between the structure of Vopěnka's Principle and weak Vopěnka's Principle further and provide a model-theoretic characterisation of weak Vopěnka's Principle. Our goal is to show that the answer is positive. We will see that analogously to how below VP the $C^{(n)}$-extendible cardinals correspond to Henkin-compactness-properties of logics, the Π_n-strong cardinals correspond to (different) Henkin-compactness-properties of logics as well. Note that independently Holy, Lücke, and Müller in [HLM24, Theorem 1.7] provide a characterisation of the axiom schema "Ord is Woodin" (and thus by Wilson's Theorem 2.3.5 of weak Vopěnka's Principle) in terms of model-theoretic properties of logics (though the properties they consider are entirely different ones than those we will employ).

We will first give some background on weak Vopěnka's Principle and Π_n-strong cardinals before presenting our results in Section 2.3.2. To motivate the formulation of weak Vopěnka's Principle, let us consider some of the category-theoretic formulations of Vopěnka's Principle, without the need to give precise definitions of the involved notions. For example, two of the statements that were singled out to be equivalent to Vopěnka's Principle (cf. [AR94, Section 6.D and Lemma 6.3]) are:

(1) If $\mathcal{K}$ is a locally presentable category, then all of its full subcategories closed under colimits are coreflective in $\mathcal{K}$.

(2) The category Ord of ordinals does not fully embed into the category of graphs, i.e., if $(G_\alpha \colon \alpha \in \mathrm{Ord})$ is a sequence of graphs such that for any $\alpha \leq \beta$ there is exactly one homomorphism $G_\alpha \to G_\beta$, then there are ordinals $\alpha < \beta$ and a homomorphism $G_\beta \to G_\alpha$.

Here the category Ord has as objects the ordinals, and as morphisms the initial segment relation $\leq$. Recall that in category theory, the *dual* category $\mathcal{C}^{\mathrm{op}}$ of some category $\mathcal{C}$ is obtained by taking the same collection of objects and (informally speaking) reversing all morphisms. In the case of Ord, the dual category $\mathrm{Ord}^{\mathrm{op}}$ thus consists of all ordinals with the reverse initial segment relation $\geq$. Other category-theoretic notions can be dualised in similar ways. It was shown (cf. [AR94, Section 6.D]) that the following natural variants of the statements (1) and (2) about dual notions are also equivalent to each other:

(a) If $\mathcal{K}$ is a locally presentable category, then all of its full subcategories closed under limits are reflective in $\mathcal{K}$.

(b) The category $\mathrm{Ord}^{\mathrm{op}}$ does not fully embed into the category of graphs, i.e., if $(G_\alpha \colon \alpha \in \mathrm{Ord})$ is a sequence of graphs such that for any $\alpha \leq \beta$ there is exactly one homomorphism $G_\beta \to G_\alpha$, then there are ordinals $\alpha < \beta$ and a homomorphism $G_\alpha \to G_\beta$.

In [ART88], the authors proved that Vopěnka's Principle implies (b), and asked whether the converse holds. So the name weak *weak* Vopěnka's Principle was given to the statement (b) (or equivalently to (a)). The question remained open until Wilson in a series of articles first showed that weak Vopěnka's Principle does not imply Vopěnka's Principle (cf. [Wil20]), and then that the former is equivalent to the large cardinal axiom

"Ord is Woodin" (cf. [Wil22b] and Theorem 2.3.5), and therefore strictly weaker than Vopěnka's Principle, both in terms of direct implication and of consistency strength.

The above analyses, showing relations between category theory, versions of Vopěnka's Principle, and large cardinals, are naturally carried out using a class theory like GBC. In these terms, Vopěnka's Principle is not interpreted as an axiom schema, but as a single statement positing the existence of elementary embeddings in every class by using quantification over proper classes. Similarly, the statements about categories, about sequences of graphs indexed by the ordinals, and "Ord is Woodin" can be interpreted in such a theory. Contrastingly, in [BW23] Bagaria and Wilson carried out an analysis of weak Vopěnka's Principle for definable classes, under ZFC alone. In the following, we will also take this standpoint. Our official definition of weak Vopěnka's Principle is thus:

Definition 2.3.1. *Weak Vopěnka's Principle* (WVP) is the axiom schema positing that if $(G_\alpha\colon \alpha \in \mathrm{Ord})$ is a sequence of graphs, which is definable possibly using set parameters, such that for every $\alpha \leq \beta$ there is exactly one homomorphism $G_\beta \to G_\alpha$, then there are ordinals $\alpha < \beta$ and a homomorphism $G_\alpha \to G_\beta$.

We further write WVP(Π_n) for the schema restricting WVP to sequences of graphs definable by Π_n formulas of the Lévy hierarchy without parameters, and WVP($\boldsymbol{\Pi}_n$) for the schema restricting WVP to sequences of graphs definable by Π_n formulas with parameters. Then WVP is equivalent to WVP($\boldsymbol{\Pi}_n$) holding for every n.

To formulate Wilson's Theorem about WVP, consider the following standard notions.

Definition 2.3.2. Let κ be a cardinal.

(i) For a class A, definable with possible set parameters, and $\lambda > \kappa$, κ is λ-A-*strong* iff there is an elementary embedding $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $j(A \cap V_\lambda) \cap V_\lambda = A \cap V_\lambda$.

(ii) The cardinal κ is called A-*strong* iff it is λ-A-strong for every $\lambda > \kappa$.

(iii) We use the phrase "Ord is Woodin" for the schema expressing "For every class A, there is an A-strong cardinal."

Bagaria's and Wilson's analysis of WVP was carried out using Π_n-strong cardinals, which are a slight variant of A-strong cardinals.

Definition 2.3.3 (Bagaria & Wilson [BW23, Definition 5.1])**.** Let κ be a cardinal.

(i) For an ordinal λ, κ is λ-Π_n-*strong* iff for every class A which is Π_n definable *without* parameters there is an elementary embedding $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $A \cap V_\lambda \subseteq A^M$.

(ii) The cardinal κ is Π_n-*strong* iff it is λ-Π_n-strong for every λ.

It can be shown that strong cardinals are Π_1-strong (combine [BW23, Proposition 5.2] and [BW23, Corollary 5.4]). The characterisations of WVP, level-by-level and global, can be stated as follows.

Theorem 2.3.4 (Bagaria & Wilson [BW23, Theorem 5.11])**.** The following are equivalent for every $n \geq 1$:

(1) $\mathrm{WVP}(\Pi_n)$.

(2) There exists a Π_n-strong cardinal.

Theorem 2.3.5 (Bagaria & Wilson)**.** The following are equivalent:

(1) WVP

(2) Ord is Woodin.

(3) For every n, there is a Π_n-strong cardinal.

(4) For every n, there is a proper class of Π_n-strong cardinals.

Proof. The equivalence of (1) and (2) is already due to Wilson [Wil22b], while the equivalence of both (1) and (2) to (3) and (4) is due to Bagaria and Wilson [BW23, Section 5]. □

We will review some of the proofs in the following Section 2.3.2 to show how the lightface statements (3) and (4) are equivalent to the boldface assertions (1) and (2). If δ is a Woodin cardinal, then $(V_\delta, V_{\delta+1}, \in) \models$ "Ord is Woodin", and so the consistency strength of WVP is well below that of, for example, a supercompact cardinal, so in particular below that of VP or even weak forms like $\mathrm{VP}(\Pi_1)$.

2.3.2. Weak Henkin compactness and Π_n-strong cardinals

In this section we will show how Boney's Theorem 2.2.2 characterising strong cardinals can be generalised to sort logics to give a characterisation of Π_n-strong cardinals, and as a result of WVP. We will first state our main theorems, then review some of the necessary background on Π_n-strong cardinals, and finally proceed with proofs of the main results.

Statement of the results

Recall the finite fragments ZFC^*_n we fixed in Section 2.2 for each natural number $n > 1$, containing the statement that $\Phi^{(n)} \in \mathcal{L}^{s,n}$ is true in precisely those models which are isomorphic to some V_α for $\alpha \in C^{(n)}$, and such that $\Phi^{s,*} \in \mathcal{L}^{s,n}$ is true in precisely those models which are isomorphic to some V_α for α any ordinal.

Theorem 2.3.6. The following are equivalent for every cardinal κ and $n \geq 2$:

(1) κ is Π_n-strong

(2) For every λ which is a limit point of $C^{(n)}$ and every theory $T \subseteq \mathcal{L}^{s,n}_{\kappa\omega}$ that can be written as an increasing union $T = \bigcup_{\alpha<\kappa} T_\alpha$ of theories T_α that each have models of size $\geq \kappa$, there is a weak $\mathcal{L}^{s,n}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $|A| \geq \lambda$, $M \models \mathrm{ZFC}^*_n$, and $V_\lambda \prec_{\Sigma_n} M$.

Proof. Cf. Proof 2.3.21. □

The theorem can also be understood as a local variant of the already cited [BDGM24, Theorem 3.6] which gives a characterisation of Woodin cardinals in terms of compactness for weak Henkin models.

The smallest Π_n-strong cardinal can be described as the smallest cardinal witnessing the above property (2) for finitary sort logic $\mathcal{L}^{s,n}$.

Theorem 2.3.7. The following are equivalent for a cardinal κ.

(1) κ is the smallest Π_n-strong cardinal.

(2) κ is the smallest cardinal such that for any limit λ of $C^{(n)}$ and any theory $T \subseteq \mathcal{L}^{s,n}$ that can be written as an increasing union $T = \bigcup_{\alpha<\kappa} T_\alpha$ of theories T_α that each have models of size $\geq \kappa$, there is a weak $\mathcal{L}^{s,n}$-Henkin model $(M, \mathcal{A})$ of T such that $|A| \geq \lambda$, $M \models \mathrm{ZFC}^*_n$, and $V_\lambda \prec_{\Sigma_n} M$.

Proof. Cf. Proof 2.3.22. □

As for n-strong-Henkin-compactness (cf. Definition 2.2.7), let us fix the property expressed by (2) of the above theorems.

Definition 2.3.8. Let $\mathcal{L}$ be a logic, n a natural number, and κ a cardinal. We say that κ is an *n-Henkin-chain-compactness* (n-HCC) number of $\mathcal{L}$ iff for any $\lambda \in C^{(n)}$ and theory $T \subseteq \mathcal{L}$ which can be written as in increasing union $T = \bigcup_{\alpha<\kappa} T_\alpha$ of theories T_α which each have a model of size $\geq \kappa$, there is a weak $\mathcal{L}$-Henkin model (M, A) of T such that $M \models \mathrm{ZFC}^*_n$, $|A| \geq \lambda$, and $V_\lambda \prec_{\Sigma_n} M$.

Again (cf. Corollary 2.2.10), we get a characterisation of local forms of WVP.

Corollary 2.3.9. For any $n \geq 1$ the following are equivalent:

(1) $\mathrm{WVP}(\Pi_n)$.

(2) $\mathcal{L}^{s,n}$ has an n-HCC number.

Proof. This immediately follows from Theorem 2.3.7 and Theorem 2.3.4. □

Because strong cardinals are Π_1-strong, Boney's Theorem 2.2.2 (with a minor extra argument to adjoin the condition that $V_\lambda \prec_{\Sigma_1} M$ for $\lambda \in C^{(1)}$) shows:

Proposition 2.3.10. The following are equivalent:

(1) $\mathrm{WVP}(\Pi_1)$.

(2) $\mathcal{L}^2$ has a 1-HCC number.

Finally, we get a model-theoretic characterisation of WVP.

Corollary 2.3.11. The following are equivalent:

(1) WVP.

(2) For any logic $\mathcal{L}$ and any natural number n, there is an n-HCC number of $\mathcal{L}$.

Proof. Cf. Proof 2.3.23. □

Note that Lemma 2.3.12 and the formulation of $C^{(n)}$-extendible cardinals as being witnessed by specific supercompactness embeddings (cf. Theorem 2.2.6), show that the definitions of Π_{n+1}-strong and $C^{(n)}$-extendible cardinals result from each other, simply by replacing the clause "$V_\lambda \subseteq M$" by "$M^\lambda \subseteq M$". This shows how the strengths of the stratifications of WVP and VP along the Lévy hierarchy differ in slight (but crucial) variances in formulation of our large cardinal assumptions. The Corollaries 2.2.10 and 2.3.9 show how the same is true for our model-theoretic characterisations: the stratifications of WVP and VP they describe can be obtained from each other by replacing n-Henkin-chain compactness of $\mathcal{L}^{s,n}$ by n-strong-Henkin compactness of $\mathcal{L}^{s,n}$. The situation for WVP is summarised in Figure 2.2.

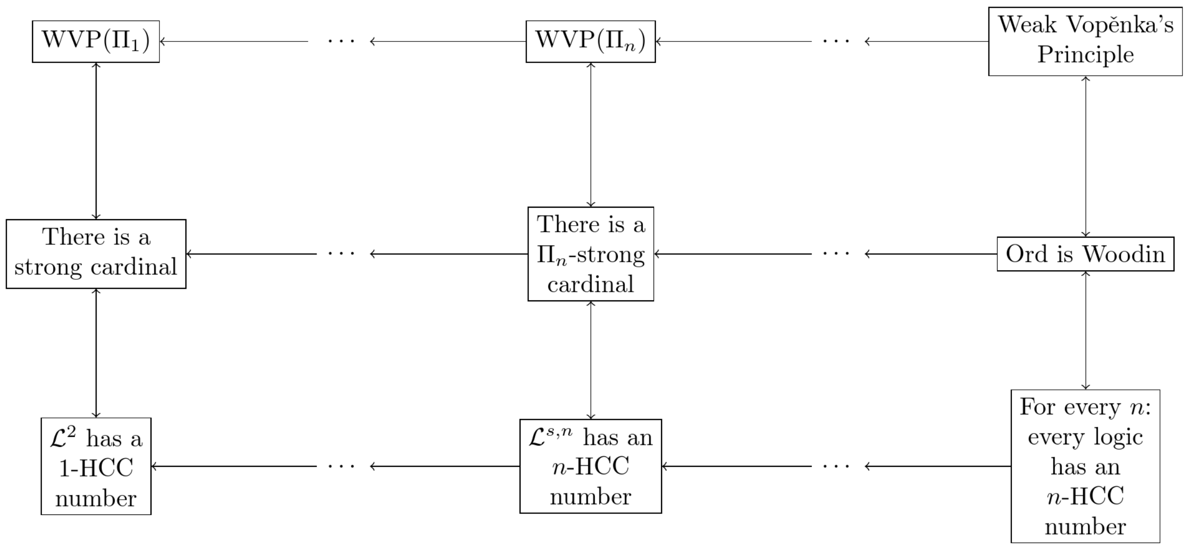

Figure 2.2.: Relations between WVP, Π_n-strong cardinals, and HCC numbers.

More on Π_n-strong cardinals

To work towards a proof of Theorems 2.3.6 and 2.3.7, we start by giving some relevant results on Π_n-strong and A-strong cardinals. In particular, we will show that being Π_n-strong can be witnessed by embeddings between set-sized structures (cf. Lemma 2.3.14).

First of all, note that our definition of κ being a λ-strong cardinal contains the condition that $j(\kappa) > \lambda$, while the definitions of Π_n-strong and A-strong omit it. It is well known that for the global notion of being strong, whether one omits or includes this condition, one arrives at equivalent notions: Call a cardinal κ *weakly λ-strong* if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_\lambda \subseteq M$. Then one can show that κ is strong if and only if κ is weakly λ-strong for every λ (cf., e.g., [Kan03, Exercise 26.7]). As Bagaria and Wilson point out, the argument carries over to Π_n-strong (cf. [BW23, p. 162]) and A-strong cardinals (cf. [Wil22b]). Thus, κ is Π_n-strong iff for every Π_n-definable class A and every λ there is an elementary embedding $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$, $A \cap V_\lambda \subseteq A^M$, *and* $j(\kappa) > \lambda$, i.e., we can equivalently assume for a Π_n-strong cardinal that its critical point gets in each case pushed beyond λ and analogously for A-strong cardinals. Note that if κ is λ-A-strong and there is some ordinal $\kappa \leq \delta < \lambda$ such that $j(\delta) \geq \lambda$, then the A-strong condition $j(A \cap V_\lambda) \cap V_\lambda = A \cap V_\lambda$ is equivalent to $j(A \cap V_\delta) \cap V_\lambda = A \cap V_\lambda$. We will use this observation tacitly below.

If $A = C^{(n)}$, Π_n-strong cardinals and A-strong cardinals coincide, as formulated in the following lemma. Note that by "$C^{(n)}$-strong", we mean here and in the following an A-strong cardinal for $A = C^{(n)}$. The term is also used to refer to a strong cardinal for which we can demand $j(\kappa) \in C^{(n)}$. It is known that this latter notion is equivalent to merely being strong (cf. [Bag12, Proposition 1.2]).

Lemma 2.3.12 (Bagaria & Wilson [BW23, Proposition 5.9])**.** Let $n \geq 1$ and λ be a limit point of $C^{(n)}$. Then the following are equivalent:

(1) κ is λ-Π_n-strong.

(2) There is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $M \models \lambda \in C^{(n)}$.

(3) κ is λ-$C^{(n)}$-strong, i.e., there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $V_\lambda \subseteq M$ and $j(C^{(n)} \cap V_\lambda) \cap V_\lambda = C^{(n)} \cap V_\lambda$.

(4) κ is λ-A-strong for every Π_n-definable class A.

Item (3) is not made explicit in Bagaria's and Wilson's formulation of the lemma but its equivalence to the other statements is argued for in the proof they provide. In particular, the lemma implies:

Corollary 2.3.13. Let $n \geq 1$. The following are equivalent:

(1) κ is Π_n-strong.

(2) κ is $C^{(n)}$-strong.

Let us use Bagaria's and Wilson's result to show how Π_n-strength can be witnessed by embeddings between set models. We will employ this result later in the proof of our model-theoretic characterisation of Π_n-strength (cf. Theorem 2.3.6).

Lemma 2.3.14. Let λ be a limit of $C^{(n)}$. If there is an elementary embedding $j : V_{\kappa+1} \to M$ into a transitive set such that $\mathrm{crit}(j) = \delta \leq \kappa$, $j(\kappa) \geq \lambda$, $V_\lambda \subseteq M$ and $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$, then δ is λ-Π_n-strong.

Proof. Under the above assumptions, note that there is $\zeta \leq \kappa$ the smallest ordinal such that $j(\zeta) \geq \lambda$. We can derive a (δ, λ)-extender $E = (E_a \colon a \in [\lambda]^{<\omega})$ by letting for each a and $X \subseteq [\zeta]^{|a|}$:

$$X \in E_a \text{ iff } a \in j(X).$$

We proceed to build the extender power m_E of $V_{\kappa+1}$ by $(E_a \colon a \in [\lambda]^{<\omega})$. By Theorem 1.3.39, this comes with the standard elementary embeddings $j_{E,m} : V_{\kappa+1} \to m_E$ and $k_{E,m} : m_E \to M$ such that $\text{crit}(j_{E,m}) = \delta$, $j_{E,m}(\zeta) \geq \lambda$, $V_\lambda \subseteq m_E$, $j = k_{E,m} \circ j_{E,m}$ and $k_{E,m} \restriction \lambda = \text{id}$. Further, by Theorem 1.3.40, building the extender power $j_E : V \to M_E$, we get $V_\lambda \subseteq m_E \subseteq M_E$ and $j_E \restriction V_{\kappa+1} = j_{E,m}$. Therefore $\text{crit}(j_E) = \delta$ and $j_E(\zeta) \geq \lambda$. Because λ is a limit of $C^{(n)}$, using Lemma 2.3.12, it is therefore sufficient to check that $j_E(C^{(n)} \cap V_\zeta) \cap V_\lambda = C^{(n)} \cap V_\lambda$ to show that δ is λ-Π_n-strong. Using again that $j_E \restriction V_{\kappa+1} = j_{E,m}$, we thus only have to show the following claim.

Claim 2.3.15. $j_{E,m}(C^{(n)} \cap V_\zeta) \cap V_\lambda = C^{(n)} \cap V_\lambda$.

By elementarity and because $k_{E,m} \circ j_{E,m} = j$, we get:

$$\alpha \in j_{E,m}(C^{(n)} \cap V_\zeta) \text{ iff } k_{E,m}(\alpha) \in k_{E,m}(j_{E,m}(C^{(n)} \cap V_\zeta)) = j(C^{(n)} \cap V_\zeta).$$

Now $k_{E,m} \restriction \lambda = \text{id}$, so if $\alpha < \lambda$, then $k_{E,m}(\alpha) = \alpha$. Together we have:

$$\alpha \in j_{E,m}(C^{(n)} \cap V_\zeta) \cap V_\lambda \text{ iff } \alpha \in j(C^{(n)} \cap V_\zeta) \cap V_\lambda = C^{(n)} \cap V_\lambda.$$

□

We state a few more properties noted by Bagaria and Wilson, adding the proofs they omit.

Proposition 2.3.16 (Bagaria & Wilson [BW23, p. 164])**.** Being Π_n-strong is a Π_{n+1}-assertion.

Proof. Note that for ordinals κ, λ, and μ, the following is a Σ_2 assertion:

$$\varphi(\kappa, \lambda, \mu) = \exists j \exists M (j : V_\mu \to M \text{ is elementary} \wedge \text{crit}(j) = \kappa \wedge j(\kappa) \geq \lambda \wedge V_\lambda \prec_{\Sigma_n} M).$$

Thus, as "$x \in C^{(n)}$" is Π_n, the following is a Π_{n+1} property of κ:

$$\psi(\kappa) = \forall \lambda, \mu(\lambda, \mu \in C^{(n)} \wedge \lambda < \mu \to \varphi(\kappa, \lambda, \mu)).$$

We claim that $\psi(\kappa)$ holds iff κ is Π_n-strong.

For the backward direction, assume that κ is Π_n-strong and take some $\lambda < \mu$ both in $C^{(n)}$. Let $\gamma > \mu$ be a limit point of $C^{(n)}$. If κ is Π_n-strong, by Corollary 2.3.13, it is $C^{(n)}$-strong. Then we can take $j : V \to N$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \gamma$, $V_\gamma \subseteq N$, and $j(C^{(n)} \cap V_\kappa) \cap V_\gamma = C^{(n)} \cap V_\gamma$. Then N sees that γ is a limit point of $C^{(n)}$, so in particular $\gamma \in (C^{(n)})^N$. Thus, $V_\gamma \prec_{\Sigma_n} N$. Because V_γ is correct about $C^{(n)}$, this implies $\lambda \in (C^{(n)})^N$, and because $V_\lambda \subseteq N$, thus $V_\lambda \prec_{\Sigma_n} N$. Furthermore, as $\mu \in C^{(n)}$, by elementarity $j(\mu) \in (C^{(n)})^N$. As $\lambda < j(\mu)$, thus $V_\lambda \prec_{\Sigma_n} V^N_{j(\mu)}$. Now note that, letting $M = V^N_{j(\mu)}$, j restricts to an embedding $j \restriction V_\mu : V_\mu \to M$ as desired.

For the forward direction, let λ be a limit point of $C^{(n)}$. It is sufficient to show that κ is λ-Π_n-strong. We claim that there is an elementary embedding $j : V_{\kappa+1} \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) \geq \lambda$, $V_\lambda \subseteq M$ and $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$. Then we are done by Lemma 2.3.14. Let $\mu > \lambda$ be a limit point of $C^{(n)}$. Then by $\psi(\kappa)$, there is an elementary embedding $j : V_\mu \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) \geq \lambda$, and $V_\lambda \prec_{\Sigma_n} N$. This implies that $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$. But then letting $M = V^N_{j(\kappa)}$, j restricts to an embedding $j \restriction V_{\kappa+1} : V_{\kappa+1} \to M$ as desired. □

Proposition 2.3.17 (Bagaria & Wilson [BW23, p. 164])**.** If κ is Π_n-strong, then $\kappa \in C^{(n+1)}$.

Proof. By induction on n. For the case $n = 1$, this follows as strong cardinals are easily seen to be Π_1-strong by downward absoluteness of Π_1 formulas and its a classic result that they are in $C^{(2)}$ (cf., e.g., [Kan03, Exercise 26.6]). So assume every Π_n-strong cardinal is in $C^{(n+1)}$ and let κ be Π_{n+1}-strong. We have to show that $\kappa \in C^{(n+2)}$. So take $\Phi(x) = \exists y \Psi(x, y)$, a Σ_{n+2} formula. Because κ is in particular Π_n-strong and thus by induction hypothesis in $C^{(n+1)}$, it follows that Σ_{n+2} formulas are upward absolute from V_κ. It is thus sufficient to show that if $a \in V_\kappa$ such that $\Phi(a)$ holds in V, then $V_\kappa \models \Phi(a)$. By $V \models \Phi(a)$, there is a b such that $V \models \Psi(a, b)$. Take some $\lambda \in C^{(n+1)}$ such that $b \in V_\lambda$ and a $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $M \models \lambda \in C^{(n+1)}$. Then we get that V, V_λ and M agree on Π_{n+1} formulas and therefore $M \models \Psi(a, b)$. Thus $M \models \exists y(\mathrm{rk}(y) < j(\kappa) \wedge \Psi(a, y))$. By elementarity thus $V \models \exists y(\mathrm{rk}(y) < \kappa \wedge \Psi(a, y))$. So there is $c \in V_\kappa$ such that $\Psi(a, c)$ holds. Because $\kappa \in C^{(n+1)}$, also $V_\kappa \models \Psi(a, c)$ and therefore $V_\kappa \models \Phi(a)$. □

Proposition 2.3.18 (Bagaria & Wilson [BW23, p. 164])**.** If κ is Π_{n+1}-strong, then it is a limit of Π_n-strong cardinals.

Proof. Take any $\alpha < \kappa$. Then because κ is Π_n-strong, the statement

$$\exists x(x > \alpha \wedge x \text{ is } \Pi_n\text{-strong})$$

holds in V. Because being Π_n-strong is a Π_{n+1}-property, this is a Σ_{n+2}-statement. As $\kappa \in C^{(n+2)}$, also V_κ believes that there is some Π_n-strong cardinal $\lambda > \alpha$, and it is correct about this. □

This easily implies:

Corollary 2.3.19. If there is a Π_{n+1}-strong cardinal, then there is a proper class of Π_n-strong cardinals.

Proof. The statement $\forall \alpha \exists \lambda(\lambda > \alpha \wedge \lambda \text{ is } \Pi_n\text{-strong})$ is Π_{n+2}. If κ is Π_{n+1}-strong, V_κ satisfies this statement, as it is a limit of Π_n-strong cardinals. We further have $\kappa \in C^{(n+2)}$ and so V_κ reflects it to V. □

Putting everything together, Bagaria and Wilson get the result we already partially quoted above:

Corollary 2.3.20 (Bagaria & Wilson[BW23])**.** The following are equivalent:

(1) WVP.

(2) Ord is Woodin.

(3) For every n, there is a Π_n-strong cardinal.

(4) For every n, there is a $C^{(n)}$-strong cardinal.

(5) For every n, there is a proper class of Π_n-strong cardinals.

(6) For every n, there is a proper class of $C^{(n)}$-strong cardinals.

Proof. As promised, let us show how the lightface statements (3)-(6) imply (2), i.e., the existence of A-strong cardinals for classes A that are defined with parameters. This holds, as if κ is Π_n-strong, then it is A-strong for every class which is Π_n-definable with parameters in V_κ: Let A be a class defined by a Π_n formula $\Phi(x,p)$ with $p \in V_\kappa$ and let $\lambda \in C^{(n)}$. Take $j : V \to M$, $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $M \models \lambda \in C^{(n)}$. Then V_λ, V and M agree on Π_n formulas. We want to see that $j(A \cap V_\lambda) \cap V_\lambda = A \cap V_\lambda$. So fix $a \in V_\lambda$. We have $a \in A$ iff $\Phi(a,p)$ iff $V_\lambda \models \Phi(a,p)$ iff $M \models \Phi(a,p)$. Now as $j(p) = p$ and thus $j(A \cap V_\lambda) = \{y \in M \colon \text{rk}(y) < j(\lambda) \wedge \Phi^M(y,p)\}$, we get $M \models \Phi(a,p)$ iff $a \in j(A \cap V_\lambda)$. Since by (5), we get arbitrarily large Π_n-strong cardinals, we can therefore cover definitions with any parameter. □

Proofs of the main results

Proof 2.3.21 (Proof of Theorem 2.3.6)**.** For the forward direction, suppose we have a setup like in (2), i.e., a theory $T \subseteq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$ and an increasing union $\bigcup_{\alpha<\kappa} T_\alpha = T$ such that every T_α has a model $\mathcal{A}_\alpha$ of size $\geq \kappa$ and some $\lambda \in C^{(n)}$. Then we can pick a function f on κ such that every $f(\alpha) = (V_{\beta_\alpha}, \mathcal{A}_\alpha)$ is a weak $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$-Henkin model of T_α. Without loss of generality we can choose $\kappa < \beta_\alpha \in C^{(n)}$ such that $V_{\beta_\alpha} \models \text{ZFC}^*_n$.

By κ being Π_n-strong, take an elementary embedding $j : V \to N$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq N$ and $N \models \lambda \in C^{(n)}$. Computing $j(T)$, we get that $j(T) = \bigcup_{\alpha<j(\kappa)} T^*_\alpha$ for some theories $T^*_\alpha \subseteq \mathcal{L}^{\mathrm{s},n}_{j(\kappa)\omega}$. Then $j(f)$ is a function on $j(\kappa)$ such that, in N, $j(f)(\alpha)$ is a weak $\mathcal{L}^{\mathrm{s},n}_{j(\kappa)\omega}$-Henkin model of T^*_α. Thus, in N, we have that $j(f)(\kappa) = (M^*_\kappa, \mathcal{A}^*_\kappa)$ is an $\mathcal{L}^{\mathrm{s},n}_{j(\kappa)\omega}$-Henkin model of T^*_κ. Note that $j``T = \bigcup_{\alpha<\kappa} j``T_\alpha \subseteq \bigcup_{\alpha<\kappa} j(T_\alpha) \subseteq \bigcup_{\alpha<\kappa} T^*_\alpha \subseteq T^*_\kappa$, and so from the outside, we see that $(M^*_\kappa, \mathcal{A}^*_\kappa)$ is really a weak Henkin model of $j``T$. Because $\text{crit}(j) = \kappa$, $j``T \subseteq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$ and so $j``T$ is really an $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$-theory. Furthermore, it is a copy of T and so $(M^*_\kappa, \mathcal{A}^*_\kappa)$ really is a weak $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$-Henkin model of T.

Note that $f(\alpha) = (V_{\beta_\alpha}, \mathcal{A}_\alpha)$ and $V_\kappa \subseteq V_{\beta_\alpha}$ for every α. Thus N believes that $V^N_{j(\kappa)} \subseteq M^*_\kappa$. Because $V_\lambda \subseteq N$ and $j(\kappa) > \lambda$, we get that $V_\lambda \subseteq M^*_\kappa$. Also $|\mathcal{A}_\alpha| \geq \kappa$ for all α

and so N believes that $|\mathcal{A}^*_\kappa| \geq \lambda$. Because λ is a cardinal, this really holds. Since all V_{β_α} satisfy ZFC^*_n, N believes M^*_κ to satisfy ZFC^*_n, and, by absoluteness of first-order satisfaction, it is correct about this. Further, $\kappa < \beta_\alpha \in C^{(n)}$ for all α and so $M^*_\kappa = V^N_\beta$ for some $\beta > j(\kappa) > \lambda$ with $\beta \in (C^{(n)})^N$, i.e., $M^*_\kappa = V^N_\beta \prec_{\Sigma_n} N$. Now by assumption, $N \models \lambda \in C^{(n)}$ and $V^N_\lambda = V_\lambda$, so $V_\lambda \prec_{\Sigma_n} N$. As $\lambda < \beta$, together this implies $V_\lambda \prec_{\Sigma_n} V^N_\beta = M^*_\kappa$. Summarising, $(M^*_\kappa, \mathcal{A}^*_\kappa)$ is a weak Henkin model of T as promised.

For the backwards direction, suppose we have the compactness property from (2). By Lemma 2.3.14, it is sufficient to provide, for λ a limit point of $C^{(n)}$, an elementary embedding $j : V_{\kappa+1} \to M$ with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) \geq \lambda$, $V_\lambda \subseteq M$ and $C^{(n)} \cap V_\lambda = j(C^{(n)} \cap V_\kappa) \cap V_\lambda$. Recall that the class $\{(M, E) \colon \exists\alpha((M, E) \cong (V_\alpha, \in))\}$ is Σ_2-definable, and therefore in particular, it is axiomatisable by some $\Phi^{\mathrm{s},*} \in \mathcal{L}^{\mathrm{s},2}$ (cf. Corollary 1.2.18). Consider the following theory:

$$\begin{aligned} T = {} & \mathrm{ElDiag}_{\mathcal{L}_{\kappa\omega}}(V_{\kappa+1}, \in) \cup \{c_i < c < c_\kappa \colon i < \kappa\} \cup \{\Phi^{\mathrm{s},*}\} \\ & \cup \{\forall x(x \in c_{C^{(n)} \cap V_\kappa} \to (\Phi^{(n)})^{\{y \colon y \in V_x\}}\} \cup \\ & \cup \{\forall x((\Phi^{(n)})^{\{y \colon y \in V_x\}} \wedge x \in c_{V_\kappa} \to x \in c_{C^{(n)} \cap V_\kappa}\}, \end{aligned}$$

where $(\Phi^{(n)})^{\{y \colon y \in V_x\}}$ is the relativisation of $\Phi^{(n)}$ to the structure which consists of the elements of what the structure believes to be V_x, thus coding that $x \in C^{(n)}$. Clearly, this theory can be written as an increasing union of satisfiable theories T_α for $\alpha < \kappa$ by considering those bits of T that include only the sentences $c_i < c < c_\kappa$ for $i < \alpha$ and using $V_{\kappa+1}$ as a model. Then by (b), T has an $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ such that $M \models \mathrm{ZFC}^*_n$, $V_\lambda \prec_{\Sigma_n} M$ and $|A| \geq \lambda$. We have that M believes that $\mathcal{A}$ satisfies $\Phi^{\mathrm{s},*}$, so we may assume that $A = V^M_\beta$ for some β. As $\mathcal{L}_{\kappa\omega}$-satisfaction is absolute, from the outside we see that $\mathcal{A} \models \mathrm{ElDiag}_{\mathcal{L}_{\kappa\omega}}(V_{\kappa+1}, \in)$. Therefore there is an elementary embedding $j : V_{\kappa+1} \to V^M_\beta$ and we have that this has a critical point $\mathrm{crit}(j) \geq \kappa$, as $\mathcal{L}_{\kappa\omega}$ can define all ordinals below κ. Further $\mathrm{crit}(j) \leq \kappa$ by the sentences $c_i < c < c_\kappa$ holding in N. Thus $\mathrm{crit}(j) = \kappa$. To see that $V_\lambda \subseteq V^M_\beta$, note that because $|V^M_\beta| = |A| \geq \lambda$ and $\lambda = \beth_\lambda$ by being a member of $C^{(n)}$, we have to have $\beta \geq \lambda$. And because $V_\lambda \subseteq M$, therefore $V_\lambda \subseteq (V_\beta)^M$. Because $V_{\kappa+1}$ satisfies the sentence that there is a largest cardinal, also $\mathcal{A}$ satisfies this. In particular, we have to have $\lambda \in V^M_\beta$. Because κ is the largest cardinal of $V_{\kappa+1}$, $j(\kappa)$ has to be the largest cardinal of V^M_β. Thus $j(\kappa) \geq \lambda$.

Finally, the theory T implies that for $\alpha < \lambda$, we have

$$M \models \text{“}\mathcal{A} \models \alpha \in c_{C^{(n)} \cap V_\kappa}\text{” iff } M \models \text{“}\mathcal{A} \models (\Phi^{(n)})^{\{y \colon y \in V_\alpha\}}\text{”}.$$

The first part means that $\alpha \in j(C^{(n)} \cap V_\kappa)$, while, as $M \models \mathrm{ZFC}^*_n$, the second is equivalent to $M \models \alpha \in C^{(n)}$. Because $V_\lambda \prec_{\Sigma_n} M$, for $\alpha < \lambda$, this implies $V_\lambda \models \alpha \in C^{(n)}$. Since λ really is a member of $C^{(n)}$, V_λ is correct about membership in $C^{(n)}$ and so this implies $\alpha \in C^{(n)}$. Thus $\alpha \in j(C^{(n)} \cap V_\kappa) \cap V_\lambda$ iff $\alpha \in C^{(n)} \cap V_\lambda$. □

Proof 2.3.22 (Proof of Theorem 2.3.7)**.** Let κ be the cardinal as designated by (2). It is sufficient to show that there is a smallest Π_n-strong cardinal $\gamma \leq \kappa$: By Theorem 2.3.6, then the above Henkin compactness property holds for theories $T \subseteq \mathcal{L}^{\mathrm{s},n}_{\gamma\omega}$, so in particular for theories of $\mathcal{L}^{\mathrm{s},n} \subseteq \mathcal{L}^{\mathrm{s},n}_{\gamma\omega}$. As κ is the smallest cardinal for which this holds, we thus get $\kappa \leq \gamma$ and hence $\gamma = \kappa$.

Consider the $\mathcal{L}^{\mathrm{s},n}$-theory

$$\begin{aligned} T =& \mathrm{ElDiag}(V_{\kappa+1}, \in) \cup \{c_i < c < c_\kappa \colon i < \kappa\} \cup \{\Phi^{\mathrm{s},*}\} \\ &\cup \{\forall x(x \in c_{C^{(n)} \cap V_\kappa} \to (\Phi^{(n)})^{\{y \colon y \in V_x\}}\} \cup \\ &\cup \{\forall x((\Phi^{(n)})^{\{y \colon y \in V_x\}} \wedge x \in c_{V_\kappa} \to x \in c_{C^{(n)} \cap V_\kappa}\}. \end{aligned}$$

This is basically the same theory as considered above, with the only difference being that we consider the first-order elementary diagram instead of the $\mathcal{L}_{\kappa\omega}$-diagram. Exactly the same argument as above gives us that there is an elementary embedding $j : V_{\kappa+1} \to M$ with M transitive, $\mathrm{crit}(j) \leq \kappa$, $j(\kappa) \geq \lambda$, $V_\lambda \subseteq M$ and $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$. The only difference is that we cannot use $\mathcal{L}_{\kappa\omega}$ to show that the critical point $\mathrm{crit}(j)$ is exactly κ. Let $\delta = \mathrm{crit}(j)$. Then Lemma 2.3.14 shows that δ is λ-Π_n-strong. We can conclude that there is a Π_n-strong cardinal $\leq \kappa$ as there is a proper class of $\lambda > \kappa$, but only κ-many possible values for critical points $\leq \kappa$, so some fixed cardinal will have to be the critical point for arbitrarily large λ. □

Proof 2.3.23 (Proof of Corollary 2.3.11)**.** Assume WVP and let $\mathcal{L}$ be a logic and n a natural number. There is a cardinal κ and a natural number m such that $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},m}_{\kappa\omega}$. Let $k = \max\{n, m\}$. By Corollary 2.3.20, from WVP we get a Π_k-strong cardinal $\delta > \kappa$. By Theorem 2.3.6, δ is a k-HCC number of $\mathcal{L}^{\mathrm{s},k}_{\delta\omega}$. In particular, δ is an n-HCC number for $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},k}_{\delta\omega}$.

And now assume (2). In particular it follows that for any n, $\mathcal{L}^{\mathrm{s},n}$ has an n-HCC number. Theorem 2.3.6 then implies that there is a Π_n-strong cardinal. Therefore by Corollary 2.3.20, WVP holds. □

2.3.3. Variations of jointly strong and strongly compact cardinals

We proceed by presenting some further applications of weak Henkin models by considering other compactness properties. Note that there are two essential differences between n-Henkin-chain-compactness and n-strong-Henkin compactness. First, the latter provides strong Henkin models while the former provides weak Henkin models. And second, the latter assumes $< \kappa$-satisfiability of theories, while the former assumes satisfiability along a κ-chain. It is thus natural to ask what we get from a compactness property which provides weak Henkin models under the assumption of $< \kappa$-satisfiability. Boney considered this question for second-order logic with the classical Henkin semantics, stating without proof that in this case, this amounts to a jointly strong and strongly compact cardinal (cf. [Bon20, p. 159]). With our framework, his remarks can be formulated in the following way.

Theorem 2.3.24 (Boney)**.** The following are equivalent for a cardinal κ:

(1) κ is jointly strong and strongly compact, i.e., for every $\lambda > \kappa$, there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and with a $d \in M$ such that $j``\lambda \subseteq d$ and $M \models |d| < j(\kappa)$.

(2) For every $<\kappa$-satisfiable theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ there is a weak $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ such that $M \models \text{ZFC}^*$ and $V_\lambda \subseteq M$.

We refrain from giving a proof of the above result, as it can be shown with similar arguments as the following theorem, which provides a generalisation to the sort logic case.

Theorem 2.3.25. The following are equivalent for any cardinal κ and natural number $n \geq 2$:

(1) κ is jointly Π_n-strong and strongly compact, i.e., for every $\lambda \in C^{(n)}$, there is an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, $M \models \lambda \in C^{(n)}$ and with a $d \in M$ such that $j``\lambda \subseteq d$ and $M \models |d| < j(\kappa)$.

(2) For every $\lambda \in C^{(n)}$ and every $<\kappa$-satisfiable theory $T \subseteq \mathcal{L}^{s,n}_{\kappa\omega}$ there is a weak $\mathcal{L}^{s,n}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ such that $M \models \text{ZFC}^*_n$ and $V_\lambda \prec_{\Sigma_n} M$.

Proof. Assume (1). It is sufficient to show (2) for λ a limit point of $C^{(n)}$. Let $T \subseteq \mathcal{L}^{s,n}_{\kappa\omega}$ be $<\kappa$-satisfiable. Let $\gamma > \lambda$ be a limit point of $C^{(n)}$ such that V_γ satisfies ZFC^*_n and verifies that T is $<\kappa$-satisfiable. Take an elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \gamma$, $V_\gamma \subseteq M$, $M \models \gamma \in C^{(n)}$ and with a $d \in M$ such that $j``\gamma \subseteq d \subseteq j(\gamma)$ and $M \models |d| < j(\kappa)$. The following claim is standard.

Claim 2.3.26. There is a $d_0 \in M$ such that $j``T \subseteq d_0 \subseteq j(T)$ and $M \models |d_0| < j(\kappa)$.

Proof. Notice that there is a surjection $g : \gamma \to T$. Now if $j(\varphi) \in j``T$, there is an $\alpha < \gamma$ such that $g(\alpha) = \varphi$ and so $j(\varphi) = j(g(\alpha)) = j(g)(j(\alpha)) \in j(g)``d$. Let $d_0 = \text{ran}(j(g) \restriction d)$. Then $j``T \subseteq d_0 = \text{ran}(j(g) \restriction d) \in M$. Further $M \models |d_0| \leq |d| < j(\kappa)$. □

By elementarity $j(T)$ is $<j(\kappa)$-satisfiable and so M believes that there is a model $\mathcal{A} \models d_0$, and further we can let $\mathcal{A} \in V^M_{j(\gamma)}$. Then because $j``T \subseteq d_0$, from the outside we see that $(V^M_{j(\gamma)}, \mathcal{A})$ is a weak $\mathcal{L}^{s,n}_{\kappa\omega}$-Henkin model of T. Using elementarity again, $V^M_{j(\gamma)}$ satisfies ZFC^*_n. Furthermore, $\gamma \in C^{(n)}$, so by elementarity, $M \models j(\gamma) \in C^{(n)}$ and by assumption also $M \models \gamma \in C^{(n)}$. Since $V_\gamma \subseteq M$, this implies $V_\gamma \prec_{\Sigma_n} V^M_{j(\gamma)}$. Therefore $(V^M_{j(\gamma)}, \mathcal{A})$ is as desired.

And now assume (2). It is sufficient to show (1) for λ a limit of $C^{(n)}$ of cofinality $\text{cof}(\lambda) \geq \kappa$. Again, let $\Phi^{s,*} \in \mathcal{L}^{s,n}$ be the sentence axiomatising the class of models (M, E) isomorphic to some $(V_\alpha, \in)$, and $\Phi^{(n)}$ the sentence axiomatising the class of models isomorphic to some $(V_\alpha, \in)$ such that $\alpha \in C^{(n)}$ (cf. Corollaries 1.2.18 and 1.2.17). Take a new constant symbol d and consider the theory

$$
\begin{aligned}
T = {} & \mathrm{ElDiag}_{\mathcal{L}_{\kappa\omega}}(V_{\lambda+1}, \in) \cup \{c_i < d \wedge |d| < c_\kappa \colon i < \lambda\} \cup \{\Phi^{\mathrm{s},*}\} \\
& \cup \{\forall x(x \in c_{C^{(n)} \cap V_\kappa} \rightarrow (\Phi^{(n)})^{\{y\colon y \in V_x\}}\} \cup \\
& \cup \{\forall x((\Phi^{(n)})^{\{y\colon y \in V_x\}} \wedge x \in c_{V_\kappa} \rightarrow x \in c_{C^{(n)} \cap V_\kappa}\}.
\end{aligned}
$$

It is easily seen that $V_{\lambda+1}$ satisfies every $< \kappa$-sized subset of T and so T has by assumption a weak $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ such that $V_\lambda \prec_{\Sigma_n} M$, satisfying ZFC^*_n. From the outside we see that $\mathcal{A} \models \mathrm{ElDiag}_{\mathcal{L}_{\kappa\omega}}(V_{\lambda+1}, \in)$ and that M believes that $A = V^M_\gamma$ is some rank initial segment by virtue of $\Phi^{\mathrm{s},*}$ and so we get an elementary embedding $j : V_{\lambda+1} \rightarrow V^M_\gamma$ with $\mathrm{crit}(j) \geq \kappa$. In particular, $\gamma = \beta + 1$ for some β. By the theory, $j``\lambda \subseteq d^{\mathcal{A}}$ and $|d^{\mathcal{A}}|^{\mathcal{A}} < j(\kappa)$ so $j(\kappa) > \lambda$ and thus, by usage of $\mathcal{L}_{\kappa\omega}$, $\mathrm{crit}(j) = \kappa$. Because $V_\lambda \subseteq M$, that $j(\kappa) > \lambda$, implies $V_\lambda \subseteq V^M_{\beta+1}$. Exactly as in the proof of Theorem 2.3.6, the last parts of the theory and that $V_\lambda \prec_{\Sigma_n} M \models \mathrm{ZFC}^*_n$ implies $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$.

We want to see that the properties of j carry over to an embedding defined on V. We have $j(\lambda) = \beta$ so we can derive a long (κ, β)-extender $E = (E_a \colon a \in [\beta]^{<\omega})$ by letting for $X \subseteq [\lambda]^{|a|}$:

$$X \in E_a \text{ iff } a \in j(X).$$

Theorem 1.3.39 implies that taking the extender power m_E of $V_{\lambda+1}$ by E we get an elementary embedding $j_{E,m} : V_{\lambda+1} \rightarrow m_E$ with $\mathrm{crit}(j_{E,m}) = \kappa$, $j_{E,m}(\kappa) > \lambda$, $V_\lambda \subseteq m_E$. Using that the factor embedding does not move ordinals $< \lambda$, as in the proof of Lemma 2.3.14, we get that $j_{E,m}(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$. And taking the extender power M_E of V, by Theorem 1.3.40, $m_E \subseteq M_E$ and these properties of $j_{E,m}$ carry over to $j_E : V \rightarrow M_E$, as $j_E \restriction V_{\lambda+1} = j_{E,m}$. In particular, $j_E(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$, so if $\alpha < \lambda$ and $\alpha \in C^{(n)}$, then $M_E \models \alpha \in C^{(n)}$. Because λ is a limit point of $C^{(n)}$, M_E thus sees that $C^{(n)}$ is unbounded below λ and therefore $M_E \models \lambda \in C^{(n)}$.

Thus, left to show is that there is $d^* \in M_E$ such that $j_E``\lambda \subseteq d^*$ and $M \models |d^*| < j_E(\kappa)$. Note that $\lambda = \beth_\lambda$ and so $j(\lambda) = \beta$ is a $\beth$-fixed point in $V^M_{\beta+1}$. Therefore $V^M_{\beta+1} \models |V_\beta| \leq \beta$ and so by Theorem 1.3.39, $V^{m_E}_\beta = V^M_\beta$. Consider $d^{\mathcal{A}}$. We may assume that $d^{\mathcal{A}} \subseteq \beta$ by otherwise letting $d_0 = d^{\mathcal{A}} \cap \beta \in V^M_{\beta+1}$. Notice that, in $V^M_{\beta+1}$, we have $|d^{\mathcal{A}}| < j(\kappa)$ and, because λ really has cofinality at least κ, in $V^M_{\beta+1}$, $\mathrm{cof}(\beta) = \mathrm{cof}(j(\lambda)) \geq j(\kappa)$. Thus $d^{\mathcal{A}}$ is bounded in $j(\lambda) = \beta$. Therefore $d^{\mathcal{A}} \in V^M_\eta$ for some $\eta < \beta$, and hence, in particular, $d^{\mathcal{A}} \in V^M_\beta = V^{m_E}_\beta$. Recall that $m_E \subseteq M_E$ and thus $d^{\mathcal{A}} \in M_E$. Further, considering the factor map $k_{E,m} : m_E \rightarrow V^M_{\beta+1}$, we have $k_{E,m} \restriction \lambda = \mathrm{id}$ and so for $\alpha < \lambda$, $j(\alpha) = j_{E,m}(\alpha) = j_E(\alpha)$. Thus, $d^{\mathcal{A}}$ covers $j_E``\lambda$. Because $V^M_\beta \models |d^{\mathcal{A}}| < j(\kappa)$ then also $|d^{\mathcal{A}}|^{M_E} < j(\kappa) = j_E(\kappa)$ □

We would like to make some remarks about these and related results. First, an argument by Dimopoulos shows that cardinals which are both strong and strongly compact are already jointly strong and strongly compact (cf. [Dim19, Proposition 2.3]). Theorem 2.3.24 therefore simply characterises cardinals which are both strong and strongly compact. Recall that any supercompact cardinal is strong and strongly compact.

Moreover, Apter and Hamkins showed that it is consistent that a cardinal is both strong and strongly compact, but not supercompact (cf. [AH03, Theorem 1.2]). Being strong and strongly compact is therefore strictly weaker in terms of direct implication than supercomapctness. In particular, this shows that the usage of strong Henkin models in Theorem 2.2.4 is necessary.

For the stronger notion of jointly Π_n-strong and strongly compact cardinals, their precise strength is not so apparent. Notice that as an upper bound, Bagaria's and Goldberg's Theorem 2.2.6 implies that any $C^{(n)}$-extendible cardinal is jointly Π_{n+1}-strong and strongly compact.

Our results leave open what kind of cardinals can be characterised by asking for *strong* Henkin models of theories which are satisfiable along a chain.

Question 2.3.27. What kind of cardinal κ is characterised by the following compactness property?

> For any λ and any theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ that can be written as an increasing union $T = \bigcup_{\alpha\in\kappa} T_\alpha$ of theories T_α which each have a model of size $\geq \kappa$, there is a strong $\mathcal{L}^2$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$, $V_\lambda \subseteq M$ and $|A| \geq \lambda$.

2.3.4. Superstrong cardinals

As a last note on Henkin models, let us show that a version of Henkin-chain-compactness gives a characterisation of superstrong cardinals. Recall that a cardinal κ is *superstrong with target* λ (cf. [Kan03, Section 26]), if there is an elementary embedding $j : V \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, and $V_\lambda \subseteq N$. To our best knowledge, we give the first known model-theoretic formulation of superstrength.

Theorem 2.3.28. The following are equivalent:

(1) κ is superstrong with target λ.

(2) For any theory $T \subseteq \mathcal{L}^2_{\kappa\omega}$ such that $\mathrm{rk}(T) < \kappa + \omega$ and that can be written as an increasing union $T = \bigcup_{\alpha\in\kappa} T_\alpha$ of theories T_α which each have a model of rank $< \kappa + \omega$ and of size $\geq \kappa$, there is a weak $\mathcal{L}^2_{\kappa\omega}$-Henkin model $(M, \mathcal{A})$ of T such that $M \models \mathrm{ZFC}^*$, $V_\lambda \subseteq M \subseteq V_{\lambda+\omega}$, $|A| \geq \lambda$, and $M \models \lambda = \beth^M_\lambda$.

Proof. First assume (1) and suppose we have a setup as in (2). Then there is a function f with domain κ such that $f(\alpha) \models T_\alpha$, $\mathrm{rk}(f(\alpha)) < \kappa+\omega$ and $|f(\alpha)| \geq \kappa$. Take an elementary embedding $j : V \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $V_{j(\kappa)} \subseteq N$. Consider the sequence $(T_\alpha \colon \alpha < \kappa)$. Evaluating it via j leads to a sequence $j((T_\alpha \colon \alpha < \kappa)) = (T^*_\alpha \colon \alpha < j(\kappa))$ such that $j``T \subseteq T^*_\kappa$. Note that $j``T$ is a copy of T. By elementarity, in N, we have that $j(f)(\kappa) \models T^*_\kappa$ and so in particular, for every $\varphi \in j``T$, $N \models$ "$j(f)(\kappa) \models \varphi$". Further, $\mathrm{rk}(j(f)(\kappa)) < j(\kappa)+\omega = \lambda+\omega$ and thus $j(f)(\kappa) \in V^N_{\lambda+\omega}$. Then $(M, \mathcal{A}) = (V^N_{\lambda+\omega}, j(f)(\kappa))$ gives our desired Henkin model: $V^N_{\lambda+\omega}$ satisfies ZFC^*, as $\lambda+\omega$ is a limit ordinal. Because $V_\lambda \subseteq N$, we have $V_\lambda \subseteq V^N_{\lambda+\omega} \subseteq V_{\lambda+\omega}$. Because $V^N_{\lambda+\omega}$ and N agree on second-order satisfaction, we have $V^N_{\lambda+\omega} \models$ "$\mathcal{A} \models \varphi$" for every $\varphi \in j$"T. By elementarity, N, and hence $V^N_{\lambda+\omega}$ believes that $j(\kappa) = \lambda$ is a $\beth$-fixed point. Finally, note that λ is actually a (strong limit) cardinal as the target of a superstrong embedding, and so because by elementarity $N \models |j(f)(\kappa)| \geq j(\kappa) = \lambda$, that $|j(f)(\kappa)| \geq \lambda$ really holds in V.

And now assume (2). We show that κ is superstrong with target λ. Using Theorems 1.3.39 and 1.3.40, if $j : V_{\kappa+1} \to N$ is an elementary embedding such that $\text{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $V_{j(\kappa)} \subseteq N$, and we derive an extender by letting for $a \in [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{<\omega}$,

$$X \in E_a \text{ iff } a \in j(X),$$

then the extender power of the universe witnesses that κ is superstrong with target λ. So it is sufficient to derive an embedding as above. For this, consider the following theory:

$$T = \text{ElDiag}_{\mathcal{L}^2_{\kappa\omega}}(V_{\kappa+1}, \in) \cup \{c_i < c < c_\kappa \colon i < \kappa\},$$

where c is a new constant and the c_i are the constants from the elementary diagram. Clearly, T can be considered to have rank $< \kappa + \omega$ and can be written as an increasing union of length κ of theories T_α for $\alpha < \kappa$ by considering in T_α only those bits of the second part of T such that $i < \alpha$. Then $(V_{\kappa+1}, \in)$ gives a model of T_α of size $\geq \kappa$ and of rank $< \kappa + \omega$. By (2), we get a transitive set M and $\mathcal{A} \in M$ such that $M \models$ "$\mathcal{A} \models \varphi$" for every φ from (a copy of) T and such that $M \models \text{ZFC}^*$, $V_\lambda \subseteq M \subseteq V_{\lambda+\omega}$, $|A| \geq \lambda$ and $M \models \lambda = \beth^M_\lambda$. Because T contains Magidor's Φ, we have that $A = V^M_\beta$ for some β. By size of A and $\lambda = \beth^M_\lambda$, we get $\beta \geq \lambda$. Furthermore, in V we see that $\mathcal{A} \models T$ and there is an elementary embedding $j : V_{\kappa+1} \to A = V^M_\beta$ such that $\text{crit}(j) = \kappa$. Because $A \in M \subseteq V_{\lambda+\omega}$, this implies $\beta = \lambda + 1$ and then clearly $j(\kappa) = \lambda$. Because $V_\lambda \subseteq M$, finally $V_{j(\lambda)} = V_\lambda \subseteq V^M_{\lambda+1} = A$. $\square$

3. Cardinal Correctly Extendible Cardinals

Remarks on co-authorship. The definitions of Section 3.2, as well as the results of Sections 3.3 and 3.5 (except for 3.5.7 to 3.5.12) are joint with Victoria Gitman and appear in [GO24]. The results of Section 3.4 are joint with Alejandro Poveda. The results numbered 3.5.10 to 3.5.12 are joint with Will Boney.

3.1. Introduction

In this chapter, we introduce new large cardinal notions: *cardinal correctly extendible cardinals* and their variants. We will see in Section 3.5 and Chapter 4 that these large cardinals naturally arise from trying to characterise strong compactness cardinals and so-called *ULST numbers* for the logic $\mathcal{L}(\mathsf{I})$. They further are a natural weakening of extendible cardinals. We show that they exhibit some form of *identity crisis*: the smallest cardinal correctly extendible cardinal can consistently be equal to, and consistently be larger than the smallest supercompact cardinal.

The chapter is structured as follows. Section 3.2 states the definitions of the cardinals we will consider. In Section 3.3, we will show that cardinal correctly extendible cardinals are strongly compact, and that also the consistency of the weaker variants we consider implies the consistency of the existence of a strongly compact cardinal. Further, we separate them from supercompactness, by showing that the smallest cardinal correctly extendible may consistently be larger than the smallest supercompact cardinal. In Section 3.4, we give a general result that under certain assumptions on the relation of HOD and V, the smallest extendible cardinal is cardinal correctly extendible in HOD (Theorem 3.4.10). We then employ a model by Goldberg and Poveda, to derive a situation in which the smallest cardinal correctly extendible cardinal, the smallest strongly compact, and the smallest supercompact cardinal are all equal. In particular, this separates cardinal correctly extendible cardinals from the usual extendibles. Finally, in Section 3.5, we give a characterisation of the compactness number of the logic $\mathcal{L}(\mathsf{I})$ and their infinitary version $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$ by variations of cardinal correctly extendible cardinals. We further give two proofs that $\text{comp}(\mathcal{L}(\mathsf{I}))$ may consistently be larger than the first supercompact cardinal.

3.2. Motivation and definitions

Recall that a cardinal κ is extendible if for every $\alpha > \kappa$ there is some β and an elementary embedding $j : V_\alpha \to V_\beta$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. Cardinal correctly extendible cardinals arrive from weaker assumptions on the target models involved in the embeddings witnessing the notion.

Definition 3.2.1. Let M be a transitive set. We call M *cardinal correct* if the cardinals in the sense of M are also cardinals in the sense of V, i.e., $\mathrm{Card}^M = \mathrm{Card} \cap M$.

Definition 3.2.2. A cardinal κ is *cardinal correctly extendible* if for every $\alpha > \kappa$ there is an elementary embedding $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and M cardinal correct. A cardinal κ is *weakly cardinal correctly extendible* if we remove the requirement that $j(\kappa) > \alpha$.

Clearly, extendible cardinals are cardinal correctly extendible because the rank initial segments V_β are always cardinal correct. Recall Theorem 1.3.23, that the property that $j(\kappa) > \alpha$ comes for free in the case of extendible cardinals, but it is not known to us whether this is the case for cardinal correctly extendibles.

Question 3.2.3. Are weakly cardinal correctly extendible cardinals and cardinal correctly extendible cardinals equivalent?

For our applications in model theory, the following notion will further be relevant:

Definition 3.2.4. A cardinal κ is called *cardinal correctly extendible pushing up some* $\delta \geq \kappa$ iff for every $\alpha > \kappa$ there is an elementary embedding $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\delta) > \alpha$, and M cardinal correct.

Note that if κ is cardinal correctly extendible pushing up some δ, then it is weakly cardinal correctly extendible, and if it is cardinal correctly extendible pushing up κ, then it is cardinal correctly extendible.

Recall that Magidor showed that the smallest strongly compact cardinal can consistently be the smallest measurable cardinal, but also the smallest supercompact cardinal (cf. [Mag76]). This phenomenon was dubbed the *identity crisis* of strongly compact cardinals. Our results will imply that cardinal correctly extendibles also exhibit some form of identity crisis.

3.3. Relations to supercompactness and strong compactness

We will first mention some results on strongly compact cardinals before studying their relation to (weakly) cardinal correctly extendibles (cf. Propositions 3.3.2 and 3.3.4). We then show that the smallest cardinal correctly extendible may exceed the smallest supercompact cardinal (cf. Theorem 3.3.6).

Recall that a cardinal κ is λ-*compact* for some $\lambda \geq \kappa$ if there is a fine κ-complete ultrafilter over $P_\kappa(\lambda)$ and a cardinal κ is strongly compact if it is λ-compact for every $\lambda \geq \kappa$. By a theorem of Ketonen, if κ and λ are regular, then κ is λ-compact if and only if every regular α in the interval $[\kappa, \lambda]$ carries a uniform κ-complete ultrafilter [Ket72, Theorem 5.9]. As Goldberg points out in [Gol21], using a theorem of Kunen and Příkrý, if λ is a successor cardinal, it suffices to show this only for successor cardinals in the interval $[\kappa, \lambda]$. Kunen and Příkrý showed that if κ is regular and U is a κ^+-descendingly incomplete ultrafilter on some set, then U is already κ-descendingly incomplete [KP71, Theorem 0.2]. An ultrafilter U is δ-*descendingly incomplete* if there is a decreasing sequence of sets in U whose intersection is empty. If an ultrafilter U is κ-complete and δ-descendingly incomplete, we claim that:

$$\text{There is a uniform } \kappa\text{-complete ultrafilter W over } \delta. \qquad (*)$$

That U is δ-descendingly incomplete means that there is $\{A_i : i < \delta\} \subseteq U$ such that $A_i \supseteq A_j$ for $i < j$ and $\bigcap_{i<\delta} = \emptyset$. Suppose without loss of generality that U is an ultrafilter over A_0. Now define a function

$$f : A_0 \to \delta, a \mapsto \text{least } i \text{ such that } a \notin A_i,$$

and let $W = \{X \subseteq \delta : f^{-1}[X] \in U\}$. Then it is standard to check that W is the desired uniform κ-complete ultrafilter, and hence $(*)$ holds.

Now suppose there is a uniform κ-complete ultrafilter over β^+ that is κ-complete. In particular, U is β^+-descendingly incomplete, and hence by Kunen's and Příkrý's theorem, it is β-descendingly incomplete. Thus, by $(*)$, there is a uniform κ-complete ultrafilter W over β. Thus:

Theorem 3.3.1 (Goldberg)**.** If κ is regular and λ is a successor cardinal, then κ is λ-compact if and only if every successor cardinal in the interval $[\kappa, \lambda]$ carries a uniform κ-complete ultrafilter.

We can now show our results on the relation of cardinal correctly extendibles to strongly compact cardinals.

Proposition 3.3.2. If κ is cardinal correctly extendible, then κ is strongly compact.

Proof. By Theorem 3.3.1, it suffices to argue that every successor cardinal $\beta^+ > \kappa$ carries a uniform κ-complete ultrafilter. Let $\alpha > \beta^+$ and take $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and M cardinal correct. Since $j(\kappa) > \alpha > \beta^+$, it follows that $j(\beta) > \beta$. Because M is cardinal correct, we have that $j(\beta)$ is a cardinal and

$$(j(\beta)^+)^M = j(\beta^+) = j(\beta)^+ > \beta^+.$$

This means that $j(\beta^+)$ is regular, and so, in particular, j is discontinuous at β^+, i.e., $j``\beta^+$ is bounded in $j(\beta^+)$. Thus, we can let $\gamma = \sup(j``\beta^+) < j(\beta^+)$. It is then easy to check that we can define a uniform κ-complete ultrafilter U on β^+ by letting $X \in U$ if and only if $\gamma \in j(X)$. $\square$

For the relation of strong compactness to weakly cardinal correctly extendibles, we will use the following lemma which is implicit in [Gol21, Theorem 2.10].[1]

Lemma 3.3.3 (Goldberg)**.** If $j : V_\lambda \to M$ is an elementary embedding with M cardinal correct, $\mathrm{crit}(j) = \kappa$ and such that $\lambda = \sup\{j^n(\kappa) \colon n \in \omega\}$, then $j(\kappa)$ is inaccessible and $V_{j(\kappa)} \models$ "κ is strongly compact".

Proposition 3.3.4. If κ is weakly cardinal correctly extendible, then κ is strongly compact or there is an inaccessible cardinal α such that V_α satisfies that κ is a strongly compact cardinal.

Proof. Suppose that κ is not strongly compact. Choose some successor $\lambda > \kappa$ such that κ is not λ-compact. Let $j : V_{\lambda^+} \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and M cardinal correct. If $j(\kappa) \geq \lambda$, then the argument of the proof of Proposition 3.3.2 would show that κ is λ-compact. Thus, we have $j(\kappa) < \lambda$. This means we can apply j to $j(\kappa) = \gamma$ to get $j^2(\kappa) = j(\gamma)$. Again, assuming that $j(\gamma) \geq \lambda$, we will argue that κ must be λ-strongly compact, and so will be able to conclude that $j(\gamma) < \lambda$. By the same argument as before, we get a discontinuity for successors of $\gamma \leq \beta < \lambda$. But if $\kappa \leq \beta < \gamma$, then $\beta < \gamma = j(\kappa) \leq j(\beta)$. Repeating this argument, we get that $j^n(\kappa) < \lambda$ for all $n < \omega$. Letting $\gamma = \sup\{j^n(\kappa) \colon n < \omega\}$ we get that j restricts to $j : V_\gamma \to V_\gamma^M$ and the latter is cardinal correct, because M is. By Lemma 3.3.3, $j(\kappa)$ is inaccessible and $V_{j(\kappa)}$ satisfies that κ is a strongly compact cardinal. Thus, we proved what we promised. □

Thus, a weakly cardinal correctly extendible κ is either strongly compact or there is some ordinal λ such that for cofinally many $\alpha > \kappa$, if $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$ and M cardinal correct, then $j^n(\kappa) < \lambda$ for all $n < \omega$. It is not known to us whether this situation is consistent.

We will use the following lemma to separate supercompact cardinals from cardinal correctly extendible cardinals.

Lemma 3.3.5. If κ is cardinal correctly extendible pushing up δ and the GCH fails at some cardinal $\gamma \geq \delta$, then the GCH fails cofinally often.

Proof. Suppose $\delta \leq \gamma$ and $2^\gamma > \gamma^+$. Fix an ordinal $\lambda > 2^\gamma$ and let $\alpha > \lambda$. We will show that the GCH fails somewhere above λ. Let $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$, M cardinal correct, and $j(\delta) > \alpha$. We have $j(\gamma) \geq j(\delta) > \alpha > \lambda$. Since $V_\alpha \models 2^\gamma > \gamma^+$, by elementarity, $M \models 2^{j(\gamma)} > j(\gamma)^+$. Since M is cardinal correct, the $j(\gamma)^+$ of M is the real $j(\gamma)^+$ and $(2^{j(\gamma)})^M$ is a cardinal. Thus, the GCH really must fail at $j(\gamma)^+$. Since λ was chosen arbitrarily, it follows that the GCH fails cofinally often. □

Theorem 3.3.6. It is consistent, relative to an extendible cardinal, that there is an extendible cardinal and for every pair $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ, δ is bigger than the least supercompact cardinal.

[1]Goldberg formulates his result for embeddings $j : V \to M$ such that M is cardinal correct, but his argument considers the restriction $j \restriction V_\lambda$ for $\lambda = \sup\{j^n(\kappa) \colon n \in \omega\}$. Our Lemma 3.3.3 thus implicitly follows from his arguments.

Proof. We can force the GCH to hold at all regular cardinals while preserving an extendible cardinal [Tsa13]. So we can suppose that we start in a model V in which the GCH holds at all regular cardinals and there is an extendible cardinal χ. Let ν be the least supercompact cardinal, and note that $\nu < \chi$. We force with the Laver preparation [Lav78] to make the supercompactness of ν indestructible by all $<\nu$-directed closed forcing and let $V[G]$ be the resulting forcing extension. Since the Laver preparation has size ν, the GCH still holds above ν in $V[G]$ (while it fails badly below ν) and χ remains extendible. Now fix any cardinal $\nu < \gamma < \chi$ and force with $\mathrm{Add}(\gamma, \gamma^{++})$, the forcing to add γ^{++}-many Cohen subsets to γ, and let $V[G][g]$ be the forcing extension. By the indestructibility of ν in $V[G]$, since $\mathrm{Add}(\gamma, \gamma^{++})$ is $<\nu$-directed closed, ν remains supercompact in $V[G][g]$. Also, the GCH holds above γ in $V[G][g]$ and χ remains extendible. Thus, by Lemma 3.3.5, in $V[G][g]$, there cannot be a pair $\kappa \leq \delta$ such that $\delta \leq \gamma$ and κ is cardinal correctly extendible pushing up δ. □

Corollary 3.3.7. It is consistent that a supercompact cardinal is not cardinal correctly extendible.[2]

3.4. Cardinal correctly extendible cardinals in HOD

In this section, we will show that it is consistent that the first cardinal correctly extendible cardinal is equal to the first supercompact cardinal. Our main Theorem 3.4.10 shows that extendible cardinals, under certain assumptions, are cardinal correctly extendible in HOD. To show this, we will first discuss some relevant results on HOD. We then use a model provided by Goldberg and Poveda which satisfies the assumptions of our Theorem 3.4.10, and in which the first extendible cardinal is the first supercompact cardinal in HOD. In particular, it is consistent that smallest cardinal correctly extendible is simultaneously the smallest supercompact and strongly compact cardinal (cf. Corollary 3.4.14), and not extendible (cf. Corollary 3.4.15).

Recall (cf., e.g., [Kun80, Ch.5, §2]) that a set a is *ordinal definable* if there are ordinals $\alpha_1 < \cdots < \alpha_n < \alpha$ and $\varphi(x, x_1, \ldots, x_n) \in \mathcal{L}_{\omega\omega}[\{\in\}]$ such that

$$V_\alpha \models \forall x(x \in a \leftrightarrow \varphi(x, \alpha_1, \ldots, \alpha_n)).$$

We call the class of ordinal definable sets OD. Further, a is called *hereditarily ordinal definable* if $\mathrm{tcl}(\{a\}) \subseteq \mathrm{OD}$, and the class of all such sets is called HOD. The more interesting of these classes is HOD, as this is a transitive model of ZFC (again, cf. [Kun80, Ch.5, §2]).

It is easy to see that

$$a \in \mathrm{HOD} \text{ iff } \exists\alpha(V_\alpha \models \text{“}a \in \mathrm{HOD}\text{”}).$$

In particular, HOD is Σ_2-definable by the above formula. We fix the following useful facts:

[2]This result was first pointed out to us by Alejandro Poveda with a more complicated argument using Radin forcing (cf. [Pov24, Theorem 5.2]).

Proposition 3.4.1. (i) Let $\alpha \in C^{(2)}$. Then $\mathrm{HOD}^{V_\alpha} = V_\alpha^{\mathrm{HOD}}$.

(ii) For all α, HOD^{V_α} is transitive.

(iii) For all α, $\mathrm{HOD}^{V_\alpha} \in \mathrm{HOD}$.

Proof. For (i), if $V_\alpha \models a \in \mathrm{HOD}$, then this means that for every $b \in \mathrm{tcl}(\{a\})$, there are ordinals $\beta_1, \ldots, \beta_n, \beta < \alpha$ and a formula $\varphi(x, x_1, \ldots, x_n)$ such that

$$V_\alpha \models \text{“}V_\beta \models \forall x(x \in b \leftrightarrow \varphi(x, \beta_1, \ldots, \beta_n))\text{”}.$$

Clearly, V_α is correct about V_β satisfying this, so really $b \in \mathrm{OD}$. Because V_α can compute the transitive closure of $\{a\}$, this implies that really $\mathrm{tcl}(\{a\}) \subseteq \mathrm{OD}$ and thus $a \in V_\alpha \cap \mathrm{HOD} = V_\alpha^{\mathrm{HOD}}$. Thus $\mathrm{HOD}^{V_\alpha} \subseteq V_\alpha^{\mathrm{HOD}}$. On the other hand if $a \in V_\alpha^{\mathrm{HOD}} = V_\alpha \cap \mathrm{HOD}$, then by Σ_2-correctness, $V_\alpha \models a \in \mathrm{HOD}$ and so $a \in \mathrm{HOD}^{V_\alpha}$.

For (ii), let $b \in a \in \mathrm{HOD}^{V_\alpha}$. Then $V_\alpha \models \mathrm{tcl}(\{a\}) \subseteq \mathrm{OD}$. Because $b \in a$, clearly $\mathrm{tcl}(\{b\}) \subseteq \mathrm{tcl}(\{a\})$, and further V_α can compute all these transitive closures. Thus $V_\alpha \models \mathrm{tcl}(\{b\}) \subseteq \mathrm{OD}$, and therefore $b \in \mathrm{HOD}^{V_\beta}$.

For (iii), if $a \in \mathrm{HOD}^{V_\alpha}$, then $V_\alpha \models a \in \mathrm{HOD}$ and so $a \in \mathrm{HOD}$ by our remarks above. As HOD^{V_α} is transitive, it is thus sufficient to show that $\mathrm{HOD}^{V_\alpha} \in \mathrm{OD}$. But any $\beta > \alpha$ has the following ordinal definition of HOD^{V_α}: $V_\beta \models \forall x(x \in \mathrm{HOD}^{V_\alpha} \leftrightarrow V_\alpha \models x \in \mathrm{HOD})$. □

We will use some results about closeness of HOD and V. Woodin isolated desiderata for an inner model with a supercompact cardinal by studying the relationship of supercompact cardinals of V and HOD. In particular, he introduced the following notion:

Definition 3.4.2 (Woodin [Woo17, Definition 3.5])**.** Let $N \models \mathrm{ZFC}$ be a transitive class and δ some ordinal. We call N a *weak extender model for “δ is supercompact”* iff for every $\gamma > \delta$ there is a fine, normal, δ-complete ultrafilter U over $\mathcal{P}_\delta\gamma$ such that

(i) $N \cap \mathcal{P}_\delta\gamma \in U$ and

(ii) $U \cap N \in N$.

For us, weak extender models for “δ is supercompact” are important, as we will employ the following theorem which tells us about N-extenders for some inner model N belonging to N itself.

Theorem 3.4.3 (Universality Theorem; Woodin [Woo17, Theorem 3.26])**.** Let N be a weak extender model for “δ is supercompact”. Suppose there is an N-extender of length β with critical point $\kappa \geq \delta$. Let $j_E : N \to N_E$ be the extender power embedding. Then the following are equivalent:

(1) For all $A \in \mathcal{P}^N(\beta)$: $j_E(A) \cap \beta \in N$.

(2) $E \in N$.

Interesting for inner model theory are closeness results between some inner model and V. For HOD, the following is a relevant closeness measure.

Definition 3.4.4 (Woodin [Woo17, Definition 3.35])**.** Let λ be an uncountable regular cardinal. We say that λ is *ω-strongly measurable in HOD* iff there exists $\kappa < \lambda$ such that

(1) $(2^\kappa)^{\mathrm{HOD}} < \lambda$ and

(2) The stationary set $\{\alpha < \lambda \colon \mathrm{cof}(\alpha) = \omega\} \in V$ does not have a partition $(S_\alpha \colon \alpha < \kappa)$ into stationary sets such that $(S_\alpha \colon \alpha < \kappa) \in \mathrm{HOD}$.

Note that by Solovay's Theorem (cf. [Sol71, Theorem 9], or [Jec03, Theorem 8.10]) such a partition of $\{\alpha < \lambda \colon \mathrm{cof}(\alpha) = \omega\}$ exists in V. Thus, λ being ω-strongly measurable in HOD means that HOD is in this sense far away from V. The following axiom studied by Woodin is therefore a positive closeness assumption about V and HOD.

Definition 3.4.5 (Woodin [Woo17, Definition 3.42])**.** The *HOD Hypothesis* is the statement "There exists a proper class of regular cardinals which are *not* ω-strongly measurable in HOD."

The result relevant for our analysis is:

Theorem 3.4.6 (Woodin [Woo17, Theorem 3.44])**.** Let δ be an extendible cardinal and suppose that the HOD Hypothesis holds. Then HOD is a weak extender model for "δ is supercompact".

Another result we will use is that under the assumptions of this theorem, nicely approximated subsets of HOD, are already in HOD.

Definition 3.4.7 (Hamkins [Ham03])**.** Let $N \models \mathrm{ZFC}$ be transitive and κ a cardinal. N is said to have the *κ-approximation property* iff for all sets $A \subseteq N$, if for all $\sigma \in N$ such that $|\sigma|^N < \kappa$ also $A \cap \sigma \in N$, then also $A \in N$.

Theorem 3.4.8 (Woodin [Woo, Theorem 6.26])**.** Let δ be an extendible cardinal and suppose that N is a weak extender model for "δ is supercompact". Then N has the δ-approximation property. In particular, if δ is extendible and the HOD Hypothesis holds, then HOD has the δ-approximation property.

The situation, in which δ is both the first supercompact and the first cardinal correctly extendible cardinal is given by the following yet unpublished theorem.

Theorem 3.4.9 (Goldberg & Poveda)**.** Suppose it is consistent that there is an extendible cardinal. Then there is a model of ZFC in which the following hold:

(1) There exists a smallest extendible cardinal δ.

(2) The HOD Hypothesis.

(3) δ is the smallest supercompact cardinal in HOD.

(4) δ is the smallest strongly compact cardinal in HOD.

(5) Every HOD-cardinal is a cardinal.

We are now ready to show our theorem.

Theorem 3.4.10. Suppose that there is a smallest extendible cardinal δ, that the HOD Hypothesis holds, and that every HOD-cardinal is a cardinal. Then δ is cardinal correctly extendible in HOD.

Notice that the consistency of the assumption of Theorem 3.4.10 is provided by Goldberg's and Poveda's Theorem 3.4.9.

Proof. We will show that for a proper class of ordinals $\alpha > \delta$, HOD has an elementary embedding $i : V_\alpha^{\mathrm{HOD}} \to M$ such that $\mathrm{crit}(i) = \delta$, $i(\delta) > \alpha$, and M and HOD agree on cardinals. Let $\alpha > \delta$ be in $C^{(2)}$ of cofinality $\mathrm{cof}(\alpha) > \delta$. By extendibility of δ, there is an elementary embedding $j : V_{\alpha+1} \to V_{\beta+1}$ such that $\mathrm{crit}(j) = \delta$ and $j(\delta) > \alpha + 1$. Notice that $j(\alpha) = \beta$, so we can derive an extender $(E_a \colon a \in [\beta]^{<\omega})$ by letting for $X \subseteq [\alpha]^{|a|}$:

$$X \in E_a \text{ iff } a \in j(X).$$

It is easy to check that letting $F = (E_a \cap \mathrm{HOD} \colon a \in [\beta]^{<\omega})$ gives us a HOD-extender. So we can consider the extender power $j_F : \mathrm{HOD} \to M_F$. Our goal is to show that $F \in \mathrm{HOD}$ and that $j_F \restriction V_\alpha^{\mathrm{HOD}}$ is our desired embedding. We first prove some claims about the relationship between j and j_F, and between F and HOD.

Claim 3.4.11. $j_F \restriction V_\alpha^{\mathrm{HOD}} = j \restriction V_\alpha^{\mathrm{HOD}}$ and $V_\beta^{M_F} = \mathrm{HOD}^{V_\beta}$.

Proof. Standard results imply that

$$M_F = \{j_F(f)(a) \colon a \in [\beta]^{<\omega} \wedge f \in {}^{|\alpha|^{|a|}}\mathrm{HOD} \cap \mathrm{HOD}\},$$

$j_F(\alpha) \geq \beta$ and $j_F(f)(a) = [a, [f]]$ for every $a \in [\beta]^{<\omega}$ and $f \in {}^{|\alpha|^{|a|}}\mathrm{HOD} \cap \mathrm{HOD}$ (cf. [Kan03, Lemma 26.2], where the latter assertion is not explicitly stated but shown in the proof of part (c)). Let us define a map $k : V_\beta^{M_F} \to \mathrm{HOD}^{V_\beta}$ by

$$j_F(f)(a) \mapsto j(f)(a).$$

Our goal is to show that k is a bijection.

We first argue that k is well-defined. As an extender power, M_F is the direct limit of the ultrapowers $\mathrm{Ult}(\mathrm{HOD}, F_a)$ which come with elementary embeddings $j_a : \mathrm{HOD} \to \mathrm{Ult}(\mathrm{HOD}, F_a)$ defined by

$$x \mapsto [c_x]_{F_a}$$

and $k_a : \mathrm{Ult}(\mathrm{HOD}, F_a) \to M_F$ defined by

$$[f]_{F_a} \mapsto [a, [f]] = j_F(f)(a),$$

and further $j_F = k_a \circ j_a$. Now let $j_F(f)(a) \in V_\beta^{M_F}$. We have $j_F(f)(a) = k_a([f]_{F_a})$. Note that $\beta \leq j_F(\alpha) = k_a(j_a(\alpha))$. Therefore $M_F \models k_a([f]_{F_a}) \in V_{k_a(j_a(\alpha))}$. Thus by elementarity, $\mathrm{Ult}(\mathrm{HOD}, F_a) \models [f]_{F_a} \in V_{j_a(\alpha)}$. Note that $j_a(\alpha) = [c_\alpha]_{F_a}$ and so by Łos,

$$\{s \in [\alpha]^{|a|} \colon \mathrm{HOD} \models f(s) \in V_\alpha\} = \{s \in [\alpha]^{|a|} \colon \mathrm{HOD} \models f(s) \in V_{c_\alpha(s)}\} \in F_a.$$

Therefore without loss of generality, f is a function $f : [\alpha]^{|a|} \to V_\alpha^{\text{HOD}}$. Recall that $\alpha \in C^{(2)}$ and so $V_\alpha^{\text{HOD}} = \text{HOD}^{V_\alpha}$. Further, $f \in V_{\alpha+1}$. Therefore it makes sense to apply j to f and then by elementarity, $j(f) : [\beta]^{|a|} \to \text{HOD}^{V_\beta}$. Thus $j(f)(a) \in \text{HOD}^{V_\beta}$ and hence k is well-defined.

The following chain of equivalences shows that k is elementary and, in particular, injective.

$$\begin{aligned} V_\beta^{M_F} \models \varphi(j_F(f)(a)) &\text{ iff } M_F \models \text{“}V_{k_a(j_a(\alpha))} \models \varphi(k_a([f]_{f_a}))\text{”} \\ &\text{ iff } \text{Ult}(\text{HOD}, F_a) \models \text{“}V_{[c_\alpha]_{F_a}} \models \varphi([f]_{F_a})\text{”} \\ &\text{ iff } \{s \in [\alpha]^{|a|} \colon \text{HOD} \models \text{“}V_\alpha \models \varphi(f(s))\text{”}\} \in F_a \\ &\text{ iff } \{s \in [\alpha]^{|a|} \colon V_\alpha^{\text{HOD}} \models \varphi(f(s))\} \in F_a \\ &\text{ iff } \{s \in [\alpha]^{|a|} \colon \text{HOD}^{V_\alpha} \models \varphi(f(s))\} \in F_a \\ &\text{ iff } a \in j(\{s \in [\alpha]^{|a|} \colon \text{HOD}^{V_\alpha} \models \varphi(f(s))\}) \\ &\text{ iff } \text{HOD}^{V_\beta} \models \varphi(j(f)(a)). \end{aligned}$$

Here we use again that $\text{HOD}^{V_\alpha} = V_\alpha^{\text{HOD}}$.

And now we argue for surjectivity of k. Because $\alpha \in C^{(2)}$, $\alpha = \beth_\alpha$, and this is downwards absolute to HOD. So HOD has a bijection $g : [\alpha]^1 \to V_\alpha^{\text{HOD}} = \text{HOD}^{V_\alpha}$. Further $g \in V_{\alpha+1}$. Thus, by elementarity, $V_{\beta+1} \models$ “$j(g)$ is a bijection $[\beta]^1 \to \text{HOD}^{V_\beta}$”. Therefore, for $x \in \text{HOD}^{V_\beta}$, there is $\eta < \beta$ such that $j(g)(\{\eta\}) = x$. Thus $k(j_F(g)(\{\eta\})) = x$ and so we showed that k is surjective.

We showed that k is an isomorphism between transitive structures and therefore k is the identity. In particular, $V_\beta^{M_F} = \text{HOD}^{V_\beta}$. Finally, $j_F(x) = [a, [c_x]_{F_a}] = j_F(c_x)(a) = k(j_F(c_x)(a)) = j(c_x)(a) = j(x)$. □

Claim 3.4.12. $F \in \text{HOD}$.

Proof. Here we use Woodin's machinery on HOD. Because the HOD Hypothesis holds, by Theorem 3.4.6, HOD is a weak extender model for “δ is supercompact”. By the Universality Theorem 3.4.3, it is thus sufficient to show that for any $A \in \mathcal{P}^{\text{HOD}}(\beta)$, $j_F(A) \cap \beta \in \text{HOD}$. First consider $\gamma < \beta$. Then $j_F(A) \cap \gamma \in V_\beta^{M_F}$. By the previous Claim 3.4.11, $V_\beta^{M_F} = \text{HOD}^{V_\beta}$. Therefore $j_F(A) \cap \gamma \in \text{HOD}^{V_\beta} \in \text{HOD}$. This implies $j_F(A) \cap \gamma \in \text{HOD}$, as HOD is transitive.

By Theorem 3.4.8, in our situation HOD has the δ approximation property. Consider $\sigma \in \text{HOD}$ such that $|\sigma|^{\text{HOD}} < \delta$. If we can show that $j_F(A) \cap \beta \cap \sigma \in \text{HOD}$, then the approximation property tells us that $j_F(A) \cap \beta \in \text{HOD}$ and we are done. So take such a σ. We took α of cofinality $> \delta$, so by elementarity of j, $\text{cof}^{V_{\beta+1}}(\beta) > j(\delta) > \delta$. Of course, $V_{\beta+1}$ is correct about cofinalities and so really $\text{cof}(\beta) > \delta$. Thus, if $|\sigma|^{\text{HOD}} < \delta$, then $j_F(A) \cap \beta \cap \sigma = j_F(A) \cap \gamma \cap \sigma$ for some $\gamma < \beta$. But we already showed above that $j_F(A) \cap \gamma \in \text{HOD}$, and as $\sigma \in \text{HOD}$, thus also $j_F(A) \cap \gamma \cap \sigma \in \text{HOD}$. □

Now we will piece everything together. Because $F \in \mathrm{HOD}$, we get that HOD can compute $j_F \restriction V_\alpha^{\mathrm{HOD}}$. Furthermore, we assumed $\alpha \in C^{(2)}$ and so $V_\alpha^{\mathrm{HOD}} = \mathrm{HOD}^{V_\alpha}$. Notice that if $a \in \mathrm{HOD}^{V_\alpha}$, then by elementarity, $j(a) \in \mathrm{HOD}^{V_\beta}$. By Claim 3.4.11, it follows that $j \restriction V_\alpha^{\mathrm{HOD}} = j_F \restriction V_\alpha^{\mathrm{HOD}}$. Thus $j_F \restriction V_\alpha^{\mathrm{HOD}} = j \restriction V_\alpha^{\mathrm{HOD}} : V_\alpha^{\mathrm{HOD}} \to \mathrm{HOD}^{V_\beta}$. Further, by Proposition 3.4.1, $\mathrm{HOD}^{V_\beta} \in \mathrm{HOD}$, so HOD knows of the elementary embedding

$$j_F \restriction V_\alpha^{\mathrm{HOD}} : V_\alpha^{\mathrm{HOD}} \to \mathrm{HOD}^{V_\beta}.$$

We have $\mathrm{crit}(j_F) = \mathrm{crit}(j) = \delta$ and $j_F(\delta) = j(\delta) > \alpha$. So we are done, if we can show that HOD^{V_β} is cardinal correct in HOD.

Claim 3.4.13. HOD^{V_β} is cardinal correct in HOD.

Proof. If $\mathrm{HOD}^{V_\alpha} \models$ "γ is a cardinal", then as $\mathrm{HOD}^{V_\alpha} = V_\alpha^{\mathrm{HOD}}$, γ is a cardinal in V_α^{HOD}, and thus in particular in HOD. Because by assumption every HOD-cardinal is a cardinal, then γ really is a cardinal. Then in particular, $V_\alpha \models$ "γ is a cardinal". This shows that

$$V_\alpha \models \text{"Every HOD-cardinal is a cardinal"}.$$

By elementarity of j, V_β satisfies this statement, and so every HOD^{V_β}-cardinal is a cardinal, and hence in particular a HOD-cardinal. □

□

Corollary 3.4.14. It is consistent, relative to consistency of the existence of an extendible cardinal, that the smallest strongly compact, supercompact and cardinal correctly extendible cardinal all coincide.

Proof. This follows immediately from considering the model of Theorem 3.4.9 and applying Theorem 3.4.10. □

Corollary 3.4.15. If the existence of a cardinal correctly extendible cardinal is consistent, then it is consistent that there is a cardinal correctly extendible cardinal which is not extendible.

Proof. This follows as the smallest supercompact cardinal cannot be extendible. □

Theorem 3.3.6 and Corollary 3.4.14 together show that the first cardinal correctly extendible cardinals exhibits some form of identity crisis. The following problems remain open.

Question 3.4.16. Is it consistent, relative to large cardinals, that the first cardinal correctly extendible cardinal is smaller than the first supercomapct cardinal?

Question 3.4.17. Is it consistent, relative to large cardinals, that the first cardinal correctly extendible cardinal is the first extendible cardinal?

3.5. Compactness numbers of the equicardinality logic

In this section, we introduce a large cardinal variant of cardinal correctly extendibles characterising being a compactness number for the logic $\mathcal{L}(\mathsf{I})$ (cf. Theorem 3.5.2). We then consider the relation of $\mathrm{comp}(\mathcal{L}(\mathsf{I}))$ to strongly compact cardinals (cf. Corollaries 3.5.3 and 3.5.4). We give two proofs that $\mathrm{comp}(\mathcal{L}(\mathsf{I}))$ may consistently be larger than the smallest supercompact cardinal (cf. Corollaries 3.5.6 and 3.5.12). Finally, we consider $\mathrm{comp}(\mathcal{L}_{\delta\delta}(\mathsf{I}))$ (cf. Theorem 3.5.13).

In order to obtain our theorem, we first need to argue that under certain circumstances, we can express well-foundedness in the logic $\mathcal{L}(\mathsf{I})$, which we cannot generally do. However, it turns out that we can express well-foundedness over models of a sufficiently large fragment of set theory that are cardinal correct. Recall that $\mathcal{L}(\mathsf{I})$ can express that a transitive set is cardinal correct via the sentence

$$\varphi_{\mathrm{Card}} = \forall x(\mathrm{Card}(x) \leftrightarrow (\mathrm{Ord}(x) \wedge \forall y(y \in x \to \neg \mathsf{I} zz(z \in y, z \in x))).$$

Let ZFC^*_a be a sufficiently large finite fragment of ZFC such that ZFC^*_a contains the statements:

(i) Every set has a rank, i.e., for every ordinal α, V_α exists and every set is a member of V_α for some minimal ordinal α.

(ii) For every ordinal α, the α-th cardinal $\aleph_\alpha$ exists.

The following result was pointed out to us by Gabriel Goldberg.

Theorem 3.5.1 (Goldberg)**.** If $(M, E) \models \mathrm{ZFC}^*_a$ and $(M, E) \models_{\mathcal{L}(\mathsf{I})} \varphi_{\mathrm{Card}}$, then E is well-founded.

Proof. Suppose that E is not well-founded. Then there is a set $\{x_i : i \in \omega\} \subseteq M$ such that from the outside we see that $x_{i+1} E x_i$ for all $i \in \omega$. Since M believes that every set has some rank, for every x_i there is some minimal $\alpha_i \in \mathrm{Ord}^M$ such that $x_i E V^M_{\alpha_i}$. Then $\alpha_{i+1} E \alpha_i$ for all $i \in \omega$, and therefore also $\aleph^M_{\alpha_{i+1}} E \aleph^M_{\alpha_i}$ for all $i \in \omega$. But note that then for all $i \in \omega$, by φ_{Card},

$$|\{y \in M : y E \aleph^M_{\alpha_{i+1}}\}| < |\{y \in M : y E \aleph^M_{\alpha_i}\}|.$$

This is an infinite decreasing sequence of V-cardinals, which is impossible. □

Theorem 3.5.2. The following are equivalent for a cardinal δ:

(a) δ is a strong compactness cardinal for $\mathcal{L}(\mathsf{I})$.

(b) For every $\gamma > \delta$ there is $\alpha > \gamma$, a transitive set M and an elementary embedding $j : V_\alpha \to M$ such that M is cardinal correct, $\mathrm{crit}(j) \leq \delta$ and there exists $d \in M$ such that $j``\gamma \subseteq d$ and $M \models |d| < j(\delta)$.

Proof. First assume (a) and fix $\gamma > \delta$. Take any $\aleph$-fixed point $\alpha > \gamma$ of cofinality $\mathrm{cof}(\alpha) \geq \delta$. Take a constant symbol d and let T be the following $\mathcal{L}(\mathsf{I})$-theory, where the c_i are the constants from the elementary diagram:

$$T = \mathrm{ElDiag}_{\mathcal{L}(\mathsf{I})}(V_\alpha, \in) \cup \{c_i \in d \wedge |d| < c_\delta \colon i < \gamma\}.$$

Clearly, any theory $T_0 \in \mathcal{P}_\delta T$ is satisfiable, as witnessed by the model $(V_\alpha, \in)$ itself, interpreting the constant d by the set X of all $i < \gamma$ such that the sentence $c_i \in d \wedge |d| < c_\delta$ is in T_0 (because $\mathrm{cof}(\alpha) \geq \delta$, $X \in V_\alpha$, so we may interpret d by X). Thus, since δ is a strong compactness cardinal of $\mathcal{L}(\mathsf{I})$, we can fix a model

$$\mathcal{N} = (N, E, c_x^N, d)_{x \in V_\alpha} \models T.$$

The model N is well-founded by Theorem 3.5.1 because ZFC_a^* and φ_{Card} are in the elementary diagram. Thus, by collapsing we can assume that N is transitive and $E = \in$. By φ_{Card}, N is cardinal correct. We get an elementary embedding $j : V_\alpha \to N$ with $\mathrm{crit}(j) \leq \delta$ given by $x \mapsto c_x^N$. Since $\mathcal{N} \models T$, it follows that $|d| < j(\delta)$ and $j``\gamma \subseteq d$.

Next, we assume (b). Let τ be a vocabulary and let T be a $<\delta$-satisfiable $\mathcal{L}(\mathsf{I})[\tau]$-theory. Assume without loss of generality that $|T| = \gamma > \delta$. Let $\gamma' > \gamma$ be large enough that $T, \tau, \delta \in V_{\gamma'}$. By (b), take $\alpha > \gamma'$ and $j : V_\alpha \to M$ such that M is transitive and cardinal correct, $\mathrm{crit}(j) \leq \delta$ and there is a $d \in M$ such that $j``\gamma \subseteq d$ and $|d|^M < j(\delta)$. Using that $V_\alpha$ has a surjection $\gamma \to T$, we find a $d_0 \in M$ such that $j``T \subseteq d_0 \subseteq j(T)$ and $M \models |d_0| < j(\delta)$ (cf. the proof of Theorem 2.3.25). By elementarity, M has a $j(\tau)$-structure $\mathcal{A}$, which it thinks satisfies the $\mathcal{L}(\mathsf{I})[j(\tau)]$-theory d_0. Since M is cardinal correct, it is correct about $\mathcal{L}(\mathsf{I})$-satisfaction, and so $\mathcal{A}$ is really a model of d_0. Thus, in particular, (the reduct of) $\mathcal{A}$ is a $j``\tau$-structure which satisfies $j``T$. Then because j is a renaming $j : T \to j``T$ (cf. Section 1.3.2), $\mathcal{A}$ can be renamed to a τ-structure satisfying the theory T. □

Note that, as there is a proper class of $\gamma > \delta$ but only boundedly many possible critical points $\leq \delta$, if δ is a strong compactness cardinal for $\mathcal{L}(\mathsf{I})$, then there is a fixed $\kappa \leq \delta$ such that for any $\gamma > \delta$ there is $\alpha > \gamma$ and an elementary embedding $j : V_\alpha \to M$ with M transitive and cardinal correct, $\mathrm{crit}(j) = \kappa$ and $d \in M$ such that $j``\gamma \subseteq d$ and $|d|^M < j(\delta)$. We therefore get:

Corollary 3.5.3. If δ is a strong compactness cardinal for $\mathcal{L}(\mathsf{I})$, then there is $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ and δ is κ-strongly compact.

Proof. Take $\kappa \leq \delta$ as a critical point of unboundedly many embeddings witnessing (b) of Theorem 3.5.2 as pointed out above. Then clearly, κ is cardinal correctly extendible pushing up δ. And it follows from the elementary embedding characterisation of κ-strongly compact cardinals (cf. Theorem 1.3.20) and (b) of Theorem 3.5.2 that δ is κ-strongly compact. □

Corollary 3.5.4. Consistency of a strong compactness cardinal for $\mathcal{L}(\mathsf{I})$ implies the consistency of a strongly compact cardinal.

Proof. This follows immediately from the fact that κ from the previous theorem is cardinal correctly extendible pushing up δ and Proposition 3.3.4. $\square$

Question 3.5.5. If δ is a cardinal and there is a cardinal correctly extendible cardinal $\kappa \leq \delta$ pushing up δ, then is δ a strong compactness cardinal for the logic $\mathcal{L}(\mathsf{I})$?

We will show that $\mathrm{comp}(\mathcal{L}(\mathsf{I}))$ can consistently be larger than the first supercompact cardinal.

Corollary 3.5.6. It is consistent that the least strong compactness cardinal for $\mathcal{L}(\mathsf{I})$ is above the least supercompact cardinal.

Proof. Consider the model M from Theorem 3.3.6. Since it has an extendible cardinal, by Theorem 1.3.28 we know that there is a strong compactness cardinal for $\mathcal{L}^2$ and therefore in particular for $\mathcal{L}(\mathsf{I}) \leq \mathcal{L}^2$. But in M, the least cardinal δ such that there is a cardinal correctly extendible $\kappa \leq \delta$ pushing up δ must be above the least supercompact cardinal ν. By Corollary 3.5.3, in particular $\mathrm{comp}(\mathcal{L}(\mathsf{I})) > \nu$. $\square$

This result was first obtained by Will Boney and the author employing a different proof we will present now. For this, we have to consider some results by Magdior and Väänänen connecting model theory of $\mathcal{L}(\mathsf{I})$ to combinatorial principles. More concretely, they reprove results on *good scales* from Shelah's pcf theory (cf. [She94]) for weaker notions tailored to their applications. For this, if $f, g \in \mathrm{Ord}^\omega$, we write

$$f <^* g \text{ iff } f(n) < g(n) \text{ for all but finitely many } n \in \omega.$$

Definition 3.5.7 ([MV11, Definition 12])**.** Let $(f_\alpha \colon \alpha < \mu)$ be a $<^*$-increasing sequence of $f_\alpha \in \mathrm{Ord}^\omega$. We make the following conventions.

(a) An ordinal $\delta \in \mu$ is called a *good point of the sequence* $(f_\alpha \colon \alpha < \mu)$ iff there is a cofinal set $C \subseteq \delta$ and a function $C \to \omega$, $\alpha \mapsto n_\alpha$, such that $\alpha < \beta$ in C and $k > \max(n_\alpha, n_\beta)$ implies $f_\alpha(k) < f_\beta(k)$.

(b) The sequenece $(f_\alpha \colon \alpha < \mu)$ is called *good* if there is a club subset $D \subseteq \mu$ such that all members of D are good points of the sequence.

Lemma 3.5.8 (Magidor & Väänänen ([MV11, Theorem 15]))**.** If $\kappa = \mathrm{LST}(\mathcal{L}(\mathsf{I}))$, then there is no good sequence $(f_\alpha \colon \alpha < \lambda^+)$ of functions $f : \lambda \to \omega$ for any $\lambda \geq \kappa$ with $\mathrm{cof}(\lambda) = \omega$.

Recall that the *Singular Cardinal Hypothesis* (SCH) is the statement:

If λ is singular and $2^{\mathrm{cof}(\lambda)} < \lambda$, then $2^\lambda = \lambda^+$.

Failure of SCH is connected to the existence of good sequences.

Lemma 3.5.9 (Shelah (cf., e.g., [MV11, Lemma 17]))**.** If λ is a singular cardinal of $\mathrm{cof}(\lambda) = \omega$ such that SCH fails at λ, then there is a good sequence $(f_\alpha \colon \alpha < \lambda^+)$ of functions $f_\alpha : \omega \to \lambda$.

We show that the existence of good sequences can be expressed by a sentence of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}, \mathsf{I})$.

Lemma 3.5.10. There is a vocabulary τ and a sentence $\varphi_{\mathrm{good}} \in \mathcal{L}(\mathsf{Q}^{\mathrm{WF}}, \mathsf{I})[\tau]$ such that for any set M, the following are equivalent:

(1) $|M|$ is a successor cardinal λ^+ such that $\mathrm{cof}(\lambda) = \omega$ and there is a good sequence $(f_\alpha \colon \alpha < \lambda^+)$ of functions $f_\alpha : \omega \to \lambda$.

(2) There is a τ-structure $\mathcal{M}$ with universe M such that $\mathcal{M} \models \varphi_{\mathrm{good}}$.

Proof. Consider the vocabulary $\{E, F, G, D, C, S\}$, where D is unary, E, G, C are binary, and F and S are ternary. Then let φ_{good} be the conjunction of the following sentences, where (i) uses Q^{WF} for the well-foundedness assertion:

(i) "E is a well-order without a largest element."

(ii) $\mathrm{ot}(E)$ is a cardinal: $\neg\exists x \mathsf{I} yz(yEx, z = z)$.

(iii) There is a largest cardinal λ:

$$\exists\lambda\forall x((xE\lambda \to \neg\exists w I yz(yEw, zE\lambda)) \wedge (\lambda E x \to \mathsf{I} yz(yE\lambda, zEx))).$$

(iv) "$G(\cdot,\cdot)$ is a function with domain the smallest limit ordinal (i.e., ω) into λ with unbounded range."

(v) "For every α, $F(\alpha,\cdot,\cdot)$ is a function with domain ω and range $\subseteq \lambda$."

(vi) "If $\alpha < \beta$, there is an $n < \omega$ such that $\forall m \geq n$: $F(\alpha, m) < F(\beta, m)$."

(vii) "D is a club subset of the model and $\forall\delta \in D$, $C(\delta,\cdot)$ is good, i.e., $C(\delta,\cdot)$ is a club subset of δ and $S(\delta,\cdot,\cdot)$ is a function with domain $C(\delta,\cdot)$ and range ω such that $\forall\alpha < \beta$ both in $C(\delta,\cdot)$: $\forall k > \max(S(\delta,\alpha), S(\delta,\beta))(F(\alpha,k) < F(\beta,k))$."

Then φ_{good} is as desired. For (vii), note that we can express that, for example, D is club by saying:

$$\forall x\exists y(D(y) \wedge xEy) \qquad \text{("}D\text{ is unbounded")}$$
$$\forall x(\forall y(yEx \to \exists z(yEz \wedge zEx \wedge D(z)) \to D(x)) \qquad \text{("}D\text{ is closed")}$$

□

Lemma 3.5.11. If for every $\kappa < \delta = \mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}, \mathsf{I}))$ there is $\lambda \geq \kappa$ of cofinality ω such that there is a good sequence $(f_\alpha \colon \alpha < \lambda^+)$ of functions $f_\alpha : \omega \to \lambda$, then there are unboundedly many such λ in the ordinals.

Proof. Let ρ be any cardinal. It suffices to show that there is $\gamma \geq \rho$ of cofinality ω such that there is a good sequence of functions $(f_\alpha \in \gamma^\omega : \alpha < \gamma^+)$. Consider the theory $\{\varphi_{\text{good}}\} \cup \{c_i \neq c_j : i < j < \rho^+\}$, where the c_i are new constants. Let $T_0 \subseteq T$ be of size $|T_0| < \delta$. Then for some $\kappa < \delta$, there are κ-many sentences of the form "$c_i \neq c_j$" appearing in T_0. By the assumption take $\lambda \geq \kappa$ such that there is a good sequence $(f_\alpha \in \lambda^\omega : \alpha < \lambda^+)$. Then by Lemma 3.5.10, φ_{good} has a model of with universe λ^+, so of size $> \kappa$. Clearly, this can be expanded to a model of T_0. We showed that T is $< \delta$-satisfiable. Hence, T has a model, which gives rise to a model M of φ_{good} of size $\geq \rho^+$. Then with $|M| = \gamma^+$, γ is as desired, by Lemma 3.5.10. □

Corollary 3.5.12. If there is an extendible cardinal, then it is consistent that $\text{comp}(\mathcal{L}(\mathsf{I}))$ is larger than the first supercompact cardinal.

Proof. Let η be the smallest extendible cardinal and ν the smallest supercompact one. We can use Magidor's forcing from [Mag76, Section 4] to go to a model N in which ν becomes simultaneously the smallest supercompact and the smallest strongly compact cardinal, by introducing unboundedly many points $(\lambda_i)_{i<\nu}$ below ν of cofinality $\text{cof}(\lambda_i) = \omega$ such that SCH fails at λ_i. Because the forcing used has size $< \eta$, η remains extendible in N. Let us work in N. Recall Theorem 1.3.14 that the smallest supercompact $\nu = \text{LST}(\mathcal{L}^2)$. In particular, $\text{LST}(\mathcal{L}(\mathsf{I})) \leq \mathcal{L}^2 \leq \nu$. Thus, by Lemma 3.5.8, for no $\lambda \geq \nu$ of cofinality ω there is a good sequence $(f_\alpha : \alpha < \lambda^+)$ of functions $f_\alpha : \omega \to \lambda$. Furthermore, by Lemma 3.5.9, for all λ_i, $i < \nu$, there is such a good sequence. Thus, using Lemma 3.5.11, it is impossible that $\text{comp}(\mathcal{L}(\mathsf{Q}^{\text{WF}}, \mathsf{I})) \leq \nu$. Because there is an extendible cardinal, $\text{comp}(\mathcal{L}^2)$ exists by Theorem 1.3.28. Therefore also $\text{comp}(\mathcal{L}(\mathsf{Q}^{\text{WF}}, \mathsf{I})) \leq \text{comp}(\mathcal{L}^2)$ exists and is thus larger than ν. But it follows from Theorem 6.2.2 and Proposition 6.2.4 that $\text{comp}(\mathcal{L}(\mathsf{Q}^{\text{WF}}, \mathsf{I})) = \text{comp}(\mathcal{L}(\mathsf{I}))$. Thus, $\text{comp}(\mathcal{L}(\mathsf{I})) > \nu$. □

For a cardinal δ, the logic $\mathcal{L}_{\delta\delta}(\mathsf{I})$ is obtained by adding conjunctions and disjunctions and first-order quantifiers of size $< \delta$ to $\mathcal{L}(\mathsf{I})$. Considering strong compactness cardinals of this logic gives us a sharper version of Corollary 3.5.3.

Theorem 3.5.13. If δ is a strong compactness cardinal for $\mathcal{L}_{\delta\delta}(\mathsf{I})$, then δ is cardinal correctly extendible. Moreover, the embeddings witnessing cardinal correct extendibility also witness the strong compactness of δ.

Proof. Since every ordinal $<\delta$ is definable in the logic $\mathcal{L}_{\delta\delta}(\mathsf{I})$, we can use the same argument as in the proof of Theorem 3.5.2 to show that for every $\gamma > \delta$ there is $\alpha > \gamma$, a transitive set M and an elementary embedding $j : V_\alpha \to M$ such that M is cardinal correct, $\text{crit}(j) = \delta$ and there exists $d \in M$ such that $j``\gamma \subseteq d$ and $M \models |d| < j(\delta)$. □

Question 3.5.14. If δ is cardinal correctly extendible, then is δ a strong compactness cardinal for $\mathcal{L}_{\delta\delta}(\mathsf{I})$?

4. Upward Löwenheim-Skolem-Tarski Numbers

Remarks on co-authorship. The results of this chapter are joint with Victoria Gitman and appear in [GO24]. Exceptions are Sections 4.5 and 4.6 whose results are joint with Will Boney, and Section 4.9 which is solely due to the author.

4.1. Introduction

Galeotti, Khomskii and Väänänen recently introduced the notion of the upward Löwenheim-Skolem-Tarski number for a logic, strengthening the classical notion of a Hanf number. A cardinal κ is the *upward Löwenheim-Skolem-Tarski number* (ULST) of a logic $\mathcal{L}$ if it is the least cardinal with the property that whenever M is a model of size at least κ satisfying a sentence φ in $\mathcal{L}$, then there are arbitrarily large models satisfying φ and having M as a substructure. The substructure requirement is what differentiates the ULST number from what is known as the *Hanf number* and gives the notion large cardinal strength. While it is a theorem of ZFC that every logic has a Hanf number, Galeotti, Khomskii and Väänänen showed that the existence of the ULST number for second-order logic implies the existence of a partially extendible cardinal. We answer positively their conjecture that the ULST number for second-order logic is the least extendible cardinal.

We define the *strong ULST* number of $\mathcal{L}$ by strengthening the substructure requirement to $\mathcal{L}$-elementary substructure. We investigate the ULST and strong ULST numbers for several strong logics. We show that the ULST and the strong ULST numbers are characterised in some cases by classical large cardinals and in some cases by natural new large cardinal notions. We show that for some logics the notions of the ULST number, strong ULST number and compactness number coincide, while for others, it is consistent that they can be separated.

The chapter is structured as follows. Section 4.2 reviews the definition of the ULST number, as well as some related notions and some known results, and introduces the strong ULST number. Section 4.3 discusses the technical device of *truth predicates*, we will use throughout the other sections. The following sections each discuss one logic and the relations of their ULST and strong ULST numbers to large cardinals. The logics we consider are the well-foundedness logic (Section 4.4), second-order logic (Section 4.5), sort logics (Section 4.6), infinitary logics (Section 4.7), and the equicardinality logic (Section 4.8). We will see that they are related, respectively, to measurable cardinals, supercompact cardinals, $C^{(n)}$-extendible cardinals and VP, tall cardinals, and cardinal correctly extendible cardinals. Finally, Section 4.9 introduces a general notion of $\mathcal{L}$-extendible cardinals for an arbitrary logic $\mathcal{L}$ and shows that they are systematically related to ULST numbers of $\mathcal{L}$.

4.2. Motivation and definitions

In the previous chapters, our attention was focused on compactness properties of logics. We further saw that downward Löwenheim-Skolem numbers are similarly able to give rise to a wide range of large cardinal notions. Another of the most important results from first-order model theory is the *upward* Löwenheim-Skolem Theorem: Any infinite structure $\mathcal{A}$ has arbitrarily large elementary superstructures. For strong logics, the following weak version of such a property is a classic realm of study.

Definition 4.2.1. Let $\mathcal{L}$ be a logic. The *Hanf number* of $\mathcal{L}$ is the smallest cardinal κ such that any sentence $\varphi \in \mathcal{L}$ that has a model of size at least κ has arbitrarily large models.

While it can be challenging to compute Hanf numbers of logics, the following fact is well-known and shows that Hanf numbers do not carry any large cardinal strength (cf., e.g., [BF85, Chapter II, Theorem 6.1.4]).

Proposition 4.2.2. Every logic has a Hanf number.

Proof. Let $\mathcal{L}$ be a logic. By definition, $\mathcal{L}$ has a strong dependence number $\operatorname{dep}^*(\mathcal{L}) = \lambda$. Then every sentence of $\mathcal{L}$ is equivalent up to renaming to a member of $\mathcal{L} \cap H_\lambda$. To analyse the size of models of sentences of $\mathcal{L}$ it is thus sufficient to restrict attention to sentences coming from H_λ. For every $\varphi \in \mathcal{L} \cap H_\lambda$ such that φ does *not* have arbitrarily large models, let δ_φ be

$$\delta_\varphi = \sup\{|\mathcal{A}| \colon \mathcal{A} \models \varphi\}.$$

Clearly, for the φ considered, δ_φ is a cardinal. Now let

$$\kappa = \sup\{\delta_\varphi \colon \varphi \in \mathcal{L} \cap H_\lambda \text{ and } \varphi \text{ does not have arbitrarily large models}\}.$$

Then κ is the supremum of a set of cardinals, and thus a cardinal. Clearly, $\mathcal{L}$ has Hanf number at most κ^+. □

Recall that the LS number of logics does not carry any large cardinal strength either, but that adding a substructure requirement to obtain the LST number brings large cardinals into the fray (cf. Proposition 1.2.10 and Theorem 1.3.14). This provided motivation for Galeotti, Khomskii, and Väänänen to strengthen the notion of the Hanf number in a similar vein to study certain upwards directed set-theoretic reflection principles (cf. [Gal19, GKV20]).

Definition 4.2.3 (Galeotti, Khomskii & Väänänen [GKV20])**.** Fix a logic $\mathcal{L}$. The *upward Löwenheim-Skolem-Tarski number* $\mathrm{ULST}(\mathcal{L})$ of $\mathcal{L}$, if it exists, is the least cardinal δ such that for any vocabulary τ and $\varphi \in \mathcal{L}[\tau]$, if a τ-structure $\mathcal{A} \models_{\mathcal{L}} \varphi$ and has size $|A| \geq \delta$, then for every cardinal $\gamma \geq |A|$, there is a τ-structure $\mathcal{B}$ of size at least γ such that $\mathcal{B} \models_{\mathcal{L}} \varphi$ and $\mathcal{A} \subseteq \mathcal{B}$ is a substructure of $\mathcal{B}$.

Similar to this gain in strength that occurs when switching attention from LS numbers to LST numbers, Galeotti, Khomskii and Väänänen showed that the existence of the ULST number for second-order logic, $\mathrm{ULST}(\mathcal{L}^2)$, implies the existence of very strong large cardinals.

Theorem 4.2.4 (Galeotti, Khomskii & Väänänen [GKV20, Theorem 7.4])**.** If $\mathrm{ULST}(\mathcal{L}^2)$ exists, then for every $n \in \omega$ there is an n-extendible cardinal $\lambda \leq \mathrm{ULST}(\mathcal{L}^2)$.

They further conjectured that the strength of the existence of $\mathrm{ULST}(\mathcal{L}^2)$ is exactly that of an extendible cardinal (cf. [GKV20, Conjecture 7.7]). We will confirm their conjecture.

Theorem 4.2.5. Let κ be a cardinal. Then $\mathrm{ULST}(\mathcal{L}^2) = \kappa$ iff κ is the least extendible cardinal.

We can further strengthen the notion of the ULST number to capture the full power of the upward Löwenheim-Skolem Theorem.

Definition 4.2.6. Fix a logic $\mathcal{L}$. The *strong upward Löwenheim-Skolem-Tarski number* $\mathrm{SULST}(\mathcal{L})$ of $\mathcal{L}$, if it exists, is the least cardinal δ such that for any vocabulary τ and any τ-structure $\mathcal{A}$ of size $|A| \geq \delta$, for every cardinal $\gamma \geq |A|$, there is a τ-structure $\mathcal{B}$ of size at least γ such that $\mathcal{A} \prec_{\mathcal{L}} \mathcal{B}$ is an $\mathcal{L}$-elementary substructure of $\mathcal{B}$.

Note that we could equivalently define the strong upward Löwenheim-Skolem-Tarski number analogously to the upward Löwenheim-Skolem-Tarski number but preserving theories instead of single sentences, as being an $\mathcal{L}$-elementary substructure simply comes down to being a model of an $\mathcal{L}$-elementary diagram. Clearly, $\mathrm{ULST}(\mathcal{L}) \leq \mathrm{SULST}(\mathcal{L})$.

Note that Theorem 4.2.5 shows that $\mathrm{ULST}(\mathcal{L}^2)$ is the same as the compactness number of $\mathcal{L}^2$ (cf. Theorem 1.3.28). In general, SULST numbers are bounded by compactness numbers.

Proposition 4.2.7. Let $\mathcal{L}$ be a logic which has a compactness number $\mathrm{comp}(\mathcal{L}) = \kappa$. Then $\mathrm{SULST}(\mathcal{L})$ exists and $\mathrm{SULST}(\mathcal{L}) \leq \kappa$.

Proof. The proof goes exactly like the proof of the upward Löwenheim-Skolem Theorem from the Compactness Theorem for first-order logic. Suppose that $\kappa = \mathrm{comp}(\mathcal{L})$. Fix a τ-structure $\mathcal{A}$ of size $\gamma \geq \kappa$ and a cardinal $\bar{\gamma} > \gamma$. Consider the theory $T = \mathrm{ElDiag}_{\mathcal{L}}(\mathcal{A}) \cup \{c_i \neq c_j \colon i < j < \bar{\gamma}\}$, where $\{c_i \colon i < \bar{\gamma}\}$ is a set of new, distinct constant symbols. Clearly, T is $< \kappa$-satisfiable because it holds true in $\mathcal{A}$ (with the distinct constants c_i interpreted by distinct elements of A). Thus, T has a model $\mathcal{B}$. By construction, $\mathcal{B}$ has size at least $\bar{\gamma}$ and is an $\mathcal{L}$-elementary superstructure of $\mathcal{A}$. $\square$

In particular, together with Theorems 1.3.34 and 4.2.5 this shows that $\mathrm{comp}(\mathcal{L}^2) = \mathrm{ULST}(\mathcal{L}^2) = \mathrm{SULST}(\mathcal{L}^2)$ and that all these cardinals, should they exist, are equal to the first extendible cardinal.

4.3. Truth predicates

A central ingredient to extract large cardinal strength from the existence of a ULST number of a logic $\mathcal{L}$ is $\mathcal{L}$'s potential ability to define well-foundedness, combined with the availability of *truth predicates* for transitive models. Suppose that $(M, \in)$ is a transitive set which is a model of some finite fragment ZFC^*_b of ZFC including the extensionality, pairing and union axioms and which is large enough such that any transitive model of ZFC^*_b can carry out the usual definition of the syntax of first-order logic.[1] The essentials of the following construction were obtained by Magidor in [Mag71]. We say that $\mathrm{Tr}^M \subseteq M$ is a *truth predicate* for M if for all first-order formulas $\varphi(x_1, \dots, x_n) \in \mathcal{L}_{\omega\omega}[\{\in\}]$ and all tuples $(a_1, \dots, a_n)$:

$$(M, \in) \models \varphi(a_1, \dots, a_n) \text{ if and only if } (\varphi, (a_1, \dots, a_n)) \in \mathrm{Tr}^M.$$

Note that there is a sentence φ_{Truth} of first-order logic in the expanded language $\{\in, T\}$ such that for any $T \subseteq M^2$,

$$(M, \in, T) \models \varphi_{\mathrm{Truth}} \text{ if and only if } T = \mathrm{Tr}^M.$$

The sentence φ_{Truth} can be obtained by taking the conjunction of the following sentences, which go through Tarski's truth definition in the usual way and fix that the model satisfies a sufficient part of ZFC:

(i) ZFC^*_b.

(ii) $\forall x(x = (\ulcorner x_i \in x_j \urcorner, a_1, \dots, a_n) \to (T(x) \leftrightarrow a_i \in a_j))$.

(iii) $\forall x(x = (\ulcorner x_i = x_j \urcorner, a_1, \dots, a_n) \to (T(x) \leftrightarrow a_i = a_j))$.

(iv) $\forall x(x = (\ulcorner \psi \wedge \chi \urcorner, a_1, \dots, a_n) \to$
$(T(x) \leftrightarrow [T((\ulcorner \psi \urcorner, a_1, \dots, a_n)) \wedge T((\ulcorner \chi \urcorner, a_1, \dots, a_n))])$.

(v) $\forall x(x = (\ulcorner \neg\psi \urcorner, a_1, \dots, a_n) \to (T(x) \leftrightarrow \neg T((\ulcorner \psi \urcorner, a_1, \dots, a_n))))$.

(vi) $\forall x(x = (\ulcorner \exists x \psi \urcorner, a_1, \dots, a_n) \to (T(x) \leftrightarrow \exists y T((\ulcorner \psi \urcorner, a_1, \dots, a_n, y))))$.

We show that in specific situations, embeddings between transitive sets including a truth predicate, are already elementary embeddings for the structure without the truth predicate. We work in the vocabulary $\{\in, c, T, S, P\}$, where c is a constant symbol, and T, S, and P are unary, binary, and ternary relation symbols, respectively. Consider the following sentences:

(vii) $\varphi_{\emptyset} = \neg\exists x(x \in c)$.

(viii) $\varphi_{\mathrm{Succ}} = \forall x, y(S(x, y) \leftrightarrow \forall z(z \in y \leftrightarrow x = z \vee x \in z))$.

[1] We use the index b to distinguish ZFC^*_b from ZFC^*_a considered in Chapter 3, and in Section 4.8 below.

(ix) $\varphi_{\text{Pair}} = \forall x, y, z(P(x,y,z) \leftrightarrow \exists v, w(v, w \in z \wedge \forall v_0(v_0 \in v \leftrightarrow v_0 = x) \wedge \forall v_0(v_0 \in w \leftrightarrow v_0 = x \vee v_0 = y) \wedge \forall v_1(v_1 \in z \to v_1 = v \vee v_1 = w)))$,

Let $\mathcal{M} = (M, \in, c^M, T^M, S^M, P^M)$ be transitive. We already noted that φ_{Truth} codes that $T^M = \text{Tr}^M$. The sentence $\varphi_\emptyset$ codes that c is the emptyset, i.e., $\mathcal{M} \models c^M$ iff $c^M = \emptyset$. Further, φ_{Succ} expresses that S^M codes the successor function, i.e., $(a,b) \in S^M$ iff $b = a \cup \{a\}$, and φ_{Pair} expresses that P^M codes the pairing function, i.e., $(a,b,c) \in P^M$ iff $c = (a,b)$.

Lemma 4.3.1. Let $(M, \in, c^M, T^M, S^M, P^M)$ and $(N, \in, c^N, T^N, S^N, P^N)$ be transitive and assume both structures satisfy φ_{Truth}, $\varphi_\emptyset$, φ_{Succ} and φ_{Pair}. Suppose there is an embedding

$$j : (M, \in, c^M, T^M, S^M, P^M) \to (N, \in, c^N, T^N, S^N, P^N).$$

Then j is an elementary embedding between the structures $(M, \in)$ and $(N, \in)$.

Proof. Let $(M, \in) \models \varphi(a_1, \ldots, a_n)$. Then $(\varphi, (a_1, \ldots, a_n)) \in \text{Tr}^M$ by φ_{Truth}. Because j is an embedding, $j((\varphi, (a_1, \ldots, a_n))) \in \text{Tr}^N$. We claim that $j((\varphi, (a_1, \ldots, a_n))) = (\varphi, (j(a_1), \ldots, j(a_n)))$. Then, again by φ_{Truth},

$$(N, \in) \models \varphi(j(a_1), \ldots, j(a_n)),$$

which is sufficient. To prove our claim, let us first argue that $j(n) = n$ for all $n \in \omega$. We have $j(0) = 0$, as j is an embedding and so $j(\emptyset) = j(c^M) = c^N = \emptyset$. Suppose that $j(n) = n$. Then $(n, n+1) \in S^M$. Because j is an embedding, we get $(n, j(n+1)) = (j(n), j(n+1)) \in S^N$. Since N satisfies that S^N is the successor function, and transitive sets are correct about this, therefore $j(n+1) = n+1$. Now let us argue that $j((a_1, \ldots, a_n)) = (j(a_1), \ldots, j(a_n))$ for all $a_1, \ldots, a_n \in M$. Because $(a_1, \ldots, a_n) = (a_1, (a_2, \ldots, a_n))$ it is sufficient to argue that $j((a,b)) = (j(a), j(b))$ for all $a, b \in M$, since the claim then follows inductively. We have that $(a, b, (a,b)) \in P^M$. This implies $(j(a), j(b), j((a,b))) \in P^N$, as j is an embedding. Because N believes that P^N codes the pairing function, therefore $N \models j((a,b)) = (j(a), j(b))$. The model N is correct about this as a transitive set. Thus $j((\varphi, (a_1, \ldots, a_n))) = (j(\varphi), j(a_1), \ldots, j(a_n))$. It is thus sufficient to argue that $j(\varphi) = \varphi$. Recall that we assume that any first-order formula $\varphi \in \mathcal{L}_{\omega\omega}[\{\in\}]$ is coded as a finite tuple of natural numbers (cf. Appendix A). But then $j(\varphi) = \varphi$, because j respects pairing and $j(n) = n$ for all $n \in \omega$, as just argued. □

We will use this observation repeatedly in the following situation to extract large cardinal strength from the existence of ULST numbers.

Lemma 4.3.2. Let $\mathcal{M} = (M, \in, c^M, T^M, S^M, P^M)$ be transitive. Suppose there is a superstructure $\mathcal{N} = (N, E, c^N, T^N, S^N, P^N)$ of $\mathcal{M}$ and assume that both $\mathcal{M}$ and $\mathcal{N}$ satisfy φ_{Truth}, $\varphi_\emptyset$, φ_{Succ}, and φ_{Pair}. Further assume that E is well-founded and extensional. Fix the transitive collapse $\pi : N \to \bar{N}$. Then $\pi \restriction M : (M, \in) \to (\bar{N}, \in)$ is elementary.

Proof. We may define a $\tau = \{\in, c, T, S, P\}$-structure $\bar{\mathcal{N}}$ on $\bar{N}$ by letting $c^{\bar{N}} = \pi(c^N)$, and further $T^{\bar{N}} = \{\pi(x) \colon x \in T^N\}$, $S^{\bar{N}} = \{(\pi(x), \pi(y)) \colon (x,y) \in S\}$ and $P^N =$

$\{(\pi(x), \pi(y), \pi(z)) \colon (x, y, z) \in T^N\}$. Then π is an isomorphism of $\mathcal{N}$ and $\bar{\mathcal{N}}$ as τ-structures. In particular, $\bar{\mathcal{N}}$ satisfies φ_{Truth}, $\varphi_{\emptyset}$, φ_{Succ} and φ_{Pair}. Because $\mathcal{N}$ is a superstructure of $\mathcal{M}$, then $\pi \circ \text{id} = \pi \restriction M$ is an embedding between transitive structures satisfying φ_{Truth}, $\varphi_{\emptyset}$, φ_{Succ}, and φ_{Pair}. Hence, by Lemma 4.3.1, $\pi \restriction M$ is an elementary embedding $(M, \in) \to (\bar{N}, \in)$. □

4.4. The well-foundedness logic

In this section we show that $\text{ULST}(\mathcal{L}(\mathsf{Q}^{\text{WF}})) = \text{SULST}(\mathcal{L}(\mathsf{Q}^{\text{WF}}))$ is the least measurable cardinal, and then consider how both these numbers relate to $\text{comp}(\mathcal{L}(\mathsf{Q}^{\text{WF}}))$.

Theorem 4.4.1. If there is a measurable cardinal κ, then $\text{SULST}(\mathcal{L}(\mathsf{Q}^{\text{WF}}))$ exists and is at most κ.

Proof. Suppose that κ is a measurable cardinal. Let $\mathcal{A}$ be a τ-structure of size $\gamma \geq \kappa$. Let $\bar{\gamma} > \gamma$ be any cardinal. Let $j : V \to M$ be an elementary embedding with $\text{crit}(j) = \kappa$ and $j(\kappa) > \bar{\gamma}^+$. We can obtain such an embedding by iterating the ultrapower construction with a κ-complete ultrafilter enough times (cf., e.g., [Jec03, Section 19]). Note that $j``\mathcal{A}$ provides a renaming of $\mathcal{A}$ to a $j``\tau$-structure. Further, $j``\tau \subseteq j(\tau)$, $j(\mathcal{A})$ is a $j(\tau)$-structure, and it is easy to see that $j``\mathcal{A}$ is a substructure of $j(\mathcal{A}) \restriction j``\tau$. Also $|\mathcal{A}| \geq \gamma \geq \kappa$, so $|j(\mathcal{A})|^M \geq j(\kappa) > \bar{\gamma}^+$, and therefore in particular in $V$, $|j(\mathcal{A})| > \bar{\gamma}$. Hence, if we can show that $j``\mathcal{A}$ is an $\mathcal{L}(\mathsf{Q}^{\text{WF}})$-elementary substructure of $j(\mathcal{A}) \restriction j``\tau$, we are done, as then we can rename $j(\mathcal{A}) \restriction j``\tau$ to an $\mathcal{L}(\mathsf{Q}^{\text{WF}})$-elementary superstructure of $\mathcal{A}$ of size $> \bar{\gamma}$.

To this end suppose that $\psi(x) \in \mathcal{L}(\mathsf{Q}^{\text{WF}})[j``\tau]$ - for simplicity in one free variable - and $j``\mathcal{A} \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} \psi(b)$ for some $b \in j``\mathcal{A}$. Recall that $j$ as an elementary embedding restricts to a renaming $j : \mathcal{L}(\mathsf{Q}^{\text{WF}})[\tau] \to \mathcal{L}(\mathsf{Q}^{\text{WF}})[j``\tau]$ (cf. Section 1.3.2). Therefore $\psi = j(\varphi)$ for some $\varphi \in \mathcal{L}(\mathsf{Q}^{\text{WF}})[\tau]$, and further $b = j(a)$ for some $a \in \mathcal{A}$. So this means that

$$j``\mathcal{A} \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} j(\varphi)(j(a)).$$

Pulling satisfaction of $j(\varphi)(j(a))$ back along the renaming $j : \mathcal{A} \to j``\mathcal{A}$ we get that $\mathcal{A} \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} \varphi(a)$. But then by elementarity of j,

$$M \models \text{“}j(\mathcal{A}) \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} j(\varphi)(j(a))\text{”}.$$

Because M is transitive, it is correct about $\mathcal{L}(\mathsf{Q}^{\text{WF}})$-satisfaction and so really $j(\mathcal{A}) \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} j(\varphi)(j(a))$. As $j(\varphi) = \psi \in \mathcal{L}(\mathsf{Q}^{\text{WF}})[j``\tau]$, then $j(\mathcal{A}) \restriction j``\tau \models_{\mathcal{L}(\mathsf{Q}^{\text{WF}})} \psi(j(a))$, which is what we promised. □

Theorem 4.4.2. If $\text{ULST}(\mathcal{L}(\mathsf{Q}^{\text{WF}})) = \delta$, then there is a measurable cardinal $\leq\delta$.

Proof. Suppose $\text{ULST}(\mathcal{L}(\mathsf{Q}^{\text{WF}})) = \delta$. Consider the model

$$\mathcal{M} = (V_{\delta^+}, \in, \emptyset, \delta, \text{Tr}, S, P),$$

where Tr is a truth predicate for $(V_{\delta^+}, \in)$, $\emptyset$ and δ are added as constants, S codes the successor function and P codes the pairing function. Then $\mathcal{M}$ satisfies the sentence φ in the logic $\mathcal{L}(\mathsf{Q}^{\text{WF}})$ over the vocabulary $\tau = \{\in, c, d, T, S, P\}$, which is the conjunction of the sentences:

(i) $\mathsf{Q}^{\mathrm{WF}}xy(x \in y) \wedge \mathrm{Ext}$.

(ii) $\varphi_{\mathrm{Truth}} \wedge \varphi_{\emptyset} \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

(iii) "d is the largest cardinal": $\mathrm{Card}(d) \wedge \forall x(\mathrm{Card}(x) \rightarrow x \leq d)$.

Since $\mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) = \delta$, there is a structure

$$\mathcal{N} = (N, E, c^N, d^N, T^N, S^N, P^N)$$

of size larger than $\beth_{\delta^+}$ having $\mathcal{M}$ as a substructure and which is a model of the above sentences (i) to (iii). Because E is well-founded and extensional by (i), we may assume that N is given by its transitive collapse and that $E = \in$. Because N satisfies (ii), by Lemma 4.3.2 the transitive collapse is an elementary embedding $j : V_{\delta^+} \rightarrow N$. Notice that $j(\delta) = j(d^{V_{\delta^+}}) = d^N$ and, by (iii), the latter is the largest N-cardinal. Because $|N| > \beth_{\delta^+}$, N has to contain a cardinal larger than δ and so $j(\delta) > \delta$. In particular, j has a critical point $\kappa = \mathrm{crit}(j) \leq \delta$. Notice that $\mathcal{P}(\kappa) \subseteq V_{\delta^+}$ and so we can define for $X \subseteq \kappa$:

$$X \in U \text{ iff } \kappa \in j(X).$$

It is standard to check that U is a κ-complete ultrafilter and hence κ is measurable. □

Corollary 4.4.3. The following are equivalent for a cardinal κ.

(1) κ is the least measurable cardinal.

(2) $\kappa = \mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

(3) $\kappa = \mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

Next, we would like to understand the relationship between the compactness number of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ and $\mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$. Recall that $\mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ is the smallest ω_1-strongly compact cardinal (cf. Theorem 1.3.21). Note that the proof of this result shows that if δ is ω_1-strongly compact, then there is a measurable cardinal $\leq \delta$. Magidor showed in [Mag76] that it is consistent, relative to a supercompact cardinal, that the least measurable cardinal is the least strongly compact cardinal, and hence, in particular, the least measurable cardinal can be the least ω_1-strongly compact cardinal. Bagaria and Magidor in [BM14a] showed that it is consistent, relative to a supercompact cardinal, that the least ω_1-strongly compact cardinal is singular of cofinality greater than or equal to the least measurable cardinal. In this situation, $\mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ exists, but is greater than the least measurable cardinal.

Further, consider the canonical model $L[U]$ for a normal ultrafilter over a measurable cardinal (cf., e.g., [Jec03, Chapter 19]). This model cannot have an ω_1-strongly compact cardinal δ: If δ is ω_1-strongly compact, then by Theorem 1.3.20 for every $\gamma > \delta$ there is an ω_1-complete fine ultrafilter W over $\mathcal{P}_\delta\gamma$. Consider the ultrapower $j_W : V \rightarrow M_W$. Then with $\mathrm{id} : \mathcal{P}_\delta\gamma \rightarrow V$, we have that $j``\gamma \subseteq [\mathrm{id}]_W$, as for every $\alpha < \gamma$, by fineness, $\{s \in \mathcal{P}_\delta\gamma \colon c_\alpha(s) = \alpha \in s = \mathrm{id}(s)\} \in W$. Further, $\{s \in \mathcal{P}_\delta\gamma \colon |s| = |\mathrm{id}(s)| < |c_\delta(s)| = \delta\} \in W$

and therefore $j_W(\delta) > |[\mathrm{id}]_W| \geq \gamma$. Furthermore, M_W is closed under ω_1-sequences, as the ultrapower by an ω_1-complete ultrafilter (cf. [Ham09, Theorem 2.11]). Thus, for every ordinal γ there is an elementary embedding $j : V \to M$ such that $j(\delta) > \gamma$ and $M^{\omega_1} \subseteq M$. But in $L[U]$, the only elementary embeddings of the universe are iterates of the ultrapower by the unique measure on the unique measurable cardinal and only finite iterates have a target that is closed under ω-sequences. Thus, $L[U]$ does not have a compactness nuber of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, while it has a measurable cardinal, and thus $\mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ exists. Combining Corollary 4.4.3 with the above results, we get the following corollary.

Corollary 4.4.4. The consistency of the existence of a supercompact cardinal implies the consistency of the following situations:

(1) $\mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) = \mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) = \mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

(2) $\mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) = \mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) < \mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

(3) $\mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})) = \mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ exists, but $\mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$ does not exist.

Thus, we have an example of a logic for which the ULST number is always equal to the strong ULST number, but consistently it is possible that either the compactness number does not exist, or it exists and is larger than the the strong ULST number, or it exists and is equal to the strong ULST number.

4.5. Second-order logic

Recall that the least extendible cardinal is the compactness number of $\mathcal{L}^2$ and thus, $\mathrm{SULST}(\mathcal{L}^2)$ is bounded by the least extendible cardinal by Proposition 4.2.7. In this section, we show that $\mathrm{ULST}(\mathcal{L}^2) = \mathrm{SULST}(\mathcal{L}^2)$ is precisely the least extendible cardinal.

Theorem 4.5.1. Let δ be a cardinal. Then $\delta = \mathrm{ULST}(\mathcal{L}^2)$ iff δ is the least extendible cardinal.

We will use the following version of Fodor's Lemma for definable classes.

Lemma 4.5.2 (The very weak class Fodor principle (definable version); Gitman, Hamkins & Karagila [GHK19])**.** Let $S \subseteq \mathrm{Ord}$ be a definable stationary class and $F : S \to \mathrm{Ord}$ a definable regressive function. Then there is an unbounded class $A \subseteq S$ such that F is constant on A.

Gitman, Hamkins and Karagila proved a version of this statement in a weak version of GBC class theory. We check that their proof carries over to the present ZFC setting.

Proof. Let S be a stationary class of ordinals and $F : S \to \mathrm{Ord}$ a function such that $F(\gamma) \in \gamma$ for all $\gamma \in S$. Suppose for any unbounded class $A \subseteq S$ that F is not constant on A. Then for every ordinal γ there is a least $\beta_\gamma > \gamma$ such that $F(\alpha) > \gamma$ for all $\alpha \geq \beta$: If this would not be the case, there would be a γ such that for all $\beta > \gamma$ there is some $\alpha \geq \beta$ such that $F(\alpha) \leq \gamma$. Then we could define α_0 as the least $\alpha > \gamma$ such that $F(\alpha) \leq \gamma$ and, recursively for $\beta > 0$, α_β the least $\alpha > \sup\{\alpha_i : i < \beta\}$ such that $F(\alpha) \leq \gamma$. Then α_β is an increasing proper class sequence of ordinals with $F(\alpha_\beta) \leq \gamma$ for all β and then, by the pigeon hole principle, there is a fixed $\delta < \gamma$ such that $F(\alpha_\beta) = \delta$ for unboundedly many of those.

Note that the function $\beta \mapsto \beta_\gamma$ is definable. Consider the class C of closure points of this function, i.e., $C = \{\theta : \gamma < \theta \to \beta_\gamma < \theta\}$. We claim that C is club. For closedness, if $(\theta_i)_{i<\delta} \subseteq C$ is some increasing sequence and $\theta = \sup\{\theta_i : i < \delta\}$. Then if $\gamma < \theta$, then $\gamma < \theta_i$ for some i. So $\beta_\gamma < \theta_i$ and thus $\beta_\gamma < \theta$. And for unboundedness, if γ is any ordinal, define $\eta_0 = \beta_\gamma$ and, recursively for $n+1 \in \omega$, let $\beta_{n+1} = \beta_{\eta_n}$. Consider $\eta = \sup\{\beta_n : n \in \omega\}$. If $\delta < \eta$, then $\delta < \beta_n$ for some n. But then $\beta_\delta < \beta_n < \eta$. So $\eta \in C$ and clearly $\eta > \gamma$.

As C is club, stationarity of S gives us some $\theta \in C \cap S$. But then for any $\gamma < \theta$, also $\beta_\gamma < \theta$ and so $F(\theta) \neq \gamma$. This contradicts that F is regressive. □

Proof of Theorem 4.5.1. [2] Let $\mathrm{ULST}(\mathcal{L}^2) = \delta$. By Proposition 4.2.7 and the leastness property of δ, it suffices to show that there is an extendible cardinal $\leq \delta$. Consider any ordinal $\alpha \geq \delta$ of cofinality ω. Fix a function f_α with domain ω that is cofinal in α, a truth predicate T_α for $(V_\alpha, \in)$ and relations S_α and P_α coding the successor and pairing functions. Then the structure $(V_\alpha, \in, f_\alpha, T_\alpha, \emptyset, S_\alpha, P_\alpha)$ is a model of the conjunction of the following sentences in the language $\{\in, f, T, c, S, P\}$, where f is a two place predicate

(i) Magidor's Φ (cf. Lemma 1.2.4).

(ii) $\varphi_{\text{Truth}} \wedge \varphi_{\emptyset} \wedge \varphi_{\text{Succ}} \wedge \varphi_{\text{Pair}}$.

(iii) "f is a function with domain ω which is cofinal in the ordinals":

$$\forall x, y, z(f(x,y) \wedge f(x,z) \to y = z \wedge \mathrm{Ord}(y)) \wedge$$
$$\forall x(x \in \omega \leftrightarrow \exists y f(x,y)) \wedge \forall \alpha(\mathrm{Ord}(\alpha) \to \exists x, \beta(\alpha < f(x,\beta))).$$

Because $\delta = \mathrm{ULST}(\mathcal{L}^2)$ we find a superstructure $\mathcal{A}_\alpha = (A_\alpha, E_\alpha, f^*_\alpha, T^*_\alpha)$ of size $> |V_\alpha|$ satisfying the above sentences (i) to (iii). By Φ, we can collapse $\mathcal{A}_\alpha$ to a structure of the form $(V_{\beta_\alpha}, \in, f_{\beta_\alpha}, T_{\beta_\alpha})$. By (ii) and Lemma 4.3.2, the collapse isomorphism gives an elementary embedding $j_\alpha : (V_\alpha, \in) \to (V_{\beta_\alpha}, \in)$. By (iii), f_{β_α} is a function with domain ω which is cofinal in β_α. Notice that $j_\alpha(f_\alpha(n)) = f_{\beta_\alpha}(j_\alpha(n)) = f_{\beta_\alpha}(n)$. Thus for some n, $j_\alpha(f_\alpha(n)) > f_\alpha(n)$ and so j_α has some critical point $\mathrm{crit}(j_\alpha) < \alpha$. Therefore, the function F which sends α to the smallest value of a critical point $\mathrm{crit}(j_\alpha)$ of some elementary embedding $j_\alpha : V_\alpha \to V_\gamma$ is a definable function on the stationary class $S = \{\alpha > \delta : \mathrm{cof}(\alpha) = \omega\}$. By the very weak class Fodor principle 4.5.2, F is constant

[2]The presented proof is due to Will Boney and the author. Independently, Yair Hayut provided a different proof of this result, which remains unpublished, and Victoria Gitman and the author in [GO24, Theorem 6.1] provide yet another proof.

on an unbounded subclass of S, say with value κ. Then κ is the critical point of $j_\alpha : (V_\alpha, \in) \to (V_\gamma, \in)$ for a proper class of α and therefore extendible.

Hence, we can let η be the smallest extendible cardinal. We claim $\eta \leq \delta$. Suppose $\eta > \delta$. Notice that the assertion that δ is a ULST number of $\mathcal{L}^2$ can be formalised by the following Π_3 formula:

$$\forall \mathcal{A} \forall \varphi \forall \bar{\gamma} \exists \mathcal{B} (|\mathcal{A}| \geq \delta \wedge \mathcal{A} \models_{\mathcal{L}^2} \varphi \to \mathcal{A} \subseteq \mathcal{B} \wedge |\mathcal{B}| \geq \bar{\gamma} \wedge \mathcal{B} \models_{\mathcal{L}^2} \varphi).$$

This is a Π_3-statement as $\models_{\mathcal{L}^2}$ is Δ_2-definable (cf., e.g., [GKV20, Proposition 3.6]). By extendibility, $\eta \in C^{(3)}$ and so η satisfies that $\delta = \mathrm{ULST}(\mathcal{L}^2)$. Then we can repeat our argument in V_η and find a cardinal $\nu < \eta$ such that $V_\eta \models$ "ν is extendible". Because also being extendible is a Π_3-statement, V_η is correct about this fact, and so ν is really extendible. But this contradicts minimality of η. □

Corollary 4.5.3. The following are equivalent for a cardinal κ.

(1) κ is the least extendible cardinal.

(2) $\kappa = \mathrm{comp}(\mathcal{L}^2)$.

(3) $\kappa = \mathrm{SULST}(\mathcal{L}^2)$.

(4) $\kappa = \mathrm{ULST}(\mathcal{L}^2)$.

We further get a characterisation of extendibility by varying the above proof:

Theorem 4.5.4. The following are equivalent for a cardinal κ:

(1) κ is extendible.

(2) $\kappa = \mathrm{comp}(\mathcal{L}^2_{\kappa\omega})$.

(3) $\kappa = \mathrm{comp}(\mathcal{L}^2_{\kappa\kappa})$.

(4) $\kappa = \mathrm{SULST}(\mathcal{L}^2_{\kappa\omega})$.

(5) $\kappa = \mathrm{SULST}(\mathcal{L}^2_{\kappa\kappa})$.

(6) $\kappa = \mathrm{ULST}(\mathcal{L}^2_{\kappa\omega})$.

(7) $\kappa = \mathrm{ULST}(\mathcal{L}^2_{\kappa\kappa})$.

Proof. Because extendibility of κ implies that $\kappa = \mathrm{comp}(\mathcal{L}^2_{\kappa\kappa})$ by Magidor's Theorem 1.3.28, it is sufficient to show that if $\kappa = \mathrm{ULST}(\mathcal{L}^2_{\kappa\omega})$, then for any $\beta < \kappa$ there is an extendible cardinal η such that $\beta < \eta \leq \kappa$. To show this, we can use the same proof as for Theorem 4.5.1, only changing that for α of cofinality ω we consider an expanded structure $(V_\alpha, \in, f_\alpha, T_\alpha, \emptyset, S_\alpha, P_\alpha, c_i)_{i \leq \beta}$ with constants c_i interpreted as $c_i = i$ for $i \leq \beta$. Recall that $\mathcal{L}_{\kappa\omega}$ has for every ordinal $i < \kappa$ the formula $\sigma_i(x)$ defining i (cf. Lemma 1.2.4). Then adding the sentence

(iv) $\bigwedge_{i\leq\beta}\sigma_i(c_i)$,

to the sentences (i) to (iii) from the previous proof results in an elementary embedding $j_\alpha : V_\alpha \to V_{\beta_\alpha}$ with $\mathrm{crit}(j) < \alpha$ and additionally fixing all ordinals $\leq \beta$. Thus $\mathrm{crit}(j_\alpha) > \beta$. The analogous argument using the very weak class Fodor principle gives us an extendible cardinal $> \beta$. Then we can show that the smallest extendible cardinal which is larger than β has to be at most κ, using that the fact that $\kappa = \mathrm{ULST}(\mathcal{L}^2_{\kappa\omega})$ is Π_3-expressible with κ as a parameter. □

Thus, we have an example of logics stronger than first-order logic for which the ULST number is same as the strong ULST number and the same as the compactness number.

4.6. Sort logics

In this section we show that analogously to how the $C^{(n)}$-extendible cardinals are equivalent to the existence of compactness numbers of $\mathcal{L}^{s,n}$ (see Boney's Theorem 1.3.34), they are related to ULST numbers of $\mathcal{L}^{s,n}$. ULST numbers hence provide yet another stratification of VP.

Theorem 4.6.1. The following are equivalent for every natural number n and every cardinal κ:

(1) κ is the least $C^{(n)}$-extendible cardinal.

(2) $\kappa = \mathrm{comp}(\mathcal{L}^{s,n})$.

(3) $\kappa = \mathrm{SULST}(\mathcal{L}^{s,n})$.

(4) $\kappa = \mathrm{ULST}(\mathcal{L}^{s,n})$.

Proof. [3] Because κ being the least $C^{(n)}$-extendible implies that $\kappa = \mathrm{comp}(\mathcal{L}^{s,n})$, by minimality of the ULST number it is sufficient to show that if $\kappa = \mathrm{ULST}(\mathcal{L}^{s,n})$, there is a $C^{(n)}$-extendible cardinal $\leq \kappa$. The proof goes mostly analogously to that of Theorem 4.5.1. Let $\alpha > \kappa$ be an ordinal in $C^{(n)}$ and of cofinality ω. Consider again the structure $(V_\alpha, \in, f_\alpha, T_\alpha, \emptyset, S_\alpha, P_\alpha)$ where f_α is a cofinal function in α with domain ω, T_α is a truth predicate, S_α codes the successor function and P_α codes the pairing function. This structure satisfies the sentences

(i) $\Phi^{(n)}$

(ii) $\varphi_{\mathrm{Truth}} \wedge \varphi_\emptyset \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

(iii) "f is a function with domain ω which is cofinal in the ordinals".

[3]As with Theorem 4.5.1, the same result was obtained by Gitman and the author using different methods (cf. [GO24, Corollary 7.2].

Here $\Phi^{(n)}$ is the sentence of $\mathcal{L}^{s,n}$ axiomatising the class of models (M, E) isomorphic to some $(V_\lambda, \in)$ where $\lambda \in C^{(n)}$ (cf. Corollary 1.2.17). The same argument as in the proof of Theorem 4.5.1 gives us an elementary embedding

$$j_\alpha : (V_\alpha, \in) \to (V_{\beta_\alpha}, \in),$$

but using $\Phi^{(n)}$ we may assume that $\beta_\alpha \in C^{(n)}$. Then the function F which sends α to the smallest value of a critical point $\text{crit}(j_\alpha)$ of some elementary embedding $j_\alpha : V_\alpha \to V_{\beta_\alpha}$ with $\beta_\alpha \in C^{(n)}$ is a definable regressive function on the stationary class $S = \{\alpha \colon \text{cof}(\alpha) = \omega \wedge \alpha \in C^{(n)}\}$. By the very weak class Fodor principle, F is constant on an unbounded class $A \subseteq S$, say with value κ. Then κ is $C^{(n)}$-extendible, using the characterisation of Theorem 1.3.23.

Let η be the smallest $C^{(n)}$-extendible cardinal. If $\eta > \kappa$, because $\eta \in C^{(n+2)}$ as a $C^{(n)}$-extendible cardinal (cf. Theorem 1.3.26), it satisfies the Π_{n+2} formula

$$\forall \mathcal{A} \forall \varphi \forall \bar{\gamma} \exists \mathcal{B} (|\mathcal{A}| \geq \kappa \wedge \mathcal{A} \models_{\mathcal{L}^{s,n}} \varphi \to \mathcal{A} \subseteq \mathcal{B} \wedge |\mathcal{B}| \geq \bar{\gamma} \wedge \mathcal{B} \models_{\mathcal{L}^{s,n}} \varphi).$$

This is Π_{n+2} because $\models_{\mathcal{L}^{s,n}}$ is Δ_{n+1}-definable (cf. Corollary 1.2.22). Therefore η believes that $\kappa = \text{ULST}(\mathcal{L}^{s,n})$. We can therefore repeat our argument in V_η to get a cardinal ν which V_η believes to be $C^{(n)}$-extendible. Because being $C^{(n)}$-extendible is a Π_{n+2}-statement (cf. Section 1.3.5), η is correct about this, contradicting minimality of η. □

Adapting the proof exactly as the adaptation needed between Theorems 4.5.1 and 4.5.4, we get the following results.

Theorem 4.6.2. The following are equivalent any natural number n and any cardinal κ:

(1) κ is extendible.

(2) $\kappa = \text{comp}(\mathcal{L}^{s,n}_{\kappa\omega})$.

(4) $\kappa = \text{SULST}(\mathcal{L}^{s,n}_{\kappa\omega})$.

(6) $\kappa = \text{ULST}(\mathcal{L}^{s,n}_{\kappa\omega})$.

Because of the stratification of Vopěnka's Principle by $C^{(n)}$-extendible cardinals (cf. Theorem 1.3.30), we further get:

Corollary 4.6.3. The following are equivalent for every natural number n:

(1) $\text{VP}(\Pi_{n+1})$

(2) $\mathcal{L}^{s,n}$ has a ULST number.

(3) $\mathcal{L}^{s,n}$ has an SULST number.

Corollary 4.6.4. The following are equivalent:

(1) VP.

(2) Every logic has a ULST number.

(3) Every logic has an SULST number.

Figure 4.1 summarises the relation of ULST and SULST numbers to the notions considered in Chapters 1 and 2.

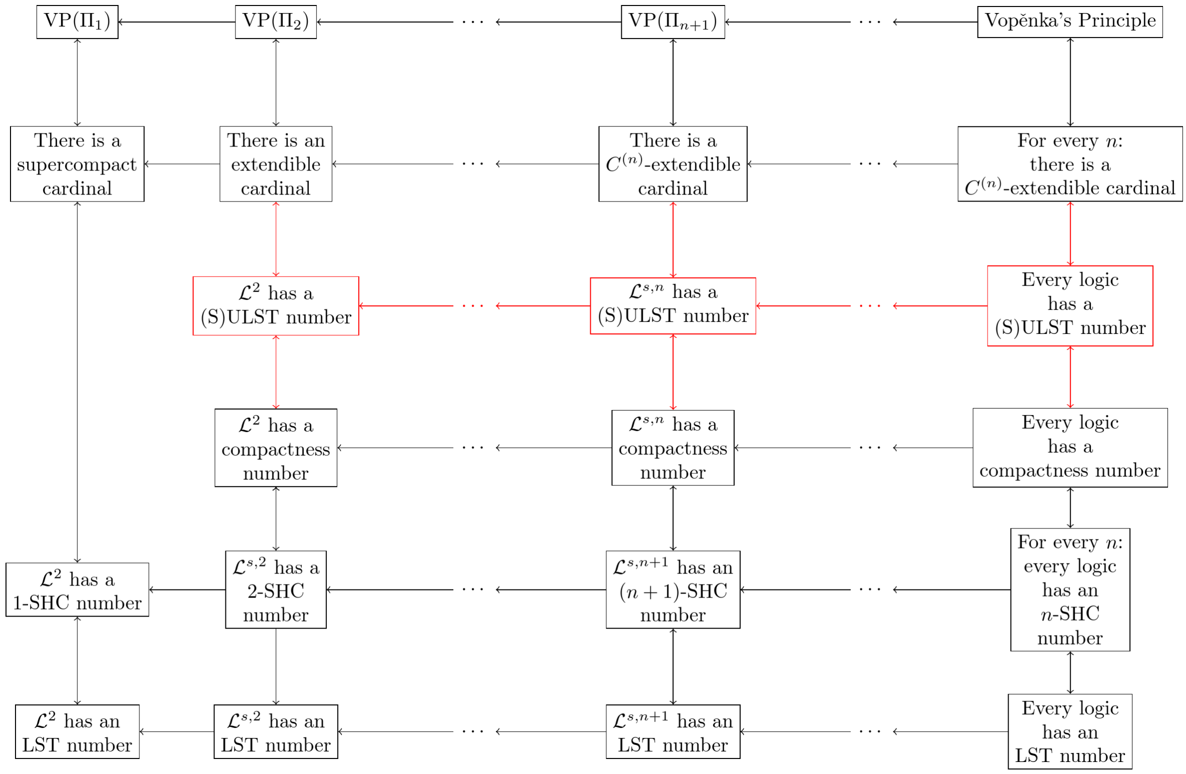

Figure 4.1.: Relations between VP, $C^{(n)}$-extendible cardinals, (S)ULST numbers, compactness numbers, SCH numbers, and LST numbers.

4.7. Infinitary logics

In this section, we consider infinitary logics $\mathcal{L}_{\eta\eta}$ with η an uncountable regular cardinal. We will see that ULST and SULST numbers are related to variations of *tall* cardinals. In Section 4.7.1, we first review this notion and show that the existence of tall cardinals can be witnessed by set-sized embeddings. We then proceed to analyse SULST numbers in Section 4.7.2. Our main results about SULST numbers are that $\text{SULST}(\mathcal{L}_{\eta\eta}) = \eta$ if and only if η is tall (Corollary 4.7.9), and further a general description of $\text{SULST}(\mathcal{L}_{\eta\eta})$ by a variation of tall cardinals (Corollary 4.7.20). Again, we will need the existence of these variations to be witnessed by set-sized embeddings (Corollary 4.7.16). In Section 4.7.3, to analyse ULST numbers we introduce the notion of *supreme for tallness*, and show that $\text{ULST}(\mathcal{L}_{\eta\eta}) = \eta$ if and only if η is supreme for tallness (Corollary 4.7.24). Finally, in Section 4.7.4, we show that the existence ULST numbers and SULST numbers of $\mathcal{L}_{\eta\eta}$ can consistently be separated (Theorems 4.7.25 and 4.7.26).

Before considering tall cardinals, let us start with an easy observation: that $\text{ULST}(\mathcal{L}_{\eta\eta})$ is bound by η. Recall that for every ordinal $\alpha < \eta$ there is the formula $\sigma_\alpha(x) \in \mathcal{L}_{\eta\omega}$ such that in any transitive model $(M, \in)$ we have $M \models \sigma_\alpha(a)$ iff $\alpha = a$. Recall further that for every $\alpha < \eta$, there is a sentence ψ_α in $\mathcal{L}_{\eta\eta}$, which over a transitive model of set theory N, expresses closure under α-sequences, $N^\alpha \subseteq N$, and that $\mathcal{L}_{\eta\eta}$ can define well-foundedness by the sentence $\varphi_{\text{WF}} = \neg\exists(x_i : i \in \omega) \bigwedge_{i<\omega} x_{i+1} \in x_i$ (cf. Lemma 1.2.4).

Proposition 4.7.1. If $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ exists, then $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) \geq \eta$.

Proof. Suppose that $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) = \delta < \eta$. Consider the model $\mathcal{M} = (\delta, \in)$ and let $\varphi = \neg\exists x \sigma_\delta(x) \wedge \mathrm{Ext} \wedge \varphi_{\mathrm{WF}} \wedge$ " $\in$ is a linear order". Then $\mathcal{M}$ satisfies φ. But, by $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) = \delta$, take a superstructure $(N, \in)$ of M of size $|N| > \delta$ satisfying φ. By collapsing we can without loss of generality assume N to be transitive. Then N is well-ordered by $\in$ and therefore an ordinal. Because $|N| \geq \delta^+$, it follows that $\delta \in N$. Thus $N \models \exists x \sigma_\delta(x)$ and $(N, \in) \models \neg\varphi$. Contradiction. □

4.7.1. Tall cardinals

Let us introduce the basic large cardinal notion relevant for $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ and $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$. A cardinal κ is *θ-tall*, for $\theta > \kappa$, if there exists an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \theta$ and $M^\kappa \subseteq M$, and a cardinal κ is *tall* if it is θ-tall for every $\theta > \kappa$. The difference between a tallness embedding and an iterated measurability embedding is the closure of the target model M. More generally, for some λ with $\omega \leq \lambda \leq \kappa$, the cardinal κ is *θ-tall with closure* λ if there exists an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \theta$ and $M^\lambda \subseteq M$, and a cardinal κ is *tall with closure* λ if it is θ-tall with closure λ for every $\theta > \kappa$. If the target model M has closure $M^{<\lambda} \subseteq M$, then we say that κ is (θ-)*tall with closure* $<\lambda$. All these cardinals were introduced by Hamkins in [Ham09]. Closure of size $<\kappa$ is sufficient for full tallness:

Proposition 4.7.2 (Hamkins [Ham09, Theorem 5.1])**.** If a cardinal κ is tall with closure $<\kappa$, then κ is tall.

We will use that tallness with closure $\lambda < \kappa$ can already be witnessed by set-sized embeddings. Before we show this, let us first argue that it is witnessed by extender embeddings. This was shown for full tallness in [Ham09, Lemma 2.9] and our argument is similar. Suppose $j : V \to M$ is elementary with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \theta$ and $M^\lambda \subseteq M$, witnessing that κ is θ-tall with closure $\lambda < \kappa$. Note that for any $\alpha > \kappa$, j restricts to an elementary embedding $j : V_\alpha \to V^M_{j(\alpha)}$. Let us derive a $(\kappa, j(\kappa))$-extender $E = (E_a \colon a \in [j(\kappa)]^{<\omega})$ by letting for $X \subseteq [\kappa]^{|a|}$:

$$X \in E_a \text{ iff } a \in j(X).$$

Let M_E be the extender power of V by E. By Theorem 1.3.36 this comes with the canonical elementary embedding $j_E : V \to M_E$ such that $\mathrm{crit}(j_E) = \kappa$ and $j_E(\kappa) \geq j(\kappa) > \theta$. We show the following assertion.

Claim 4.7.3. The map j_E witnesses that κ is θ-tall with closure λ.

Proof. The only thing left to check is the closure. Recall that $M_E = \{j_E(f)(a) \colon a \in [j(\kappa)]^{<\omega}, f : [\kappa]^{|a|} \to V\}$. So fix

$$\{j_E(f_\alpha)(a_\alpha) \colon \alpha < \lambda\} \subseteq M_E,$$

where each $a_\alpha \in [j(\kappa)]^{<\omega}$ and $f_\alpha : [\kappa]^{|a|} \to V$. Note that because $\lambda < \kappa = \mathrm{crit}(j_E)$,

$$j_E((f_\alpha \colon \alpha < \lambda)) = (j_E(f_\alpha) \colon \alpha < \lambda) \in M_E.$$

So if we can show that $(a_\alpha \colon \alpha < \lambda) \in M_E$ we are done with the proof of Claim 4.7.3, as then the pointwise evaluation $(j_E(f_\alpha)(a_\alpha) \colon \alpha < \lambda)$ will also be in M_E. To this end, fix the canonical factor map $k : M_E \to M$ such that $j = k \circ j_E$ and with $\mathrm{crit}(\kappa) > j(\kappa)$ (cf. Theorem 1.3.36). Recall that k is the inverse collapsing isomorphism. Since M is closed under λ-sequences, ${}^\lambda([j(\kappa)]^{<\omega}) \subseteq M$ and so M's version of ${}^\lambda([j(\kappa)]^{<\omega})$ is the real ${}^\lambda([j(\kappa)]^{<\omega})$. Because $\mathrm{crit}(k) > j(\kappa)$, we have $\lambda, j(\kappa) \in \mathrm{ran}(k)$. Because ${}^\lambda([j(\kappa)]^{<\omega})$ is definable from λ and $j(\kappa)$, by elementarity of k, we have ${}^\lambda([j(\kappa)]^{<\omega}) \in \mathrm{ran}(k)$. Further, M has an enumeration g of ${}^\lambda([j(\kappa)]^{<\omega}) \in \mathrm{ran}(k)$ and g has domain $(j(\kappa)^\lambda)^M$. Note that as κ is inaccessible in M, $(j(\kappa)^\lambda)^M = j(\kappa)$. So g has domain $j(\kappa)$. Again, by elementarity, $\mathrm{ran}(k)$ also has such an enumeration with domain $j(\kappa)$, say h. Because $j(\kappa) \subseteq \mathrm{ran}(k)$, we get that the evaluation of $h(\beta)$ at any $\beta < j(\kappa)$ is in $\mathrm{ran}(k)$. Thus, as $(a_\alpha \colon \alpha < \lambda) \in {}^\lambda([j(\kappa)]^{<\omega})$, we have that $(a_\alpha \colon \alpha < \lambda) = h(\beta) \in \mathrm{ran}(k)$ for some $\beta < j(\kappa)$. So we can consider $k^{-1}((a_\alpha \colon \alpha < \lambda)) = (k^{-1}(a_\alpha) \colon \alpha < \lambda) = (a_\alpha \colon \alpha < \lambda)$, where the equalities hold because $\mathrm{crit}(k) > j(\kappa)$ and $a_\alpha \in [j(\kappa)]^{<\omega}$. Therefore $(a_\alpha \colon \alpha < \lambda) \in M_E$. □

We thus know that being θ-tall with closure λ is witnessed by extender embeddings. Towards our goal of witnessing this by set-sized embeddings, we use this fact to prove the following lemma.

Lemma 4.7.4. Let κ be a cardinal, $\theta > \lambda$ and $\lambda < \kappa$. Suppose for some $\alpha > \kappa$, which is a successor ordinal or of cofinality $\mathrm{cof}(\alpha) > \kappa$, there is an elementary embedding $j : V_\alpha \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \theta$ and such that the set of all functions $\lambda \to [j(\kappa)]^{<\omega}$ is contained in N, i.e., ${}^\lambda([j(\kappa)]^{<\omega}) \subseteq N$. Then κ is θ-tall with closure λ.

Proof. We can use j to derive an extender $E = (E_a \colon a \in [j(\kappa)]^{<\omega})$ by letting for $X \subseteq [\kappa]^{|a|}$:

$$X \in E \text{ iff } a \in j(X).$$

We again claim that the extender power $j_E : V \to M_E$ witnesses that κ is θ-tall with closure λ. By Theorem 1.3.40, $\mathrm{crit}(j_E) = \kappa$ and $j_E(\kappa) \geq j(\kappa) > \theta$. So again, left to show is closure of M_E under λ-sequences. As in the proof of Claim 4.7.3, it is sufficient to argue that any sequence $(a_\alpha \colon \alpha < \lambda)$ belongs to M_E, where $a_\alpha \in [j(\kappa)]^{<\omega}$. We would like to argue as in that proof, but we do not have the factor map k around, as j is not into a proper class target. Nevertheless, we can argue nearly analogously: By Theorem 1.3.40, j_E restricts to the canonical map $j_{E,m} : V_\alpha \to m_E$ into the extender power m_E of V_α by E, and further $m_E \subseteq M_E$. For $j_{E,m}$, we have the canonical map $k_{E,m} : m_E \to N$ such that $\mathrm{crit}(j) = k_{E,m} \circ j_{E,m}$ with $k_{E,m}(\beta) = \beta$ for all $\beta \leq j(\kappa)$. We did not assume closure of N under sequences, but by assumption, N has all functions $\lambda \to [j(\kappa)]^{<\omega}$. So N's version of ${}^\lambda([j(\kappa)])^{<\omega}$ is the real ${}^\lambda([j(\kappa)])^{<\omega}$. Using that $\lambda, \kappa \in \mathrm{ran}(k)$ (as $k_{E,m}(\beta) = \beta$ for all $\beta \leq j(\kappa)$), we can argue exactly as in the proof of Claim 4.7.3 to show that ${}^\lambda([j(\kappa)])^{<\omega} \subseteq m_E$. Because $m_E \subseteq M_E$, we are done. □

Summarising, we get the following characterisation.

Lemma 4.7.5. The following are equivalent for a cardinal κ, $\lambda < \kappa$ and $\theta > \kappa$.

(1) κ is θ-tall with closure λ.

(2) For some $\alpha > \kappa$, which is a successor ordinal or of $\mathrm{cof}(\alpha) > \kappa$, there is an elementary embedding $h : V_\alpha \to N$ with $\mathrm{crit}(h) = \kappa$, $h(\kappa) > \theta$ and such that N has all functions $\lambda \to [j(\kappa)]^{<\omega}$.

Further, κ is θ-tall with closure $< \kappa$ iff for some $\alpha > \kappa$, which is a successor ordinal or of $\mathrm{cof}(\alpha) > \kappa$, there is an elementary embedding $h : V_\alpha \to N$ with $\mathrm{crit}(h) = \kappa$, $h(\kappa) > \theta$ and such that N has, for all $\lambda < \kappa$, all functions $\lambda \to [j(\kappa)]^{<\omega}$.

Proof. That (2) implies (1) is exactly the assertion of Lemma 4.7.4. In particular, for the backward direction of the *further* part, if there is such an embedding $j : V_\alpha \to N$ such that N has all functions $\lambda \to [j(\kappa)]^{<\omega}$ for *all* $\lambda < \kappa$, then κ is θ-tall with closure λ for all $\lambda < \kappa$, so θ-tall with closure $< \kappa$. On the other hand, if (1) holds, so κ is θ-tall with closure λ, as, say, witnessed by $j : V \to M$ such that $M^\lambda \subseteq M$. Then clearly, M has all functions $\lambda \to [j(\kappa)]^{<\omega} \subseteq V^M_{j(\alpha)}$ for any $\alpha > j(\kappa)$ as assumed. Because j restricts to an elementary embedding $j : V_\alpha \to V^M_{j(\alpha)}$, this shows (2). If M has closure $< \kappa$, then $V^M_{j(\alpha)}$ has all functions $\lambda \to [j(\kappa)]^{<\omega}$ for *all* $\lambda < \kappa$ and so the *further* part follows. □

We can therefore witness tallness by embeddings between set-sized structures.

4.7.2. SULST numbers of infinitary logics

We proceed to consider the relationship of $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$ with tall cardinals.

Theorem 4.7.6. If there is a tall cardinal $\kappa \geq \eta$ with closure $<\eta$, then $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$ exists and is at most κ. In particular, if κ is tall, then

$$\mathrm{SULST}(\mathcal{L}_{\kappa\kappa}) = \mathrm{ULST}(\mathcal{L}_{\kappa\kappa}) = \kappa.$$

Proof. Suppose that $\mathcal{A}$ is a τ-structure of size $|\mathcal{A}| \geq \kappa$. Let $\bar{\gamma} > |\mathcal{A}|$ be a cardinal. By tallness, let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \bar{\gamma}^+$ and $M^{<\eta} \subseteq M$. We now argue as in the proof of Theorem 4.4.1. In M, $j(\mathcal{A})$ is a $j(\tau)$-structure of size $|j(\mathcal{A})|^M \geq j(\kappa) > \bar{\gamma}^+$. In particular, in V, $|j(\mathcal{A})| \geq \bar{\gamma}^+ > \bar{\gamma}$. Further, in V we see that $j(\mathcal{A})$ restricts to a $j``\tau$-structure $j(\mathcal{A}) \restriction j``\tau$ which is a superstructure of $j``\mathcal{A}$. It is therefore sufficient to show that $j``\mathcal{A}$ is an $\mathcal{L}_{\eta\eta}$-elementary substructure of $j(\mathcal{A}) \restriction j``\tau$. Note that $j : \mathcal{L}_{\eta\eta}[\tau] \to \mathcal{L}_{\eta\eta}[j``\tau]$ is a renaming (cf. Proposition 1.3.8). So suppose $j``\mathcal{A} \models j(\varphi)(j(a))$. Then via the renaming, we get $\mathcal{A} \models \varphi(a)$. And then by elementarity of j,

$$M \models \text{``}j(\mathcal{A}) \models j(\varphi)(j(a))\text{''}.$$

By M's closure, it is correct about $\mathcal{L}_{\eta\eta}$-satisfaction. In particular, in V we get that $j(\mathcal{A}) \restriction j``\tau \models j(\varphi)(j(a))$, as $j(\varphi) \in \mathcal{L}_{\eta\eta}[j``\tau]$.

We just argued that $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) \leq \kappa$. By Proposition 4.7.1, $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) \geq \eta$. Thus, in particular, if κ is tall, then $\mathrm{SULST}(\mathcal{L}_{\kappa\kappa}) = \mathrm{ULST}(\mathcal{L}_{\kappa\kappa}) = \kappa$. □

Recall that $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}) \leq \mathcal{L}_{\omega_1\omega_1}$. In particular, a ULST number of $\mathcal{L}_{\omega_1\omega_1}$ must be greater or equal than the least measurable cardinal. Since strongly compact cardinals are tall (cf. [Ham09, Theorem 2.11]) and it is consistent that the least measurable is the least strongly compact cardinal (cf. [Mag76]), we get the following:

Theorem 4.7.7. It is consistent that the following are equivalent for a cardinal κ:

(1) κ is the least measurable cardinal.

(2) $\kappa = \mathrm{comp}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

(3) κ is the least cardinal such that $\kappa = \mathrm{comp}(\mathcal{L}_{\kappa\kappa})$.

(4) $\kappa = \mathrm{SULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$.

(5) $\kappa = \mathrm{SULST}(\mathcal{L}_{\eta\eta})$ for all uncountable regular $\eta \leq \kappa$.

(6) $\kappa = \mathrm{ULST}(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$

(7) $\kappa = \mathrm{ULST}(\mathcal{L}_{\eta\eta})$ for all uncountable regular $\eta \leq \kappa$.

Theorem 4.7.8. If $\mathrm{SULST}(\mathcal{L}_{\kappa\kappa}) = \kappa$, then κ is tall.

Proof. By Proposition 4.7.2, it suffices to show that κ is tall with closure $<\kappa$. We will show that for every cardinal $\theta > \kappa$, there is an elementary embedding $j_\theta : V_{\kappa+1} \to N_\theta$ with $\mathrm{crit}(j_\theta) = \kappa$, $j_\theta(\kappa) > \theta$ and such that N_θ has all functions $\lambda \to [j(\kappa)]^{<\omega}$ for all $\lambda < \kappa$. This suffices by Lemma 4.7.5. Consider the structure

$$\mathcal{M} = (H_{\kappa^+}, \in, \kappa).$$

Fix $\theta > \kappa$. Since $\kappa = \mathrm{SULST}(\mathcal{L}_{\kappa\kappa})$, there is a model $\mathcal{N} = (N, E, \kappa)$ of size larger than the smallest $\beth$-fixed point $\beth_\rho = \rho > \theta$ and such that $\mathcal{M}$ is an $\mathcal{L}_{\kappa\kappa}$-elementary substructure of $\mathcal{N}$. As H_{κ^+} is well-founded, it satisfies φ_{WF} and the extensionality axiom. Thus, also (N, E) satisfies these sentences and so we can collapse (N, E, κ) to a model $\bar{\mathcal{N}} = (\bar{N}, \in, \bar{\kappa})$. Then the collapse isomorphism restricts to an $\mathcal{L}_{\kappa\kappa}$-elementary embedding $j : (H_\kappa^+, \in, \kappa) \to (\bar{N}, \in, \bar{\kappa})$. Because H_{κ^+} satisfies that κ is the largest cardinal, $\bar{\kappa}$ is the largest $\bar{N}$-cardinal, and in particular $j(\kappa) = \bar{\kappa}$. By size of $\bar{N}$, we have $j(\kappa) = \bar{\kappa} \geq \rho > \theta$. In particular, j has a critical point $\mathrm{crit}(j) \leq \kappa$. Using that $\mathcal{L}_{\kappa\kappa}$ can define all ordinals $< \kappa$, we see that $\mathrm{crit}(j) = \kappa$. Furthermore, notice that H_{κ^+} is closed under κ-sequences, and in particular, $\mathcal{M} \models \psi_\alpha$, witnessing that $H_{\kappa^+}^\alpha \subseteq H_{\kappa^+}$ for every $\alpha < \kappa$. Thus also $\bar{\mathcal{N}}$ satisfies these sentences, and thus $\bar{N}^{<\kappa} \subseteq \bar{N}$. Further, $\bar{N}$ believes that $V^{\bar{N}}_{j(\kappa)+1}$ has all functions $\lambda \to [j(\kappa)]^{<\omega}$ for any $\lambda < \kappa$ (as $j(\kappa)$ is regular in $\bar{N}$). By its own closure, it is correct about this. Thus the restriction $j \upharpoonright V_{\kappa+1} : V_{\kappa+1} \to V^{\bar{N}}_{j(\kappa)+1}$ has all the required properties. □

Corollary 4.7.9. A cardinal κ is tall if and only if $\mathrm{ULST}(\mathcal{L}_{\kappa\kappa}) = \mathrm{SULST}(\mathcal{L}_{\kappa\kappa}) = \kappa$.

Since consistency strength-wise strongly compact cardinals are much stronger than tall cardinals (which are equiconsistent with strong cardinals (see [Ham09, Corollary 3.14])), it is consistent to have a tall cardinal that is not strongly compact.

Corollary 4.7.10. It is consistent that $\mathrm{ULST}(\mathcal{L}_{\kappa\kappa}) = \mathrm{SULST}(\mathcal{L}_{\kappa\kappa}) = \kappa$, but κ is not a compactness number for $\mathcal{L}_{\kappa\kappa}$.

Next, we introduce a version of tall cardinals κ, where the defining embeddings, instead of mapping κ as high as desired, map some fixed ordinal $\delta \geq \kappa$ as high as desired. Note the similarity to our Definition 3.2.4.

Definition 4.7.11. A cardinal $\kappa \leq \delta$ is *θ-tall pushing up δ with closure $\lambda \leq \kappa$* if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $M^\lambda \subseteq M$, and $j(\delta) > \theta$. A cardinal $\kappa \leq \delta$ is *tall pushing up δ with closure λ* if it is θ-tall pushing up δ with closure λ for all $\theta > \kappa$. If the target model M has closure $M^{<\lambda} \subseteq M$, then we say that κ is (θ-)*tall pushing up δ with closure $<\lambda$*.

Observe that a θ-tall cardinal κ with closure λ is θ-tall pushing up κ with closure λ. But in our more general definition it might be a larger ordinal than κ that gets mapped beyond θ.

We would like to thank Joel David Hamkins for pointing out the following result separating tall cardinals from tall cardinal pushing up some δ.

Proposition 4.7.12 (Hamkins)**.** It is consistent that there is a cardinal κ which is not tall, but for which there is an ordinal $\delta > \kappa$ such that κ is tall pushing up δ.

Proof. Suppose we have a model in which κ is measurable but not tall and $\delta > \kappa$ is tall. Let us argue that κ is tall pushing up δ. Fix an ordinal θ and let $j : V \to M$ be an elementary embedding with $M^\delta \subseteq M$, $\mathrm{crit}(j) = \delta$, and $j(\delta) > \theta$, witnessing the θ-tallness of δ. Let $h : V \to N$ be the ultrapower embedding by a κ-complete ultrafilter on κ, so that we have $\mathrm{crit}(h) = \kappa$. Recall that for any such ultrapower, $N^\kappa \subseteq N$ (cf., e.g., [Kan03, Proposition 5.7(d)]). Let $j : N \to \bar{N}$ be the restriction of j to N. Since $N^\kappa \subseteq N$, by elementarity, we get that $\bar{N}^{j(\kappa)} \subseteq \bar{N}$ in M. By the closure of M, we get that $\bar{N}^\kappa \subseteq \bar{N}$. The composition $j \circ h : V \to \bar{N}$ now witnesses that κ is θ-tall pushing up δ. □

Question 4.7.13. Is the existence of a tall cardinal κ pushing up some $\delta > \kappa$ equiconsistent with a tall cardinal?

We want to show that κ being tall pushing up δ with closure $<\eta$ is witnessed by extenders and, thus, by set-sized embeddings. This can be shown by similar arguments as for the case of being tall with closure $< \kappa$, but the technical details are somewhat more involved, so let us give the argument. We assume that δ is a strong limit cardinal, $\eta \leq \kappa$ is regular, and $\theta^\omega > \theta$ (for example, by assuming $\mathrm{cof}(\theta) = \omega$). Suppose we have an embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\delta) > \theta$ and such that M is $<\eta$-closed witnessing that κ is θ-tall pushing up δ with closure $<\eta$. By the closure of M, for any $\beta < \eta$ we get that ${}^\beta\theta = ({}^\beta\theta)^M$ and thus $\theta^\beta \leq (\theta^\beta)^M$. Further, we have that $\theta^\beta \leq (\theta^\beta)^M < j(\delta)$, because M believes that $j(\delta)$ is a strong limit cardinal and

$\beta < \theta < j(\delta)$. Let $\gamma = \sup\{(\theta^\beta)^M \colon \beta < \eta\}$. Notice that by our remarks we have that $\gamma \leq j(\delta)$. We can therefore fix a smallest ordinal $\zeta \leq \delta$ such that $j(\zeta) \geq \gamma$ and derive an extender E from j by letting for $a \in [\gamma]^{<\omega}$ and $X \subseteq [\zeta]^{|a|}$:

$$X \in E_a \text{ iff } a \in j(X).$$

Let M_a be the ultrapower of V by E_a and let M_E be the direct limit of the M_a. By Theorem 1.3.36 we get the canonical embeddings $j_E : V \to M_E$ and $k : M_E \to M$ such that $j = k \circ j_E$ and with $\operatorname{crit}(j_E) = \kappa$, $j_E(\delta) \geq j_E(\zeta) \geq \gamma \geq \theta^\omega > \theta$ and $\operatorname{crit}(k) \geq \gamma$, where k is the inverse transitive collapse. Further

$$M_E = \{j_E(f)(a) \colon a \in [\gamma]^{<\omega}, f : [\zeta]^{|a|} \to V\}$$

and

$$\operatorname{ran}(k) = \{j(f)(a) \colon a \in [\gamma]^{<\omega}, f : [\zeta]^{|a|} \to V\}.$$

Note that $\operatorname{ran}(k)$ is an elementary substructure of M. Again, we want to show the following claim.

Claim 4.7.14. The map $j_E : V \to M_E$ witnesses that κ is θ-tall pushing up δ with closure $<\eta$.

Proof. It remains to check the closure of M_E. So let $\nu < \eta$. Exactly as in the proof of Claim 4.7.3, it is sufficient to show for any $\{a_\alpha \colon \alpha < \nu\} \subseteq M_E$ where each $a_\alpha \in [\gamma]^{<\omega}$, that the sequence $(a_\alpha \colon \alpha < \lambda)$ belongs to M_E. Because

$$\operatorname{crit}(k) \geq \gamma \geq (\theta^\nu)^M \geq \theta^\nu$$

we know that $\gamma \subseteq \operatorname{ran}(k)$ and as we assumed $\theta^\nu \geq \theta^\omega > \theta > \eta > \nu$, we have $\theta, \eta, \nu \in \operatorname{ran}(k)$. Because M is closed under ν-sequences, ${}^\nu([\gamma]^{<\omega}) \subseteq M$, and so M's version of ${}^\nu([\gamma]^{<\omega})$ is the real ${}^\nu([\gamma]^{<\omega})$. Now note that γ is definable from η and θ in M and, thus, by elementarity, γ must be in $\operatorname{ran}(k)$. Similarly, ${}^\nu([\gamma]^{<\omega})$ is definable from ν and γ and so again by elementarity we get that ${}^\nu([\gamma]^{<\omega})$ has to be in $\operatorname{ran}(k)$ as well. Further, M believes that there is an enumeration g of ${}^\nu([\gamma]^{<\omega})$ and g has domain $(\gamma^\nu)^M$. Thus, $\operatorname{ran}(k)$ also has such an enumeration, say h. We claim that $(\gamma^\nu)^M = \gamma$. Note that by definition of γ and regularity of η, we either have that $\gamma = (\theta^\beta)^M = (\theta^\mu)^M$ for some $\beta < \eta$ and any $\beta \leq \mu < \eta$, or $\operatorname{cof}(\gamma) = \eta$. In the first case, in M, we have $\gamma^\nu = (\theta^\beta)^\nu = \theta^\beta = \gamma$. So let us assume that $\operatorname{cof}(\gamma) = \eta$. Thus, because $\nu < \eta$, if $f : \nu \to \gamma$ is a function in M, then $\operatorname{ran}(f) \subseteq (\theta^\beta)^M$ for some $\beta < \eta$. Then we have

$$(\gamma^\nu)^M \leq \sup\{((\theta^\beta)^\nu)^M \colon \beta < \eta\} = \sup\{(\theta^\beta)^M \colon \beta < \eta\} = \gamma.$$

Thus $(\gamma^\nu)^M = \gamma$. Therefore h is an enumeration of ${}^\nu([\gamma]^{<\omega})$ with domain $\gamma \subseteq \operatorname{ran}(k)$ and we get that the evaluation $h(\alpha)$ at any $\alpha < \gamma \subseteq \operatorname{ran}(k)$ is in $\operatorname{ran}(k)$. So ${}^\nu([\gamma]^{<\omega}) \subseteq \operatorname{ran}(k)$. In particular, $(a_\alpha \colon \alpha < \nu) \in \operatorname{ran}(k)$. Fix $f \in {}^\nu([\gamma]^{<\omega}) \subseteq \operatorname{ran}(k)$. Our argument above also shows that $f : \nu \to [(\theta^\beta)^M]^{<\omega}$ for some $\beta < \eta$. Since k fixes β and θ, it follows that $k(\theta^\beta) = (\theta^\beta)^M$. If $(\theta^\beta)^M < \gamma$, then $f \in V_\gamma$ and so $k^{-1}(f) = f$. And if $\theta^\beta = \gamma$, then $\gamma \in \operatorname{ran}(k)$, hence $\operatorname{crit}(k) > \gamma$, and so also, $k^{-1}(f) = f$. It follows that

$$(a_\alpha \colon \alpha < \nu) = k^{-1}((a_\alpha \colon \alpha < \nu\})) \in M_E.$$

$\square$

Note that in the above argument, the fact that $V^M_{j(\delta)+1}$ has all functions $f : \nu \to [\gamma]^{<\omega}$ is sufficient to show that M_E is $<\eta$-closed. Thus, analogously to how we showed Lemma 4.7.4, we can conclude that θ-tallness pushing up δ with closure $<\eta$ is, under the above conditions, already witnessed by set-sized embeddings:

Lemma 4.7.15. Suppose that δ is a strong limit cardinal, $\eta \leq \kappa$ is regular and $\theta > \delta$ is such that $\theta^\omega > \theta$. Assume that for some $\alpha > \delta$, which is a successor ordinal or of $\operatorname{cof}(\alpha) > \delta$, there is an elementary embedding $j : V_\alpha \to N$ with $\operatorname{crit}(j) = \kappa$ and $j(\delta) > \theta$. Further let $\gamma = \sup\{(\theta^\beta)^N \colon \beta < \eta\}$ and assume that N has, for all $\nu < \eta$, all functions $\nu \to [\gamma]^{<\omega}$. Let $E = (E_a \colon a \in [\gamma]^{<\omega})$ be the extender derived from j consisting of ultrafilters over $[\zeta]^{|a|}$, where ζ is the smallest ordinal such that $j(\zeta) \geq \gamma$. Then the canonical embedding $j_E : V \to M_E$ witnesses that κ is θ-tall pushing up δ with closure $<\eta$.

Because the extender embedding j_E from Claim 4.7.14 restricts to an embedding $j_E : V_{\delta+1} \to V^{M_E}_{j(\delta)+1}$, we therefore get:

Corollary 4.7.16. Suppose that δ is a strong limit cardinal, $\eta \leq \kappa$ is regular and $\theta > \delta$ is such that $\theta^\omega > \theta$. Then the following are equivalent:

(1) κ is θ-tall pushing up δ with closure $<\eta$.

(2) For some $\alpha > \delta$, which is a successor ordinal or of $\operatorname{cof}(\alpha) > \delta$, there is an elementary embedding $h : V_\alpha \to N$ with $\operatorname{crit}(h) = \kappa$, $h(\delta) > \theta$, and such that for all $\nu < \eta$, N has all functions $\nu \to [\gamma]^{<\omega}$, where $\gamma = \sup\{(\theta^\beta)^N \colon \beta < \eta\}$.

Note that the following characterisation follows as a special case, letting $\eta = \lambda^+$.

Corollary 4.7.17. Suppose that δ is a strong limit cardinal, $\lambda < \kappa$ and $\theta > \delta$ is such that $\theta^\omega > \theta$. Then the following are equivalent:

(1) κ is θ-tall pushing up δ with closure λ.

(2) For some $\alpha > \delta$, which is a successor ordinal or of $\operatorname{cof}(\alpha) > \delta$, there is an elementary embedding $h : V_\alpha \to N$ with $\operatorname{crit}(h) = \kappa$, $h(\delta) > \theta$, and such that N has all functions $\lambda \to [(\theta^\lambda)^N]^{<\omega}$.

We will use these results to get more general versions of Theorems 4.7.6 and 4.7.8 as well as results about the ULST numbers.

Theorem 4.7.18. If there is a tall cardinal κ pushing up δ with closure $<\eta$ for some $\delta \geq \kappa$ and regular $\eta \leq \kappa$, then $\operatorname{SULST}(\mathcal{L}_{\eta\eta})$ exists and is at most δ.

The proof of the result above is completely analogous to the proof of Theorem 4.7.6 and indeed, Theorem 4.7.6 can be derived as a corollary.

Theorem 4.7.19. If $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) = \delta$, then there is a cardinal κ with $\eta \le \kappa \le \delta$ and such that κ is tall pushing up δ with closure $<\eta$.

Proof. Let ρ be the first $\beth$-fixed point above δ. We first argue that for any $\beth$-fixed point $\theta > \delta$ there is an elementary embedding $j_\theta : V_{\rho+1} \to N_\theta$ with $\eta \le \mathrm{crit}(j_\theta) \le \delta$, $j_\theta(\delta) > \theta$, and such that N_θ has all functions $f : \nu \to [\alpha]^{<\omega}$ for all $\nu < \eta$ and all ordinals $\alpha < j_\theta(\rho)$. We consider the structure $\mathcal{M} = (V_\rho, \in, \delta)$. Fix ϑ_1, the first $\beth$-fixed point above θ, and ϑ_2, the first $\beth$-fixed point above ϑ_1. Since $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) = \delta$, there is a model $\mathcal{N} = (N, E, \bar{\delta})$, with $|N| > \vartheta_2$, and such that $\mathcal{M} \prec_{\mathcal{L}_{\eta\eta}} \mathcal{N}$. Because $\mathcal{M}$ satisfies the $\mathcal{L}_{\eta\eta}$-sentence asserting well-foundedness, we can assume that N is transitive, $E = \in$ and $j : M \to N$ is $\mathcal{L}_{\eta\eta}$-elementary with $j(\delta) = \bar{\delta}$. Write largest(x) for the formula expressing that x is the largest ordinal. The model $\mathcal{M}$ satisfies, for every $\nu < \eta$, the following sentence χ_ν of $\mathcal{L}_{\eta\eta}$, truthfully asserting that $V_{\rho+1}$ has all functions $\nu \to [\alpha]^{<\omega}$ for all ordinals $\alpha < \rho$:

$$\begin{aligned}\chi_\nu = &\forall\alpha\forall(x_\beta\colon \beta<\nu)[\exists x(\mathrm{largest}(x) \wedge \alpha < x \wedge \bigwedge_{\beta<\nu}(x_\beta \in [\alpha]^{<\omega}))\\ &\to \exists f \exists y(\mathrm{func}(f) \wedge \mathrm{dom}(f) = y \wedge \sigma_\nu(y) \wedge \bigwedge_{\beta<\nu}(\forall z(\sigma_\beta(z) \to f(z) = x_\beta)))].\end{aligned}$$

Because $|N| > \vartheta_2$ and $\vartheta_2 = \beth_{\vartheta_2}$, we get that $\vartheta_2 \in N$. By elementarity, $\mathcal{N}$ believes that there is exactly one $\beth$-fixed point above $\bar{\delta}$, so since $\mathcal{N}$ sees that ϑ_1 and ϑ_2 are $\beth$-fixed points, it follows that $\bar{\delta} \ge \vartheta_1 > \theta > \delta$. In particular, $\mathrm{crit}(j) \le \delta$. Since every ordinal $\beta < \eta$ is definable in the logic $\mathcal{L}_{\eta\eta}$, we must have $\eta \le \mathrm{crit}(j)$. Because $\mathcal{N}$ satisfies the sentences χ_ν for all $\nu < \eta$, it has all the required functions. So j is how we promised.

As there are unboundedly many $\theta > \delta$ but boundedly many $\kappa \le \delta$, we can fix a single κ with $\eta \le \kappa \le \delta$ and such that for any $\theta > \delta$, there is an elementary embedding $j_\theta : V_{\rho+1} \to N_\theta$ with $\mathrm{crit}(j_\theta) = \kappa$, $j_\theta(\delta) > \theta$, and such that N_θ has all functions $\nu \to [\alpha]^{<\omega}$ for all $\nu < \eta$ and all ordinals $\alpha < j_\theta(\rho)$. Let δ^* be the least cardinal with $\kappa \le \delta^* \le \delta$ and such that for any $\theta > \delta^*$, there is an elementary embedding $j_\theta : V_{\rho^*+1} \to N_\theta$, where ρ^* is the least $\beth$-fixed point above δ^*, with $\mathrm{crit}(j_\theta) = \kappa$, $j_\theta(\delta^*) > \theta$, and such that N_θ has all functions $\nu \to [\alpha]^{<\omega}$ for all $\nu < \eta$ and all ordinals $\alpha < j_\theta(\rho^*)$. Let us argue that δ^* is a strong limit. If δ^* is not a strong limit, then there is $\gamma < \delta^*$ with $\rho^* > 2^\gamma \ge \delta^*$. Note that ρ^* is the least $\beth$-fixed point above γ. Consider any strong limit $\theta > \rho^*$. By assumption, there is an elementary embedding $j : V_{\rho^*+1} \to N_\theta$ with $\mathrm{crit}(j) = \kappa$ and $j(\delta^*) > \theta$. Now because $2^\gamma \ge \delta^*$, by elementarity we get,

$$N_\theta \models 2^{j(\gamma)} \ge j(\delta^*) > \theta.$$

But then because θ is a strong limit, also $N_\theta \models j(\gamma) \ge \theta$. Because this works for any θ, this is a contradiction to the minimality of δ^*, verifying that δ^* is a strong limit cardinal.

We claim that κ is tall pushing up δ^* with closure $< \eta$. Then in particular κ is tall pushing up $\delta^* > \delta$ and so we showed what we promised. By Corollary 4.7.16, it is sufficient to show that for $\theta^\omega > \theta > \eta$, if $j_\theta : V_{\rho^*+1} \to N_\theta$ is one of the embeddings we produced, then N_θ has, for any $\nu < \eta$, all functions $\nu \to [\gamma]^{<\omega}$, where $\gamma = \sup\{(\theta^\beta)^{N_\theta} : \beta < \eta\}$. We know that N_θ has all functions $\nu \to \alpha$ for all $\alpha < j_\theta(\rho^*)$, so it suffices to argue $\gamma < j_\theta(\rho^*)$. By assumption on the embedding j_θ, we have $\theta < j_\theta(\delta^*) < j_\theta(\rho^*)$. As δ^* is a strong limit, $j_\theta(\delta^*)$ is a strong limit in the sense of N_θ. Thus $(\theta^\beta)^{N_\theta} < j_\theta(\delta^*)$ for any $\beta < \eta$. It follows that $\gamma \leq j_\theta(\delta^*) < j(\rho^*)$, and so we are done. □

Corollary 4.7.20. $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) = \delta$ if and only if δ is the smallest cardinal $\geq \eta$ such that there is a tall cardinal $\eta \leq \kappa \leq \delta$ pushing up δ with closure $< \eta$.

Observe that in the canonical model $L[U]$, there are no tall cardinals $\gamma \leq \delta$ pushing up δ with closure ω for the same reason that there are no ω_1-strongly compact cardinals (cf. the discussion after Corollary 4.4.3). Thus, in particular, it is consistent that there is a measurable cardinal, but there is no pair $\gamma \leq \delta$ such that γ is a tall cardinal pushing up δ with closure ω. In particular, by Theorem 4.7.19, if $\mathrm{SULST}(\mathcal{L}_{\omega_1\omega_1})$ exists, we must already have a pair $\kappa \leq \delta$ such that κ is a tall cardinal pushing up δ with closure ω. So having an SULST number for $\mathcal{L}_{\omega_1\omega_1}$ is stronger than having an SULST number for $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$.

4.7.3. ULST numbers of infinitary logics

To consider situations where $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) = \eta$, we introduce the following concept.

Definition 4.7.21. We say that a cardinal δ is *supreme for tallness* iff for all $\lambda < \delta$ and ordinals θ, there is a cardinal κ with $\lambda < \kappa \leq \delta$ and such that κ is θ-tall pushing up δ with closure λ.

Observe that a cardinal δ is supreme for tallness if and only if for every $\lambda < \delta$, there is a cardinal $\lambda < \kappa \leq \delta$ that is tall pushing up δ with closure λ. This follows because there are proper class many θ and the cardinals κ are bounded by δ. Observe also that a tall cardinal is trivially supreme for tallness. A non-tall cardinal that is a limit of tall cardinals is also supreme for tallness. Thus, a supreme for tallness cardinal can be singular. On the other hand, a regular supreme for tallness cardinal is inaccessible because it is a limit of measurable cardinals. But we show below that it need not be weakly compact (Theorem 4.7.25).

Theorem 4.7.22. If δ is supreme for tallness, then for every regular $\eta \leq \delta$, $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ exists and is at most δ. In particular, if δ is regular, then $\mathrm{ULST}(\mathcal{L}_{\delta\delta}) = \delta$.

Proof. Suppose that $\mathcal{A}$ is a τ-structure of size $|\mathcal{A}| \geq \delta$, $\eta \leq \delta$, and φ is a sentence in $\mathcal{L}_{\eta\eta}[\tau]$ such that $\mathcal{A} \models_{\mathcal{L}_{\eta\eta}} \varphi$. Since η is regular, there is $\lambda < \eta$ such that the length of all conjunctions and quantifiers in φ is smaller than λ. Let $\bar{\gamma} > |\mathcal{A}|$ be any cardinal. By our assumption there exists a cardinal κ with $\lambda < \kappa \leq \delta$ such that κ is $\bar{\gamma}^+$-tall pushing up δ with closure λ. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\delta) > \bar{\gamma}^+$, and $M^\lambda \subseteq M$. Consider the $j(\tau)$-structure $j(\mathcal{A})$. This restricts to a $j``\tau$-structure $j(\mathcal{A}) \upharpoonright j``\tau$, which is a superstructure of $j``\mathcal{A}$, which in turn is a renamed version of $\mathcal{A}$ via the renaming $j : \tau \to j``\tau$. Because $|j(\mathcal{A})|^M \geq j(\delta) > \bar{\gamma}^+$, we get $|j(\mathcal{A})| \geq \bar{\gamma}^+$. Since $\mathrm{crit}(j) = \kappa > \lambda$ and $\varphi \in \mathcal{L}_{\lambda\lambda}$, we get that $j(\varphi) \in \mathcal{L}_{\lambda\lambda}[j``\tau]$. By elementarity, $M$ satisfies that $j(\mathcal{A}) \models j(\varphi)$. Because $M$ is closed under $\lambda$-sequences, it is correct about $\mathcal{L}_{\lambda\lambda}$-satisfaction and so really $j(\mathcal{A}) \upharpoonright j``\tau \models j(\varphi)$. Then by renaming $j(\mathcal{A}) \upharpoonright j``\tau$ to a τ-structure, we found a superstructure of $\mathcal{A}$ of size $|A| > \bar{\gamma}$ satisfying φ. □

Theorem 4.7.23. If $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) = \eta$, then η is supreme for tallness.

Proof. Because $\mathcal{L}_{\eta\eta}$ can define well-foundedness and all ordinals $< \eta$, it is easy to see that η is either measurable or a limit of measurables. In particular it follows that η is a strong limit cardinal. Now let $\lambda < \eta$ and let $\theta > \eta$ be an ordinal with $\theta^\omega > \theta$. We need to find a cardinal $\lambda < \kappa \leq \eta$ that is θ-tall pushing up η with closure λ. We produce an embedding $j : V_{\eta^+} \to N$ with $\lambda < \mathrm{crit}(j) \leq \eta$, $j(\eta) > \theta$, and such that $N^\lambda \subseteq N$. By Corollary 4.7.17 this is sufficient. For $\alpha \leq \lambda$ take constant symbols c_α, let $c_\alpha^M = \alpha$ and consider the structure $M = (V_{\eta^+}, \in, \eta, c_\alpha^M, \mathrm{Tr}, \emptyset, S, P)_{\alpha \leq \lambda}$, where Tr is a truth predicate for $(V_{\eta^+}, \in)$, S codes the successor function, and P codes the pairing function. Note that because η^+ has cofinality greater than λ, V_ν is closed under λ-sequences. Then M satisfies the sentence φ of $\mathcal{L}_{\eta\eta}$ which is the conjunction of the following sentences:

(i) $\varphi_{\mathrm{WF}} \wedge \mathrm{Ext}$.

(ii) $\varphi_{\mathrm{Truth}} \wedge \varphi_\emptyset \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

(iii) "η is the largest cardinal."

(iv) ψ_λ.

(v) $\bigwedge_{\alpha \leq \lambda} \sigma_\alpha(c_\alpha)$.

Since $\eta = \mathrm{ULS}(\mathcal{L}_{\eta\eta})$, there is a model $N = (N, E, \bar{\eta}, c_\xi^N, \bar{\mathrm{Tr}})_{\xi \leq \lambda}$, with $|N| > \beth_\vartheta$, where $\vartheta > \theta$ and $\beth_\vartheta = \vartheta$, satisfying the above sentences and having M as a substructure. It follows that E is well-founded, so we can, by collapsing, assume that $\in = E$ and N is transitive. By (ii) and Lemma 4.3.2, the collapse isomorphism restricts to an elementary embedding $j : M \to N$. Because $\bar{\eta}$ is the largest N-cardinal, $j(\eta) = \bar{\eta}$. Since $|N| > \beth_\vartheta$, we have that $\eta < \theta < \vartheta \leq j(\eta) = \bar{\eta}$. In particular, $\mathrm{crit}(j) \leq \eta$. Because $N \models_{\mathcal{L}_{\eta\eta}} \bigwedge_{\alpha \leq \lambda} \sigma_\alpha(c_\alpha)$, it follows that $c_\alpha^N = \alpha$, and so $j(\alpha) = \alpha$ for all $\alpha \leq \lambda$. It follows that $\mathrm{crit}(j) > \lambda$. Finally, $N \models \psi_\lambda$ and therefore $N^\lambda \subseteq N$. □

Corollary 4.7.24. For regular cardinals η, $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) = \eta$ if and only if η is supreme for tallness.

4.7.4. Separating ULST and SULST numbers

Finally, we consistently separate the existence of $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ and $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$. Note that this differentiates $\mathcal{L}_{\eta\eta}$ from the other logics we considered.

Theorem 4.7.25. It is consistent that η is an inaccessible cardinal, $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ exists, but $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$ does not exist.

Proof. Let η be a supercompact cardinal with an inaccessible cardinal ν above it, and assume that ν is the least such inaccessible. Then η is a limit of strong cardinals, and hence a limit of tall cardinals (as strong cardinals are tall by [Ham09, Theorem 2.10]). In V_ν, η is also a supercompact limit of tall cardinals. Thus, we can assume without loss of generality that $V = V_\nu$, so that there are no inaccessible cardinals above η. First, we go to a forcing extension $V[c]$ by Cohen forcing. Since small forcing preserves tall cardinals (cf. [Ham09, Theorem 2.13]) and supercompact cardinals (cf., e.g., [Jec03, Theorem 21.2]) by standard embedding lifting arguments, η is still a supercompact limit of tall cardinals in $V[c]$. Next, we go to a forcing extension $V[c][G]$ by $\mathrm{Add}(\eta, 1)$. The forcing $\mathrm{Add}(\eta, 1)$ is $<\eta$-closed and hence, in particular, $\leq\kappa$-distributive for every tall cardinal $\kappa < \eta$ in $V[c][G]$. Thus, every tall cardinal $\kappa < \eta$ remains tall in $V[c][G]$ by [Ham09, Theorem 3.1]. The cardinal η remains inaccessible by the closure of $\mathrm{Add}(\eta, 1)$, but since the Cohen forcing makes η super destructible, it is not even weakly compact in $V[c][G]$ (cf. [Ham98, Main Theorem]). In particular, η is not tall. Thus, in $V[c][G]$, η cannot be $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$, and since there are no inaccessible cardinals above η, $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$ does not exist. But since η is a limit of tall cardinals in $V[c][G]$, it is, in particular, supreme for tallness there, and hence, in $V[c][G]$, $\eta = \mathrm{ULST}(\mathcal{L}_{\eta\eta})$. □

Next, we show that consistently we can have $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) < \mathrm{SULST}(\mathcal{L}_{\eta\eta})$.

Theorem 4.7.26. It is consistent that η is an inaccessible cardinal, $\mathrm{ULST}(\mathcal{L}_{\eta\eta})$ and $\mathrm{SULST}(\mathcal{L}_{\eta\eta})$ both exists, and $\mathrm{ULST}(\mathcal{L}_{\eta\eta}) < \mathrm{SULST}(\mathcal{L}_{\eta\eta})$.

Proof. We will argue as in the proof of Theorem 4.7.25, but start with a model in which there is a supercompact η and a tall cardinal ν above the supercompact cardinal. We again go to the forcing extension $V[c][G]$, in which $\eta = \mathrm{ULST}(\mathcal{L}_{\eta\eta})$, but $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) \neq \eta$. Next, observe that since tall cardinals are preserved by small forcing and $\mathrm{Add}(\eta, 1)$ is small relative to ν, the latter remains a tall cardinal in $V[c][G]$. Thus, $\mathrm{SULST}(\mathcal{L}_{\eta\eta}) \leq \mathrm{SULST}(\mathcal{L}_{\nu\nu}) = \nu$ exists. □

4.8. The equicardinality logic

Recall the notion of cardinal correctly extendible cardinals and its variants from Chapter 3, and that the models of the sentence $\mathrm{ZFC}^*_a \wedge \varphi_{\mathrm{Card}}$ are well-founded (cf. Theorem 3.5.1) and cardinal correct. We show that if there is a pair $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ, then $\mathrm{SULST}(\mathcal{L}(\mathsf{I}))$ exists and is bounded by δ. Almost conversely, we show that if $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ exists, then there is a pair $\kappa \leq \gamma$ such that κ is cardinal correctly extendible pushing up γ. We further show that a strongly compact cardinal is a lower bound on the consistency strength of the existence of a $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ (Theorem 4.8.6), and that $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ may be above the least supercompact cardinal (Theorem 4.8.5).

Theorem 4.8.1. If there exists a pair $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ, then SULST($\mathcal{L}(\mathsf{I})$) exists and is at most δ.

Proof. Let $\mathcal{A}$ be a τ-structure of size $|\mathcal{A}| \geq \delta$ and $\bar{\delta} > \delta$. Take $\alpha > \bar{\delta}^+$ with $\mathcal{A} \in V_\alpha$ and an elementary embedding $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\delta) > \alpha$ such that M is cardinal correct. Consider the $j(\tau)$-structure $j(\mathcal{A}) \in M$. Since $j``\tau \subseteq j(\tau)$, $j(\mathcal{A})$ is a $\tau$-structure modulo the renaming which takes $\tau$ to $j``\tau$ and contains $j``\mathcal{A}$ as a $j``\tau$-substructure. Note that j also restricts to a renaming $j : \mathcal{L}(\mathsf{I})[\tau] \to \mathcal{L}(\mathsf{I})[j``\tau]$ (cf. Section 1.3.2). If $j``\mathcal{A} \models j(\varphi)(j(a))$ for some $a \in A$, then $\mathcal{A} \models \varphi(a)$ via the renaming j. Then by elementarity, M satisfies that $j(\mathcal{A}) \models j(\varphi)(j(a))$, and is correct about this by cardinal correctness. This shows that $j``\mathcal{A}$ is an $\mathcal{L}(\mathsf{I})$-elementary substructure of $j(\mathcal{A}) \restriction j``\tau$. We can therefore rename the latter to an $\mathcal{L}(\mathsf{I})$-elementary superstructure of $\mathcal{A}$. Since $|\mathcal{A}| \geq \delta$, in M, by elementarity, we have $|j(A)| \geq j(\delta) > \alpha > \bar{\delta}^+$, as desired, and all these computations are correct by cardinal correctness. □

Theorem 4.8.2. If ULST($\mathcal{L}(\mathsf{I})$) exists, then there is a pair $\kappa \leq \gamma$ such that κ is cardinal correctly extendible cardinal pushing up γ.

Note that the following proof goes similar to that of Theorem 4.5.1. We will generalise the argument in Section 4.9 to show a general correspondence between ULST numbers and large cardinals for arbitrary logics.

Proof. Let $\delta = \mathrm{ULST}(\mathcal{L}(\mathsf{I}))$. Suppose $\alpha > \delta$ is a cardinal of cofinality ω, as witnessed by a cofinal function f. Let ρ_α be the least $\aleph$-fixed point above α. Consider the structure

$$\mathcal{M} = (V_{\rho_\alpha}, \in, f, \alpha, \mathrm{Tr}, \emptyset, S, P),$$

where Tr is a truth predicate, S codes the successor function and P codes the pairing function. Then $\mathcal{M}$ satisfies the sentence φ in the logic $\mathcal{L}(\mathsf{I})$, which is the conjunction of the sentences:

(i) ZFC^*_a.

(ii) φ_{Card}.

(iii) There are no $\aleph$-fixed points above α.

(iv) $\varphi_{\mathrm{Truth}} \wedge \varphi_\emptyset \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

(v) "f is a function with domain omega which is cofinal in α".

Since $\delta = \mathrm{ULST}(\mathcal{L}(\mathsf{I}))$, there is a model

$$\mathcal{N} = (N, E, \bar{f}, \bar{\alpha}, \bar{\mathrm{Tr}}, \bar{\emptyset}, \bar{S}, \bar{P})$$

of size larger than the smallest $\beth$-fixed point above ρ_α with $\mathcal{M}$ as a substructure. It follows, by Theorem 3.5.1, that E is well-founded. We can therefore assume that N is given by its transitive collapse and that $E = \in$, and further, by (iv) and Lemma 4.3.2, that the collapse restricts to an elementary embedding $j : V_{\rho_\alpha} \to N$. Note that N satisfies that there are no $\aleph$-fixed points above $\bar{\alpha}$ and so by its size, we get that $j(\alpha) = \bar{\alpha} > \alpha$. We have $j(f(n)) = \bar{f}(j(n)) = \bar{f}(n)$ for all $n \in \omega$ and further, f is cofinal in $\bar{\alpha}$ and thus there is $n \in \omega$ such that $j(f(n)) > \alpha > f(n)$. In particular $\mathrm{crit}(j) < \alpha$.

By what we showed, we can define the proper class function F on the stationary class $S = \{\alpha\colon \mathrm{Card}(\alpha) \text{ and } \mathrm{cof}(\alpha) = \omega\}$ such that $F(\alpha)$ is the least (ordinal coding a) pair $(\kappa_\alpha, \gamma_\alpha)$ such that for ρ_α the least $\aleph$-fixed point above α, κ_α is the least critical point $< \alpha$ of any elementary embedding $j : V_{\rho_\alpha} \to N$ such that N is cardinal correct and there is $\beta < \alpha$ such that $j(\beta) > \alpha$ and γ_α is the least such β. Since the (ordinal coding the) pair $(\kappa_\alpha, \gamma_\alpha)$ is smaller than α, F is regressive on the stationary class S. Thus, by the very weak class Fodor principle 4.5.2, F is constant on a proper class. Let (κ, γ) be this constant value. Then κ is cardinal correctly extendible pushing up γ. □

Corollary 4.8.3. The following are equivalent:

(1) There is a pair $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ.

(2) $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ exists.

(3) $\mathrm{SULST}(\mathcal{L}(\mathsf{I}))$ exists.

Question 4.8.4. If $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) = \delta$ exists, is there a cardinal $\kappa \leq \delta$ such that κ is cardinal correctly extendible pushing up δ?

Theorem 4.8.5. It is consistent, relative to an extendible cardinal, that $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ is above the least supercompact cardinal.

Proof. We work in the model $V[G][g]$ from the proof of Theorem 3.3.6, where ν was the least supercompact cardinal, $\chi > \nu$ was extendible, $2^\gamma = \gamma^{++}$ for a cardinal $\nu < \gamma < \chi$, and the GCH held above γ. Note that since we have an extendible cardinal in this model, $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ exists. Suppose that $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) \leq \nu$. Consider the model $(V_\rho, \in, \gamma)$, where ρ is the least $\aleph$-fixed point above γ. Then by our usual arguments, there is a model $\mathcal{N} = (N, \in, \bar{\gamma})$ of size much larger than ρ that is cardinal correct and we have an elementary embedding $j : V_\rho \to N$ such that $j(\gamma) = \bar{\gamma} \gg \gamma$. It follows, by elementarity, that N satisfies that $2^{\bar{\gamma}} > \bar{\gamma}^+$ and it must be correct about this by cardinal correctness. Thus, we have reached a contradiction showing that $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) > \gamma > \nu$. □

It follows from combining Proposition 3.3.4 and Theorem 4.8.2 that if $\mathrm{ULST}(\mathcal{L}(\mathsf{I}))$ exists, then either there is a strongly compact cardinal or there is an inaccessible α such that V_α satisfies that there is strongly compact cardinal. Below we slightly improve this result by showing that if $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) = \delta$, then either there is a strongly compact cardinal $\leq\delta$ or there is an inaccessible α such that V_α satisfies that there is a strongly compact cardinal. This appears to be a step in the direction of getting a positive answer to Question 4.8.4.

We will use the following observations. Suppose that $j : V \to M$ is the ultrapower by a fine, κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$. Consider the restriction $j : V_{\lambda+2} \to V^M_{j(\lambda)+2}$, and observe that it suffices to recover U since $\mathcal{P}(\mathcal{P}_\kappa(\lambda)) \subseteq V_{\lambda+2}$. Using a flat pairing function, we may assume that $V_{\lambda+2}$ is closed under functions $f : \mathcal{P}_\kappa\lambda \to V_{\lambda+2}$, as was carried out in Section 1.3.7. We can thus conclude that all the information about a λ-compactness embedding is already captured by the restriction of the embedding to $V_{\lambda+2}$. Further, suppose that $\alpha = \beta + 1$ is some successor ordinal $\geq \lambda + 2$. Recall that building the ultrapower m of V_α by U we get that $m = V^M_{j(\alpha)}$. Then $m = \{[f]_U \colon f : \mathcal{P}_\kappa\lambda \to V_\alpha\}$, and by coding, we can assume that every $f \in V_\alpha$. Thus $|V^M_{j(\alpha)}| = |m| \leq |V_\alpha|$.

Theorem 4.8.6. If $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) = \delta$, then there is a strongly compact cardinal $\kappa \leq \delta$ or there is an inaccessible cardinal α such that V_α satisfies that there is a strongly compact cardinal.

Proof. First, let us suppose that no cardinal $\kappa \leq \delta$ is δ^+-compact. Consider the structure

$$\mathcal{M} = (V_\rho, \in, \delta, \mathrm{Tr}, \emptyset, S, P),$$

where ρ is the least $\aleph$-fixed point above δ, Tr is a truth predicate for $(V_\rho, \in)$, S codes the successor function and P codes the pairing function. Then $\mathcal{M}$ satisfies the sentence φ in the logic $\mathcal{L}(\mathsf{I})$, which is the conjunction of the sentences:

(i) ZFC^*_a.

(ii) φ_{Card}.

(iii) There are no $\aleph$-fixed points above δ.

(iv) $\varphi_{\mathrm{Truth}} \wedge \varphi_\emptyset \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

Since $\mathrm{ULST}(\mathcal{L}(\mathsf{I})) = \delta$, there is a model

$$\mathcal{N} = (N, E, \bar{\delta}, \bar{\mathrm{Tr}}, \bar{\emptyset}, \bar{S}, \bar{P}) \models_{\mathcal{L}(\mathsf{I})} \varphi$$

of size much larger than ρ with $\mathcal{M}$ as a substructure. It follows by Theorem 3.5.1 that E is well-founded. Hence by collapsing and our usual argument using (iv) and Lemma 4.3.2, we can assume without loss of generality that $E = \in$, N is transitive, and we get an elementary embedding

$$j : V_\rho \to N$$

such that $j(\delta) = \bar{\delta}$ and N is cardinal correct. Since $|N|$ is much larger than ρ and N believes that there are no $\aleph$-fixed points above $\bar{\delta}$, it follows that $\bar{\delta} > \delta$, which means j has a critical point and $\mathrm{crit}(j) = \kappa \leq \delta$. If $\mathrm{crit}(j) = \delta$, then $j(\delta) > \delta$, and by cardinal correctness we get that $j(\delta)$ is some limit cardinal. Hence $\delta^+ < j(\delta) < j(\delta^+) = j(\delta)^+$, again using cardinal correctness. Thus, $j(\delta^+)$ is a successor cardinal and in particular regular. Then as in the proof of Proposition 3.3.2, j is discontinuous at δ^+, i.e., $\sup(j``\delta^+) < \delta^+$, and we can use $\sup(j``\delta^+)$ to obtain a uniform δ-complete ultrafilter on δ^+, letting $X \in U$ iff $\sup(j``\delta^+) \in j(\delta^+)$. By Theorem 3.3.1, this means that δ is δ^+-compact, contradicting our assumption. Thus, $\mathrm{crit}(j) = \kappa < \delta$.

First, suppose that $j(\kappa) \geq \delta$, and let us argue that κ is δ^+-compact, which would again contradict our assumption. By Theorem 3.3.1, it suffices to show that every successor cardinal β^+ in the interval $[\kappa, \delta^+]$ carries a uniform κ-complete ultrafilter. And again, for this it suffices to show that $\beta^+ < j(\beta^+) = j(\beta)^+$: then j is discontinuous at β^+, namely $j``\beta^+$ is bounded in $j(\beta^+)$, and we can argue as before. This is clearly true for $\beta = \delta$. So fix $\kappa \leq \beta < \delta$ and consider β^+. Since $j(\beta) \geq j(\kappa) \geq \delta$, it follows that $j(\beta) > \beta$, and hence $j(\beta)^+ = j(\beta^+) > \beta^+$. This concludes the argument that κ is δ^+-compact, which is the desired contradiction, showing that $j(\kappa) < \delta$. This means we can apply j to $j(\kappa) = \gamma$ to get $j^2(\kappa) = j(\gamma)$. Again, assuming that $j(\gamma) \geq \delta$, we will argue that κ must be δ^+-compact, and so will be able to conclude that $j(\gamma) < \delta$. By the same argument as before, we get a discontinuity for successors of $\gamma \leq \beta \leq \delta$. But if $\kappa \leq \beta < \gamma$, then $\beta < \gamma = j(\kappa) \leq j(\beta)$. Repeating this argument, we get that $j^n(\kappa) < \delta$ for all $n < \omega$. Thus, by Lemma 3.3.3, we get that $j(\kappa)$ is inaccessible and $V_{j(\kappa)}$ satisfies that there is a strongly compact cardinal. Thus, we proved what we promised.

We assumed that there is no cardinal $\kappa \leq \delta$ which is δ^+ compact. So let us now suppose that for some $\kappa \leq \delta$, κ is δ^+-compact. So we can let $\lambda > \delta^+$ be the least successor cardinal such that no $\kappa \leq \delta$ is λ-compact. Let ρ be the least $\aleph$-fixed point above $|V_{\lambda+2}|$. In particular, for every successor cardinal θ with $\delta^+ \leq \theta < \lambda$, V_ρ will have an embedding $j_\theta : V_{\lambda+2} \to M_\theta$ by a fine κ_θ-complete ultrafilter over $\mathcal{P}_{\kappa_\theta}(\theta)$ for some $\kappa_\theta \leq \delta$. This is the case because each M_θ has size at most $|V_{\lambda+2}|$, and thus $M_\theta \in H_{|V_{\lambda+2}|^+} \subseteq H_\rho \subseteq V_\theta$. Consider the structure

$$\mathcal{M} = (V_\rho, \in, \delta, \lambda, \mathrm{Tr}, \emptyset, S, P),$$

where Tr is a truth predicate for $(V_\rho, \in)$, S codes the successor function and P the pairing function. Then $\mathcal{M}$ satisfies the sentence φ in the logic $\mathcal{L}(\mathsf{I})$, which is the conjunction of the sentences:

(i) ZFC^*_a.

(ii) φ_{Card}.

(iii) There are no $\aleph$-fixed points above $|V_{\lambda+2}|$.

(iv) For every successor $\delta^+ \leq \theta < \lambda$, there is an elementary embedding $j_\theta : V_{\lambda+2} \to M_\theta$ with $\mathrm{crit}(j_\theta) = \kappa_\theta \leq \delta$, and $j_\theta``\theta \subseteq d$ with $|d|^M < j(\kappa_\theta)$.

(v) $\varphi_{\mathrm{Truth}} \wedge \varphi_\emptyset \wedge \varphi_{\mathrm{Succ}} \wedge \varphi_{\mathrm{Pair}}$.

Since $\delta = \mathrm{ULST}(\mathcal{L}(\mathsf{I}))$, there is a model

$$\mathcal{N} = (N, E, \bar{\delta}, \bar{\lambda}, \bar{\mathrm{Tr}}) \models_{\mathcal{L}(\mathsf{I})} \varphi$$

of size much larger than ρ with $\mathcal{M}$ as a substructure. It follows by Theorem 3.5.1 that E is well-founded. Hence by collapsing, (v), and Lemma 4.3.2, we can assume without loss of generality that $E = \in$, N is transitive and cardinal correct, and we get an elementary embedding

$$j : V_\rho \to N$$

such that $j(\delta) = \bar{\delta}$ and $j(\lambda) = \bar{\lambda}$. Since $|N|$ is much larger than ρ and N believes that there are no $\aleph$-fixed points above $|V_{\bar{\lambda}+2}|$, it follows that $\bar{\lambda} > \lambda$, and thus $\mathrm{crit}(j) \leq \lambda$.

Let us suppose that $\mathrm{crit}(j) = \lambda$. Then N is correct about $\mathcal{P}(\lambda)$ and $j(\kappa) = \kappa$ for every $\kappa \leq \delta$. By elementarity, $\mathcal{N}$ satisfies that $\bar{\lambda} > \lambda$ is the least successor cardinal that is a counterexample for compactness for cardinals $\kappa \leq \delta$. Thus, $\mathcal{N}$ satisfies that there is $\kappa \leq \delta$ that is λ-compact, so that it has a fine, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$. But since N has the correct powerset of λ, this object really is a fine, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$ which contradicts that κ is not λ-compact. Thus, $\chi = \mathrm{crit}(j) < \lambda$. The rest of the argument splits into two cases based on whether $\chi > \delta$.

We first suppose that $\delta < \chi < \lambda$. In this case, δ is not moved by j and so there is $\kappa \leq \delta$ such that N thinks that κ is λ-compact and, by (iv), N has an elementary embedding

$$j_{\bar{\lambda}} : V^N_{\bar{\lambda}+2} \to M_{\bar{\lambda}}$$

with $\mathrm{crit}(j_{\bar{\lambda}}) = \kappa$ and $j_{\bar{\lambda}}``\lambda \subseteq d$ in $M_{\bar{\lambda}}$ with $|d|^{M_{\bar{\lambda}}} < j_{\lambda}(\kappa)$. Consider the composition

$$j_{\bar{\lambda}} \circ j : V_{\lambda+2} \to M_{\bar{\lambda}}.$$

This makes sense as $j(V_{\lambda+2}) = V^N_{\bar{\lambda}+2}$. Observe that $\mathrm{crit}(j_{\bar{\lambda}} \circ j) = \kappa$ by our assumption that $\delta < \chi$.

The argument splits into two further cases. Let us consider the case that $j``\lambda \subseteq \lambda$. If $\beta < \lambda$, then $j_{\bar{\lambda}} \circ j(\beta) = j_{\bar{\lambda}}(j(\beta))$, where $j(\beta) = \beta' < \lambda$ by our assumption. So $(j_{\bar{\lambda}} \circ j)(\beta) = j_{\bar{\lambda}}(\beta')$ for some $\beta' < \lambda$. Thus $(j_{\bar{\lambda}} \circ j)``\lambda \subseteq j_{\bar{\lambda}}``\lambda \subseteq d$ and $|d|^{M_{\bar{\lambda}}} < j_{\bar{\lambda}}(\kappa) = (j_{\bar{\lambda}} \circ j)(\kappa)$. Thus, we can use the embedding $j_{\bar{\lambda}} \circ j$ to derive a fine, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$, contradicting our assumption that κ is not λ-compact.

In the other case, there is a least $\gamma < \lambda$ such that $j(\gamma) \geq \lambda$. Again, we will aim to show that κ is λ-compact, deriving a contradiction. For $\gamma \leq \beta^+ \leq \lambda$, we have $\beta < j(\beta) \leq (j_{\bar{\lambda}} \circ j)(\beta)$ and so we get a discontinuity of $j_{\bar{\lambda}} \circ j$. As above, this gives rise to a uniform, κ-complete ultrafilter over β^+. So it remains to show that we have a uniform, κ-complete ultrafilters on successors $\kappa < \beta^+ < \gamma$. For this it suffices to show that κ is β^+-compact, but this follows, by using $j_{\bar{\lambda}} \circ j$ and observing that $(j_{\bar{\lambda}} \circ j)``\beta^+ \subseteq j_{\bar{\lambda}}``\lambda \subseteq d$. So we have again derived a contradiction. This ends the case where $\chi > \delta$.

So let us finally assume that $\chi \leq \delta < \lambda$. Then it follows that χ is not λ-compact. Now if $j(\chi) \geq \lambda$, we get for $\chi \leq \beta^+ \leq \lambda$ that $\beta < j(\beta)$ and thus a discontinuity of j at β^+, allowing us to derive a uniform, κ-complete ultrafilter over β^+. In particular, χ is λ-strongly compact, which is again a contradiction. And if $j(\chi) < \lambda$, we can reason as before to show that $j^n(\chi) < \lambda$ for all $n \in \omega$. Thus, with $\gamma = \sup\{j^n(\chi) : n \in \omega\} \leq \lambda$, we have that j restricts to $j : V_\gamma \to V^N_\gamma$ and the latter is cardinal correct by correctness of N and hence, by Lemma 3.3.3, we get that $j(\chi)$ is inaccessible and $V_{j(\chi)}$ believes that χ is a strongly compact cardinal. $\square$

4.9. A meta-result on ULST numbers and large cardinals

Note that our results indicate that the ULST numbers of logics are related to large cardinal notions which are witnessed by elementary embeddings which have targets that are correct about the logic in question. The ULST number of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ gives rise to measurable cardinals, which have as targets transitive models, which are in turn correct about $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$-satisfaction; ULST($\mathcal{L}^2$) amounts to extendible cardinals, which involve levels of the von Neumann hierarchy V_α, which are in turn correct about $\mathcal{L}^2$-satisfaction; $C^{(n)}$-extendible cardinals involve some V_α for $\alpha \in C^{(n)}$, which are correct about $\mathcal{L}^{\mathrm{s},n}$-satisfaction; and cardinal correctly extendible cardinals have as targets cardinal correct sets, which compute $\mathcal{L}(\mathsf{I})$-satisfaction correctly. In this section we want to build on this observation to show a general result, connecting some template large cardinal notion to the existence of the ULST number of a given logic.

For this we introduce the new large cardinal notion of $\mathcal{L}$-*extendible* cardinals pushing up some δ (cf. Definition 4.9.1). We restrict attention to logics that *behave well under elementary embeddings*, and which *can pick out their own correct models*, two notions we will motivate and introduce shortly. For logics with these properties, we then relate the existence of $\mathcal{L}$-extendible cardinals to ULST numbers of $\mathcal{L}$. The main results are summarised by Corollary 4.9.6. Our result will further show that for such logics, the existence of ULST and SULST numbers is equivalent.

As suggested by our informal comments above, the relevant large cardinal notions in play involve transitive sets which are correct about the logic in question itself. Recall from Section 1.2 that our logics $\mathcal{L}$ involve a definable satisfaction relation, i.e., $\models_\mathcal{L}$ is a formula in the language of set theory. Let us say that a transitive set M is $\mathcal{L}$-*correct* if for every vocabulary $\tau \in M$, every $\varphi \in \mathcal{L}[\tau] \cap M$ and every τ-structure $\mathcal{A} \in M$:

$$\mathcal{A} \models_\mathcal{L} \varphi \text{ iff } M \models \text{“}\mathcal{A} \models_\mathcal{L} \varphi\text{”}.$$

Here we implicitly assume that if M is $\mathcal{L}$-correct and $\models_\mathcal{L}$ is defined via some parameter p, then $p \in M$ (as otherwise, even stating $M \models$ “$\mathcal{A} \models_\mathcal{L} \varphi$” does not make any sense). Let us consider the following variation of extendibility.

Definition 4.9.1. Let $\mathcal{L}$ be a logic. A cardinal κ is $\mathcal{L}$-*extendible* iff for every $\alpha > \kappa$, if V_α is $\mathcal{L}$-correct, then there is an elementary embedding $j : V_\alpha \to M$ such that M is a transitive and $\mathcal{L}$-correct set, $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

If δ is some ordinal, we further say κ is $\mathcal{L}$-*extendible pushing up* δ iff for every $\alpha > \kappa$, if V_α is $\mathcal{L}$-correct, then there is an elementary embedding $j : V_\alpha \to M$ such that M is an $\mathcal{L}$-correct set, $\mathrm{crit}(j) = \kappa$ and $j(\delta) > \alpha$.

Recall that a copy S of some $\mathcal{L}$-theory T is simply the image $S = f\text{“}T$ under some renaming f (cf. Definition 1.2.1). For the concrete logics we considered before, we frequently used that their syntax interacts well with elementary embeddings j, in the sense that copies of theories are provided by the images of elementary embeddings (cf.

Section 1.3.2). To make $\mathcal{L}$-extendible cardinals fruitful, we restrict ourselves to logics which share this property.

To make this precise, let $\mathcal{L}$ be a logic. Suppose we have an elementary embedding $j : V_\alpha \to M$ with a critical point $\mathrm{crit}(j) \geq \mathrm{dep}^*(\mathcal{L})$, and a vocabulary $\tau \in V_\alpha$. Then $j : \tau \to j``\tau$ is a renaming and further we can turn any $\tau$-structure $\mathcal{A}$ into a $j``\tau$-structure $\mathcal{A}^*$ with universe A and interpreting, for instance, $j(r)^{\mathcal{A}^*}$ as $r^{\mathcal{A}}$. Recall that as a renaming, j comes with a bijection $f : \mathcal{L}[\tau] \to \mathcal{L}[j``\tau]$ such that for any $\varphi \in \mathcal{L}[\tau]$:

$$\mathcal{A} \models \varphi \text{ iff } \mathcal{A}^* \models f(\varphi).$$

Let us say that $\mathcal{L}$ *syntactically behaves well under elementary embeddings* if for any elementary embedding as above, $j \restriction \mathcal{L}[\tau] \cap V_\alpha$ itself is the restriction of such a map f, i.e., for any $\varphi \in \mathcal{L}[\tau] \cap V_\alpha$, we have $j(\varphi) \in \mathcal{L}[j``\tau]$ and

$$\mathcal{A} \models \varphi \text{ iff } \mathcal{A}^* \models j(\varphi).$$

For logics with this property, the existence of ULST numbers of any logic $\mathcal{L}$ is implied by the existence of $\mathcal{L}$-extendible cardinals.

Theorem 4.9.2. Let $\mathcal{L}$ be a logic which syntactically behaves well under elementary embeddings. If there is a cardinal κ which is $\mathcal{L}$-extendible pushing up some δ, then $\mathrm{SULST}(\mathcal{L}) \leq \delta$.

Proof. Let $\mathcal{A}$ be a τ-structure of size $|\mathcal{A}| \geq \delta$ and let $\bar{\delta} > \delta$. Take, by the Reflection Theorem, some cardinal $\alpha = \beth_\alpha$ such that V_α is $\mathcal{L}$-correct, $\alpha > \bar{\delta}$, $\mathcal{A} \in V_\alpha$ and $\mathcal{L}[\tau] \in V_\alpha$. There is an elementary embedding $j : V_\alpha \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\delta) > \alpha$ and such that M is $\mathcal{L}$-correct. Consider the $j(\tau)$-structure $j(\mathcal{A}) \in M$, and the $j``\tau$-structure $j``\mathcal{A}$. We can rename $j(\mathcal{A}) \restriction j``\tau$ back along j to a τ-structure $\mathcal{B}$. Let us argue that $\mathcal{A}$ is an $\mathcal{L}$-elementary substructure of $\mathcal{B}$. Fix $\varphi \in \mathrm{ElDiag}_{\mathcal{L}}(\mathcal{A})$. Note that φ is an $\mathcal{L}[\tau^*]$-sentence for $\tau^* = \tau \cup \{c_x \colon x \in A\}$ with constants c_x and that $\mathcal{A}^* = (\mathcal{A}, c_x)_{x \in A} \models \varphi$, with the standard interpretation of the c_x by x itself. We may assume $\tau^* \in V_\alpha$ and because V_α is $\mathcal{L}$-correct, it believes that $\mathcal{A}^* \models \varphi$. By elementarity and because M is $\mathcal{L}$-correct, $j(\mathcal{A}^*) \models j(\varphi)$. Furthermore, because $\mathcal{L}$ behaves well under elementary embeddings, $j(\varphi) \in \mathcal{L}[j``\tau^*]$. In particular, $j(\mathcal{A}^*) \restriction j``\tau^* \models j(\varphi)$. Note that $\mathcal{B}$ naturally expands to a $j``\tau^*$-structure $\mathcal{B}^*$, interpreting $c_x^{\mathcal{B}^*} = j(c_x)^{\mathcal{A}^*}$ and then $\mathcal{B}^*$ is simply the induced renamed version of $j(\mathcal{A}^*) \restriction j``\tau^*$. Then using syntactical well-behaviour under elementary embeddings, we get that $\mathcal{B}^* \models \varphi$. This shows that $\mathcal{B}^*$ satisfies $\mathrm{ElDiag}_{\mathcal{L}}(\mathcal{A})$, and therefore that $\mathcal{B}$ is an $\mathcal{L}$-elementary superstructure of $\mathcal{A}$. Because $|B| = |j(\mathcal{A})| \geq |j(\delta)| \geq \alpha > \bar{\delta}$, this shows what we promised. ◻

To show a converse of this result, we have to restrict the class of logics we are considering further.

Definition 4.9.3. Let $\mathcal{L}$ be a logic, possibly defined using a parameter p. We say that $\mathcal{L}$ *can pick out its own correct models* iff there is a sentence $\varphi \in \mathcal{L}[\{\in\}]$ such that:

(i) $V_\alpha \models \varphi$ for a club class of ordinals α.

(ii) For any structure (N, E), if $(N, E) \models \varphi$, then N is well-founded and extensional and its transitive collapse $\bar{N}$ is $\mathcal{L}$-correct.

The first condition of the above definition ensures that φ is a not a trivial statement, for example, a contradiction. The second statement is the relevant one. Notice that the notion further implies that the logic can define any parameter used in its own definition. A logic being able to pick out its own correct models resembles the classically studied notion of *adequacy to truth in itself* (cf. [BF85, Chapter XVII]). It is, however, much simpler and adequate for our purposes, so we will employ it here.

Most of the logics we are typically considering in this thesis can pick out their own correct models:

Example 4.9.4. The logics $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, $\mathcal{L}(\mathsf{I})$, $\mathcal{L}^2$, $\mathcal{L}^{\mathrm{s},n}$, and $\mathcal{L}_{\eta\eta}$ for any *successor* cardinal η, can all pick out their own correct models.

Proof. For $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, let $\varphi = \mathsf{Q}^{\mathrm{WF}}xy(x \in y)$. For $\mathcal{L}(\mathsf{I})$, let $\varphi = \mathrm{ZFC}^*_a \wedge \varphi_{\mathrm{card}}$ (cf. Section 3.5). For $\mathcal{L}^2$, let φ be Magidor's Φ (cf. Lemma 1.2.4). For $\mathcal{L}^{\mathrm{s},n}$, let $\varphi = \Phi^{(n)}$ (cf. Corollary 1.2.17). For $\mathcal{L}_{\eta\eta}$, if $\eta = \nu^+$, let φ be a sentence that expresses well-foundedness and furthermore, that ν exists, i.e., $\exists x \sigma_\nu(x)$ and that for any set x, the model has all functions $\nu \to x$:

$$\forall x \forall (x_\beta \colon \beta < \nu)[\bigwedge_{\beta<\nu}(x_\beta \in x) \to$$
$$\exists f \exists y(\mathrm{func}(f) \wedge \mathrm{dom}(f) = y \wedge \sigma_\nu(y) \wedge \bigwedge_{\beta<\nu}(\forall z(\sigma_\beta(z) \to f(z) = x_\beta)))].$$

□

However, we will see that for η weakly inaccessible, $\mathcal{L}_{\eta\eta}$ need not be able to pick out its own correct models.

Theorem 4.9.5. Let $\mathcal{L}$ be a logic that can pick out its own correct models. If $\mathrm{ULST}(\mathcal{L})$ exists, then there is a pair $\kappa \leq \gamma$ such that κ is $\mathcal{L}$-extendible pushing up γ.

Proof. The proof is a generalisation of by now familiar arguments, used, for example, for the proof of Theorem 4.5.1. Let $\delta = \mathrm{ULST}(\mathcal{L})$ and φ be the sentence that witnesses that $\mathcal{L}$ can pick out its own correct models. Suppose $\alpha > \delta$ is an ordinal of cofinality ω such that $V_\alpha \models \varphi$. Note that $V_\beta \models \varphi$ for a club class of β and thus there are many such α. Consider the structure

$$\mathcal{M} = (V_\alpha, \in, f, \mathrm{Tr}, \emptyset, S, P)$$

where $f : \omega \to \alpha$ is cofinal, Tr is a truth predicate for $(V_\alpha, \in)$, S codes the successor function and P codes the pairing function. Then $\mathcal{M}$ satisfies the conjunction of the following sentences:

(i) φ.

(ii) $\varphi_{\text{Truth}} \wedge \varphi_{\emptyset} \wedge \varphi_{\text{Succ}} \wedge \varphi_{\text{Pair}}$.

(iii) "f is a function with domain ω that is cofinal in the ordinals."

By $\delta = \text{ULST}(\mathcal{L})$ there is a superstructure

$$(N, E, \bar{f}, \bar{\text{Tr}}, \bar{\emptyset}, \bar{S}, \bar{P}) = \mathcal{N} \supseteq \mathcal{M}$$

satisfying (i) to (iii) and such that $|N| > |V_\alpha|$. Because $\mathcal{N}$ satisfies φ, we know that E is well-founded and extensional and thus by collapsing, we can assume that $E = \in$. Using (ii) and Lemma 4.3.2, the collapse restricts to an elementary embedding $j : \mathcal{M} \to \mathcal{N}$. By (i), N is $\mathcal{L}$-correct. Further by (iii), $\bar{f}$ is a function with domain ω which is cofinal in the ordinals of N. Because of N's size, $\alpha \in N$ and there is therefore an n such that $\bar{f}(n) > \alpha$. Because $j(f(n)) = \bar{f}(j(n)) = \bar{f}(n) > \alpha$, we get that j has some critical point $\kappa_\alpha < \alpha$ and there is a $\delta_\alpha < \alpha$ such that $j(\delta_\alpha) > \alpha$. Now define the proper class function F on the ordinals α of cofinality ω with $V_\alpha \models \varphi$ such that $F(\alpha)$ is the least (ordinal coding a pair) $(\kappa_\alpha, \delta_\alpha)$ where κ_α is the critical point of an elementary embedding $j : V_\alpha \to N$ with N an $\mathcal{L}$-correct transitive model and δ_α the smallest ordinal such that $j(\delta_\alpha) > \alpha$. Our argument above shows that F is a regressive function on a stationary class of ordinals $S = \{\alpha \colon \text{cof}(\alpha) = \omega \wedge V_\alpha \models \varphi\}$. This is stationary because the α such that $V_\alpha \models \varphi$ form a club class. Hence, by the very weak class Fodor principle 4.5.2, the function F is constant on an unbounded subclass of S. Let (κ, δ) be the constant value. Then κ is $\mathcal{L}$-extendible pushing up δ. □

Corollary 4.9.6. Let $\mathcal{L}$ be a logic which syntactically behaves well under elementary embeddings and which is able to pick out its own correct models. Then the following are equivalent:

(1) There exists a pair $\kappa \leq \delta$ such that κ is $\mathcal{L}$-extendible pushing up δ.

(2) ULST($\mathcal{L}$) exists.

(3) SULST($\mathcal{L}$) exists.

Recall Theorem 4.7.25, that it is consistent for η to be weakly inaccessible and that ULST($\mathcal{L}_{\eta\eta}$) exists but SULST($\mathcal{L}_{\eta\eta}$) does not exist. As $\mathcal{L}_{\eta\eta}$ syntactically behaves well under elementary embeddings, this shows that $\mathcal{L}_{\eta\eta}$ need not be able to pick out its own correct models.

To demonstrate the usefulness of the above corollary, let us collect some more logics for which it finds application.

Proposition 4.9.7. The following logics behave syntactically well under elementary embeddings and are able to pick out their own correct models.

(a) $\mathcal{L}(\mathsf{I}, \mathsf{Q}^{\text{WI}})$ which expands $\mathcal{L}(\mathsf{I})$ by the quantifier with the semantics:

$$\mathcal{A} \models \mathsf{Q}^{\text{WI}} x \varphi(x) \text{ iff } |\{a \in A \colon \mathcal{A} \models \varphi(a)\}| \text{ is weakly inaccessible.}$$

(b) $\mathcal{L}(\mathsf{I}, \mathsf{Q}^{\mathrm{Rg}})$ which expands $\mathcal{L}(\mathsf{I})$ by the quantifier with the semantics:

$$\mathcal{A} \models \mathsf{Q}^{\mathrm{Rg}} x, y \varphi(x, y) \text{ iff } \{(a, b) \in A^2 \colon \mathcal{A} \models \varphi(a, b)\}$$
$$\text{has the order type of a regular cardinal.}$$

(c) $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}, \mathsf{Q}_1)$ which expands $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ by the quantifier with the semantics:

$$\mathcal{A} \models \mathsf{Q}_1 x \varphi(x) \text{ iff } \{a \in A \colon \mathcal{A} \models \varphi(a)\} \text{ is uncountable.}$$

Proof. Because these logics are finitary, they all behave syntactically well under elementary embeddings. For ability to pick out correct models, for (a), let φ be the conjunction of ZFC^*_a and φ_{Card} as well as the following sentence:

$$\forall x(\mathrm{Card}(x) \wedge \text{“}x \text{ is weakly inaccessible”} \leftrightarrow \mathsf{Q}^{\mathrm{WI}} y(y \in x)).$$

For (b), let φ be the conjunction of ZFC^*_a and φ_{Card} as well as the following sentence:

$$\forall x(\mathrm{Card}(x) \wedge \text{“}x \text{ is regular”} \leftrightarrow \mathsf{Q}^{\mathrm{Rg}} yz(y \in x \wedge z \in x \wedge y \in z)).$$

For (c), let φ be the sentence

$$\mathsf{Q}^{\mathrm{WF}} xy(x \in y) \wedge \mathrm{Ext} \wedge \forall x(\text{“there is no surjection } f : \omega \to x\text{”} \leftrightarrow \mathsf{Q}_1 y(y \in x)).$$

□

5. Model Theory of Class Logics

Remarks on co-authorship. The results of this chapter are joint with Trevor Wilson.

5.1. Introduction

In this chapter we consider logics which have a proper class of sentences over set-sized vocabularies. The most straightforward example of such a logic is $\mathcal{L}_{\infty\infty}$, and it is easy to see that it is inconsistent for $\mathcal{L}_{\infty\infty}$ to have much of a meaningful model theory. However, considering some restricted sublogics of $\mathcal{L}_{\infty\infty}$, we will see that properties of class logics have varying consistency strengths, and sometimes can even be proven to hold in ZFC.

The chapter is structured as follows. In Section 5.2, we give the definitions of the class logics we will study. Section 5.3 has three parts, which in turn analyse compactness properties of class extensions of first-order logic, infinitary logics, and second-order logic and sort logics. In Section 5.4 we will show that Π_n-strong cardinals are characterised by Löwenheim-Skolem numbers of class versions of sort logics, and therefore provide a second stratification of WVP in terms of model-theoretic properties. In Section 5.5, we will provide a characterisation of *Shelah cardinals* by properties of second-order class logic.

5.2. Motivation and definitions

Recall that $\mathcal{L}_{\infty\infty} = \bigcup_{\kappa\in\mathrm{Card}} \mathcal{L}_{\kappa\kappa}$ and that, while $\mathcal{L}_{\infty\infty}$ is an abstract logic in the sense of Definition 1.2.1, it is not a *logic* in the formal sense of our definition, as for any vocabulary τ, the collection of sentences $\mathcal{L}_{\infty\infty}[\tau]$ forms a proper class. As all the abstract logics we will consider in the current chapter have this characteristic, to emphasise this fact we will also use the term *class logic* to refer to abstract logics. Recall that we also dub class logics $\mathcal{L}$ for which $\mathcal{L}[\tau]$ really can form a proper class *proper* class logics.

It is easy to see that already for $\mathcal{L}_{\infty\infty}$ – arguably the simplest proper class logic available – it is inconsistent to have any of the model-theoretic properties we are usually considering. Recall that the Hanf number of a logic (cf. Definition 4.2.1) describes a weak upward Löwenheim-Skolem property, and that ZFC proves the existence of the Hanf number of any (set-sized) logic (cf. Proposition 4.2.2). Similarly, ZFC proves the existence of a Löwenheim-Skolem number of any strong logic (cf. Proposition 1.2.10). For $\mathcal{L}_{\infty\infty}$ however, already these weak properties are inconsistent.

Proposition 5.2.1. The logic $\mathcal{L}_{\infty\infty}$ has neither a compactness number, nor a Hanf number, nor an LS number.

Proof. Notice that for any cardinal κ, the sentence

$$\varphi_{\geq\kappa} = \exists(x_i : i < \kappa) \bigwedge_{i<j<\kappa} x_i \neq x_j.$$

holds in a structure $\mathcal{A}$ iff $|A| \geq \kappa$. As $\varphi_{\geq\kappa} \in \mathcal{L}_{\infty\infty}$ for all cardinals κ, $\mathcal{L}_{\infty\infty}$ cannot have an LS number.

Consider further the sentence

$$\varphi_{<\kappa} = \forall(x_i : i < \kappa) \bigvee_{i<j<\kappa} x_i = x_j,$$

which holds in a structure $\mathcal{A}$ iff $|A| < \kappa$. Because $\varphi_{<\kappa} \in \mathcal{L}_{\infty\infty}$ for all cardinals κ, $\mathcal{L}_{\infty\infty}$ cannot have a Hanf number. This means that $\mathcal{L}_{\infty\infty}$ also cannot have a compactness number, as the existence of a compactness number implies the existence of a Hanf number. □

We will consider fragments of $\mathcal{L}_{\infty\infty}$ and its extensions like $\mathcal{L}^2_{\infty\infty}$. Notice that the sentences $\varphi_{\geq\kappa}$ showing that $\mathcal{L}_{\infty\infty}$ cannot have an LS number use both infinite conjunctions and existential quantification, but neither infinite disjunctions nor universal quantification. And dually, this holds for the sentences $\varphi_{<\kappa}$ witnessing that $\mathcal{L}_{\infty\infty}$ cannot have a Hanf number. This is no coincidence. In fact, we will see that banning the combination of infinite conjunctions and existential quantification can lead to proper class logics that *do* have Löwenheim-Skolem properties. Similarly, banning the combination of infinite disjunctions and universal quantification can lead to proper class logics that *do* have upward Löwenheim-Skolem and compactness properties.

To make this more precise, let us introduce some notation. Let us call a formula $\chi \in \mathcal{L}_{\infty\infty}$ *negated* iff $\chi = \neg\chi_0$ for some $\chi_0 \in \mathcal{L}_{\infty\infty}$. Let us say χ is an *infinite disjunction* iff $\chi = \bigvee T$ for some set T of formulas such that $|T| \geq \omega$. And let us say that χ is an *infinite universal quantification* iff $\chi = \forall(x_i : i \in X)\psi$ for some set X of size $|X| \geq \omega$. We call $|T|$ and $|X|$ the *size* of the disjunction or universal quantification, respectively. Analogously, define *infinite conjunctions* and *infinite existential quantifications*. Recall that every sentence $\varphi \in \mathcal{L}_{\infty\infty}$ is equivalent to a sentence ψ which is in *negation normal form*, i.e., in which no non-atomic subformula of ψ is negated. Let λ be a cardinal. Let us write $\mathcal{L}(\wedge^\lambda, \exists^\lambda)$ for the sublogic of $\mathcal{L}_{\lambda\lambda}$ such that for any vocabulary τ, the set of sentences $\mathcal{L}(\wedge^\lambda, \exists^\lambda)[\tau]$ consists of all sentences $\psi \in \mathcal{L}_{\lambda\lambda}[\tau]$ such that:

(1) ψ is in negation normal form, and

(2) no subformula of ψ is an infinite disjunction, and

(3) no subformula of ψ is an infinite universal quantification.

The satisfaction relation of $\mathcal{L}(\wedge^\lambda, \exists^\lambda)$ is simply defined as the one of $\mathcal{L}_{\lambda\lambda}$, restricted to formulas in $\mathcal{L}(\wedge^\lambda, \exists^\lambda)$. Now let $\mathcal{L}(\wedge^\infty, \exists^\infty) = \bigcup_{\lambda\in\mathrm{Card}} \mathcal{L}(\wedge^\lambda, \exists^\lambda)$. Intuitively, this is the the sub-proper-class-logic of $\mathcal{L}_{\infty\infty}$ which allows arbitrary infinite conjunctions and arbitrary infinite existential quantifications, but only finitary disjunctions and universal quantification. Note that $\mathcal{L}(\wedge^\infty, \exists^\infty)$ is *not* closed under negation.

We will also consider class versions of stronger logics than first-order: we can define $\mathcal{L}^2(\wedge^\infty, \exists^\infty)$ as the sublogic of $\mathcal{L}^2_{\infty\infty}$ which allows arbitrary infinite conjunctions and infinite existential (first- and second-order) quantification, but only finitary disjunctions and universal quantification, in a similar way making use of negation normal form. Further we will consider the class logics $\mathcal{L}(\vee^\infty, \forall^\infty)$ and $\mathcal{L}^2(\vee^\infty, \forall^\infty)$ which are dually defined excluding infinite conjunctions and infinite existential quantification but allowing arbitrary disjunctions and universal quantification. Note that these class logics are dual: for every $\varphi \in \mathcal{L}(\wedge^\infty, \exists^\infty)$ there is a $\psi \in \mathcal{L}(\vee^\infty, \forall^\infty)$ such that $\mathcal{A} \models \varphi$ iff $\mathcal{A} \not\models \psi$. More precisely, ψ can be constructed by considering $\neg\varphi \in \mathcal{L}_{\infty\infty}$ and then building its negation normal form by pushing the negation through all quantifiers and boolean connectives. We will further want to consider class logics such as $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$ which are similarly build as $\mathcal{L}(\wedge^\infty, \exists^\infty)$, simply omitting clause (3) above. And similarly, omitting clause (3) and changing clause (2) to only exclude infinite disjunctions of size $\geq \lambda$ for some cardinal λ gives the class logic $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty, \vee^\lambda)$ for some cardinal λ, which allows for arbitrarily sized conjunctions, existential- and universal quantification, and additionally disjunctions of size $< \lambda$ (but not any larger sizes of disjunctions). It should be obvious how these definitions and notations can be varied to consider class logics which allow for various combinations of infinite boolean connectives and quantifiers of different sizes.

We will also consider class versions of sort logic. We construct this in the following way. For κ a regular cardinal and $\lambda > \kappa$, consider $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\wedge^\lambda, \exists^\lambda)$ which expands our usual sort logic $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$ – which, to recall, is based on $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ – by adding conjunctions of size $< \lambda$ and first- and second-order quantifiers of size $< \lambda$. Again, we impose some restriction and only consider formulas ψ which adhere to the following rules:

(1) If $\widetilde{Q} \in \{\widetilde{\exists}, \widetilde{\forall}\}$ and $\widetilde{Q}X\chi$ is a subformula of ψ, then $\chi \in \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$, and

(2) if $\mathsf{Q}^{\mathrm{WF}}xy\chi$ is a subformula of ψ, then $\chi \in \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$, and

(3) if $\neg\chi$ is a subformula of ψ, then $\chi \in \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$.

These restrictions mean that sort quantifiers cannot take any formulas that contain any disjunctions of size $\geq \kappa$, nor any infinite quantification, nor any second-order quantification. And the same is true for Q^{WF} and negations. In particular, also the formulas of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\wedge^\lambda, \exists^\lambda)$ are in some sort of negation normal form, only that the negation is allowed to stand in front of any $\chi \in \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$, and is not restricted to atomic formulas. We obtain the proper class version by letting $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\wedge^\infty, \exists^\infty) = \bigcup_{\lambda\in\mathrm{Card}} \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\wedge^\lambda, \exists^\lambda)$. The satisfaction relation is defined in the obvious way. Again, dually we define the class logic $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^\infty, \forall^\infty)$.

That, consistently up to large cardinals, proper class logics can have some interesting model theory was first noted by Trevor Wilson. The results were presented, for example, in [Wil22a], but remain unpublished. Wilson noted the following:

Theorem 5.2.2 (Wilson)**.** Let κ be a cardinal.

(i) κ is the smallest supercompact cardinal iff $\kappa = \mathrm{LST}(\mathcal{L}(\vee^\infty, \forall^\infty, \exists^\infty))$.

(ii) κ is *huge with target* λ iff for every vocabulary τ of size $< \kappa$ and every $\varphi \in \mathcal{L}(\vee^\infty, \forall^\infty, \exists^\infty)[\tau]$, if $\mathcal{A}$ is a τ-structure of size $|A| = \lambda$, then there is a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $|B| = \kappa$ and $\mathcal{B} \models \varphi$.

Recall Magidor's Theorem, that the first supercompact cardinal is the LST number of second-order logic (cf. Theorem 1.3.14). Item (i) of Wilson's result gives another characterisation of supercompact cardinals in terms of LST numbers, but note that here the LST number of a first-order class logic is sufficient. For item (ii), recall that a cardinal is huge with target λ if there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $M^\lambda \subseteq M$, and $j(\kappa) = \lambda$. It is well known that the existence of a huge cardinal exceeds VP in consistency strength (cf., e.g., [Jec03, Lemma 20.27]). Thus, item (ii) shows that even large cardinals whose existence has consistency strength stronger than VP can be characterised by Löwenheim-Skolem properties of first-order class logics.

Let us make some further remarks about notation. We will consider proofs by induction on the structure of some formula φ. We consider the semantics of $\mathcal{A} \models_{\mathcal{L}} \varphi$ for the class logics $\mathcal{L}$ described above to be given via variable assignments, analogously to how we presented the semantics of sort logic in Section 1.2.4. In particular, recall the notion of a variable assignment presented right before Definition 1.2.12. Then, for instance, if $\varphi = \exists W \psi \in \mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$, where W is some set of variables and the set of variables used in φ is given by X, then given some structure $\mathcal{A}$ and a variable assignment $f : X \to A$, we say that

$$\mathcal{A} \models_{\mathcal{L}(\wedge^\infty,\exists^\infty,\forall^\infty)} \varphi[f] \text{ iff there is some } W\text{-variant } g \text{ of } f \text{ such that } \mathcal{A} \models_{\mathcal{L}(\wedge^\infty,\exists^\infty,\forall^\infty)} \psi[g].$$

It should be clear, given our treatment of sort logic, how the semantics of any of the logics of interest in this chapter can be stated in these terms. Note that for second-order logic, we have to consider free relational variables x, so a variable assignment f might take a value $f(x) \subseteq A^n$ for a structure $\mathcal{A}$, i.e., $\mathrm{ran}(f) \subseteq \bigcup_{n\in\omega} \mathcal{P}(A^n)$. For simplicity, we will restrict attention to structures in a relational vocabulary (but note that we can code function and constant symbols by relation symbols).

5.3. Compactness of class logics

5.3.1. Class versions of first-order logic

In this section, we will show, in ZFC, that the compactness number of $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$ is ω. Note that this means that switching from first-order logic to this class logic does have no effect on the compactness number. The proof of this theorem is a refinement of the ultraproduct proof of first-order logic's Compactness Theorem. To obtain this, we show that the relevant usage of Łos' Theorem carries over to $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$, and afterwards derive our theorem.

Lemma 5.3.1. Suppose τ is a vocabulary, U is an ultrafilter over some set I, and for each $i \in I$, there is a τ-structure M_i. Let φ be a formula of $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$ over τ with free variables among a set X, and for $i \in I$, let $f_i : X \to M_i$ be variable assignments. Consider $f : X \to \prod_{i\in I} M_i/U$, the variable assignment into the ultraproduct of the M_i given by $x \mapsto [i \mapsto f_i(x)]_U$. Then if $\{i \in I \colon M_i \models \varphi[f_i]\} \in U$, then $M = \prod_{i\in I} M_i/U \models \varphi[f]$.

Proof. We proceed by induction on the structure of φ. The cases in which φ is an atomic formula, or a negation of an atomic formula simply follow by the definition of the ultraproduct. The case in which φ is a finitary disjunction is simply the usual step from Łos Theorem.

So suppose that S is some set of formulas and $\varphi = \bigwedge S$ is an infinite conjunction such that $\{i \in I \colon M_i \models \varphi[f_i]\} \in U$. Let $\psi \in S$. Then $U \ni \{i \in I \colon M_i \models \varphi[f_i]\} \subseteq \{i \in I \colon M_i \models \psi[f_i]\}$ and so the latter is in U. By induction hypothesis, therefore $M \models \psi[f]$.

And now let us consider $\varphi = \exists W\psi$, where W is a set of variables of any size and assume for U-many i, that we have $M_i \models \exists W\psi[f_i]$, where the f_i are assignments on the set X of variables of φ. Then for U-many i there is a W-variant g_i of f_i with $M_i \models \psi[g_i]$. Define g as the function that sends $v \in X$ to $[i \mapsto g_i(v)]_U$. By induction hypothesis it follows that $M \models \psi[g]$. Now if $v \in X \setminus W$, then $g(v) = [i \mapsto g_i(v)]_U = [i \mapsto f_i(v)]_U = f(v)$, because the g_i are W-variants of the f_i. This shows that g is a W-variant of f and therefore $M \models \exists W\psi[f]$.

And finally, let $\varphi = \forall W\psi$ and assume for U-many i that $M_i \models \forall W\psi[f_i]$. For those U-many i thus for all W-variants g_i of f_i it holds that $M_i \models \psi[g_i]$. Let g be any W-variant of f. We want to show that $M \models \psi[g]$. Note that by induction hypothesis it is sufficient to show that for U-many i we have

$$M_i \models \psi[v \mapsto h_v(i)], \tag{$*$}$$

where $h_v : I \to \bigcup_{i\in I} M_i$ is the function representing $g(v)$, i.e. with $[h_v]_U = g(v)$. Now if $v \in X \setminus W$, then $[h_v]_U = g(v) = f(v) = [i \mapsto f_i(v)]_U$, because g is a W-variant of f. So for $v \in X \setminus W$ we can without loss of generality assume that h_v is given by the function $i \mapsto f_i(v)$. Then the functions defined by $v \mapsto h_v(i)$ are W-variants of the f_i. In particular, for U-many i, we get $M_i \models \psi[v \mapsto h_v(i)]$, so $(*)$ holds and we are done. □

Recall that a filter F over $\mathcal{P}_\omega X$ is called *fine* iff for any $x \in X$, $\{s \in \mathcal{P}_\omega X \colon x \in s\} \in F$. For every non-empty set X there is a fine ultrafilter over $\mathcal{P}_\omega X$. It is straightforward to check that for any set X there is a fine filter F over $\mathcal{P}_\omega X$, defined by:

$$Y \in F \text{ iff } Y \subseteq \mathcal{P}_\omega X \text{ and } \exists x_1, \ldots, x_n \in X \colon \{s \in \mathcal{P}_\omega X \colon x_1, \ldots, x_n \in s\} \subseteq Y$$

It is a well known theorem by Tarski that any filter can be expanded to an ultrafilter over the same set (cf., e.g., [Jec03, Theorem 7.5]). Expanding F as above to an ultrafilter U over $\mathcal{P}_\omega X$ results in a fine ultrafilter.

Theorem 5.3.2. $\mathrm{comp}(\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)) = \omega$.

Proof. Let $T \subseteq \mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$ be finitely satisfiable. Then for every $s \in \mathcal{P}_\omega T$ there is a model $M_s \models s$. Let U be any fine ultrafilter over $\mathcal{P}_\omega T$. Then for every $\varphi \in T$, the set $\{s \in \mathcal{P}_\omega T \colon \varphi \in s\} \in U$. But $\{s \in \mathcal{P}_\omega T \colon \varphi \in s\} \subseteq \{s \in \mathcal{P}_\omega T \colon M_s \models \varphi\}$, and so the latter is in U. Therefore by Lemma 5.3.1, $\prod_{s \in \mathcal{P}_\omega T} M_s/U \models \varphi$ and so this ultraproduct is a model of T. □

Recall that the *weak compactness number* of a logic, if it exists, is the smallest cardinal κ such that any $<\kappa$-satisfiable $\mathcal{L}$-theory T of size $|T| = \kappa$ has a model. Also recall the notions of ULST and SULST number from Chapter 4.

Corollary 5.3.3. The weak compactness number, the Hanf number, the ULST number, and the SULST number of $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty)$ are all ω.

5.3.2. Class versions of infinitary logics

In the last section we saw that adding arbitrary conjunctions, existential- and universal quantifiers to first-order logic does not change its compactness properties. In this section we will see that a similar assertion is true for infinitary first-order logic $\mathcal{L}_{\kappa\kappa}$. Recall that a regular uncountable cardinal κ is strongly compact iff $\mathrm{comp}(\mathcal{L}_{\kappa\kappa}) = \kappa$. Further, this is equivalent to the assertion that every κ-complete filter over any set can be expanded to a κ-complete ultrafilter (cf., e.g., [Kan03, Proposition 4.1]). Completeness of an ultrafilter lets us extend Lemma 5.3.1 to cover proper class expansions of $\mathcal{L}_{\kappa\kappa}$ in the to be expected way, expanding the usual Łos-like theorem for $\mathcal{L}_{\kappa\kappa}$.

Lemma 5.3.4. Suppose τ is a vocabulary, κ is a regular cardinal, U is a κ-complete ultrafilter over some set I, and for each $i \in I$, there is a τ-structure M_i. Let φ be a formula of $\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty, \vee^\kappa)$ over τ with free variables among a set X, and for $i \in I$, let $f_i : X \to M_i$ be variable assignments. Consider $f : X \to \prod_{i \in I} M_i/U$, the variable assignment into the ultraproduct of the M_i given by $x \mapsto [i \mapsto f_i(x)]_U$. Then if $\{i \in I \colon M_i \models \varphi[f_i]\} \in U$, then $M = \prod_{i \in I} M_i/U \models \varphi[f]$.

Proof. Exactly like in the proof of Lemma 5.3.1, proceed by induction on φ. The only case not covered there is $\varphi = \bigvee S$ for some set of formulas S of size $< \kappa$. This is taken care of by κ-completeness: If $\{i \in I \colon M_i \models \bigvee S[f_i]\} \in U$, note that $\{i \in I \colon M_i \models \bigvee S[f_i]\} = \bigcup_{\psi \in S} \{i \in I \colon M_i \models \psi[f_i]\}$. By κ-completeness of U, there is thus a fixed $\psi \in S$ such that $\{i \in I \colon M_i \models \psi[f_i]\} \in U$. By induction hypothesis therefore $\prod_{i \in I} M_i/U \models \psi[f]$ and hence $\prod_{i \in I} M_i/U \models \bigvee S[f]$. □

Theorem 5.3.5. Let κ be a regular uncountable cardinal. Then κ is strongly compact iff $\mathrm{comp}(\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty, \vee^\kappa)) = \kappa$.

Proof. Clearly, if $\mathrm{comp}(\mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty, \vee^\kappa)) = \kappa$, then $\mathrm{comp}(\mathcal{L}_{\kappa\kappa}) = \kappa$, and thus κ is strongly compact. On, the other hand, if κ is strongly compact and $T \subseteq \mathcal{L}(\wedge^\infty, \exists^\infty, \forall^\infty, \vee^\kappa)$ is $<\kappa$-satisfiable, then for any $s \in \mathcal{P}_\kappa T$ there is a model $M_s \models s$. Note that

$$Y \in F \text{ iff } Y \subseteq \mathcal{P}_\kappa T \text{ and } \exists t \in \mathcal{P}_\kappa T : \{s \in \mathcal{P}_\kappa T \colon t \subseteq s\} \subseteq Y$$

defines a κ-complete, fine filter over $\mathcal{P}_\kappa T$. By strong compactness of κ, there is a κ-complete, fine ultrafilter U over $\mathcal{P}_\kappa T$ extending F. Then Lemma 5.3.4 implies that $\prod_{s\in\mathcal{P}_\kappa T} M_s/U \models T$. $\square$

5.3.3. Class versions of second-order logic and sort logics

Magidor's Theorem 1.3.28 shows that the compactness number of second-order logic is the first extendible cardinal. Again, as for first-order logic, we show that switching to appropriate proper class versions of second-order logic does not increase the compactness number. Contrastingly though, we show that the switch drastically increases the Hanf and weak compactness numbers. Recall that ZFC proves the existence of Hanf numbers of any set-sized strong logic. Further, the existence of weak compactness numbers of any set-sized strong logic is weaker than a measurable cardinal.[1] Our theorem therefore shows that the existence of a weak compactness number, and of a Hanf number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$, are both much stronger than the existence of the respective numbers for $\mathcal{L}^2$.

Theorem 5.3.6. The following are equivalent for a cardinal κ:

(1) κ is the smallest extendible cardinal.

(2) κ is the compactness number of $\mathcal{L}^2$.

(3) κ is the compactness number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$.

(4) κ is the weak compactness number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$.

(5) κ is the Hanf number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$.

(6) κ is the ULST number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$.

(7) κ is the SULST number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$.

Proof. The first two items are equivalent by Magidor's theorem 1.3.28. If λ is a compactness number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$ it is a bound on the weak compactness number, ULST number, and SULST number of the same logic. Further, if κ is an SULST number, then it is a ULST number, and if it is a ULST number, then it is a Hanf number. So it is sufficient to show (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (1), and (5) $\Rightarrow$ (1).

For these implications it suffices to show that if κ is the first extendible cardinal, then κ is a strong compactness cardinal for $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$ and that if κ is either the weak compactness cardinal or the Hanf number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$, then there is an extendible cardinal $\leq \kappa$. Then by minimality of the properties we are considering, our theorem follows.

[1]This was shown by Philipp Lücke. The results remain unpublished as of yet, but were, for example, part of a talk at the XVII International Luminy Workshop in Set Theory [Lüc23].

So first assume that κ is the weak compactness cardinal of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$. We aim to show that there is an extendible cardinal $\lambda \leq \kappa$. For this purpose let $\alpha > \kappa$ be any limit ordinal. We want to show that there is a δ_α and an elementary embedding $j_\alpha : V_\alpha \to V_{\delta_\alpha}$ with $\mathrm{crit}(j_\alpha) \leq \kappa$. If we showed this, as there are class many ordinals above κ but at most κ many cardinals below κ, there is then a fixed $\lambda \leq \kappa$ that is the critical point of j_α for unboundedly many α. This λ is then extendible. To this end, let $\gamma = |V_\alpha|$ and $(a_i \colon i < \gamma)$ be an enumeration of V_α. Assume without loss of generality that $a_0 = \kappa$. Take a constant symbol c and consider the following sentence ψ of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$:

$$\psi = \exists (b_i \colon i < \gamma) \bigwedge_{\substack{n \in \omega, \varphi(x_1, \ldots, x_n) \in \mathcal{L}_{\omega\omega}[\{\in\}], \\ i_1, \ldots, i_n < \gamma, \\ V_\alpha \models \varphi(a_{i_1}, \ldots, a_{i_n})}} \varphi(b_{i_1}, \ldots, b_{i_n}) \wedge b_0 = c.$$

Note that the big conjunction above codes the elementary diagram of $(V_\alpha, \in)$ into a single sentence of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$. Hence, if $M \models \psi$, then a sequence $(b_i^M \colon i < \gamma)$ witnessing this gives rise to an elementary embedding $j : V_\alpha \to M$ by letting $a_i \mapsto b_i^M$. Note that $j(\kappa) = b_0^M = c^M$. Recall Magidor's Φ, axiomatising the class of structures (M, E) isomorphic to some $(V_\alpha, \in)$ for a limit ordinal α (cf. Lemma 1.2.4). Take additional constants c_α for $\alpha \leq \kappa$ and consider the theory

$$T = \{\psi\} \cup \{\Phi\} \cup \{c_i < c_j < c \colon i < j \leq \kappa\} \cup \{\mathrm{Ord}(c_i) \colon i \leq \kappa\}.$$

If T is satisfiable, say by $(M, E^M, c^M, c_i^M)_{i \leq \kappa} \models T$, we have that without loss of generality it is of the form $(V_\delta, \in, c^{V_\delta}, c_i^{V_\delta})_{i \leq \kappa}$ for some δ by the usage of Magidor's Φ. Further, because $\psi \in T$, there is an elementary embedding $j : V_\alpha \to V_\delta$ as described above. In particular, $j(\kappa) = c^{V_\delta}$. Now because $c_i^{V_\delta} < c_j^{V_\delta} < c^{V_\delta} = j(\kappa)$ for all $i < j \leq \kappa$, we have that $j(\kappa) > \kappa$. In particular, j has a critical point $\leq \kappa$. So to show satisfiability of T suffices. Now clearly T has size exactly κ and is $<\kappa$-satisfiable (by V_κ). By our assumption, it is therefore satisfiable.

Now let κ be the Hanf number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$. Again, we want to show that there is an extendible cardinal $\leq \kappa$. The proof goes similar to that of Theorem 4.5.1. Let $\alpha > \kappa$ be an ordinal of cofinality ω. Fix a function F_α with domain ω that is cofinal in α. Let ψ be defined similarly to above, but this time using formulas over the language $\{\in, F\}$ where F is a binary relation symbol. Then $(V_\alpha, \in, F_\alpha)$ is a model of the conjunction of the sentences ψ, Magidor's Φ and a formula χ saying "F is a function with domain ω that is cofinal in the ordinals". Since $|V_\alpha| \geq \kappa$, the sentence $\psi \wedge \Phi \wedge \chi$ has a model M of size $> |V_\alpha|$. Because of the usage of Φ, M is without loss of generality of the form $(V_{\beta_\alpha}, \in, F_{\beta_\alpha})$ for some ordinal $\beta_\alpha > \alpha$. As $M \models \psi$, there is an elementary embedding $j : (V_\alpha, \in, F_\alpha) \to (V_{\beta_\alpha}, \in, F_{\beta_\alpha})$. And because F_{β_α} is a function with domain ω which is cofinal in β_α by χ, j has a critical point κ_α. This shows that the function $\alpha \mapsto \kappa_\alpha$ is regressive on the stationary class $\{\alpha \in \mathrm{Ord} \colon \mathrm{cof}(\alpha) = \omega\}$. Then by the very weak class Fodor principle (Theorem 4.5.2) this function is constant on an unbounded class, say with value δ. Then δ is extendible. We can therefore take the smallest extendible cardinal η. Now if $\eta \leq \kappa$ we are done. If $\eta > \kappa$, then because η is extendible it is in $C^{(3)}$ (cf. Theorem

1.3.26). It follows that $V_\eta \models$ "κ is the Hanf number of $\mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$" (note that this is a Π_3-statement). Thus we can repeat our argument to get, in V_η, an extendible cardinal $\nu < \eta$. Being extendible is Π_3-definable (cf. Section 1.3.5), and thus, as $\eta \in C^{(3)}$, it is correct about extendibility. So ν is really extendible. But this contradicts that η is the smallest extendible cardinal.

Finally, assume (1) and let $T \subseteq \mathcal{L}^2(\wedge^\infty, \exists^\infty, \forall^\infty)$ be a $<\kappa$-satisfiable theory over some vocabulary τ. For simplicity, let us assume that τ contains only relation symbols and only a single sort symbol. Let $\beth_\alpha = \alpha > \kappa$ be such that V_α verifies the satisfiability of all $T_0 \in \mathcal{P}_\kappa T$. Note that $T \subseteq \mathcal{L}^2(\wedge^\lambda, \exists^\lambda, \forall^\lambda)$ for some $\lambda < \alpha$. By extendibility find some β such that $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. Then by elementarity $V_\beta \models$ "$j(T)$ is a $<j(\kappa)$-satisfiable theory" and further $j``T \subseteq j(T)$ and $|j``T| = |T| < \alpha < j(\kappa)$. Thus $V_\beta \models$ "$j``T$ has a model $\mathcal{B}$". Now let $\mathcal{B}'$ be the τ-structure with universe B and with $R^{\mathcal{B}'} = j(R)^{\mathcal{B}}$ for all $R \in \tau$. We will show that $\mathcal{B}' \models T$. For this it is sufficient to show that $V_\beta \models$ "$\mathcal{B} \models j(\varphi)$" implies $\mathcal{B}' \models \varphi$ for all $\varphi \in \mathcal{L}^2_{\wedge^\lambda, \exists^\lambda, \forall^\lambda}$. We will show this by proving the following claim:

Claim 5.3.7. For every $\varphi \in \mathcal{L}^2(\wedge^\lambda, \exists^\lambda, \forall^\lambda)$ and every variable assignment $f : j(S) \to B \cup \bigcup_{n\in\omega} \mathcal{P}(B^n)$ where S is the set of variables in φ:

$$V_\beta \models \text{“}\mathcal{B} \models j(\varphi)[f]\text{” implies } \mathcal{B}' \models \varphi[f \circ j].$$

Note that $j \restriction S$ is a map $S \to j(S)$, so $f \circ j$ is a sensible assignment on S, and that because V_β is a rank initial segment, any assignment $f : j(S) \to B \cup \bigcup_{n\in\omega} \mathcal{P}(B^n)$ belongs to it.

For the base case, if $\varphi = R(x_1, \dots, x_n)$ for some $R \in \tau$ and variables $x_1, \dots, x_n \in S$, then $j(\varphi) = j(R)(j(x_1), \dots, j(x_n))$. We assume that

$$V_\beta \models \text{“}\mathcal{M} \models j(R)(f(j(x_1)), \dots, f(j(x_n)))\text{”},$$

which by definition means $(f(j(x_1)), \dots, f(j(x_n))) \in j(R)^{\mathcal{B}}$. But then, by definition of $R^{\mathcal{B}'}$, we have $(f(j(x_1)), \dots, f(j(x_n))) \in R^{\mathcal{B}'}$, which, again by definition, means $\mathcal{B}' \models R(x_1, \dots, x_m)[f \circ j]$. The case of negated atomic formulas goes similar. The case of finite applications of $\vee$ is trivial.

Consider $\varphi = \bigwedge_{i<\gamma} \chi_i$ and assume $V_\beta \models$ "$\mathcal{B} \models j(\varphi)[f]$". We have that $j(\varphi) = \bigwedge_{i<j(\gamma)} \chi_i^*$ for some formulas χ_i^*. Further, for any $i < \gamma$, $j(\chi_i) = \chi_{j(i)}^*$. We get that $V_\beta \models$ "$\mathcal{B} \models j(\chi_i)[f]$" and therefore by induction hypothesis $\mathcal{B}' \models \chi_i[f \circ j]$, for any of the χ_i. Thus, $\mathcal{B}' \models \varphi[f \circ j]$.

If $\varphi = \exists W \chi$, then $j(\varphi) = \exists j(W) j(\chi)$. Because $V_\beta \models$ "$\mathcal{B} \models \exists j(W) j(\chi)[f]$", there is a $j(W)$-variant g of f such that $V_\beta \models$ "$\mathcal{B} \models j(\chi)[g]$". By induction hypothesis, $\mathcal{B}' \models \chi[g \circ j]$. We claim that $g \circ j$ is a W-variant of $f \circ j$, and so $\mathcal{B}' \models \exists W \chi[f \circ j]$. For $x \in S \setminus W$, we have $j(x) \in j(W)$. Since g is a $j(W)$-variant of f, thus $g(j(x)) = f(j(x))$.

Finally, if $\varphi = \forall W \chi$, then $j(\varphi) = \forall j(W) j(\chi)$. We assume that

$$V_\beta \models \text{“}\mathcal{B} \models \forall j(W) j(\chi)[f]\text{”},$$

and want to show that $\mathcal{B}' \models \forall W\chi[f \circ j]$. So let g be a W-variant of $f \circ j$. Define $h : j(S) \to B$ by

$$h(y) = \begin{cases} g(x), & \text{if } y \in j``S \text{ and } j(x) = y \\ f(y), & \text{otherwise.} \end{cases}$$

We claim that h is a $j(W)$-variant of f. Let $y \in j(S) \setminus j(W)$. If $y \notin j``S$, then $h(y) = f(y)$ by definition. And if $y \in j``S$, let $j(x) = y$. Then $h(y) = g(x)$. Because $y \notin j(W)$, $x \notin W$, so $h(y) = g(x) = f \circ j(x) = f(y)$ where the middle equality holds because g is a W-variant of $f \circ j$. So in any case, $h(y) = f(y)$ and h is thus an $j(W)$-variant of f. Thus, $V_\beta \models$ "$\mathcal{B} \models j(\chi)[h]$". Hence, by induction hypothesis, $\mathcal{B}' \models \chi[h \circ j]$. But $h \circ j = g$. Therefore $\mathcal{B}' \models \chi[g]$. □

By a similar proof to that of Theorem 5.3.6, we can also give a characterisation of extendibility. The only things in need to be changed are that usage of $\mathcal{L}_{\kappa\kappa}$ can force critical points to be precisely κ and that embeddings with critical point κ fix formulas of $\mathcal{L}_{\kappa\kappa}$ up to renaming.

Theorem 5.3.8. The following are equivalent for a cardinal κ.

(1) κ is extendible.

(2) κ is the compactness number of $\mathcal{L}^2_{\kappa\kappa}$.

(3) κ is the compactness cardinal of $\mathcal{L}^2(\bigwedge^\infty, \exists^\infty, \forall^\infty, \bigvee^\kappa)$.

(4) κ is the weak compactness number of $\mathcal{L}^2(\bigwedge^\infty, \exists^\infty, \forall^\infty, \bigvee^\kappa)$.

(5) κ is the Hanf number of $\mathcal{L}^2(\bigwedge^\infty, \exists^\infty, \forall^\infty, \bigvee^\kappa)$.

(6) κ is the ULST number of $\mathcal{L}^2(\bigwedge^\infty, \exists^\infty, \forall^\infty, \bigvee^\kappa)$.

(7) κ is the SULST number of $\mathcal{L}^2(\bigwedge^\infty, \exists^\infty, \forall^\infty, \bigvee^\kappa)$.

Finally, adding the power of sort logic to express that some α is in $C^{(n)}$, a similar argument shows:

Theorem 5.3.9. The following are equivalent for a cardinal κ:

(1) κ is $C^{(n)}$-extendible.

(2) κ is the compactness number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}$.

(3) κ is the compactness number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\bigwedge^\infty, \exists^\infty, \forall^\infty)$.

(4) κ is the weak compactness number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\bigwedge^\infty, \exists^\infty, \forall^\infty)$.

(5) κ is the Hanf number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\bigwedge^\infty, \exists^\infty, \forall^\infty)$.

(6) κ is the ULST number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\bigwedge^\infty, \exists^\infty, \forall^\infty)$.

(7) κ is the SULST number of $\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\bigwedge^\infty, \exists^\infty, \forall^\infty)$.

5.4. Weak Vopěnka's Principle and Löwenheim-Skolem properties

We saw in Chapter 2 that there are two analogous stratifications of VP and WVP in terms of strong- and weak-Henkin-compactness properties (cf. Corollaries 2.2.12 and 2.3.11). On the other hand, VP is not only characterised by compactness properties, but it has a further stratification by downward Löwenheim-Skolem-Tarski properties of sort logics, which also correspond to the $C^{(n)}$-extendible cardinals (cf. Theorem 1.3.35). It is therefore natural to ask whether a similar statement is true for the hierarchy of Π_n-strong cardinals below WVP. In this section we will show that this is indeed the case, pushing the analogy of the structure of model-theoretic assumptions below VP and WVP further. We will consider certain Löwenheim-Skolem numbers of the class logic $\mathcal{L}^{s,n}(\vee^\infty, \forall^\infty)$ and show that their existence is equivalent to that of Π_n-strong cardinals (cf. Theorem 5.4.3). We will first state our main results in Section 5.4.1. The main proof is then presented in Section 5.4.2.

5.4.1. Statement of the main results

Recall that if $\mathrm{LST}(\mathcal{L}) = \kappa$ for some logic $\mathcal{L}$, we demand that for any $\varphi \in \mathcal{L}$ and any $\mathcal{A} \models \varphi$, there is some substructure $\mathcal{B} \models \varphi$ such that $|B| < \kappa$, *provided* $\mathcal{A}$ is a τ-structure for some vocabulary τ of size $|\tau| < \kappa$. For the LS number, we usually do not need such a restriction in the size of vocabularies, as for set-sized strong logics, the amount of non-logical symbols that a sentence can use is bounded anyway. But for class logics this is not the case.

We therefore consider the following version of the LS number.

Definition 5.4.1. Let $\mathcal{L}$ be an abstract logic and λ a cardinal. A cardinal κ is an *LS^λ number of $\mathcal{L}$* iff any satisfiable sentence of $\mathcal{L}$ over a vocabulary of size $< \lambda$ has a model of size $< \kappa$. Should such a cardinal exist, we call the smallest such $\mathrm{LS}^\lambda(\mathcal{L})$.

We first mention the following unpublished result by Wilson. We omit the proof, as it can be carried out with similar arguments as that of Theorem 5.4.3.

Theorem 5.4.2 (Wilson)**.** The following are equivalent for a cardinal κ:

(1) κ is the smallest strong cardinal.

(2) $\kappa = \mathrm{LS}^\omega(\mathcal{L}^2(\vee^\infty, \forall^\infty))$.

(3) κ is the smallest cardinal such that $\kappa = \mathrm{LS}^\kappa(\mathcal{L}^2(\vee^\infty, \forall^\infty))$.

We will show the following theorem that connects LS numbers of class logics of the form $\mathcal{L}^{s,n}(\vee^\infty, \forall^\infty)$ to local forms of "Ord is Woodin". For the large cardinal notions involved, and also for the definition of weak Vopěnka's Principle referred to in the corollaries below, recall Section 2.3.1.

Theorem 5.4.3. The following are equivalent for every natural number $n \geq 1$ and a cardinal κ:

(1) κ is the smallest Π_n-strong cardinal.

(2) κ is the smallest $C^{(n)}$-strong cardinal, i.e., the smallest A-strong cardinal with $A = C^{(n)}$.

(3) $\kappa = \mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$.

(4) κ is the smallest cardinal such that $\kappa = \mathrm{LS}^{\kappa}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$.

Proof. Cf. Proof 5.4.10. □

In particular, by Theorem 2.3.4, the above implies that we get a local equivalence of the existence of $\mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$ to fragments of WVP.

Corollary 5.4.4. The following are equivalent:

(1) $\mathrm{WVP}(\Pi_n)$.

(2) $\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty})$ has an LS^{ω} number.

As for the case of compactness numbers of $\mathcal{L}^2(\wedge^{\infty}, \exists^{\infty}, \forall^{\infty})$, by minor adaptations to the proof of Theorem 5.4.3, we obtain the following result.

Theorem 5.4.5. The following are equivalent for a cardinal κ:

(1) κ is Π_n-strong.

(3) $\kappa = \mathrm{LS}^{\kappa}(\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^{\infty}, \forall^{\infty}))$.

Combining this with Corollary 2.3.20, we get a characterisation of WVP in terms of Löwenheim-Skolem properties.

Corollary 5.4.6. The following are equivalent.

(1) WVP.

(2) Every class logic $\mathcal{L}$ with $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^{\infty}, \forall^{\infty})$ for some κ has an LS^{ω} number.

(3) Every class logic $\mathcal{L}$ with $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^{\infty}, \forall^{\infty})$ for some κ has an LS^{λ} number for every λ.

Proof. Clearly (3) implies (2). By Corollary 2.3.20, WVP is equivalent to the existence of a Π_n-strong cardinal for every n, and of a proper class of Π_n-strong cardinals for every n. In particular, (2) implies (1) by Theorem 5.4.3. So left to show is that (1) implies (3). Assume (1) and take any cardinal λ and a class logic $\mathcal{L} \leq \mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^{\infty}, \forall^{\infty})$ for some κ. By WVP, we get a Π_n-strong cardinal $\gamma > \kappa + \lambda$. By Theorem 5.4.5, $\gamma = \mathrm{LS}^{\gamma}(\mathcal{L}^{\mathrm{s},n}_{\gamma\omega}(\vee^{\infty}, \forall^{\infty}))$. Then $\mathrm{LS}^{\lambda}(\mathcal{L}) \leq \mathrm{LS}^{\lambda}(\mathcal{L}^{\mathrm{s},n}_{\kappa\omega}(\vee^{\infty}, \forall^{\infty})) \leq \mathrm{LS}^{\gamma}(\mathcal{L}^{\mathrm{s},n}_{\gamma\omega}(\vee^{\infty}, \forall^{\infty}))$. □

We update Figure 2.2 from Section 2.3.2 by the Löwenheim-Skolem characterisations of WVP and its fragments.

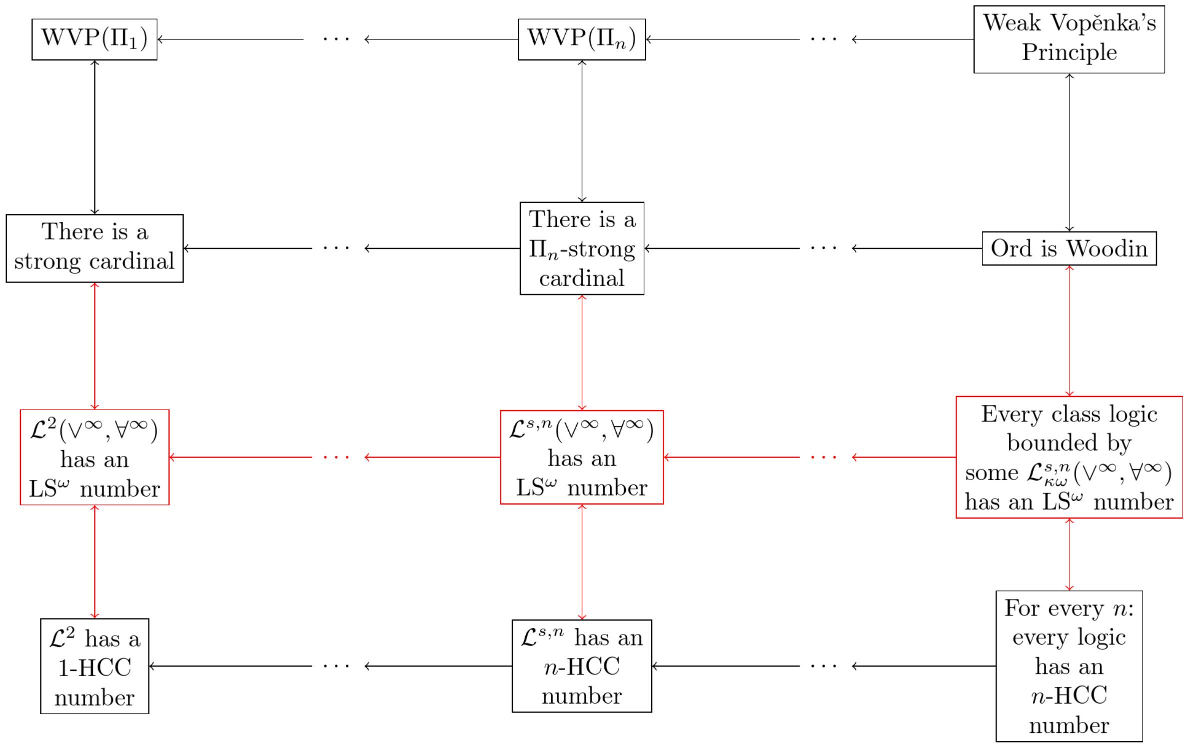

Figure 5.1.: Relations between WVP, Π_n-strong cardinals, HCC numbers, and Löwenheim-Skolem properties of class logics.

5.4.2. Proof of the main theorem

In this section, we give a proof of Theorem 5.4.3. We will need the framework of $\mathscr{P}$-*structures*, which we will present first. It constitutes an alternative approach to extenders and similarly allows to approximate elementary embeddings of the universe.

$\mathscr{P}$-structures as an alternative to extenders

The technicalities of the framework presented below are due to Wilson [Wil22b], and similar constructions can already be found in [Nee04] and [Zem02]. Instead of sequences of ultrafilters as for extenders, this approach constructs elementary embeddings from homomorphism of certain Boolean algebras called $\mathscr{P}$-*structures*. For a set X, we write $X^{<\omega}$ for the set

$$\{(a_1, \ldots, a_k) \colon k \in \omega \text{ and } a_1, \ldots, a_k \in X\}$$

of finite sequences of members of X. We write $\subsetneq$ for the proper initial segment relation, i.e., $(a_1, \ldots, a_k) \subsetneq (b_1, \ldots, b_l)$ iff $k < l$ and $a_i = b_i$ for all $i \leq k$. We further write $\supsetneq$ for the reverse of the relation $\subsetneq$.

Definition 5.4.7 (Wilson [Wil22b])**.** Let X be a transitive set. A $\mathscr{P}$-*structure* is a structure

$$\mathscr{P}_X = (\mathcal{P}(X^{<\omega}), \cap, \setminus, X^k, \mathrm{WF}, \pi^{-1}_{k,(i_1,\ldots,i_j)}, \mathrm{BP}_k)_{j,k<\omega, 1 \leq i_1,\ldots,i_j \leq k}.$$

such that

(1) $\cap$ is intersection, interpreted as a binary operation.

(2) $\setminus$ is complementation, interpreted as a unary operation.

(3) $X^k \in \mathcal{P}(X^{<\omega})$ is a constant.

(4) WF is a unary relation such that $A \in \mathrm{WF}$ iff $A \subseteq P(X^{<\omega})$ and $(A, \supsetneq)$ is well-founded.

(5) $\pi^{-1}_{k,(i_1,\ldots,i_j)}$ is a function $\mathcal{P}(X^j) \to \mathcal{P}(X^k)$ defined by

$$\pi^{-1}_{k,(i_1,\ldots,i_j)}(A) = \{(a_1, \ldots, a_k) \in X^k \colon (x_{i_1}, \ldots, x_{i_j}) \in A\}.$$

(6) BP_k is a function $\mathcal{P}(X^{k+1}) \to \mathcal{P}(X^{k+1})$ defined by

$$\mathrm{BP}_k(A) = \{(a_1, \ldots, a_{k+1}) \in X^k \colon \exists z \in x_{k+1}((x_1, \ldots, x_k, z) \in A)\}.$$

If further the structure is of the following form, for an additional constant $c^{<\omega}$ where $c \subseteq X$, we call $\mathcal{P}_{X,c}$ a *pointed $\mathscr{P}$-structure.*

$$\mathscr{P}_{X,c} = (\mathcal{P}(X^{<\omega}), \cap, \setminus, X^k, \mathrm{WF}, \pi^{-1}_{k,(i_1,\ldots,i_j)}, \mathrm{BP}_k, c^{<\omega})_{j,k<\omega, 1\leq i_1,\ldots,i_j \leq k}.$$

A *homomorphism of $\mathscr{P}$-structures* $h : \mathscr{P}_X \to \mathscr{P}_Y$ is simply a homomorphism in the usual model-theoretic sense, i.e., a map preserving the relations, constants and functions defined on $\mathscr{P}_X$. For instance, $h(A \cap B) = h(A) \cap h(B)$, or $A \in \mathrm{WF}^{\mathscr{P}_X}$ implies $A \in \mathrm{WF}^{\mathscr{P}_Y}$. A *homomorphism of pointed $\mathscr{P}$-structures* $h : \mathscr{P}_{X,c} \to \mathscr{P}_{Y,d}$ additionally preserves the constant $c^{<\omega}$, i.e., $h(c^{<\omega}) = d^{<\omega}$.

The way in which we will use $\mathscr{P}$-structures is by the following theorem, which shows how they give rise to elementary embeddings.

Theorem 5.4.8 (Wilson [Wil22b])**.** *If X and Y are transitive, $c \subseteq X$, $d \subseteq Y$ and $h : \mathscr{P}_{X,c} \to \mathscr{P}_{Y,d}$ a homomorphism of pointed $\mathscr{P}$-structures. Then there is a transitive class M and an elementary embedding $j : V \to M$ such that $Y \subseteq j(X)$, $h(A) = j(A) \cap Y^{<\omega}$ for all $A \subseteq X^{<\omega}$ and $j(c) \cap Y = d$.*

Let us further note that there is a trivial homomorphism $\mathscr{P}_X \to \mathscr{P}_Y$ if $Y \subseteq X$.

Proposition 5.4.9 (Wilson [Wil22b])**.** *Let X and Y be transitive such that $Y \subseteq X$. Then $h : \mathcal{P}(X^{<\omega}) \to \mathcal{P}(Y^{<\omega})$ defined by $A \mapsto A \cap Y^{<\omega}$ is a homomorphism of $\mathscr{P}$-structures $h : \mathscr{P}_X \to \mathscr{P}_Y$.*

Main proof

Proof 5.4.10 (Proof of Theorem 5.4.3)**.** The first two items are equivalent by Lemma 2.3.12. It suffices to show that (3) implies (2), and that (2) implies (4). This holds, because if (4) is true, and κ is the smallest cardinal such that $\kappa = \mathrm{LS}^{\kappa}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$, then in particular, $\gamma = \mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty})) \leq \mathrm{LS}^{\kappa}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$ exists. But if (3) implies (2), then this means that γ is the smallest $C^{(n)}$-strong cardinal. And if (2) implies (4), then γ is the smallest cardinal such that $\gamma = \mathrm{LS}^{\gamma}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$. But this implies $\kappa = \gamma$ and so (4) implies (3).

So let us first show that (3) implies (2). The following claim suffices for this.

Claim 5.4.11. If $\kappa = \mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$, then for all $\lambda > \kappa$ which are limits of $C^{(n)}$, there is a λ-$C^{(n)}$-strong cardinal $\leq \kappa$.

If we showed Claim 5.4.11, then we showed that (3) implies (2): as there are set many cardinals $\leq \kappa$ but a proper class of $\lambda > \kappa$, there then has to be such a cardinal that is λ-$C^{(n)}$-strong for arbitrarily large λ and hence $C^{(n)}$-strong.

So we can let $\lambda > \kappa$ be a limit of $C^{(n)}$, and assume that $\kappa = \mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$. To show that κ is λ-$C^{(n)}$-strong, it is sufficient to show that there is a homorophism $h : \mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\lambda, C^{(n)} \cap V_\lambda}$ of pointed $\mathscr{P}$-structures. For then, Wilson's Theorem 5.4.8 implies that there is $j : V \to M$ such that $V_\lambda \subseteq M$ and $j(C^{(n)} \cap V_\kappa) \cap V_\lambda = C^{(n)} \cap V_\lambda$. In particular, as λ is a limit of $C^{(n)}$, there is some $\alpha > \kappa$ such that $\alpha \in C^{(n)}$ and $\alpha \in j(C^{(n)} \cap V_\kappa) \cap V_\lambda$. This implies that $j(\kappa) > \kappa$ and so j has a critical point $\mathrm{crit}(j) \leq \kappa$. Then j witnesses that $\mathrm{crit}(j)$ is λ-$C^{(n)}$-strong. So we showed that it is sufficient to find a homomorphism $h : \mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\lambda, C^{(n)} \cap V_\lambda}$ of pointed $\mathscr{P}$-structures. To obtain this, it is sufficient to show the following claim.

Claim 5.4.12. There is a sentence $\varphi \in \mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty})$ such that for any limit ordinal α:

$$(V_\alpha, \in) \models \varphi \text{ iff there is no homomorphism } h : \mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\alpha, C^{(n)} \cap V_\alpha}.$$

Let us argue that showing Claim 5.4.12 is sufficient to derive the existence of a homomorphism $h : \mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\lambda, C^{(n)} \cap V_\lambda}$. Suppose there is no such homomorphism. Let φ be the sentence from Claim 5.4.12 and let φ^* be the conjunction of φ and Magidor's Φ (cf. Lemma 1.2.4). Then $V_\lambda \models \varphi^*$. Therefore, by κ being $\mathrm{LS}^{\omega}(\mathcal{L}^{\mathrm{s},n}(\vee^{\infty}, \forall^{\infty}))$ there is a structure (M, E^M) of size $< \kappa$ with $(M, E^M) \models \varphi^*$. As $(M, E^M) \models \Phi$, we can without loss of generality assume that $(M, E^M) = (V_\alpha, \in)$ for some $\alpha < \kappa$, for which in particular $V_\alpha \subseteq V_\kappa$. But by this fact and Proposition 5.4.9, the map $A \mapsto A \cap V_\alpha^{<\omega}$ is a (trivial) homomorphism $h : \mathscr{P}_{V_\kappa} \to \mathscr{P}_{V_\alpha}$ of $\mathscr{P}$-structures. Because $h((C^{(n)} \cap V_\kappa)^{<\omega}) = (C^{(n)} \cap V_\kappa)^{<\omega} \cap V_\alpha^{<\omega} = (C^{(n)} \cap V_\alpha)^{<\omega}$, this is also a homomorphism of pointed $\mathscr{P}$-structures $\mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\alpha, C^{(n)} \cap V_\alpha}$. Thus $V_\alpha \not\models \varphi$. Contradiction.

So we reduced our aim to proving Claim 5.4.12. For this, we may instead show the following assertion.

Claim 5.4.13. There is a sentence $\psi \in \mathcal{L}^{\mathrm{s},n}(\wedge^{\infty}, \exists^{\infty})$ such that for any limit ordinal α:

$$(V_\alpha, \in) \models \psi \text{ iff there is a homomorphism } \mathscr{P}_{V_\kappa, C^{(n)} \cap V_\kappa} \to \mathscr{P}_{V_\alpha, C^{(n)} \cap V_\alpha}$$

Given Claim 5.4.13, taking $\varphi = \neg\psi$ (and pushing negations through the infinitary quantifiers and conjunctions) proves Claim 5.4.12. To show Claim 5.4.13, we let

$$\psi = \exists(X_A \colon A \in \mathcal{P}(V_\kappa^{<\omega}))(\bigwedge_{i=1}^{7} \psi_i),$$

where each X_A is a unary second-order variable and each ψ_i will be specified below. The purpose of the sentences is the following. If for some V_α there is a sequence $(X_A^{V_\alpha} \colon A \in \mathcal{P}(V_\kappa^{<\omega}))$ witnessing that V_α satisfies ψ_i, then the map $h : \mathcal{P}(V_\kappa^{<\omega}) \to \mathcal{P}(V_\alpha^{<\omega})$ defined by $A \mapsto X_A^{V_\alpha}$ shall preserve the clause of Definition 5.4.7 corresponding to ψ_i. Then if $V_\alpha \models \psi$, this map is a full homomorphism. Let us go through the conjuncts of ψ, each time arguing why the map h is a homomorphism with respect to the intended part of the structure.

$$\psi_1 = \bigwedge_{\substack{A,B,C\in\mathcal{P}(V_\kappa^{<\omega})\\ A\cap B=C}} \forall x((X_A(x) \wedge X_B(x)) \leftrightarrow X_C(x)).$$

$$\psi_2 = \bigwedge_{\substack{A,B\in\mathcal{P}(V_\kappa^{<\omega})\\ V_\kappa^{<\omega}\setminus A=B}} \forall x(X_{V_\kappa^{<\omega}}(x) \wedge \neg X_A(x) \leftrightarrow X_B(x)).$$

Clearly, ψ_1 codes that $X_A^{V_\alpha} \cap X_B^{V_\alpha} = X_C^{V_\alpha}$. Because $V_\kappa^{<\omega} \setminus V_\kappa^{<\omega} = \emptyset$, by ψ_2 we get that $h(\emptyset) = \emptyset$ and $h(V_\kappa^{<\omega}) = V_\kappa^{<\omega}$. Using this, satisfaction of ψ_2 implies preservation of complements.

$$\psi_3 = \bigvee_{k\in\omega} \forall x(\text{“}x \text{ is a sequence of length } k\text{”} \leftrightarrow X_{V_\kappa^k}(x)).$$

Here we take that x is a sequence of length k simply written out in first-order logic. As V_α knows of all sequences of length k, this implies that $h(V_\kappa^k) = V_\alpha^k$. The sentence ψ_4 is a conjunction

$$\psi_4 = \bigwedge_{\substack{A\subseteq X^{<\omega}\\ (A,\supsetneq) \text{ well-founded}}} \chi_1^A,$$

where each χ_1^A is given by

$$\chi_1^A = \neg\exists F(\mathrm{dom}(F) = \omega \wedge \forall n \in \omega(X_A(F(n))) \wedge \forall n \in \omega(F(n) \subsetneq F(n+1))).$$

Note that $(A, \supsetneq)$ is well-founded iff there is a function f with domain ω and and $\mathrm{ran}(f) \subseteq A$ such that $f(n) \subsetneq f(n+1)$ for all $n \in \omega$. This is coded by the above sentence, using a binary second-order variable F which we implicitly assume to be functional (note that we could express this).

In the formula below, for some $k \in \omega$ and $1 \leq i_1, \ldots, i_j \leq k$, and further $A \in \mathcal{P}(V_\kappa^{<\omega})$, we write $\varphi_{k,i_1,\ldots,i_j}(x)$ for the formula $\exists y_1, \ldots, y_k(x = (y_1, \ldots, y_k) \wedge X_A((y_{i_1}, \ldots, y_{i_j})))$, expressing that x is some k-tuple $x = (y_1, \ldots, y_k)$ and $(y_{i_1}, \ldots, y_{i_j})$ is a member of X_A. Then let

$$\psi_5 = \bigwedge_{\substack{A,B \in \mathcal{P}(V_\kappa^\omega),\\ B=\{(x_1,\ldots,x_k)\colon (x_{i_1},\ldots,x_{i_j})\in A)\},\\ k\in\omega, 1\leq i_1,\ldots,i_j\leq k}} \forall x (X_B(x) \leftrightarrow \varphi^A_{k,i_1,\ldots,i_j}(x)).$$

Then ψ_5 simply codes the defining conditions of $\pi^{-1}_{k,(i_1,\ldots,i_j)}(A) = B$.

Now write χ^A_k for the formula

$$\exists y_1, \ldots, y_{k+1}(x = (y_1, \ldots, y_{k+1}) \wedge \exists z(E(z, x_{k+1}) \wedge X_A((y_1, \ldots, y_k, z)))),$$

expressing that $x = (y_1, \ldots, y_{k+1})$ is some $(k+1)$-tuple and there is some $z \in y_{k+1}$ such that $(y_1, \ldots, y_k, z)$ is a member of X_A, and let

$$\psi_6 = \bigwedge_{\substack{B=\{(x_1,\ldots,x_{k+1})\colon\\ \exists z\in x_{k+1}(x_1,\ldots,x_k,z)\in A\}\\ A,B\in\mathcal{P}(V_\kappa^\omega),k\in\omega}} \forall x(X_B(x) \leftrightarrow \chi^A_k(x)).$$

Again, ψ_6 simply codes the defining conditions of $\mathrm{BP}_k(A) = B$. Finally, let

$$\psi_7 = \forall x(X_{C^{(n)}\cap V_\kappa}(x) \leftrightarrow (\Phi^{(n)})^{\{y\colon E(y,V_x)\}}).$$

Here $(\Phi^{(n)})^{\{y\colon E(y,V_x)\}}$ is the relativisation of $\Phi^{(n)}$ to V_x, i.e., the rank initial segment cut off at some ordinal x, truthfully coding that $V_x \in \mathcal{K}^{(n)}$, i.e., that $x \in C^{(n)}$ (recall Corollary 1.2.17). Then ψ_7 uses that V_α is correct about V_β for $\beta < \alpha$ and the sentence $\Phi^{(n)}$ to express that $X^{V_\alpha}_{C^{(n)}\cap V_\kappa}$ is actually $V_\alpha \cap C^{(n)}$. This ends the construction of ψ and we thus showed what we promised. This ends the proof that (3) implies (2).

And now we show that (4) implies (2), i.e., if κ is $C^{(n)}$-strong, then it is an LS^κ number of $\mathcal{L}^{\mathrm{s},n}(\vee^\infty, \forall^\infty)$. So let τ be some vocabulary of size $< \kappa$, $\mathcal{A}$ a τ-structure and $\varphi \in \mathcal{L}^{\mathrm{s},n}(\vee^\infty, \forall^\infty)[\tau]$ such that $\mathcal{A} \models \varphi$. Without loss of generality we can assume that $\tau \in H_\kappa$. Our goal is to show that there is a structure of size $< \kappa$ satisfying φ. For this purpose take a $\lambda > \kappa$ which is a limit point of $C^{(n)}$ with $\mathcal{A}, \varphi \in V_\lambda$. By $C^{(n)}$-strength of κ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $C^{(n)} \cap V_\lambda = j(C^{(n)} \cap V_\kappa) \cap V_\lambda$. Note that $\mathcal{A} \in M$ and that $j(\tau) = \tau$. We will show that $M \models$ "$\mathcal{A} \models j(\varphi)$". This makes sense, because $j(\varphi)$ is a τ-sentence, as $j(\tau) = \tau$. Furthermore, this is sufficient, because then $M \models$ "$|A| < \lambda < j(\kappa) \wedge \mathcal{A} \models j(\varphi)$", and so by elementarity of j, in V there has to be some model of φ of size $< \kappa$. We show the following claim, which will be sufficient. Let S be the set of variables appearing in φ. Note that the set of variables used in $j(\varphi)$ is $j(S)$.

Claim 5.4.14. For every structure $\mathcal{B} \in V_\lambda$ and for every subformula ψ of φ and any variable assignment $f : j(S) \to V_\lambda$ with $f \in M$, if $\mathcal{B} \models \psi[f \circ (j \restriction S)]$, then $M \models$ "$\mathcal{B} \models j(\psi)[f]$".

Note that $j \restriction S$ is a map $S \to j(S)$, so $f \circ (j \restriction S)$ is a sensible assignment on S. Furthermore, φ is a sentence and so concrete assignments do not play a role in its evaluation. Thus we get that $\mathcal{A} \models \varphi$ implies that $M \models$ "$\mathcal{A} \models j(\varphi)$" by the claim, by just taking f to be any assignment and $\mathcal{B}$ to be $\mathcal{A}$.

Before we show Claim 5.4.14, we show the following stronger assertion for formulas of $\mathcal{L}^{\mathrm{s},n}$, which gives us the base case for our induction for Claim 5.4.14.

Claim 5.4.15. For every structure $\mathcal{B} \in V_\lambda$ and for every subformula ψ of φ such that $\psi \in \mathcal{L}^{\mathrm{s},n}$, and every $f : j(S) \to V_\lambda$ with $f \in M$, we have $\mathcal{B} \models \psi[f \circ (j \restriction S)]$ iff $M \models$ "$\mathcal{B} \models j(\psi)[f]$".

To show Claim 5.4.15, notice that because $\psi \in \mathcal{L}^{\mathrm{s},n}$ and $\tau \in H_\kappa$, the only difference between ψ and $j(\psi)$ is a possible renaming of variables, so, for example, application of j to $\psi = R(x_1, \ldots, x_n)$ gives us $j(\psi) = R(j(x_1), \ldots, j(x_n))$. This means that if y is some variable appearing in $j(\psi)$ it is of the form $j(x)$ for some $x \in S$. But then $(f \circ (j \restriction S))(x) = f(j(x)) = f(y)$ and so

$$\mathcal{B} \models R(f(j(x_1)), \ldots, f(j(x_n))) \text{ iff } M \models \text{“}\mathcal{B} \models R(f(y), \ldots, f(y))\text{”}.$$

These calculations of how sentences of the form $j(\psi)$ for $\psi \in \mathcal{L}^{\mathrm{s},n}$ look like carry over to all sentences from $\mathcal{L}^{\mathrm{s},n}$. Thus for Claim 5.4.15 to hold, we only have to check whether M is correct about $\mathcal{B}$'s satisfaction of $\mathcal{L}^{\mathrm{s},n}$-formulas. But this is the case: Because $\lambda \in C^{(n)}$, V_λ and V agree about $\mathcal{L}^{\mathrm{s},n}$-satisfaction (cf. Corollary 1.2.21). Further, λ is a limit point of $C^{(n)}$ and M is correct about $C^{(n)}$ below λ (as it is the image of an embedding witnessing $C^{(n)}$-strength). Thus M sees that it is a limit point of $C^{(n)}$ and so $\lambda \in (C^{(n)})^M$. Therefore also V^M_λ and M agree about $\mathcal{L}^{\mathrm{s},n}$-satisfaction. But also $V^M_\lambda = V_\lambda$ and therefore V, V_λ, and M all agree on $\mathcal{L}^{\mathrm{s},n}$-satisfaction.

Now let us start our induction to prove Claim 5.4.14. Recall that we do not allow sort quantifiers, Q^{WF} or negation to take infinitary formulas and so the base case and the cases of application of sort quantifiers, Q^{WF} and negation follow from Claim 5.4.15. The case for $\wedge$ is trivial.

So let us consider the case $\psi = \exists x \chi$ and $\mathcal{B} \models \exists x \chi[f \circ (j \restriction S)]$, where x is a single (possibly second-order) variable. Then there is an x-variant g of $f \circ (j \restriction S)$ such that $\mathcal{B} \models \chi[g]$. Further $j(\psi) = \exists j(x) j(\chi)$. Fix $a = g(x)$. In M, we have to find a $j(x)$-variant h of f with $M \models$ "$\mathcal{B} \models j(\chi)[h]$." Let h be defined by

$$h(y) = \begin{cases} a, & \text{if } y = j(x) \\ f(y), & \text{otherwise.} \end{cases}$$

For the case that x is a second-order variable, note that $\mathcal{B} \in V_\lambda \subseteq M$ and so M contains all subsets of B. Clearly, h is a $j(x)$-variant of f. Further, $h \in M$ as it was defined from $f \in M$. Also for $x \neq y \in S$, $j(y) \neq j(x)$, so $g(y) = f \circ (j \restriction S)(y) = f(j(y)) = h(j(y))$ and, by definition, $h(j(x)) = a = g(x)$. Hence $h \circ (j \restriction S) = g$. In particular $\mathcal{B} \models \chi[h \circ (j \restriction S)]$ and thus, by induction hypothesis, $M \models$ "$\mathcal{B} \models j(\chi)[h]$", as desired.

If $\psi = \bigvee_{i<\gamma} \chi_i$ and $\mathcal{B} \models \psi[f \circ (j \upharpoonright S)]$, there is some $i < \gamma$ such that $\mathcal{B} \models \chi_i[f \circ (j \upharpoonright S)]$. By induction hypothesis, $M \models$ "$\mathcal{B} \models j(\chi_i)[f]$." Now $j(\psi) = \bigvee_{k<j(\gamma)} \chi_k^*$ for some χ_k^*. Because χ_i appears as a disjunct in ψ, $j(\chi_i)$ is one of the disjuncts of $j(\psi)$. In particular, then $M \models$ "$\mathcal{B} \models j(\psi)[f]$".

If $\psi = \forall T \chi$ and $\mathcal{B} \models \forall T \chi[f \circ (j \upharpoonright S)]$. Then for any T-variant g of $f \circ (j \upharpoonright S)$, we have $\mathcal{B} \models \chi[g]$. Note that $j(\psi) = \forall j(T) j(\chi)$. Now to show that $M \models$ "$B \models \forall j(T) j(\chi)[f]$", we let h be any $j(T)$-variant of f with $h \in M$. This means $h \upharpoonright (j(S) \setminus j(T)) = f \upharpoonright (j(S) \setminus j(T))$. Consider $h' = h \circ (j \upharpoonright S)$. Then if $x \in (S \setminus T)$, we have $j(x) \in (j(S) \setminus j(T))$. And then $h'(x) = h(j(x)) = f(j(x))$, where the latter equality holds because h is a $j(T)$-variant of f. But this shows that h' is a T-variant of $f \circ (j \upharpoonright S)$. Thus $\mathcal{B} \models \chi[h']$, which just means $\mathcal{B} \models \chi[h \circ (j \upharpoonright S)]$. Then by induction hypothesis $M \models$ "$\mathcal{B} \models j(\chi)[h]$". As h was an arbitrary $j(T)$-variant of f from M we thus showed that $M \models$ "$\mathcal{B} \models \forall j(T) j(\chi)[f]$." □

5.5. Characterising Shelah cardinals

As a last note on proper class logics, let us show that we can give the following compactness characterisation of *Shelah cardinals*. To our best knowledge, this is the first known model-theoretic characterisation of Shelah cardinals.

Theorem 5.5.1. The following are equivalent for a cardinal κ:

(1) κ is Shelah, i.e., for all $f : \kappa \to \kappa$, there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$.

(2) κ is inaccessible and if $T \subseteq \mathcal{L}^2(\wedge^\infty, \exists^\infty, \vee^\kappa, \forall^\kappa)[\tau]$ is a theory of size κ such that for every $\varphi \in T$ there is $\tau_\varphi \subseteq \tau$ such that $|\tau_\varphi| < \kappa$ and $\varphi \in \mathcal{L}^2(\wedge^\infty, \exists^\infty, \vee^\kappa, \forall^\kappa)[\tau_\varphi]$ and all $< \kappa$-sized subsets of T have a model of size $< \kappa$, then T has a model.

Proof. First assume (1). Clearly, a Shelah cardinal is measurable, so in particular inaccessible. To show the second part of (2), let $T = \{\varphi_i \colon i < \kappa\} \subseteq \mathcal{L}^2(\wedge^\infty, \exists^\infty, \vee^\kappa, \forall^\kappa)$ be a theory over a vocabulary τ such that every φ_i only uses $< \kappa$ many symbols from τ and such that all its $< \kappa$-sized subsets are satisfiable by a model of size $< \kappa$. Then writing $T_\alpha = \{\varphi_i \colon i < \alpha\}$ for $\alpha < \kappa$, every T_α is a theory of size $< \kappa$ over a vocabulary τ_α of size $< \kappa$ which we can assume to be in V_κ. This implies that $|\tau| \leq \kappa$ and further we can write $\tau = \bigcup_{\alpha<\kappa} \tau_\alpha$ as an increasing union. Further, every T_α has a model M_α of size $< \kappa$ and hence without loss of generality we can assume that $M_\alpha \in V_\kappa$. Then we can let $f : \kappa \to \kappa$ be a function such that $M_\alpha \in V_{f(\alpha)}$. Because κ is Shelah, there is $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$. Then in M, $j(T)$ is of the form $\{\varphi_i^* \colon i < j(\kappa)\}$ for some formulas φ_i^*, and $T_\kappa^* = \{\varphi_i^* \colon i < \kappa\}$ is satisfiable by a model $\mathcal{B} \in V_{j(f)(\kappa)}$. Further $j``T = \{j(\varphi_i) \colon i < \kappa\} \subseteq \{\varphi_i^* \colon i < \kappa\} = T_\kappa^*$, so $M \models$ "$\mathcal{B} \models j(\varphi)$" for all $\varphi \in T$. Note that $j(\tau) \supseteq j``\tau = \tau$ and so $\mathcal{B} \upharpoonright \tau$ makes sense. We show that $\mathcal{B} \upharpoonright \tau \models T$ by showing the following claim:

Claim 5.5.2. For any formula $\varphi \in \mathcal{L}^2(\wedge^\infty, \exists^\infty, \vee^\kappa, \forall^\kappa)[\tau]$ with a set of free variables S and any assignment $f : j(S) \to B \cup \bigcup_{n\in\omega} \mathcal{P}(B^n)$ in M, if $M \models$ "$\mathcal{B} \models j(\varphi)[f]$", then $\mathcal{B} \upharpoonright \tau \models \varphi[f \circ j]$.

We show this by induction on the complexity of φ. If $\varphi = R(x_1, \dots, x_n)$ for some $R \in \tau$, then $R \in V_\kappa$ and so $j(\varphi) = R(j(x_1), \dots, j(x_n))$. Then the claim is clear by definition.

If $\varphi = \bigvee_{i<\gamma} \psi_i$ for some $\gamma < \kappa$, then $j(\varphi) = \bigvee_{i<\gamma} j(\psi_i)$. So if $\mathcal{M} \models$ "$\mathcal{B} \models j(\varphi)[f]$", then $\mathcal{M} \models$ "$\mathcal{B} \models j(\psi_i)[f]$" for some $i < \gamma$. Then by induction hypothesis $\mathcal{B} \restriction \tau \models \psi_i[f \circ j]$, so also $\mathcal{B} \restriction \tau \models \bigvee_{i<\gamma} \psi_i[f \circ j]$.

If $\varphi = \forall Q \psi$ for some set of variables Q of size $< \kappa$, we can without loss of generality assume that $Q \in V_\kappa$ and so $j(Q) = Q$. Then $j(\varphi) = \forall Q j(\psi)$. We assume $M \models$ "$\mathcal{B} \models \forall Q j(\psi)[f]$". Let g' be any Q-variant of $f \circ j$. We have to show $\mathcal{B} \restriction \tau \models \psi[g']$. Note that $g' \restriction Q : Q \to B \cup \bigcup_{n\in\omega} \mathcal{P}(B^n)$ is in M because $Q, \mathcal{B} \in V_{j(f)(\kappa)} \subseteq M$. So we can define an assignment g on $j(S)$ in M by letting

$$g(x) = \begin{cases} g'(x), & \text{if } x \in Q \\ f(x), & \text{if } x \in j(S) \setminus Q. \end{cases}$$

Then g is a Q-variant of f, so $\mathcal{M} \models$ "$\mathcal{B} \models j(\psi)[g]$". Then by induction hypothesis, $\mathcal{B} \restriction \tau \models \psi[g \circ j]$. Thus it is sufficient to show that $g \circ j = g'$. But for $v \in Q$, $j(v) = v$, so $g'(v) = g(v) = g(j(v))$. And if $v \in S \setminus Q$, then $j(v) \in j(S) \setminus Q$, so $g(j(v)) = f(j(v)) = g'(v)$ where the latter holds because g' is a Q-variant of $f \circ j$.

If $\varphi = \bigwedge_{i<\delta} \psi_i$ for δ any ordinal, then $j(\varphi) = \bigwedge_{i<j(\delta)} \psi_i^*$ for some ψ_i^*. We assume $M \models$ "$\mathcal{B} \models \bigwedge_{i<j(\delta)} \psi_i^*[f]$". Then if $k < \delta$, we have that $j(\psi_k)$ is among the ψ_i^*, so $M \models$ "$\mathcal{B} \models j(\psi_k)[f]$". Then by induction hypothesis, $\mathcal{B} \restriction \tau \models \psi_k[f \circ j]$. So overall, $\mathcal{B} \restriction \tau \models \bigwedge_{i<\delta} \psi_i[f \circ j]$.

Finally, if $\varphi = \exists Q \psi$, where Q is a set of variables of any size, then $j(\varphi) = \exists j(Q) j(\psi)$. We assume $M \models$ "$\exists j(Q) j(\psi)[f]$". Then there is a $j(Q)$-variant g of f such that $M \models$ "$\mathcal{B} \models j(\psi)[g]$". By induction hypothesis $\mathcal{B} \restriction \tau \models \psi[g \circ j]$. Now if $x \in S \setminus Q$, then $j(x) \in j(S) \setminus j(Q)$, so $g(j(x)) = f(j(x))$. So $g \circ j$ is a Q-variant of $f \circ j$ with $\mathcal{B} \models \psi[g \circ j]$. Therefore $\mathcal{B} \models \exists Q \psi[f \circ j]$.

And now assume (2) and let $f : \kappa \to \kappa$. Without loss of generality, we can assume that f is increasing. We want to produce $j : V \to M$ with $\text{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subseteq M$. For this purpose consider the sentence $\psi = \exists (X_A \colon A \in \mathcal{P}(V_\kappa^{<\omega}))(\bigwedge_{i=1}^{8} \psi_i)$ where ψ_1 to ψ_6 are as in the proof of Theorem 5.4.3 and where $\psi_7 = \forall x, y (F(x, y) \leftrightarrow X_f(x, y))$ with F a new 2-place predicate symbol and X_f the variable corresponding to $f \subseteq V_\kappa^2$. Further, take for $\alpha < \kappa$ the $\sigma_\alpha(x)$ of $\mathcal{L}_{\kappa\omega}$ that defines α (cf. Lemma 1.2.4) and let

$$\psi_8 = \bigwedge_{\alpha<\kappa} \forall x (X_\alpha(x) \leftrightarrow \bigvee_{\beta<\alpha} \sigma_\beta(x)).$$

Then if $(M, \in, F^M) \models \psi$ is a transitive model, then there is a homomorphism of $\mathscr{P}$-structures $h : \mathscr{P}_{V_\kappa} \to \mathscr{P}_M$ such that $h(f) = F^M$ by letting $h(A) = X_A^M$ for $A \in \mathcal{P}(V_\kappa^{<\omega})$ where $(X_A^M \colon A \in \mathcal{P}(V_\kappa^{<\omega}))$ is the sequence witnessing that $(M, \in, F^M) \models \psi$. Equipped with ψ, take for every $\alpha \leq \kappa$ a new constant symbol c_α and consider the theory

$$T = \{\psi\} \cup \{\Phi\} \cup \{\text{“}c_\alpha \text{ and } c_\beta \text{ are ordinals with } c_\alpha < c_\beta\text{”} \colon \alpha < \beta \leq \kappa\},$$

where Φ is Magidor's Φ (cf. Lemma 1.2.4). Then any model of T can without loss of generality be assumed to be of the form $(V_\delta, \in, f', c_\alpha^{V_\delta})_{\alpha \leq \kappa}$. Any model of T has to contain an ordinal of order type $\geq \kappa$ by the last part of the theory and so $\delta > \kappa$. Since $(V_\delta, \in, f', c_\alpha^{V_\delta})_{\alpha \leq \kappa} \models \psi$, it gives rise to a homomorphism $h : \mathscr{P}_{V_\kappa} \to \mathscr{P}_{V_\delta}$ in the manner pointed out above. Note that by ψ_7 we have $h(f) = f'$. By Theorem 5.4.8, this gives rise to an elementary embedding $j : V \to M$ such that $V_\delta \subseteq j(V_\kappa)$ and $h(A) = j(A) \cap V_\delta^{<\omega}$ for all $A \subseteq V_\kappa^{<\omega}$. In particular $f' = h(f) = j(f) \cap V_\delta^2$ and so $j(f)(\kappa) = f'(\kappa) < \delta$. Thus $V_{j(f)(\kappa)} \subseteq V_\delta \subseteq j(V_\kappa) \subseteq M$. So we only have to show that $\operatorname{crit}(j) = \kappa$ to see that κ is Shelah. By our choice of T, V_δ contains an ordinal of order type $\kappa + 1$ and so we have $\delta > \kappa$. Because $V_\delta \subseteq j(V_\kappa)$ we have to have $\operatorname{crit}(j) \leq \kappa$. And further, for $\alpha < \kappa$, we have by usage of ψ_8 that $\alpha = h(\alpha) = j(\alpha) \cap V_\delta$ and so $\operatorname{crit}(j) = \kappa$.

Left to show is that T is satisfiable. Clearly T is of size κ and every sentence in T only uses finitely many symbols from T's vocabulary. So by our assumption it is sufficient to show that every $< \kappa$ sized subset T_0 of T has a model of size $< \kappa$. So let T_0 be such a subset and $\gamma = \sup\{\beta < \kappa \colon c_\alpha < c_\beta \in T_0\} < \kappa$ and take any ordinal η with $\gamma < \eta < \kappa$ such that η is closed under f. Note that such an η exists because κ is regular. Then let $c_\alpha^{V_\eta} = \alpha$ for $\alpha \leq \gamma$ and $c_\kappa^{V_\eta} = \gamma + 1$. Then $(V_\eta, \in, f \restriction \eta, c_\alpha^{V_\eta}, c_\kappa^{V_\eta})_{\alpha \leq \gamma} \models T_0$, because by Proposition 5.4.9, $V_\eta \subseteq V_\kappa$ implies that $A \mapsto A \cap V_\eta^{<\omega}$ for $A \subseteq V_\kappa^{<\omega}$ defines a (trivial) homomorphism $\mathscr{P}_{V_\kappa} \to \mathscr{P}_{V_\eta}$ and the rest of T_0 is satisfied by our choice of the constants. Finally, $|V_\eta| < \kappa$, as κ is inaccessible. $\square$

6. Symbiosis and Reflection Properties

Remarks on co-authorship. The results of Section 6.3 are joint with Will Boney. The results of Section 6.4 are joint with Lorenzo Galeotti and Yurii Khomskii.

6.1. Introduction

Väänänen studied situations in which definability in a logic and definability in set theory conincide, and dubbed this phenomenon *symbiosis* (cf. [Vää79]). Recently, first Bagaria and Väänänen [BV16], and then Galeotti, Khomskii, and Väänänen [Gal19, GKV20] showed that in case a logic is symbiotic, there is a systematic equivalence between LST and ULST numbers, respectively, and certain set-theoretic reflection principles. In his Master's Thesis [Osi21], the author considered whether the same is true for compactness properties, and while a partial result was given, showing that compactness properties of symbiotic logics give rise to reflection principles in certain classes of partial orders, this did not lead to an equivalence. In this chapter, we will refine these results and show a full equivalence (Theorem 6.3.12).

The chapter is structured as follows. Section 6.2 reviews the basic notions relevant in the context of symbiosis and the known results about how symbiosis mediates transfer between Löwenheim-Skolem properties of logics and set-theoretic reflection principles. Section 6.3 discusses the interaction between symbiosis and compactness properties of logics, and proves the main theorem 6.3.12. Finally, for purposes of completeness, Section 6.4 gives proofs of some statements about weak forms of LST numbers and weak reflection principles (cf. Corollary 6.4.16 and Theorem 6.4.13), which were partially stated before by Bagaria and Väänänen in [BV16, Bag23].

6.2. Motivation and definitions

We saw already that certain large cardinals can be characterised by certain reflection principles. For example, the existence of a supercompact cardinal is equivalent to the existence of a cardinal κ witnessing $\mathrm{VP}(\kappa, \mathbf{\Sigma}_2)$ (cf. Theorem 1.3.32). In a series of articles, Bagaria motivated axioms all over the large cardinal hierarchy by similar properties, so-called *structural reflection principles* (cf. the survey paper [Bag23]). Note that the above cited result about supercompact cardinals gives us a reflection property for all classes definable by a Σ_2 formula. In [BV16], Bagaria and Väänänen studied reflection principles for classes of structures which are situated lower in the Lévy hierarchy. For this purpose, in this chapter we will denote by R a set-theoretic predicate, i.e., simply a formula in the language of set theory. A formula Φ is called $\Delta_0(R)$ if it is formed using the usual rules for Δ_0 formulas, but additionally is allowed to use R as a primitive formula, i.e., Φ is allowed to form boolean combinations and bounded quantification over formulas involving R. The formula Φ is called $\Sigma_1(R)$ if it is of the form $\Phi = \exists x \Psi$, and it is called $\Pi_1(R)$ if $\Phi = \forall x \Psi$ for some $\Delta_0(R)$ formula, respectively. It is $\Delta_1(R)$ if it is provably equivalent to both a $\Sigma_1(R)$ and a $\Pi_1(R)$ formula.

Recall from Section 1.2.1 that by a *model class* we mean a class of structures in some fixed vocabulary τ. Bagaria and Väänänen studied reflection principles for model classes defined by some $\Sigma_1(R)$ formula for some Π_1-predicate R. Concretely they considered the following statement, which they abbreviated as $\mathrm{SR}_R(\kappa)$ (cf. [BV16, §3]):

> For every proper model class $\mathcal{K}$ such that $\mathcal{K}$ is definable by a $\Sigma_1(R)$ formula without parameters, for any $\mathcal{A} \in \mathcal{K}$ there exists $\mathcal{B} \in \mathcal{K}$ with $|B| < \kappa$ and an elementary embedding $e : \mathcal{B} \to \mathcal{A}$.

If SR_R holds of some cardinal, the least such δ is called the *structural reflection number* of R and we write $\mathrm{SR}_R = \delta$. Bagaria and Väänänen showed that under assumption of *symbiosis* between R and a logic $\mathcal{L}$, the structural reflection number of R is precisely the LST number of $\mathcal{L}$ (recall Definition 1.2.9). Let us next introduce the definitions on the logics' side needed to define the notion of symbiosis.

Recall that sort logic $\mathcal{L}^{s,n}$ can define the model classes which are $\Sigma_n \cup \Pi_n$ definable in the Lévy hierarchy. Research on symbiosis, initiated by Väänänen in [Vää79] is interested in similar connections between definability by a logic and in set theory (and in fact, Väänänen's results on sort logic were obtained in the context of studying symbiosis). For many classical logics, the relevant level of the Lévy hierarchy is situated between Δ_1 and Δ_2, or more concretely, coinciding with the $\Delta_1(R)$ level for some Π_1-predicate R.

Intuitively, symbiosis is then said to hold between a logic $\mathcal{L}$ and the predicate R if the model classes definable in the logic are precisely the $\Delta_1(R)$ definable ones (which are closed under isomorphism).[1] It turns out that for logics relevant in this context, like $\mathcal{L}(\mathsf{I})$, this cannot literally be true. Instead, the logics that achieve this kind of equidefinability are what is known as Δ-closures of logics.

Definition 6.2.1. Let $\mathcal{L}$ be a logic, τ a vocabulary. A class $\mathcal{K}$ of τ-structures is said to be $\Sigma(\mathcal{L})$, or $\Sigma(\mathcal{L})$-*definable* iff there is an expansion $\tau^* \supseteq \tau$ by finitely many symbols and a $\varphi \in \mathcal{L}[\tau^*]$ such that

$$\{\mathcal{A} \restriction \tau \colon \mathcal{A} \models_{\mathcal{L}} \varphi\} = \mathcal{K}.$$

$\mathcal{K}$ is called $\Delta(\mathcal{L})$ or $\Delta(\mathcal{L})$-*definable* iff both $\mathcal{K}$ and the complement of $\mathcal{K}$ are $\Sigma(\mathcal{L})$.

Intuitively, a model class $\mathcal{K}$ of τ-structures is $\Sigma(\mathcal{L})$ if it can be axiomatised by a sentence $\varphi \in \mathcal{L}[\tau^*]$ using some expanded vocabulary $\tau^* \supseteq \tau$. This is not literally true,

[1]Note that we cannot do without the restriction to classes closed under isomorphism by point (iii) of Definition 1.2.1.

as models of φ are τ^*-structures, but instead, $\mathcal{K}$ as required by the definition is the collection of reducts of τ^*-structures which satisfy φ.

Based on this definition, one can define the Δ-*closure* $\Delta(\mathcal{L})$ of $\mathcal{L}$ as the logic which is able to define all model classes which are $\Delta(\mathcal{L})$-definable, i.e., both $\mathcal{K}$ and its complement are definable by some sentence using additional symbols. This is a classic construction (cf., e.g., [BF85, Chapter II, Section X]). The logic $\Delta(\mathcal{L})$ has for every $\Delta(\mathcal{L})$-definable model class $\mathcal{K}$ of τ-structures a sentence $\varphi \in \Delta(\mathcal{L})[\tau]$ axiomatising $\mathcal{K}$.

To be precise, for our official definition of the logic $\Delta(\mathcal{L})$ we refer the reader to [Osi21, Section 2.8], where the construction is carried out, including a concrete syntax. This makes sure that $\Delta(\mathcal{L})$ has the following folklore properties (cf. e.g., [BF85, Chapter II, Theorem 7.2.4] and [Osi21, Section 2.8]; for property (6), cf., e.g., [Osi21, Lemma 4.1.9] for a proof).

Theorem 6.2.2. Let $\mathcal{L}$ be a strong logic with $\text{dep}^*(\mathcal{L}) = \kappa$. Then the following hold:

(1) $\Delta(\mathcal{L})$ is a strong logic.

(2) For every τ, the logic $\Delta(\mathcal{L})$ can define precisely the model classes of τ-structures which are $\Delta(\mathcal{L})$ definable.

(3) $\text{dep}^*(\Delta(\mathcal{L})) = \kappa$.

(4) $\mathcal{L} \leq \Delta(\mathcal{L})$.

(5) $\Delta(\Delta(\mathcal{L})) \equiv \Delta(\mathcal{L})$.

(6) If $\text{comp}(\mathcal{L})$ exists, then $\text{comp}(\mathcal{L}) = \text{comp}(\Delta(\mathcal{L}))$.

When considering definability in set theory, we have to pay attention to whether we are allowing the use of parameters. In our context, if τ is a vocabulary which is not $\Delta_1(R)$ definable, then we cannot hope to find a $\Delta_1(R)$ definition without parameters for a model class $\mathcal{K}$ of τ-structures. Symbiosis therefore naturally splits into the cases considering set-theoretic definability with and set-theoretic definability without parameters. To track which of the two settings we are operating with, following [Osi21, Chapter 3], we will consider the two notions of r-*symbiosis* and p-*symbiosis* (for *restricted* and *parametrised*, respectively). For this, we say that a vocabulary τ is *restricted* iff it is finite and definable by a Δ_1 formula without parameters. The unparametrised version of symbiosis is then formulated for model classes over these types of vocabularies.

Definition 6.2.3. Let $\mathcal{L}$ be a logic and R a predicate of set theory. We say that $\mathcal{L}$ and R are r-*symbiotic* iff the following two conditions are satisfied:

(i) For any restricted vocabulary τ, if $\mathcal{K}$ is an $\mathcal{L}$-definable model class of τ-structures, then $\mathcal{K}$ is $\Delta_1(R)$ definable without parameters.

(ii) For any restricted vocabulary τ, if $\mathcal{K}$ is a model class of τ-structures which is $\Delta_1(R)$ definable without parameters, then $\mathcal{K}$ is $\Delta(\mathcal{L})$-definable.

We collect the prime examples of symbiosis.

Proposition 6.2.4 (Väänänen (cf., e.g., [BV16, Proposition 5.4]))**.** The following pairs of logics $\mathcal{L}$ and predicates R are r-symbiotic.

(1) $\mathcal{L}^2$ and the power set predicate $\mathrm{Pow}(x, y)$, true of sets x and y iff $x = \mathcal{P}(y)$.

(2) $\mathcal{L}(\mathsf{I})$ and Card.

(3) $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ and $\emptyset$.

Note that item (3) means that $\Delta(\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}))$-definability corresponds to Δ_1 definability. The following lemma is useful when checking whether some logic $\mathcal{L}$ and a predicate R are symbiotic. We will further base the definition of p-symbiosis on it. To fix some notation, let us say that for some predicate R, a transitive model $(N, \in)$ is *R-correct* if $R^N = R \cap N$. We fix the class of (models isomorphic to) R-correct models as

$$\mathcal{Q}_R = \{(M, E)\colon \text{there is some transitive } R\text{-correct model } (N, \in) \text{ with } (M, E) \cong (N, \in)\}.$$

Lemma 6.2.5 (Väänänen (cf., e.g., [BV16, Proposition 5.1]))**.** The following are equivalent:

(1) Condition (ii) of the notion of r-symbiosis.

(2) The class $\mathcal{Q}_R$ is $\Delta(\mathcal{L})$-definable.

Let us further state some useful properties of R-correct models: If $(N, \in)$ is transitive and R-correct, then it is easy to see that the usual absoluteness results for formulas in the language of set theory carry over to formulas involving R. Thus, $\Delta_0(R)$ formulas are absolute between N and V; and similarly, $\Sigma_1(R)$, $\Pi_1(R)$, and $\Delta_1(R)$ formulas are upward absolute, downward absolute, and absolute, respectively.

It turns out that for transfer results between LST properties of logics on the one side, and principles of the form $\mathrm{SR}_R(\kappa)$ on the other side, the notion of r-symbiosis is sufficient.

Theorem 6.2.6 (Bagaria & Väänänen [BV16, Theorem 5.5])**.** Let $\mathcal{L}$ be a logic and R a set-theoretic predicate such that $\mathcal{L}$ and R are r-symbiotic and let κ be a cardinal. Then $\mathrm{LST}(\mathcal{L}) = \kappa$ iff $\mathrm{SR}_R = \kappa$.

Motivated by this result, Galeotti, Khomskii and Väänänen were interested in whether a similar transfer result holds between ULST numbers and some type of upward reflection principles. Note that for the Δ-closure of a logic, expanding the vocabulary of a sentence is essential. In particular, we are allowed to add additional sorts to the vocabulary, and so a sentence φ witnessing that a class $\mathcal{K}$ of τ-structures is $\Sigma(\mathcal{L})$ might involve sort symbols not included in τ. In particular, a structure expanding $\mathcal{A}$ and satisfying φ might have additional domains that make it larger than $|A|$. It is for this reason that in general, the Δ-closure of a logic $\mathcal{L}$ does not preserve $\mathrm{ULST}(\mathcal{L})$. To obtain their result, Galeotti, Khomskii and Väänänen therefore used a stronger version of symbiosis called *bounded symbiosis*, which includes an adaption of the Δ-closure called *bounded* Δ-operation Δ^{B} (cf. [Vä83]). The symbiotic logics mentioned in Proposition 6.2.4 are also boundedly symbiotic (cf. [GKV20, Section 5]). The operation Δ^{B} involves bounds on how large a structure can get when enlarging the vocabulary by possible additional sort symbols. To make up for this change, on the set-theoretic side, bounded symbiosis includes reference to the concepts of *definably bounding functions* and $\Sigma_1^F(R)$ formulas, which binds the size of witnesses to a $\Sigma_1(R)$ formula by some definable function F (cf. [GKV20, Definitions 4.3 and 4.5]). With this they could show the following transfer result:

Theorem 6.2.7 (Galeotti, Khomskii & Väänänen [GKV20, Theorem 6.3])**.** Let $\mathcal{L}$ be a logic with $\mathrm{dep}^*(\mathcal{L}) = \omega$ and Δ_0-definable syntax. Let R be a Π_1 predicate. Further, let κ be a cardinal. If $\mathcal{L}$ and R are boundedly symbiotic, then $\mathrm{ULST}(\mathcal{L}) = \kappa$ iff κ is the smallest cardinal such that the following upward reflection principle holds:

> For every definably bounding function F and every model class $\mathcal{K}$ of structures in a restricted vocabulary such that $\mathcal{K}$ is $\Sigma_1^F(R)$ definable without parameters, if there is $\mathcal{A} \in \mathcal{K}$ such that $|A| \geq \kappa$ then for every $\lambda \geq \kappa$ there is $\mathcal{B} \in \mathcal{K}$ with $|B| \geq \lambda$ and an elementary embedding $e : \mathcal{A} \to \mathcal{B}$.

6.3. Symbiosis and compactness properties

The results of Bagaria, Galeotti, Khomskii, and Väänänen left open whether similar results could be obtained for transfer between compactness properties of logics and some set-theoretic reflection principles, and in fact, this was stated as an open problem in [GKV20, Question 8.4]. This question was considered by the author in his Master's Thesis in [Osi21], but only partially answered.

We first discuss the notion of p-symbiosis, the reflection principle for classes of partial orders introduced in [Osi21], and why it fails to give rise to an equivalence to compactness properties. We proceed by introducing a framework of *local definability* in Section 6.3.1. Finally, in Section 6.3.2, we will use local definability to refine the reflection principle from [Osi21] and prove our main result (Theorem 6.3.12), which provides a full equivalence to standard compactness properties, mediated by symbiosis.

Notice that for interesting compactness properties of strong logics, considering large vocabularies is essential. For instance, as $\mathrm{comp}(\mathcal{L}^2)$ is the smallest extendible cardinal κ, the compactness property is only useful for theories over vocabularies of size $\geq \kappa$. For that reason, a notion of symbiosis that relates definability in a logic and in set theory has to make reference to large vocabularies as well. On the set-theoretic side, we therefore have to deal with parameters to be able to talk about any such vocabularies. This is where p-*symbiosis* comes into play.

Definition 6.3.1 (Osinski [Osi21, Definition 3.2.1])**.** Let $\mathcal{L}$ be a logic and R a predicate of set theory. We say that $\mathcal{L}$ and R are p-*symbiotic* iff the following two conditions are fulfilled:

(i) If $\mathcal{K}$ is a model class over the vocabulary τ and definable by $\varphi \in \mathcal{L}[\tau]$, then $\mathcal{K}$ is $\Delta_1(R)$ definable with parameters in $\{\varphi, \tau\}$.

(ii) $\mathcal{Q}_R$ is $\Delta(\mathcal{L})$-definable.

Other variations of symbiosis that allow for the usage of parameters were considered in [Vää79], namely A-*symbiosis* for some class A. The notion allowed to connect definability in a logic to definability with parameters in A. However, this required to restrict the syntax of the logic to consider only vocabularies and sentences of $\mathcal{L}$ which are contained in A. The main result on p-symbiosis is the following:

Theorem 6.3.2 (Osinski [Osi21, Theorem 3.2.2])**.** Let $\mathcal{L}$ be a logic such that $\mathcal{L} \geq \mathcal{L}_{\kappa\omega}$ and $\mathrm{dep}^*(\mathcal{L}) = \kappa$. Let R be a predicate of set theory. If $\mathcal{L}$ and R are p-symbiotic and $\tau \in H_\kappa$ is a vocabulary then the following are equivalent for any model class $\mathcal{K}$ of τ-structures closed under isomorphism:

(1) $\mathcal{K}$ is $\Delta(\mathcal{L})$-definable.

(2) $\mathcal{K}$ is $\Delta_1(R)$ definable with parameters in H_κ.

Notice the similarity between this theorem and the main results about definablitity in sort logic (cf. Corollary 1.2.23). The author's Master's thesis [Osi21] related compactness of a logic $\mathcal{L}$ symbiotic to a predicate R to the existence of specific embeddings, called (R, λ)-embeddings for a cardinal κ, in certain classes of partial orders. We will consider a similar notion below (cf. Definition 6.3.8) and so we will refrain from giving a precise definition of (R, λ)-embedding here. To state the result, we need to fix some (standard) conventions on naming properties of partial orders.

Definition 6.3.3. Suppose τ is a vocabulary including a binary relation symbol $<$ and $(\mathcal{A}, <^{\mathcal{A}})$ and $(\mathcal{B}, <^{\mathcal{B}})$ are τ-structures such that $<^{\mathcal{A}}$ and $<^{\mathcal{B}}$ are partial orders. We say:

(1) A subset $X \subseteq A$ has *an upper bound* iff there is $a \in A$ such that $x <^{\mathcal{A}} a$ for all $x \in X$.

(2) For a cardinal κ, $<^{\mathcal{A}}$ is $<\kappa$*-directed* iff every $X \in \mathcal{P}_\kappa A$ has an upper bound.

(3) If $f : \mathcal{A} \to \mathcal{B}$ is an embedding, we say that $(\mathcal{B}, <^{\mathcal{B}})$ *contains an upper bound for* $(\mathcal{A}, <^{\mathcal{A}})$ iff $f``\mathcal{A}$ has an upper bound $b \in B$.

The result proved in [Osi21] is then:

Theorem 6.3.4 (Osinski [Osi21, Corollary 5.1.3])**.** Let $\mathcal{L}$ be a logic such that $\mathcal{L} \geq \mathcal{L}_{\lambda\omega}$ and $\mathrm{dep}^*(\mathcal{L}) = \lambda$. Let $\kappa \geq \lambda$ be a regular cardinal and assume $\mathcal{L}$ to be p-symbiotic with a predicate R. Assume that $\mathcal{L}$ is κ-compact. Then the following holds:

> For every model class $\mathcal{K}$ of structures in a vocabulary $\tau \in H_\lambda$ containing a binary relation symbol $<$, if $\mathcal{K}$ is $\Sigma_1(R)$ definable with parameters in H_λ and $\mathcal{A} = (A, <^{\mathcal{A}}, \dots) \in \mathcal{K}$ is such that $<^{\mathcal{A}}$ is a $<\kappa$-directed partial order, then there is $\mathcal{B} \in \mathcal{K}$ and an embedding $f : \mathcal{A} \to \mathcal{B}$ such that:
>
> (i) f is an (R, λ)-embedding.
>
> (ii) $(\mathcal{B}, <^{\mathcal{B}})$ contains an upper bound for $(\mathcal{A}, <^{\mathcal{A}})$.

We call this an *end extension principle*, and also write $\mathrm{EEP}^\lambda_\kappa(R)$ for this statement.

The problem when trying to prove a converse result that would show κ-compactness of a logic $\mathcal{L}$ under the assumption of $\mathrm{EEP}^\lambda_\kappa(R)$ for some set-theoretic predicate symbiotic to R is that we are missing the ability to argue about large vocabularies (note that $\mathrm{EEP}^\lambda_\kappa(R)$ restricts attention to classes of structures over vocabularies from H_λ). On the other hand, directly strengthening the statement leads to inconsistency. To be precise, let $\mathrm{EEP}^\lambda_\kappa(R)^*$ be the statement:

> For every model class $\mathcal{K}$ of structures in an *arbitrary* vocabulary τ containing a binary relation symbol $<$, if $\mathcal{K}$ is $\Sigma_1(R)$ definable with parameters in $\{\tau\}$ and $\mathcal{A} = (A, <^\mathcal{A}, \dots) \in \mathcal{K}$ is such that $<^\mathcal{A}$ is a $<\kappa$-directed partial order, then there is $\mathcal{B} \in \mathcal{K}$ and an embedding $f : \mathcal{A} \to \mathcal{B}$ such that:
>
> (i) f is an (R, λ)-embedding.
>
> (ii) $(\mathcal{B}, <^\mathcal{B})$ contains an upper bound for $(\mathcal{A}, <^\mathcal{A})$.

Proposition 6.3.5 (Osinski [Osi21, Proposition 5.5.2]). $\mathrm{EEP}^\lambda_\kappa(R)^*$ is inconsistent for all cardinals λ and κ and any R.

In fact, allowing any vocabularies of size $> \kappa$ leads to inconsistency:

Proposition 6.3.6 (Osinski [Osi21, Proposition 5.5.3]). $\mathrm{EEP}^\lambda_\kappa(R)$ is inconsistent for all cardinals $\lambda > \kappa$ and any R.

6.3.1. Local definability

We will define the principle $\mathrm{EEP}^\lambda_\kappa(R)^+$, stronger than $\mathrm{EEP}^\lambda_\kappa(R)$, that can deal with additional classes of structures, but Propositions 6.3.5 and 6.3.6 show that we have to be careful with this to not run into inconsistencies. We achieve an appropriate definability notion for structures in a large vocabulary τ by considering small bits of τ, which each come with a separate $\Sigma_1(R)$ definition. In this section, we discuss this definability notion.

To state this notion of definability, we have to fix a class of canonical renamings of vocabularies. For this purpose, note that if τ is a vocabulary of size $|\tau| < \lambda$, then there is a renaming $f : \tau \to \tau^*$ to some vocabulary $\tau^* \in H_\lambda$. Let us fix for any vocabulary τ, every cardinal λ and every $\tau_0 \in \mathcal{P}_\lambda\tau$ such a vocabulary $\tau_0^* \in H_\lambda$ and such a renaming $f_{\tau_0} : \tau_0 \to \tau_0^*$. Let us further fix a set of pairwise distinct constant symbols $C_\lambda = \{c_i \colon i < \lambda\} \subseteq H_\lambda$ such that the set $\{c \in H_\lambda \colon c \text{ is a constant symbol and } c \notin C_\lambda\}$ still has size $|H_\lambda|$. We may choose C_λ in a way that $C_\lambda \subseteq C_\mu$ for $\lambda < \mu$. We may further assume the following:

(i) If $c_i \in \tau_0$ for any i, then $f_{\tau_0}(c_i) = c_i$.

(ii) If $x \neq c_i$, then for no f_{τ_0} we have $f_{\tau_0}(x) = c_i$.

(iii) If $\gamma < \lambda$, then $f_{\tau_0 \cup \{c_i \colon i < \gamma\}} \restriction \tau_0 = f_{\tau_0}$.

Let us further call a vocabulary τ *amenable* if $\tau \cap C_\lambda = \emptyset$ for all λ. The purpose of C_λ is to have a fixed set of constant symbols which we do not have to worry about interfering with, when we rename some amenable vocabulary via some f_{τ_0}. Note that any vocabulary may be renamed to an amenable one.

We can now consider the notion of definability we will operate with. Recall that for a renaming $f : \tau \to \tau^*$ and a τ-structure $\mathcal{A}$ we write $f(\mathcal{A})$ for the renamed version of $\mathcal{A}$ to a τ^*-structure.

Definition 6.3.7. Let R be a set-theoretic predicate, λ a cardinal and τ a vocabulary. Let $\mathcal{K}$ be a class of τ-structures. We say that $\mathcal{K}$ is *locally-*λ*-*$\Sigma_1(R)$ *definable* iff there is a collection $\{\mathcal{K}_{\tau_0} \colon \tau_0 \in \mathcal{P}_\lambda\tau\}$ such that for any $\tau_0 \in \mathcal{P}_\lambda\tau$, $\mathcal{K}_{\tau_0}$ is a $\Sigma_1(R)$ definable class of τ_0^*-structures and for all τ-structures $\mathcal{A}$:

$$\mathcal{A} \in \mathcal{K} \text{ iff } \forall \tau_0 \in \mathcal{P}_\lambda\tau(f_{\tau_0}(\mathcal{A} \restriction \tau_0) \in \mathcal{K}_{\tau_0})$$

Note that because the global satisfaction relation for $\Sigma_1(R)$ formulas is definable by a single formula in the language of set theory, also the above definition can be cast as a definition by a single formula of ZFC.

The idea behind a class being locally-λ-$\Sigma_1(R)$ definable is the following. When studying whether some class is definable in a logic $\mathcal{L}$, we distinguish between definability by a sentence $\varphi \in \mathcal{L}$ or by a theory $T \subseteq \mathcal{L}$. These two notions can differ, and it can happen that $\mathrm{Mod}(T)$ is not definable by a single sentence. When studying definability in set theory, we are usually only concerned with definability by a single formula $\Phi(x, p)$, potentially with a parameter p. But note that for compactness properties of a logic $\mathcal{L}$, considering $\mathcal{L}$-theories is crucial. Local definability as above allows us to consider classes which are defined by a collection of formulas in the language of set theory, mirroring definability by a theory in a logic. It is this property that will make the notion fruitful when studying transfer between compactness properties of logics and set-theoretic reflection principles.

We can now introduce the notion of $(R, \lambda)^+$-embeddings, which we will need alongside Lemma 6.3.9 for the statement and proof of our main result (Theorem 6.3.12), respectively. Lemma 6.3.9 also shows that the introduced notion is natural by showing that for a logic $\mathcal{L}$ symbiotic to R, $(R, \lambda)^+$-embeddings correspond precisely to $\Delta(\mathcal{L})$-elementary embeddings.

Definition 6.3.8. Let R be a set-theoretic prediacte, λ a cardinal and τ an amenable vocabulary. Consider τ-structures $\mathcal{A}$ and $\mathcal{B}$ and an embedding $e : \mathcal{A} \to \mathcal{B}$. We say that e is an $(R, \lambda)^+$*-embedding* iff for every $\tau_0 \in \mathcal{P}_\lambda\tau$ and every $\gamma < \lambda$, if $\mathcal{K}$ is a model class of $\tau_0^* \cup \{c_i \colon i < \gamma\}$-structures which is $\Delta_1(R)$ definable with parameters in H_λ, then for all possible interpretations $c_i^{\mathcal{A}}$ of the constants c_i by elements of $\mathcal{A}$:

$$(f_{\tau_0}(\mathcal{A} \restriction \tau_0), c_i^{\mathcal{A}})_{i<\gamma} \in \mathcal{K} \text{ iff } (f_{\tau_0}(\mathcal{B} \restriction \tau_0), e(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathcal{K}.$$

Lemma 6.3.9. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures for some amenable vocabulary τ and $e : A \to B$ a map. Let $\mathcal{L}$ be a logic with $\mathrm{dep}^*(\mathcal{L}) = \lambda$, $\mathcal{L} \geq \mathcal{L}_{\lambda\omega}$, R a set-theoretic predicate and assume that R and $\mathcal{L}$ are p-symbiotic. Then the following are equivalent:

(1) e is an $(R,\lambda)^+$-embedding.

(2) e is a $\Delta(\mathcal{L})$-elementary embedding.

For its proof, we will use the following lemma, which immediately follows as a special case of [Osi21, Lemma 4.2.3].

Lemma 6.3.10 (Osinski)**.** Let $\mathcal{L}$ be a logic such that $\mathrm{dep}^*(\mathcal{L}) \leq \lambda$. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures for some amenable vocabulary τ and $e : A \to B$ a map. Then the following are equivalent:

(1) e is an $\mathcal{L}$-elementary embedding.

(2) For all $\gamma < \lambda$, if $c_i^{\mathcal{A}}$ is an interpretation of the constants $\{c_i \colon i < \gamma\}$ by elements of A, then for all $\varphi \in \mathcal{L}[\tau \cup \{c_i \colon i < \gamma\}]$:

$$(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \models \varphi \text{ iff } (\mathcal{B}, e(c_i^{\mathcal{A}}))_{i<\gamma} \models \varphi.$$

Proof of Lemma 6.3.9. Suppose e is an $(R,\lambda)^+$-embedding. By Lemma 6.3.10, it is sufficient to show that for any interpretation $c_i^{\mathcal{A}}$ of the constants $\{c_i \colon i < \gamma\}$ for some $\gamma < \lambda$, and any $\Delta(\mathcal{L})$-sentence φ over $\tau \cup \{c_i \colon i < \gamma\}$:

$$(\mathcal{A}, c_i^{\mathcal{A}})_{i<\gamma} \in \mathrm{Mod}(\varphi) \text{ iff } (\mathcal{B}, e(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathrm{Mod}(\varphi).$$

Recall that by $\mathrm{dep}^*(\mathcal{L}) = \lambda$, also $\mathrm{dep}^*(\Delta(\mathcal{L})) = \lambda$ (cf. Theorem 6.2.2), and so φ is really a sentence over $\tau_1 = \tau_0 \cup \{c_i \colon i < \gamma\}$ for a $\tau_0 \in \mathcal{P}_\lambda \tau$. Thus, it is sufficient to show:

$$(\mathcal{A} \restriction \tau_0, c_i^{\mathcal{A}})_{i<\gamma} \in \mathrm{Mod}(\varphi) \text{ iff } (\mathcal{B} \restriction \tau_0, e(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathrm{Mod}(\varphi).$$

We may rename φ via f_{τ_1} to a sentence $\psi \in \Delta(\mathcal{L})[\tau_1^*]$, where $\tau_1^* \in H_\lambda$. Then applying the renaming to the above equivalence, it suffices to show:

$$f_{\tau_1}((\mathcal{A} \restriction \tau_0, c_i^{\mathcal{A}})_{i<\gamma}) \in \mathrm{Mod}(\psi) \text{ iff } f_{\tau_1}((\mathcal{B} \restriction \tau_0, e(c_i^{\mathcal{A}}))_{i<\gamma}) \in \mathrm{Mod}(\psi).$$

Because $\tau_1 = \tau_0 \cup \{c_i \colon i < \gamma\}$, by assumption on the renamings f_σ, we have $f_{\tau_1} \restriction \tau_0 = f_{\tau_0}$, and $f_{\tau_1}(c_i) = c_i$. Therefore $f_{\tau_1}((\mathcal{A} \restriction \tau_0, c_i^{\mathcal{A}})_{i<\gamma}) = (f_{\tau_0}(\mathcal{A} \restriction \tau_0), c_i^{\mathcal{A}})_{i<\gamma}$, and analogously for $\mathcal{B}$. Again, applying this to the equivalence above, it is sufficient to show:

$$(f_{\tau_0}(\mathcal{A} \restriction \tau_0), c_i^{\mathcal{A}})_{i<\gamma} \in \mathrm{Mod}(\psi) \text{ iff } (f_{\tau_0}(\mathcal{B} \restriction \tau_0), e(c_i^{\mathcal{A}}))_{i<\gamma} \in \mathrm{Mod}(\psi).$$

But by p-symbiosis, Theorem 6.3.2, and because $\tau_1^* \in H_\lambda$, $\mathrm{Mod}(\psi)$ is $\Delta_1(R)$ definable with parameters in H_λ. Hence, since e is an $(R,\lambda)^+$-embedding, the above equivalence holds.

We can argue for the other direction by reversing the above argument. So assume that e is a $\Delta(\mathcal{L})$-embedding. Let $\tau_0 \in \mathcal{P}_\lambda \tau$ and fix some interpretation $c_i^{\mathcal{A}}$ of the constants $\{c_i \colon i < \gamma\}$ for some $\gamma < \lambda$. Let τ_1^* be the renamed version of $\tau_1 = \tau_0 \cup \{c_i \colon i < \gamma\}$

via f_{τ_1} and $\mathcal{K}$ some $\Delta_1(R)$-definable, with parameters in H_λ, class of τ_1-structures. By definition, for e to be an $(R,\lambda)^+$-embedding, we want to show:

$$(f_{\tau_0}(\mathcal{A}\restriction\tau_0), c_i^{\mathcal{A}})_{i<\gamma}\in\mathcal{K} \text{ iff } (f_{\tau_0}(\mathcal{B}\restriction\tau_0), e(c_i^{\mathcal{A}}))_{i<\gamma}\in\mathcal{K}.$$

Again, by p-symbiosis and Theorem 6.3.2, $\mathcal{K}=\mathrm{Mod}(\psi)$ for some $\psi\in\Delta(\mathcal{L})[\tau_1^*]$. Thus we get that it suffices to show:

$$(f_{\tau_0}(\mathcal{A}\restriction\tau_0), c_i^{\mathcal{A}})_{i<\gamma}\in\mathrm{Mod}(\psi) \text{ iff } (f_{\tau_0}(\mathcal{B}\restriction\tau_0), e(c_i^{\mathcal{A}}))_{i<\gamma}\in\mathrm{Mod}(\psi).$$

By the same argument as before, using the properties of the renamings, this is equivalent to:

$$f_{\tau_1}((\mathcal{A}\restriction\tau_0, c_i^{\mathcal{A}})_{i<\gamma})\in\mathrm{Mod}(\psi) \text{ iff } f_{\tau_1}((\mathcal{B}\restriction\tau_0, e(c_i^{\mathcal{A}}))_{i<\gamma})\in\mathrm{Mod}(\psi).$$

Let φ be $f_{\tau_1}^{-1}(\psi)$. Then $\varphi\in\Delta(\mathcal{L})[\tau_1]$. Then pulling the above equivalence back along the renaming f_{τ_1}, the following is sufficient to show:

$$(\mathcal{A}\restriction\tau_0, c_i^{\mathcal{A}})_{i<\gamma}\in\mathrm{Mod}(\varphi) \text{ iff } (\mathcal{B}\restriction\tau_0, e(c_i^{\mathcal{A}}))_{i<\gamma}\in\mathrm{Mod}(\varphi).$$

But this holds because of Lemma 6.3.10 and because e is a $\Delta(\mathcal{L})$-embedding. □

6.3.2. Upward reflection in classes of partial orders and compactness

We consider the following strengthening of $\mathrm{EEP}_\kappa^\lambda(R)$.

Definition 6.3.11. Let R be a set-theoretic predicate and λ and κ be cardinals. $\mathrm{EEP}_\kappa^\lambda(R)^+$ is the statement:

> For every model class $\mathcal{K}$ of structures in an amenable vocabulary τ containing a binary relation symbol $<$ such that $\mathcal{K}$ is locally-λ-$\Sigma_1(R)$ definable, if $\mathcal{A}\in\mathcal{K}$ and $<^{\mathcal{K}}$ is a $<\kappa$-directed partial order, then there is $\mathcal{B}\in\mathcal{K}$ such that:
>
> (i) f is an $(R,\lambda)^+$-embedding.
>
> (ii) $(\mathcal{B},<^{\mathcal{B}})$ contains an upper bound for $(\mathcal{A},<^{\mathcal{A}})$.

Notice that every class of structures in a vocabulary $\tau\in H_\lambda$, which is $\Sigma_1(R)$ definable with parameters in H_λ, is also locally-λ-$\Sigma_1(R)$ definable. In particular, $\mathrm{EEP}_\kappa^\lambda(R)^+$ implies $\mathrm{EEP}_\kappa^\lambda(R)$.

This principle gives the desired equivalence:

Theorem 6.3.12. Let $\mathcal{L}$ be a logic and R a set-theoretic predicate. Assume $\mathcal{L}\geq\mathcal{L}_{\lambda\omega}$, $\mathrm{dep}^*(\mathcal{L})=\lambda$ and that $\mathcal{L}$ and R are p-symbiotic. Then the following are equivalent for a regular cardinal κ:

(1) $\mathcal{L}$ is κ-compact.

(2) $\mathrm{EEP}_\kappa^\lambda(R)^+$.

For the proof, we will use the following lemma by Väänänen, proven in [Osi21, Theorem 4.4.3].

Lemma 6.3.13 (Väänänen)**.** Let $\mathcal{L}$ be a logic and κ a regular cardinal. Then the following are equivalent:

(1) $\mathcal{L}$ is κ-compact.

(2) If τ is a vocabulary including a binary relation symbol $<$ and $\mathcal{A}$ is a τ-structure such that $<^{\mathcal{A}}$ is a $<\kappa$-directed partial order, then there is a τ-structure $\mathcal{B}$ and an $\mathcal{L}$-elementary embedding $f : \mathcal{A} \to \mathcal{B}$ such that $(\mathcal{B}, <^{\mathcal{B}})$ contains an upper bound for $(\mathcal{A}, <^{\mathcal{A}})$.

Proof of Theorem 6.3.12. The direction from (2) to (1) is pretty much immediate, so let us start with it. Assume that $\mathrm{EEP}^{\lambda}_{\kappa}(R)^{+}$ holds. To get κ-compactness of $\mathcal{L}$, by Lemma 6.3.13, it is sufficient to find, given a τ-structure $\mathcal{A}$ with $<^{\mathcal{A}}$ a $<\kappa$-directed partial order, an $\mathcal{L}$-elementary embedding $f : \mathcal{A} \to \mathcal{B}$ into a structure that contains an upper bound for $<^{\mathcal{A}}$. We may assume that τ is amenable, as we can achieve this by renaming. Consider any class $\mathcal{K}$ containing $\mathcal{A}$ that is locally-λ-$\Sigma_1(R)$ definable, say the class where $<$ is interpreted as a partial order. By $\mathrm{EEP}^{\lambda}_{\kappa}(R)^{+}$, then $\mathcal{K}$ contains a structure $\mathcal{B}$ such that there is an $(R, \lambda)^{+}$-embedding $f : \mathcal{A} \to \mathcal{B}$ and $\mathcal{B}$ has an upper bound for $<^{\mathcal{A}}$. By Lemma 6.3.9, f is also a $\Delta(\mathcal{L})$- and thus an $\mathcal{L}$-elementary embedding.

And now suppose that $\mathcal{L}$ is κ-compact. Then also $\Delta(\mathcal{L})$ is κ-compact (cf. Theorem 6.2.2). Take a set up of $\mathrm{EEP}^{\lambda}_{\kappa}(R)^{+}$, where that $\mathcal{K}$ is locally-λ-$\Sigma_1(R)$ definable is witnessed by a collection $\{\Phi_{\tau_0}(x, p_{\tau_0})\colon \tau_0 \in \mathcal{P}_\lambda\tau\}$ of $\Sigma_1(R)$ formulas Φ_{τ_0} and parameters $p_{\tau_0} \in H_\lambda$. Assume without loss of generality that $<\in H_\lambda$ and that $f_{\tau_0}(<) =<$ for all $\tau_0 \in \mathcal{P}_\lambda\tau$. Take new binary relation symbols E and $\prec$. For every $\tau_0 \in \mathcal{P}_\lambda\tau$ construct a sentence φ_{τ_0} of $\Delta(\mathcal{L})$ in the following way, using E to write down all set-theoretic expressions. Let $\chi_{\tau_0}(x)$ be an $\mathcal{L}_{\lambda\omega}$-formula that says that x is an $f_{\tau_0}(\tau_0)$-structure (where $f_{\tau_0}(\tau_0) \in H_\lambda$ is hard coded into the sentence by usage of $\mathcal{L}_{\lambda\omega}$). Add a constant c_{τ_0}. By p-symbiosis, take a sentence φ_R of $\Delta(\mathcal{L})$ that axiomatises the class of (models isomorphic to) transitive R-correct models. Let φ_{τ_0} be the conjunction of the following sentences:

(i) φ_R.

(ii) $\chi_{\tau_0}(c_{\tau_0})$.

(iii) $\Phi_{\tau_0}(c_{\tau_0}, p_{\tau_0})$.

For (iii), note that $p_{\tau_0} \in H_\lambda$ so is definable by a $\Delta(\mathcal{L})$-formula. If $<\in \tau_0$ add the following conjuncts to φ_{τ_0}.

(iv) $\prec$ is a partial order such that $\{x\colon x \in c_{\tau_0}\}$ is cofinal in it.

(v) $\prec\restriction c_{\tau_0} =<^{c_{\tau_0}}$.

Further, for every pair $\tau_0, \tau_1 \in \mathcal{P}_\lambda\tau$, let ψ_{τ_0,τ_1} be the $\Delta(\mathcal{L})$-sentence that is the conjunct of the following:

(vi) “The universe of c_{τ_0} and c_{τ_1} coincide.”

(vii) $\bigwedge_{r\in\tau_0\cap\tau_1}$ “the interpretation of $(f_{\tau_0}(r))^{c_{\tau_0}}$ and $(f_{\tau_1}(r))^{c_{\tau_1}}$ coincide.”

For (vii), note that for any r, $f_{\tau_i}(r) \in H_\lambda$ and so is $\mathcal{L}_{\lambda\omega}$-definable.

Now let $T = \{\varphi_{\tau_0} : \tau_0 \in \mathcal{P}_\lambda\tau\} \cup \{\psi_{\tau_0,\tau_1} : \tau_0, \tau_1 \in \mathcal{P}_\lambda\tau\}$. Take m large enough such that every $\Sigma_1(R)$ formula is Σ_m. By the Reflection Theorem, take $\alpha > \max(\mathrm{rk}(\mathcal{A}), \lambda)$ such that $V_\alpha \prec_{\Sigma_m} V$. Note that the $\Sigma_1(R)$ formulas are then absolute between V and V_α and, in particular, V_α is R-correct. Take the structure M with universe V_α and interpret $c^M_{\tau_0}$ as $f_{\tau_0}(\mathcal{A} \restriction \tau_0)$ and $\prec^M$ as any $<\kappa$-directed partial order extending $<^{\mathcal{A}}$ in which $<^{\mathcal{A}}$ is cofinal. Then clearly $M \models T$.

By Lemma 6.3.13 and κ-compactness of $\Delta(\mathcal{L})$, there is a structure N and a $\Delta(\mathcal{L})$-elementary embedding $e : M \to N$ such that $(N, \prec^N)$ contains an upper bound for $(M, \prec^M)$. Because e is a $\Delta(\mathcal{L})$-elementary embedding, $N \models T$. Then $N \models \varphi_R$, so it is well-founded and by collapsing we can without loss of generality assume that N is transitive and R-correct. As $\Delta(\mathcal{L}) \geq \mathcal{L}_{\lambda\omega}$, e fixes all ordinals $< \lambda$ and so $e \restriction M \cap V_\lambda = \mathrm{id}$. By ψ_{τ_0,τ_1}, the universes of the structures $c^N_{\tau_0}$ for $\tau_0 \in \mathcal{P}_\lambda\tau$ are all the same set B and the interpretations of the symbols in $\tau_0 \cap \tau_1$ coincide, so we can define a τ-structure $\mathcal{B}$ on B by interpreting $r \in \tau$ by $(f_{\tau_0}(r))^{c^N_{\tau_0}}$ for any $\tau_0 \in \mathcal{P}_\lambda\tau$ with $r \in \tau_0$. Clearly, then $f_{\tau_0}(\mathcal{B} \restriction \tau_0) = c^N_{\tau_0}$. Now because $N \models \Phi_{\tau_0}(c^N_{\tau_0}, p_{\tau_0})$, as N is R-correct, Φ_{τ_0} is upwards absolute from N to V as a $\Sigma_1(R)$ formula, and so $\Phi_{\tau_0}(c^N_{\tau_0}, p_{\tau_0})$ really holds. This shows by local-λ-$\Sigma_1(R)$ definability of $\mathcal{K}$ that $\mathcal{B} \in \mathcal{K}$. Because by (iv) and (v) of φ_{τ_0}, $<^{\mathcal{B}}$ is cofinal in $\prec^N$, and further $\prec^N$ contains an upper bound for $\prec^M \supseteq <^{\mathcal{A}}$, we have that $<^{\mathcal{B}}$ contains an upper bound for $<^{\mathcal{A}}$.

We further claim that $e \restriction A : A \to B$ is a $\Delta(\mathcal{L})$-elementary embedding of the τ-structures $\mathcal{A}$ and $\mathcal{B}$. This will finish the proof as then by Lemma 6.3.9, e is an $(R, \lambda)^+$-embedding, as τ is amenable by assumption. Because $\mathrm{dep}^*(\mathcal{L}) \leq \lambda$, every τ-formula $\varphi(\bar{x})$ of $\Delta(\mathcal{L})$ comes from $\Delta(\mathcal{L})[\tau_0]$ for a $\tau_0 \in \mathcal{P}_\lambda\tau$. Furthermore,

$$\mathcal{A} \models \varphi(\bar{a}) \text{ iff } f_{\tau_0}(\mathcal{A} \restriction \tau_0) \models f_{\tau_0}(\varphi)(\bar{a}), \tag{$*$}$$

where f_{τ_0} is the canonical renaming of φ to a $f_{\tau_0}(\tau_0)$-formula in H_λ and similar for $\mathcal{B}$. Note that e fixes $f_{\tau_0}(\varphi) \in H_\lambda$, since $e \restriction M \cap V_\lambda = \mathrm{id}$. Now

$$\begin{aligned}
\mathcal{A} \models \varphi(\bar{a}) &\text{ iff } f_{\tau_0}(\mathcal{A} \restriction \tau_0) \models f_{\tau_0}(\varphi)(\bar{a}) \\
&\text{ iff } M \models \text{“}c_{\tau_0} \models f_{\tau_0}(\varphi)(\bar{a})\text{”} \\
&\text{ iff } N \models \text{“}c_{\tau_0} \models f_{\tau_0}(\varphi)(e(\bar{a}))\text{”} \\
&\text{ iff } f_{\tau_0}(\mathcal{B} \restriction \tau_0) \models f_{\tau_0}(\varphi)(e(\bar{a})) \\
&\text{ iff } \mathcal{B} \models \varphi(e(\bar{a})).
\end{aligned}$$

The first and last “iff” hold by the comment about renamings above (cf. $(*)$). For the second “iff” recall that $f_{\tau_0}(\mathcal{A} \restriction \tau_0) = c^M_{\tau_0}$, and further, by p-symbiosis, the model class $\mathrm{Mod}(f_{\tau_0}(\varphi))$ is $\Delta_1(R)$. Because M is R-correct, satisfaction of $f_{\tau_0}(\varphi)$ is therefore absolute between M and V. The argument for the fourth “iff” is completetely analogous. The middle “iff” holds because e is a an elementary embedding between M and N. □

6.4. Weak structural reflection principles and SLST numbers

Bagaria and Väänänen also considered weaker structural reflection principles than SR_R. In particular, again for some set-theoretic predicate R, they considered the principle $\mathrm{SR}_R^-(\kappa)$ (cf. [BV16, Section 3.2]):

> For every proper model class $\mathcal{K}$ such that $\mathcal{K}$ is definable by a $\Sigma_1(R)$ formula without paramters, for any $\mathcal{A} \in \mathcal{K}$ *of size exactly* κ, there exists $\mathcal{B} \in \mathcal{K}$ with $|B| < \kappa$ and an elementary embedding $e : \mathcal{B} \to \mathcal{A}$.

We also write SR_R^- to indicate that $\mathrm{SR}_R^-(\kappa)$ is true for some cardinal κ. Bagaria and Väänänen further defined a Löwenheim-Skolem property that is designed to match $\mathrm{SR}_R^-(\kappa)$ on the logics' side.

Definition 6.4.1 (Bagaria & Väänänen [BV16, Definition 8.2])**.** Let $\mathcal{L}$ be a logic. A cardinal κ is called the *strict Löwenheim-Skolem-Tarski* (SLST) number of $\mathcal{L}$ iff it is the smallest cardinal such that for any $\varphi \in \mathcal{L}[\tau]$, where τ is a vocabulary of size $< \kappa$, if $\mathcal{A}$ is a τ-structure *of size exactly* κ such that $\mathcal{A} \models_{\mathcal{L}} \varphi$, then there is a substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \models_{\mathcal{L}} \varphi$ and $|B| < \kappa$.

However, a precise correspondence between SR_R^- and the existence of $\mathrm{SLST}(\mathcal{L})$ for symbiotic $\mathcal{L}$ and R is not possible (as noted in [Bag23, Footnote 6, p. 55]). For example, Lücke showed in [Lüc22, Theorem 1.5] that $\mathrm{SR}_{\mathrm{Card}}^-$ is equiconsistent to the existence of a *shrewd* cardinal (see Definition 6.4.2). On the other hand, Bagaria and Väänänen show that if κ is weakly inaccessible, then κ is an SLST number of $\mathcal{L}(\mathsf{I})$ (cf. [BV16, Theorem 8.4] or Theorem 6.4.8 below). Because shrewd cardinals are, for instance, weakly compact, $\mathrm{SR}_{\mathrm{Card}}^-$ has higher consistency strength than a weakly inaccessible cardinal, and thus, in particular, cannot be equivalent to the existence of $\mathrm{SLST}(\mathcal{L}(\mathsf{I}))$.

In [Bag23] though, Bagaria considers what he calls $\Sigma_1(R)^*$ *classes*, which are $\Sigma_1(R)$ definable classes with the additional requirement that membership in them can be witnessed by transitive sets which are bounded in their size (cf. Definition 6.4.9). He states that for symbiotic $\mathcal{L}$ and R, an equivalence can be shown between the existence of an SLST number of $\mathcal{L}$ and the statement of SR_R^- restricted to $\Sigma_1(R)^*$-classes, but omits a proof (cf. [Bag23, Section 6.2]).

Our aim in this section is to give a proof of Bagaria's statement under an additional assumption that the logic $\mathcal{L}$ can define the predicate R in a similarly strong sense (cf. Theorem 6.4.13). This proof will be given in Section 6.4.2. To give some background why SR_R^- is stronger than the existence of $\mathrm{SLST}(\mathcal{L})$, in Section 6.4.1 we first present some of the machinery on shrewd cardinals Lücke developed to deal with the principle SR_R^-. We will employ this (cf. Proposition 6.4.8) to show how it can be used in a new proof that weakly inaccessibles are SLST numbers of $\mathcal{L}(\mathsf{I})$.

6.4.1. Weak forms of shrewdness

The basic notion is as follows.

Definition 6.4.2 (Rathjen [Rat05]). A cardinal κ is called *shrewd* if for every formula $\varphi(x,y)$ in the language of set theory, every ordinal $\alpha > 0$ and every $A \subseteq V_\kappa$ such that $V_{\kappa+\alpha} \models \varphi(A,\kappa)$, there exist ordinals $\bar{\kappa} < \bar{\alpha} < \kappa$ such that $V_{\bar{\kappa}+\bar{\alpha}} \models \varphi(A \cap V_{\bar{\kappa}}, \bar{\kappa})$.

The definition directly implies that shrewd cardinals are totally indescribable (cf. [Kan03, Section 6]) and therefore in particular weakly compact (cf. [Kan03, Theorem 6.4]). They are thus stronger than weakly inaccessible cardinals. Consider the following result of Lücke already mentioned.

Theorem 6.4.3 (Lücke [Lüc22, Theorem 1.5]). ZFC + "there exists a shrewd cardinal" is equiconsistent to $\mathrm{ZFC} + \exists\kappa \mathrm{SR}^-_{\mathrm{Card}}(\kappa)$.

Lücke introduced the following weakening of shrewd cardinals:

Definition 6.4.4 (Lücke [Lüc22, Definition 1.6]). A cardinal κ is called *weakly shrewd* if for every formula $\varphi(x,y)$ in the language of set theory, every cardinal $\theta > \kappa$ and every $A \subseteq \kappa$ such that $H_\theta \models \varphi(A,\kappa)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ such that $\bar{\kappa} < \kappa$ and $H_{\bar{\theta}} \models \varphi(A \cap \bar{\kappa}, \bar{\kappa})$.

Lücke showed that shrewd cardinals are weakly shrewd (cf. [Lüc22, Corollary 3.2]), and that the two notions are equiconsistent (cf. [Lüc22, Corollary 1.8]), but that weakly shrewd cardinals do not need to be shrewd (cf. [Lüc22, Theorem 1.9]). Weakly shrewd cardinals are directly related to the principle SR^-_R.

Theorem 6.4.5 (Lücke [Lüc22, Theorem 1.7]). The following are equivalent for every cardinal κ:

(1) κ is the least weakly shrewd cardinal.

(2) κ is the smallest cardinal such that for every proper class $\mathcal{K}$ of structures in the same vocabulary, if $\mathcal{K}$ is Σ_2 definable with parameters in H_κ, then for any $\mathcal{A} \in \mathcal{K}$ of size $|A| = \kappa$, there exists $\mathcal{B} \in \kappa$ with $|B| < \kappa$ and an elementary embedding $e : \mathcal{B} \to \mathcal{A}$.

Note that item (2) implies that SR^-_R holds for every Π_1 predicate R. Combining Lücke's results, the consistency of $\mathrm{SR}^-_{\mathrm{Card}}$ thus implies the consistency of SR^-_R for any Π_1 predicate R. And for predicates R such that Card is $\Sigma_1(R)$ definable, the consistency strength of SR^-_R is precisely given by that of the existence of a shrewd cardinal. In particular, we get that the consistency strength of $\mathrm{SR}^-_{\mathrm{Card}}$ is higher than that of, e.g., a weakly compact cardinal. Theorem 6.4.8 below shows though that a weakly inaccessible implies the existence of $\mathrm{SLST}(\mathcal{L}(\mathsf{I}))$. In particular, the existence of $\mathrm{SLST}(\mathcal{L}(\mathsf{I}))$ and $\mathrm{SR}^-_{\mathrm{Card}}$ cannot be equivalent.

SLST numbers of symbiotic logics, are closely related to much weaker forms of weakly shrewd cardinals. We consider the special case $\mathcal{L}(\mathsf{I})$. In the following we consider an expanded language $\{\in, \dot{R}\}$ which has an additional predicate $\dot{R}$ with the same arity as a given set-theoretic predicate R. We say that a formula is $\Sigma_1(\dot{R})$ if it is a Σ_1 formula in this expanded language. Note that the difference to $\Sigma_1(R)$ formulas is, that the latter uses R written out as a formula in the language of set theory. The weak form of shrewdness we will consider is the following.

Definition 6.4.6 (Lücke [Lüc22, Definition 5.1]). Let R be a set-theoretic predicate and θ a cardinal. A cardinal $\kappa < \theta$ is called *weakly* (Σ_1, R, θ)*-shrewd* iff for every $\Sigma_1(\dot{R})$ formula $\varphi(x, y)$ and every $A \subseteq \kappa$ such that $(H_\theta, \in, R \cap H_\theta) \models \varphi(A, \kappa)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ such that $\bar{\kappa} < \kappa$ and $(H_{\bar{\theta}}, \in, R \cap H_{\bar{\theta}}) \models \varphi(A \cap \kappa, \kappa)$.

We will use the following two facts about weakly (Σ_1, R, θ)-shrewd cardinals. Note that the first one is a weak analogue of Magidor's characterisation of supercompact cardinals (cf., e.g., [Kan03, Theorem 22.10]).

Theorem 6.4.7 (Lücke [Lüc22]). Let R be a set-theoretic predicate and $\kappa < \theta$ cardinals.

(i) A cardinal κ is weakly (Σ_1, R, θ)-shrewd iff for every $z \in H_\theta$ there are cardinals $\bar{\kappa} < \bar{\theta}$ and $X \subseteq H_{\bar{\theta}}$ such that there is an elementary embedding $j : (X, \in, X \cap R) \to (H_\theta, \in, H_\theta \cap R)$ and $(X, \in, X \cap R) \prec_{\Delta_0} (H_{\bar{\theta}}, \in, R \cap H_{\bar{\theta}})$ with $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $z \in \mathrm{ran}(j)$.

(ii) κ is weakly inaccessible iff it is weakly $(\Sigma_1, \mathrm{Card}, \kappa^+)$-shrewd.

Note that the X (and H_θ and $H_{\bar{\theta}}$) from (i) is closed under pairing, so if R is an n-ary predicate, then $X \cap R = X^n \cap R$. Combining the above facts gives a new proof of Bagaria's and Väänänen's result about weakly inaccessibles being SLST numbers of $\mathcal{L}(\mathsf{I})$ (cf. [BV16, Theorem 8.4]). In fact, we give a slight improvement of Bagaria's and Väänänen's result by showing that a weakly inaccessible gives rise to the following a priori slightly stronger version of the SLST number.

> Let $\mathcal{L}$ be a logic. A cardinal κ is called the SLST^+ number of $\mathcal{L}$ iff it is the smallest cardinal such that for any $\varphi \in \mathcal{L}[\tau]$, where τ is a vocabulary of size $< \kappa$, if $\mathcal{A}$ is a τ-structure *of size exactly* κ such that $\mathcal{A} \models_{\mathcal{L}} \varphi$, then there is a first-order *elementary* substructure $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} \models_{\mathcal{L}} \varphi$ and $|B| < \kappa$.

The difference to the SLST number is the requirement that $\mathcal{B}$ is an *elementary* substructure. We will see below that our main result Theorem 6.4.13 implies that actually $\mathrm{SLST}(\mathcal{L}) = \mathrm{SLST}^+(\mathcal{L})$ for a large class of logics.

Proposition 6.4.8. If κ is weakly inaccessible, then $\mathrm{SLST}^+(\mathcal{L}(\mathsf{I})) \leq \kappa$.

Proof. Let $\varphi \in \mathcal{L}(\mathsf{I})[\tau]$ and $\mathcal{A} \models \varphi$ with $|A| = \kappa$ for κ weakly inaccessible. We can assume that $\tau \in H_\kappa$. Because κ is a limit, then $\tau \in H_\gamma$ for some $\gamma < \kappa$. By definition of the syntax of $\mathcal{L}(\mathsf{I})$ (cf. Appendix A), this implies that $\mathcal{L}(I)[\tau] \subseteq H_\gamma$, and so $\mathrm{tcl}(\{\varphi\}) \in H_\gamma$. Therefore, we get $\mathrm{subf}(\varphi) \in H_\gamma$, where $\mathrm{subf}(\varphi)$ is the set of subformulas of φ. Note that $\mathrm{subf}(\varphi)$ is finite. We can further assume that the domain $A = \kappa$. In particular, then $\mathcal{A} \in H_{\kappa^+}$. By item (ii) of Lemma 6.4.7, κ is weakly $(\Sigma_1, \mathrm{Card}, \kappa^+)$-shrewd, and thus in particular weakly $(\Sigma_1, \emptyset, \kappa^+)$-shrewd. Then item (i) gives us cardinals $\bar{\kappa} < \bar{\theta}$ and a set X with $X \prec_{\Delta_0} H_{\bar{\theta}}$ and an elementary embedding $j : (X, \in) \to (H_{\kappa^+}, \in)$ such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $\mathcal{A}, \tau \in \mathrm{ran}(j)$ and $\mathrm{subf}(\varphi) \subseteq \mathrm{ran}(j)$.

We claim that X is correct about cardinals: If $\alpha \leq \bar{\kappa}$, then $X \models$ "α is a cardinal" iff $H_{\kappa^+} \models$ "$j(\alpha)$ is a cardinal". Then as j is the identity on $\bar{\kappa}$ and H_{κ^+} is correct about cardinals, X is correct for cardinals $< \bar{\kappa}$. Now for $\bar{\kappa}$ itself, it is actually a cardinal, and X also believes so, as $j(\bar{\kappa}) = \kappa$. Furthermore, X believes that $\bar{\kappa}$ is the largest cardinal by elementarity of j. As $\bar{\kappa} + 1 \subseteq X$, the only elements of X that it can possibly believe to be cardinals are ordinals $\leq \bar{\kappa}$. For those we just argued, that X is correct about them being cardinals or not. So X really is correct about cardinals.

Recall that $\varphi \in \operatorname{ran}(j)$. Let $x \in X$ such that $j(x) = \varphi$. We claim that $x = \varphi$. Note that $X \models \operatorname{rk}(x) = \alpha$ iff $H_{\kappa_+} \models \operatorname{rk}(\varphi) = j(\alpha)$. Because really $\operatorname{rk}(\varphi) < \gamma$, we get $j(\alpha) < \gamma < \kappa$. But this together with $j(\bar{\kappa}) = \kappa$ implies that $\alpha < \bar{\kappa}$ and so $j(\alpha) = \alpha$. Therefore $X \models \operatorname{rk}(x) = \alpha < \bar{\kappa}$. But as $j \restriction \bar{\kappa} = \operatorname{id}$, this implies that $x = j(x) = \varphi$. Now note that because H_{κ^+} is correct about cardinals, it is correct about $\models_{\mathcal{L}(\mathsf{I})}$, and so $H_{\kappa^+} \models$ "$\mathcal{A} \models \varphi$". As $\mathcal{A} \in \operatorname{ran}(j)$, we may let $\mathcal{B} = j^{-1}(\mathcal{A})$, and using that $j(\varphi) = \varphi$, elementarity of j implies that $X \models$ "$\mathcal{B} \models \varphi$". We claim that $\mathcal{B}$ is a τ-structure. An analogous argument as above shows that $j(\tau) = \tau$ and so $X \models$ "$\mathcal{B}$ is a τ-structure". Since $X \prec_{\Delta_0} H_{\bar{\theta}}$, this assertion is upward absolute to $H_{\bar{\theta}}$ and clearly the latter is correct about $\mathcal{B}$ being a τ-structure. Because $A = \kappa$, by elementarity of j we get that the domain of $\mathcal{B}$ is $B = \bar{\kappa}$. In particular $|B| < \kappa$. Using that $\operatorname{subf}(\varphi) \subseteq \operatorname{ran}(j)$ an analogous argument to before shows that $\operatorname{subf}(\varphi) \subseteq X$. Also $B \subseteq X$, and so, as $X \models$ "$\mathcal{B} \models \varphi$", even though X may not be transitive, X can perform the computation whether $\mathcal{B} \models \varphi$. As X is correct about cardinals, it is correct about this. Therefore really $\mathcal{B} \models \varphi$.

We showed that $\mathcal{B}$ is a τ-structure of size $|B| < \kappa$ satisfying φ. We are done if we can show that $j \restriction B : \mathcal{B} \to \mathcal{A}$ is an elementary embedding. We invoke the Tarski-Vaught test. Let $\psi(x, x_1, \ldots, x_n)$ be a first order-formula and suppose $\mathcal{A} \models \exists x \psi(x, j(b_1), \ldots, j(b_n))$ for $b_1, \ldots, b_n \in B$. Then $H_{\kappa^+} \models \exists x \in A(\mathcal{A} \models \psi(x, j(b_1), \ldots, j(b_n)))$. By elementarity of j, thus $X \models \exists x \in B(\mathcal{B} \models \psi(x, b_1, \ldots, b_n))$. So there is $b \in B$ with $X \models$ "$\mathcal{B} \models \psi(b, b_1, \ldots, b_n)$". Then $H_{\kappa^+} \models$ "$\mathcal{A} \models \psi(j(b), j(b_1), \ldots, j(b_n))$". But then really $\mathcal{A} \models \psi(j(b), j(b_1), \ldots j(b_n))$. So $j \restriction B : \mathcal{B} \to \mathcal{A}$ really is an elementary embedding. □

6.4.2. Stronger notions of definability

We have seen that SLST numbers of logics $\mathcal{L}$ generally do not imply SR_R^- via symbiosis. For example, for the symbiotic pair $\mathcal{L}(\mathsf{I})$ and Card, the existence of an SLST number of $\mathcal{L}(\mathsf{I})$ is weaker than $\mathrm{SR}_{\mathrm{Card}}^-$. With stronger notions of definability, both on the set-theoretic, as well as on the logics side, such a transfer result is possible. We consider the following more restrictive case of definability by $\Sigma_1(R)$ formulas.

Definition 6.4.9 (Bagaria [Bag23, Definition 6.10])**.** For a predicate of set theory R, a model class of structures $\mathcal{K}$ is $\Sigma_1^*(R)$ iff it is closed under isomorphism and there is a $\Sigma_1(\dot{R})$ formula $\varphi(x)$ such that the following holds:

> $\mathcal{A} \in \mathcal{K}$ iff there is a transitive set M such that $|M| = |A|$ and an $\mathcal{A}^* \cong \mathcal{A}$ with $(M, \in, R \cap M) \models \varphi(\mathcal{A}^*)$.

Let us explain how this notion aims to solve some of the problems that occur when trying to transfer the proof of the equivalence of $\mathrm{LST}(\mathcal{L})$ and SR_R (cf. [BV16, Theorem

5.5]) to the case of SLST($\mathcal{L}$) and SR_R^-. In the proof, assuming $\kappa = \mathrm{LST}(\mathcal{L})$, one starts with a class $\mathcal{K}$ of structures definable by some $\Sigma_1(R)$ formula containing some structure $\mathcal{A}$, and one wants to find a small submodel of $\mathcal{A}$ belonging to $\mathcal{K}$. One shows that (1) in $\Delta(\mathcal{L})$ (remember we work with symbiosis) one can define a class of set-theoretic structures that are able to witness membership in $\mathcal{K}$. One then (2) takes a model of set theory M belonging to this witnessing class and containing $\mathcal{A}$. LST($\mathcal{L}$) gives a small submodel N of M and one can show that N contains a small submodel of $\mathcal{A}$, as desired. The problem when applying this to the case in which $\mathrm{SLST}(\mathcal{L}) = \kappa$ is that we have to make sure that the witnessing model of set theory has size exactly κ to be able to use the SLST number. Without additional assumptions, both in step (1) and (2), this size requirement might get violated. In step (2), when choosing a witnessing model, M's size might be larger than κ. And in step (1), the Δ-closure might bring in additional sorts which increase model sizes. Bagaria's notion of $\Sigma_1^*(R)$ definability deals with the problem from step (2). To deal with the problem from step (1), we introduce the following notion, aimed at size restrictions on the logics' side.

Definition 6.4.10. Let $\mathcal{L}$ be a logic and R a set-theoretic predicate.

(i) If M is a transitive set and $X \subseteq M$, the model $(M, \in, X)$ is said to *have R-awareness* iff $X = R \cap M$.

(ii) We say that $\mathcal{L}$ *captures* R iff there is an expansion $\tau^* \supseteq \{\in, \dot{R}\}$ and a $\varphi \in \mathcal{L}[\tau^*]$ such that the following holds:

> Any transitive $\{\in, \dot{R}\}$-structure $(M, \in, X)$ has R-awareness iff there is an expansion $\mathcal{M}^* = (M, \in, X, \dots)$ of M to a τ^*-structure with $|M^*| = |M|$ and $\mathcal{M}^* \models \varphi$.

We also say that φ *captures* R.

Note that if R is n-ary, we implicitly assume here that for M to have R-awareness, it needs to be the case that $R \cap M^n \subseteq M$.

The main examples of symbiotic logics we stated earlier (cf. Proposition 6.2.4) capture the respective predicates associated to them. Our transfer Theorem 6.4.13 can therefore be applied to them.

Proposition 6.4.11. The following hold:

(1) $\mathcal{L}^2$ captures Pow.

(2) $\mathcal{L}(\mathsf{I})$ captures Card.

(3) $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ captures $\emptyset$.

Proof. For (1), let φ be the sentence

$$\forall x, y(\dot{\mathrm{Pow}}(x,y) \leftrightarrow (\forall X[\forall w(X(w) \to w \in y) \to \exists z(z \in x \wedge \forall w(X(w) \leftrightarrow w \in z))] \\ \wedge \forall z(z \in x \to z \subseteq y) \wedge \exists z(\text{“}z = (x,y)\text{”}))),$$

codifying that $\dot{\text{Pow}}(x, y)$ is true of exactly those sets x, y in some transitive M such that $x = \mathcal{P}(y)$ and M has the pair (x, y). Then $(M, \in, P) \models \varphi$ iff $P = \text{Pow} \cap M$. For (2), let ψ be the usual sentence expressing that some ordinal is really a cardinal:

$$\forall x(\dot{\text{Card}}(x) \leftrightarrow (\text{Ord}(x) \wedge \forall y(y \in x \rightarrow \neg\mathsf{I}zz(z \in x, z \in y)))).$$

Note that for φ and ψ we do not need to expand the respective vocabularies $\{\in, \dot{\text{Pow}}\}$ and $\{\in, \dot{\text{Card}}\}$ at all to express Pow- and Card-awareness, so the condition on the size of the witnessing model is trivially fulfilled. That $\mathcal{L}(\mathsf{Q}^{\text{WF}})$ captures $\emptyset$ is trivial. □

The transfer result is now possible under assumptions of symbiosis between R and $\mathcal{L}$ and the additional assumption that $\mathcal{L}$ captures R. The transfer is between $\text{SLST}(\mathcal{L})$ and the statement of SR^- restricted to $\Sigma_1^*(R)$ classes as stated below. The proof additionally yields that for the logics considered, the SLST and SLST^+ numbers conicide.[2]

Definition 6.4.12. Let R be a set-theoretic predicate and κ a cardinal. We write $\Sigma_1^*(R)$-$\text{SR}^-(\kappa)$ for the statement:

> The cardinal κ is minimal with the property such that for every proper class $\mathcal{K}$ of structures in a joint vocabulary of size $< \kappa$ such that $\mathcal{K}$ is $\Sigma_1^*(R)$, for any $\mathcal{A} \in \mathcal{K}$ of size exactly κ, there exists $\mathcal{B} \in \mathcal{K}$ with $|B| < \kappa$ and an elementary embedding $e : \mathcal{B} \rightarrow \mathcal{A}$.

Theorem 6.4.13. Let $\mathcal{L}$ be a logic and R a set-theoretic predicate. Assume that $\mathcal{L}$ and R are r-symbiotic and that $\mathcal{L}$ captures R. Then for a cardinal κ the following are equivalent:

(1) κ is the smallest cardinal such that $\Sigma_1^*(R)$-$\text{SR}^-(\kappa)$.

(2) $\text{SLST}(\mathcal{L}) = \kappa$.

(3) $\text{SLST}^+(\mathcal{L}) = \kappa$.

Note that to prove the result it is sufficient to show that

(a) if $\Sigma_1^*(R)$-$\text{SR}^-(\kappa)$ holds, then $\text{SLST}^+(\mathcal{L}) \leq \kappa$, and

(b) if $\text{SLST}(\mathcal{L}) = \kappa$, then $\Sigma_1^*(R)$-$\text{SR}^-(\kappa)$ holds.

Proof of (b). Assume that $\text{SLST}(\mathcal{L}) = \kappa$. Let $\mathcal{K}$ be $\Sigma_1^*(R)$ and $\mathcal{A} \in \mathcal{K}$ such that $|A| = \kappa$. Then there is a $\Sigma_1(\dot{R})$ formula $\varphi(x)$ such that $\mathcal{B} \in \mathcal{K}$ iff there is a transitive set M such that $|M| = |B|$, and a $\mathcal{B}^* \cong \mathcal{B}$ such that $(M, \in, R \cap M) \models \varphi(\mathcal{B}^*)$. In particular, there is a transitive M such that $|M| = \kappa = |A|$ and $\mathcal{A}^* \cong \mathcal{A}$ such that $(M, \in, R \cap M) \models \varphi(\mathcal{A}^*)$. We need to find $\mathcal{B} \in \mathcal{K}$ such that $|B| < \kappa$ and there is an elementary embedding

[2] Note that one can similarly strengthen the LST number (cf. Definition 1.2.9) of a logic to include a condition that the small substructure provided is an elementary substructure. It is folklore that these two ways to define the LST number are for many logics equivalent. For the SLST number, we are not aware of a strengthening in the spirit of the SLST^+ number having been discussed previously.

$e : \mathcal{B} \to \mathcal{A}$. Note that it is sufficient to find some $\mathcal{B}^* \in \mathcal{K}$ of size $|B^*| < \kappa$ and an elementary embedding $\mathcal{B}^* \to \mathcal{A}^*$, as $\mathcal{A}^* \cong \mathcal{A}$.

For this, let $\chi \in \mathcal{L}[\tau^*]$ be a sentence that captures R, where $\tau^* \supseteq \{\in, \dot{R}\}$. Further, let S be the set of additional sorts of τ^* and assume that φ and $\{\in, \dot{R}\}$ are in sort $s_0 \notin S$, so that s_0 is the sort of the model $(M, \in, R \cap M)$. We construct a sentence ψ in the language $\tau^{**} = \{\in, c, \dot{R}, f, (g_s)_{s \in S}\} \cup \tau^*$, where c is a constant symbol with $\mathrm{conf}(c) = s_0$, f is a function symbol with $\mathrm{conf}(f) = (s_0, s_0)$, and the g_s are function symbols with $\mathrm{conf}(g_s) = (s, s_0)$, respectively. We let ψ be the conjunction of the following sentences:

(i) $\varphi(c)$.

(ii) The extensionality axiom Ext.

(iii) "f is a bijection with $\mathrm{ran}(f) = \{x \colon xEc\}$."

(iv) χ.

(v) For every $s \in S$: "g_s is an injection."

By definition of χ, for a transitive set N and $X \subseteq N$, the model $(N, \in, X)$ has R-awareness, i.e., $X = R \cap N$, iff there is an expansion $\mathcal{N}^*$ of $(N, \in, X)$ such that $\mathcal{N}^* \models \chi$ and $|N^*| = |N|$. Now $(M, \in, R \cap M)$ has R-awareness and χ captures R. So there is an expansion of $(M, \in, R \cap M)$ to a τ^*-structure $\mathcal{M}^*$ with $|M^*| = |M|$ and $\mathcal{M}^* \models \chi$. In, particular, the additional sorts in $\mathcal{M}^*$ cannot have larger domain M_s than the size of M, so there are injections $g_s^M : M_s \to M$. Thus, if f^M is any bijection $f : M \to A^*$, then there is an expansion $\mathcal{M}$ of $(M, \in, \mathcal{A}^*, R \cap M, f^M, g_s)_{s \in S}$ to a τ^{**}-structure $\mathcal{M}$ such that $\mathcal{M} \models \psi$. We further expand $\mathcal{M}$ to a structure that for every Δ_0 first-order formula $\theta(x, \bar{y})$ in the language $\{\in, c\}$ contains a Skolem function $f^M_{\theta(x,\bar{y})} : M \to M$ for $(M, \in, \mathcal{A}^*)$, i.e., for every $\bar{b} \in M$, if $(M, \in, \mathcal{A}^*) \models \exists x \theta(x, \bar{b})$, then $(M, \in, \mathcal{A}^*) \models \psi(f^M_{\theta(x,\bar{y})}(\bar{b}), \bar{b})$. For simplicity, we denote this expansion again by $\mathcal{M}$. As $|\mathcal{M}| = |M| = \kappa$, by $\mathrm{SLST}(\mathcal{L})$ holding at κ, there is a substructure $\mathcal{N} = (N, \in, \mathcal{A}^*, \dot{R}^N, f^N, (g_s^N)_{s \in S}, \dots)$ of $\mathcal{M}$ such that $|\mathcal{N}| < \kappa$ and $\mathcal{N} \models \psi$. Then $\mathcal{N}$ is well-founded by being a substructure of the well-founded model $\mathcal{M}$ and it is extensional by (ii), so we can consider the transitive collapse $\bar{\mathcal{N}} = (\bar{N}, \in, c^{\bar{N}}, \dot{R}^{\bar{N}}, f^{\bar{N}}, (g_s^{\bar{N}})_{s \in S}, \dots)$. Because $\bar{\mathcal{N}} \models \psi$, the function $g_s : \bar{N}_s \to \bar{N}$ is an injection for every $s \in S$ and therefore $|\bar{\mathcal{N}}| = |\bar{N}|$. Further, $\dot{R}^{\bar{N}} = \bar{N} \cap R$, because $\bar{N}$ is transitive and $\bar{\mathcal{N}} \models \chi$, as χ captures R. By (iii), we have $|\bar{N}| = |c^{\bar{N}}|$. Thus, using (i), by $\bar{\mathcal{N}} \models \varphi(c)$ and $\mathcal{K}$ being $\Sigma_1^*(R)$, we get that $c^{\bar{N}} \in \mathcal{K}$ and indeed $|c^{\bar{N}}| \leq |\bar{N}| = |N| < \kappa$.

If we can show that there is an elementary embedding $e : c^{\bar{N}} \to \mathcal{A}^*$, then we are done. Because $\bar{N}$ is transitive, we have $c^{\bar{N}} \subseteq \bar{N}$. Thus we can consider $\pi^{-1} \restriction c^{\bar{N}}$, the inverse of the transitive collapse restricted to $c^{\bar{N}}$. We claim that this is our elementary embedding. Note that if $b \in c^{\bar{N}}$, then $\pi^{-1}(b) \in c^{\mathcal{N}} = \mathcal{A}^*$, so π^{-1} is indeed a map $c^{\bar{N}} \to A^*$. Remember that we added Skolem functions for every Δ_0 formula in the language $\{\in, c\}$. Thus, $(N, \in, \mathcal{A}^*)$ is a Σ_1-elementary substructure of $(M, \in, \mathcal{A}^*)$. Further, if $\psi(\bar{x}_i)$ (with

$\bar{x}_i = (x_1, \ldots, x_n)$) is any first-order formula and $\mathcal{C}$ is any structure with $\bar{c}_i \in C$, we have that "$\mathcal{C} \models \psi(\bar{c}_i)$" is Δ_1. Hence,

$$\begin{aligned} c^{\bar{N}} \models \psi(\bar{b}_i) &\text{ iff } \bar{\mathcal{N}} \models \text{“}c^{\bar{N}} \models \psi(\bar{b}_i)\text{”} \\ &\text{ iff } \mathcal{N} \models \text{“}\pi^{-1}(c^{\bar{N}}) \models \psi(\pi^{-1}(\bar{b}_i))\text{”} \\ &\text{ iff } \mathcal{M} \models \text{“}\mathcal{A}^* \models \psi(\pi^{-1}(\bar{b}_i))\text{”} \\ &\text{ iff } \mathcal{A}^* \models \psi(\pi^{-1}(\bar{b}_i)). \end{aligned}$$

The first and last "iff" hold by absoluteness of first-order satisfaction. The second one holds as π^{-1} is an isomorphism and the third because $\pi^{-1}(c^{\bar{N}}) = \mathcal{A}^*$ and because $(N, \in, \mathcal{A}^*)$ is a Σ_1-elementary substructure of $(M, \in, \mathcal{A}^*)$. □

For the other direction (a), we will use the following, parameter-free version, of a notion by Lücke.

Definition 6.4.14 (Lücke [Lüc22, Definition 1.12])**.** For a predicate of set theory R, a model class $\mathcal{K}$ of structures over a vocabulary τ is called a *local* $\Sigma_1(R)$ class iff it is closed under isomorphisn and there is a $\Sigma_1(\dot{R})$ formula $\varphi(x)$ in the language $\{\in, \dot{R}\}$ such that for all infinite cardinals κ:

$$H_{\kappa^+} \cap \mathcal{K} = \{x \in H_{\kappa^+} : (H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models \varphi(x)\}.$$

As Bagaria remarks (cf. [Bag23, p. 54]), this coincides with being a $\Sigma_1^*(R)$ class. We want to present a proof of this fact, before we finish the proof of (a).

Proposition 6.4.15. Let R be a predicate of set theory and $\mathcal{K}$ a model class. Then $\mathcal{K}$ is a local $\Sigma_1(R)$ class iff it is $\Sigma_1^*(R)$.

Proof. First assume that $\mathcal{K}$ is $\Sigma_1^*(R)$ and that this is witnessed by a $\Sigma_1(\dot{R})$ formula $\psi(x)$. Then consider the formula

$$\begin{aligned} \varphi(x) = \exists y \exists f (&\text{“}y = (M, \in, R \cap M) \text{ is transitive”} \\ &\wedge \text{“}f : M \to x \text{ is a bijection”} \wedge y \models \psi(x)). \end{aligned}$$

Note that $\varphi(x)$ is $\Sigma_1(\dot{R})$ as well. We claim that $\varphi(x)$ witnesses that $\mathcal{K}$ is a local $\Sigma_1(\dot{R})$ class. If $\mathcal{A} \in H_{\kappa^+} \cap \mathcal{K}$. Then as $\mathcal{K}$ is $\Sigma_1^*(R)$, there is a transitive $(M, \in, R \cap M)$ such that $|M| = |A|$ and an $\mathcal{A}^* \cong \mathcal{A}$ with $(M, \in, R \cap M) \models \psi(\mathcal{A}^*)$. Then because M is transitive, $M, \mathcal{A}^* \in H_{\kappa^+}$. Further, H_{κ^+} knows about the isomorphism between $\mathcal{A}$ and $\mathcal{A}^*$, and thus also $(H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models$ "$(M, \in, R \cap M) \models \psi(\mathcal{A})$". We also have that H_{κ^+} knows about the bijection between M and A. Therefore $(H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models \varphi(\mathcal{A})$. If on the other hand $(H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models \varphi(\mathcal{A})$, then this is witnessed by some triple $(M, \in, R \cap M) \in H_{\kappa^+}$ such that $(H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models$ "$(M, \in, R \cap M) \models \psi(\mathcal{A})$". By virtue of φ, H_{κ^+} knows about a bijection $M \to \mathcal{A}$, and also it is correct about first-order satisfaction. Thus it really holds that $(M, \in, R \cap M)$ is a transitive model of the same size as $\mathcal{A}$ that has R-awareness and which believes $\psi(\mathcal{A})$. Therefore $\mathcal{A} \in \mathcal{K}$ by ψ witnessing $\Sigma_1^*(R)$-ness of $\mathcal{K}$.

For the other direction let $\mathcal{K}$ be a local $\Sigma_1(R)$ class, as witnessed by a formula $\varphi(x)$. Given a model in $\mathcal{K}$ we can assume that it has a transitive universe, by closure of $\mathcal{K}$ under isomorphism. So let $\mathcal{A}$ be a transitive structure of size κ. Then by locality, $\mathcal{A} \in \mathcal{K}$ iff $(H_{\kappa^+}, \in, R \cap H_{\kappa^+}) \models \varphi(\mathcal{A})$. Let $(X, \in, R \cap X)$ be an elementary submodel of H_{κ^+} of size κ with $\mathcal{A}, \kappa \in X$ and $\kappa \subseteq X$. Notice that such a model is automatically transitive.[3] Then $\mathcal{A} \in \mathcal{K}$ iff $(X, \in, R \cap X) \models \varphi(\mathcal{A})$. Because X is of the same size as $\mathcal{A}$, this shows that $\mathcal{K}$ is $\Sigma_1(R)^*$. $\square$

Proof of (a). Let $\varphi \in \mathcal{L}$ and $\mathcal{A} \models \varphi$ such that $|A| = \kappa$. As $\mathrm{Mod}(\varphi)$ is $\Delta_1(R)$ definable by r-symbiosis, it is absolute for all the H_{κ^+}'s. Further, it is closed under isomorphism, therefore a local $\Sigma_1(R)$ class and thus $\Sigma_1^*(R)$ by Proposition 6.4.15. By $\Sigma_1^*(R)$-$\mathrm{SR}^-(\kappa)$, we thus find a $\mathcal{B} \in \mathrm{Mod}(\varphi)$ such that $|B| < \kappa$ and an elementary embedding $e : \mathcal{B} \to \mathcal{A}$. $\square$

Note, as observed in [Bag23, p. 55], that Theorem 6.4.13 recovers the fact that $\kappa = \mathrm{SLST}(\mathcal{L}(I))$ iff κ is the smallest weakly inaccessible cardinal, which was stated in [BV16, Section 8]. Moreover, our results imply that this can be slightly improved to a characterisation of $\mathrm{SLST}^+(\kappa)$:

Corollary 6.4.16. The following are equivalent for a cardinal κ:

(1) κ is the least weakly inaccessible cardinal.

(2) $\kappa = \mathrm{SLST}(\mathcal{L}(\mathsf{I}))$.

(3) $\kappa = \mathrm{SLST}^+(\mathcal{L}(\mathsf{I}))$.

Proof. As $\mathcal{L}(\mathsf{I})$ captures Card and the two are r-symbiotic, the equivalence of (2) and (3) follows from Theorem 6.4.13. The equivalence to (1) can be seen as follows. If κ is weakly inaccessible, then $\mathrm{SLST}^+(\mathcal{L}(\mathsf{I})) \leq \kappa$ by Proposition 6.4.8. And if $\kappa = \mathrm{SLST}(\mathcal{L}(\mathsf{I}))$, then by Theorem 6.4.13 it is the least cardinal such that $\Sigma_1^*(\mathrm{Card})$-$\mathrm{SR}^-(\kappa)$ holds. Then as $\Sigma_1^*(\mathrm{Card})$ classes are local $\Sigma_1(\mathrm{Card})$ (cf. Proposition 6.4.15), results by Lücke imply that κ is weakly $(\Sigma_1, \mathrm{Card}, \kappa^+)$-shrewd (cf. [Lüc22, Corollary 6.5]). Then by Theorem 6.4.7, κ is weakly inaccessible. $\square$

[3]Let $x \in X$. As H_{κ^+} knows about a surjection from κ to x, X has this property as well. So let $f \in X$ such that $X \models$ "$f : \kappa \to x$ is a surjection." Then because $\kappa \subseteq X$ we have $\alpha \in X$ for every $\alpha \in \kappa$. Thus $f(\alpha) \in X$ for every $\alpha \in \kappa$. Therefore $x = f``\kappa \subseteq X$.

7. Some Notes on Compactness for Type Omission

Remarks on co-authorship. The results of Section 7.3 are joint with Will Boney. The results of Section 7.4 are joint with Will Boney and Victoria Gitman.

7.1. Introduction

Supercompact cardinals can be characterised by a compactness principle that provides models of a theory such that the model simultaneously omits a type. In [Bon20], Boney expanded on this result and achieved a very general result which gives an equivalence between the existence of certain ultrafilters and similar compactness for type omission principles for the logic $\mathcal{L}_{\kappa\kappa}$. In particular, he was able to characterise huge cardinals in this way, and thus showed that model-theoretic properties can have higher consistency strength than that of VP. In this chapter, we extend these results by positively answering a question by Wilson whether there is some property of a finitary logic with consistency strength exceeding that of VP. More concretely, we will show that huge cardinals can be characterised by a type omission compactness property of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ (cf. Section 7.3). We further show how a natural notion in between extendibility and supercompactness arises from type omission compactness for $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$ (cf. Section 7.4).

7.2. Motivation and definitions

In this section, we give the necessary definitions to formulate our theorems and state some of the known results on compactness for type omission. Recall the theorem going back to Benda we already stated in Chapter 1 (Theorem 1.3.18), as well as the discussion on type omission preceding it.

Theorem 7.2.1 (Benda, Boney [Ben78, Bon20])**.** The following are equivalent for a cardinal κ:

(1) κ is supercompact.

(2) For any $\lambda \geq \kappa$, if $T \subseteq \mathcal{L}_{\kappa\kappa}$ is a theory that can be written as an increasing union $T = \bigcup_{s\in\mathcal{P}_\kappa\lambda} T_s$ and $p(x) = \{\varphi_i(x)\colon i < \lambda\} \subseteq \mathcal{L}_{\kappa\kappa}$ is a type such that with $p_s = \{\varphi_i(x)\colon i \in s\}$ there is a club subset X of $\mathcal{P}_\kappa\lambda$ such that for $s \in X$, T_s has a model omitting p_s, then T has a model omitting p.

Motivated by this result, Boney considered how to characterise large cardinals witnessed by ultrafilters living on other sets than $\mathcal{P}_\kappa\lambda$. He found a very general result for the existence of ultrafilters over some set $I \subseteq \mathcal{P}(\lambda)$. Even though we will restrict attention to special cases of I, let us state his general definitions, as it lets us treat all cases we will be concerned with uniformly.

Definition 7.2.2 (Boney [Bon20, Definition 3.1])**.** Let κ be a cardinal, $\kappa \leq \lambda$ and $I \subseteq \mathcal{P}(\lambda)$.

(i) I is *κ-robust* iff for every $\alpha < \lambda$, we have $I \subseteq \{s \in \mathcal{P}(\lambda)\colon |s \cap \kappa| < \kappa\}$ and $\{s \in I\colon \alpha \in s\} \neq \emptyset$.

(ii) $C \subseteq I$ *contains a strong κ-club* iff there is a function $F : [\lambda]^2 \to \mathcal{P}_\kappa\lambda$ such that

$$C(F) = \{s \in I\colon |s| \geq \omega \wedge \forall x, y \in s(F(x, y) \subseteq s)\} \subseteq C.$$

It is a classical result of Menas (cf. [Men74] and [Kan03, Proposition 25.3]) that if $C_{\kappa,\lambda}$ is the filter generated by the club subsets of $\mathcal{P}_\kappa\lambda$, then for any $X \subseteq \mathcal{P}_\kappa\lambda$,

$$X \in C_{\kappa,\lambda} \text{ iff there is } F : [\lambda]^2 \to \mathcal{P}_\kappa\lambda \text{ such that } C(F) \subseteq X.$$

Containing a strong κ-club is therefore a generalisation to the case of arbitrary $I \subseteq \mathcal{P}(\lambda)$ of being a member of the club filter generated by the club subsets of $\mathcal{P}_\kappa\lambda$. As for ultrafilters over $\mathcal{P}_\kappa\lambda$, if U is an ultrafilter over $I \subseteq \mathcal{P}(\lambda)$, let us say that U is *fine* if for all $\alpha < \lambda$, $\{s \in I\colon \alpha \in s\} \in U$ and that U is *normal* if for all $F : I \to \lambda$ such that $\{s \in I\colon F(s) \in s\} \in U$, there is $\alpha < \lambda$ such that $\{s \in I\colon F(s) = \alpha\} \in U$. Recall that any fine, normal, κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$ contains $C_{\kappa,\lambda}$ (cf., e.g., [Kan03, Proposition 25.4]). Boney showed that the notion of containing a strong κ-club extends this result in the following way:

Proposition 7.2.3 (Boney [Bon20, Fact 3.2])**.** Let $I \subseteq \mathcal{P}(\lambda)$ be κ-robust. If U is a fine, normal, κ-complete ultrafilter over I, then $C(F) \in U$ for all $F : [\lambda]^2 \to \mathcal{P}_\kappa\lambda$.

We further need the following technical condition on how some union indexed by $s \in I$ is determined by the members of s.

Definition 7.2.4 (Boney [Bon20, Definition 3.3])**.** Let $I \subseteq \mathcal{P}(\lambda)$ and let X be a set that is written as an increasing union $X = \bigcup_{s \in I} X_s$. We say that *the union respects the index* iff there is a collection $\{X^\alpha\colon \alpha \in \lambda\}$ such that for each $s \in I$:

$$X_s = \bigcup_{\alpha \in s} X^\alpha.$$

We can now state Boney's general version of compactness for type omission.

Definition 7.2.5 (Boney [Bon20, Definition 3.4])**.** Let $\mathcal{L}$ be a logic, κ a cardinal, $\kappa \leq \lambda$, and $I \subseteq \mathcal{P}(\lambda)$ be κ-robust. We say that $\mathcal{L}$ is *I-κ-compact for type omission* iff the following holds: For any theory $T \subseteq \mathcal{L}$ which can be written as an increasing union $T = \bigcup_{s \in I} T_s$ that respects the index, and any type $p(x) = \{\varphi_i(x) \colon i < \lambda\}$ with subsets $p_s = \{\varphi_i(x) \colon i \in s\} \subseteq \mathcal{L}$ for $s \in I$, if the set

$$\{s \in I \colon T_s \text{ has a model omitting } p_s\}$$

contains a strong κ-club, then T has a model omitting p.

Note that Boney's original definition comes with a collection of types instead of a single one, but for the applications we will consider, having one type around will be sufficient.

Recall that a cardinal κ is *huge* iff there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$. Note that this implies that $j(\kappa)$ is a cardinal. If $j(\kappa) = \lambda$, we also say that κ is *huge with target* λ. It is well-known (cf., e.g., [Kan03, Theorem 24.8]) that κ is huge with target λ iff there is a fine, normal, κ-complete ultrafilter U over $\mathcal{P}(\lambda)$ such that

$$I = \{s \in \mathcal{P}(\lambda) \colon \mathrm{ot}(s) = \kappa\} \in U.$$

If κ is huge, then $(V_\kappa, \in, V_{\kappa+1})$ satisfies a second-order version of Vopěnka's Principle (cf., e.g., [Jec03, Lemma 20.27]). In particular, the existence of a huge cardinal exceeds Vopěnka's Principle in consistency strength.

Note that the I above is not κ-robust, as, for example, $\kappa \in I$, and $|\kappa \cap \kappa| = \kappa$. But note that if U is a fine, normal, κ-complete ultrafilter over I, then

$$[\lambda]^\kappa_* = \{s \in \mathcal{P}(\lambda) \colon \mathrm{ot}(s) = \kappa \text{ and } s \setminus \kappa \neq \emptyset\} \in U,$$

as, for example, for any $\kappa < \alpha < \lambda$, by fineness $\{s \in I \colon \alpha \in s\} \in U$ and further $\{s \in I \colon \alpha \in s\} \subseteq [\lambda]^\kappa_*$. But $[\lambda]^\kappa_*$ is κ-robust, and κ is huge with target λ iff there is a fine, normal, κ-complete ultrafilter W over $[\lambda]^\kappa_*$ (the forward direction follows from the argument above, the backward direction can be shown exactly as the proof of [Kan03, Theorem 24.8] by checking that taking $j_W : V \to \mathrm{Ult}(V, W)$ witnesses that κ is huge with target λ).

Let us summarise some of Boney's results relevant for us.

Theorem 7.2.6 (Boney [Bon20])**.** The following hold.

(1) κ is λ-supercompact iff $\mathcal{L}_{\kappa\kappa}$ is $\mathcal{P}_\kappa\lambda$-$\kappa$-compact for type omission.

(2) κ is extendible iff for every λ, $\mathcal{L}^2_{\kappa\kappa}$ is $\mathcal{P}_\kappa\lambda$-$\kappa$-compact for type omission.

(3) κ is huge with target λ iff $\mathcal{L}_{\kappa\kappa}$ is $[\lambda]^\kappa_*$-κ-compact for type omission.

Note that item (1) is simply Theorem 7.2.1, as for $I = \mathcal{P}_\kappa\lambda$ we may ignore the condition that an increasing union $T = \bigcup_{s \in \mathcal{P}_\kappa\lambda} T_s$ respects the index: letting $T^\alpha = T_{\{\alpha\}}$, because our fixed union is increasing, we get $T_s = \bigcup_{\alpha \in s} T^\alpha$, and so every increasing union indexed by $\mathcal{P}_\kappa\lambda$ respects the index.

7.3. Huge cardinals from omitting types for finitary logics

Recall Wilson's result Theorem 5.2.2, which characterised huge cardinals by certain Löwenheim-Skolem properties of class logics. Above we saw that huge cardinals are certain omitting types compactness cardinals for $\mathcal{L}_{\kappa\kappa}$. Wilson noted that both these known characterisations of huge cardinals utilise infinitary logics and therefore asked in a talk at the European Set Theory Conference 2022:

Question 7.3.1 (Wilson [Wil22a])**.** Is there a property P of a finitary logic such that $\mathcal{L}$ having P exceeds Vopěnka's Principle in consistency strength?

We give an affirmative answer.

Theorem 7.3.2. Let κ and λ be cardinals such that $\lambda > \kappa$. Then κ is huge with target λ iff $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ is $[\lambda]^{\kappa}_{*}$-κ-compact for type omission.

The forward direction directly follows from Boney's Theorem 7.2.6, as $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}}) \leq \mathcal{L}_{\omega_1\omega_1} \leq \mathcal{L}_{\kappa\kappa}$. For the backward direction, we use the following lemma.

Lemma 7.3.3. Let κ and λ be cardinals such that $\lambda > \kappa$. If for all transitive sets M with $\lambda \in M$ and $M^{\kappa} \subseteq M$ there is a transitive set N and an elementary embedding $j : M \to N$ such that $\mathrm{crit}(j) = \kappa$, $\lambda = j(\kappa)$ and $j``\lambda \in N$, then κ is huge with target λ.

Proof. Take some cardinal $\gamma > \lambda$ such that V_γ is closed under κ-sequences and $V_{\lambda+1} \in V_\gamma$. Then by our assumption, we have that there is a transitive N and an elementary embedding $j : V_\gamma \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $j``\lambda \in N$. Now define an ultrafilter U over $[\lambda]^{\kappa}_{*}$ by letting for $X \subseteq [\lambda]^{\kappa}_{*}$:

$$X \in U \text{ iff } j``\lambda \in j(X).$$

Using that $\mathrm{crit}(j) = \kappa$ it is easy to see that U is a κ-complete ultrafilter. We check that U is fine, normal, and that the condition on the order types is fulfilled.

For fineness, clearly for $\alpha < \lambda$ we have $j(\alpha) \in j``\lambda$ and therefore we get that $j``\lambda \in \{s \in ([j(\lambda)]^{j(\kappa)}_{*})^N : j(\alpha) \in s\}$. Thus $\{s \in [\lambda]^{\kappa}_{*} : \alpha \in s\} \in U$.

For normality, let $F : [\lambda]^{\kappa}_{*} \to \lambda$ be a function such that $\{s \in [\lambda]^{\kappa}_{*} : F(s) \in s\} \in U$. We show that F is constant on a set in U. We have $j``\lambda \in \{s \in ([j(\lambda)]^{j(\kappa)}_{*})^N : j(F)(s) \in s\}$. So $j(F)(j``\lambda) \in j``\lambda$. But this means that there is an $\alpha \in \lambda$ such that $j(\alpha) = j(F)(j``\lambda)$ and then $\{s \in [\lambda]^{\kappa} : F(s) = \alpha\} \in U$.

For the condition on the order types, note that the order type of $j``\lambda \cap j(\lambda) = \lambda = j(\kappa)$ and $N$ knows about this. Thus $j``\lambda \in \{s \in ([j(\lambda)]^{j(\kappa)}_{*})^N : N \models \mathrm{ot}(s \cap j(\lambda)) = j(\kappa)\}$. By definition of U, this means $\{s \in [\lambda]^{\kappa} : \mathrm{ot}(s \cap \lambda) = \kappa\} \in U$. □

Proof of the backwards direction of Theorem 7.3.2. Assume that $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ is $[\lambda]^\kappa_*$-κ-compact for type omission and take a transitive M with $\lambda \in M$ and such that $M^\kappa \subseteq M$. We will show that there is an elementary embedding $j : M \to N$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $j``\lambda \in N$. Let $\{c_i \colon i \in M\}$ be the collection of variables used to formulate the elementary diagram of M and let c and d be new constants. Then let

$$T = \mathrm{ElDiag}_{\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})}(M) \cup \{c_\alpha < c < c_\kappa \colon \alpha < \kappa\} \cup \{c_\alpha \in d \wedge |d| = c_\kappa \colon \alpha < \lambda\}$$

and

$$p(x) = \{x \in d \cup c\} \cup \{x \neq c_\alpha \colon \alpha < \lambda\}.$$

Let us argue that if N is a model of T omitting p, then N has all required properties by letting $j : M \to N$, $c_x^M \mapsto c_x^N$.

Because c^N has order type at least κ and is smaller than $c_\kappa^N = j(\kappa)$, we have to have $j(\kappa) > \kappa$. In particular, $\mathrm{crit}(j) \leq \kappa$. To see that $\mathrm{crit}(j) = \kappa$, the type p comes into play. Assume that $\alpha = \mathrm{crit}(j) < \kappa$ and consider $\alpha + 1$. Clearly $c_\beta^N = \beta = j(\beta) < \alpha + 1 < c^N$ for all $\beta < \alpha$. But also $j(\alpha) > \alpha + 1$ and so $c_\beta^N > \alpha + 1$ for all $\beta \geq \alpha$. This means that $\alpha + 1$ realises p, which is a contradiction. So $\mathrm{crit}(j) = \kappa$.

We also have that $d^N = j``\lambda$ and so $j``\lambda \in N$: Because $c_\alpha^N \in d^N$ for all $\alpha < \lambda$ we have $j``\lambda \subseteq d^N$. And now if $x \in d^N$ and $x \neq j(\alpha) = c_\alpha^N$ for every $\alpha < \lambda$, then $x$ would realise $p$. Thus also $d^N \subseteq j``\lambda$.

Finally, N knows that $d^N = j``\lambda$ is a set of ordinals and thus has an order type. But clearly $\mathrm{ot}(j``\lambda) = \lambda$, thus $N \models \mathrm{ot}(d^N) = \lambda$ and further $N \models j(\kappa) = c_\kappa = |d| = \lambda$ and thus $j(\kappa) = \lambda$.

So we are done if we can show that T has a model omitting p. For this purpose we show the following claim.

Claim 7.3.4. For every $s \in [\lambda]^\kappa_*$, if $s \cap \kappa$ is a limit ordinal $< \kappa$, then M can be expanded to a model of T_s omitting p_s, where

$$T_s = \mathrm{ElDiag}(M)_{\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})} \cup \{c_\alpha < c < c_\kappa \colon \alpha \in \kappa \cap s\} \cup \{c_\alpha \in d \wedge |d| = c_\kappa \colon \alpha \in s\}$$

and

$$p_s(x) = \{x \in d \cup c\} \cup \{x \neq c_\alpha \colon \alpha \in s\}.$$

To show the claim, suppose $s \in [\lambda]^\kappa_*$ such that $s \cap \kappa$ is a limit ordinal $s \cap \kappa = \beta < \kappa$. Let $c^M = \beta$ and $d^M = s$. Note that $s \in M$ by closure of M under κ sequences, so the definition of c^M and d^M make sense. Then $(M, c^M, d^M) \models T_s$: Clearly $c^M = \beta < \kappa = c_\kappa^M$. And if $\alpha \in \kappa \cap s = \beta$, then $c_\alpha^M = \alpha \in \beta$. Further $\alpha = c_\alpha^M \in s = d^M$ for all $\alpha \in s$ and $|d^M| = |s| = \kappa = c_\kappa^M$. Also, we get that (M, c^M, d^M) omits p_s. For if $x \in c^M = s \cap \kappa$, then $x = \alpha = c_\alpha^M$ for some $\alpha \in s$. And if $x \in d^M$, then $x = \alpha = c_\alpha^M$ for some $\alpha \in d^M = s$. So any x omits p.

Therefore, to get a model of T omitting p, by $[\lambda]^\kappa_*$-κ-type-omission and the claim, it is sufficient to find a function $F : [\lambda]^2 \to \mathcal{P}_\kappa\lambda$ such that for any $s \in [\lambda]^\kappa_*$ with the property that for all $x, y \in s$ we have $F(x, y) \subseteq s$, it holds that $s \cap \kappa$ is a limit ordinal $< \kappa$. Such an F is given by

$$F(x,y) = \begin{cases} x+2, & \text{if } x \in \kappa \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $s \in [\lambda]^{\kappa}_{*}$ is given such that for all $x, y \in s$ we have $F(x, y) \subseteq s$. Then if $\alpha \in s \cap \kappa$, by assumption, $F(\alpha, \alpha) = \alpha + 2 \subseteq s$. Thus $\beta \in s \cap \kappa$ for all $\beta \leq \alpha + 1$. This shows that $s \cap \kappa$ is a limit ordinal. By definition of $[\lambda]^{\kappa}_{*}$, $\mathrm{ot}(s \cap \kappa) < \kappa$. Hence, $s \cap \kappa$ is a limit ordinal less than κ. □

Let us mention some related results. In [HM22], Hayut and Magidor show that supercompact cardinals can be characterised as $\mathcal{P}_\kappa\lambda$-$\kappa$-compactness for type omission cardinals of $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$. Our result is the analogue for type omission compactness indexed by $[\lambda]^{\kappa}_{*}$. They further show that their result about supercompactness even holds when substituting $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ by first-order logic. We do not know whether the same is true for huge type omission.

Question 7.3.5. If $\mathcal{L}_{\omega\omega}$ is $[\lambda]^{\kappa}_{*}$-κ-compact for type omission, is κ huge with target λ?

7.4. Compactness for type omission for infinitary equicardinality logics

Supercompactness and extendibility are characterised by $\mathcal{P}_\kappa\lambda$-compactness for type omission for $\mathcal{L}_{\kappa\kappa}$ and $\mathcal{L}^2_{\kappa\kappa}$, respectively. Considering $\mathcal{P}_\kappa\lambda$-compactness for type omission for $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$ gives rise to a large cardinal notion in between supercompactness and cardinal correct extendibility both as lower bounds, and extendibility as an upper bound.

Recall the class of $\Sigma_1(\mathrm{Card})$ formulas introduced in Section 6.2. Let us write $C^{(1\mathrm{Card})} = \{\alpha \colon V_\alpha \prec_{\Sigma_1(\mathrm{Card})} V\}$ for the class of all ordinals α such that V_α is an elementary substructure of V with respect to the $\Sigma_1(\mathrm{Card})$ formulas. Note that $C^{(1)} \subseteq C^{(1\mathrm{Card})} \subseteq C^{(2)}$. We show the following theorem.

Theorem 7.4.1. The following are equivalent:

(1) $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$ is $\mathcal{P}_\kappa\lambda$-$\kappa$-compact for type omission for every $\lambda \in C^{(1\mathrm{Card})}$, $\lambda > \kappa$.

(2) For every $\lambda > \kappa$, $\lambda \in C^{(1\mathrm{Card})}$, there is a fine, normal, κ-complete ultrafilter U over $\mathcal{P}_\kappa\lambda$ such that $\{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in C^{(1\mathrm{Card})}\} \in U$.

(3) For every $\lambda > \kappa$, $\lambda \in C^{(1\mathrm{Card})}$, there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $M \models \lambda \in C^{(1\mathrm{Card})}$.

Proof. Assume (1) and let $\lambda > \kappa$ be in $C^{(1\mathrm{Card})}$. We show (2). Take γ, some strong limit of cofinality $\mathrm{cof}(\gamma) > |\mathcal{P}_\kappa\lambda|$, new constants c and d, and consider the theory:

$$T = \mathrm{ElDiag}_{\mathcal{L}_{\kappa\kappa(\mathsf{I})}}(V_\gamma, \in) \cup \{|d| < c_\kappa \wedge c_i \in d \colon i < \lambda\},$$

along with the type:

$$p(x) = \{x \in d\} \cup \{x \neq c_i \colon i < \lambda\}.$$

If $s \in \mathcal{P}_\kappa\lambda$, letting $T_s = \mathrm{ElDiag}_{\mathcal{L}_{\kappa\kappa}(\mathsf{I})}(V_\gamma, \in) \cup \{|d| < c_\kappa \wedge c_i \in d \colon i \in s\}$ and p_s be the type $p_s(x) = \{x \in d\} \cup \{x \neq c_i \colon i \in s\}$, then any T_s has a model omitting p_s by using V_γ, and s itself for the interpretation of d. So by assumption, there is some model M such that $M \models T$ and omits p. Because the sentence of $\mathcal{L}_{\kappa\kappa}$ axiomatising well-foundedness is in T, M is well-founded and by collapsing we may assume that it is transitive. There is an elementary embedding $j : V_\gamma \to M$ given by $x \mapsto c_x^M$. By usage of $\mathcal{L}_{\kappa\kappa}$ and the theory T, j has critical point $\mathrm{crit}(j) = \kappa$. Because M omits p, $d^M = j``\lambda$. Further, $\lambda = |d^M| \leq |d^M|^M < j(\kappa)$. It is then standard to check that for $X \subseteq \mathcal{P}_\kappa\lambda$,

$$X \in U \text{ iff } d^M \in j(X)$$

defines a κ-complete, fine, normal ultrafilter over $\mathcal{P}_\kappa\lambda$. We have to check that the set $\{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in C^{(1\mathrm{Card})}\} \in U$, i.e., by definition of U, that $V_\lambda^M \prec_{\Sigma_1(\mathrm{Card})} M$. Because M believes that V_λ^M is cardinal correct, $\Sigma_1(\mathrm{Card})$ formulas are upwards absolute from V_λ^M to M. We have to check downwards absoluteness. Note that the sentence φ_{Card} (cf. Lemma 1.2.4) is in T, and thus M is correct about cardinals. Therefore $\Sigma_1(\mathrm{Card})$ formulas are upwards absolute from M to V. Hence, if $M \models \Phi(a)$ for a $\Sigma_1(\mathrm{Card})$ formula $\Phi(x)$ and some $a \in V_\lambda^M$, then $\Phi(a)$ holds in V. Now $\lambda \in C^{(1\mathrm{Card})}$ and so this implies $V_\lambda \models \Phi(a)$. We are done if we can show that $V_\lambda = V_\lambda^M$. This basically follows from the fact that $j``\lambda \in M$. To give the details: Build the ultrapower $M_U = \mathrm{Ult}(V_\gamma, U)$. We get the standard elementary maps $j_U : V_\gamma \to M_U$, $x \mapsto [c_x]_U$, and $k : M_U \to M$, $[f]_U \mapsto j(f)(j``\lambda)$. It is standard to show that $j = k \circ j_U$, $\mathrm{crit}(k) > \lambda$ and $M_U^\lambda \subseteq M_U$ (cf., e.g., [Jec03, Chapter 20]). That M_U is closed under λ-sequences implies $V_\lambda \subseteq M_U$ and $\mathrm{crit}(k) > \lambda$ means that $k \restriction V_\lambda$ is the identity. Thus $V_\lambda \subseteq M$ and hence $V_\lambda^M = V_\lambda$.

And now assume (2) as witnessed for some $\lambda > \kappa$, $\lambda \in C^{(1\mathrm{Card})}$, by an ultrafilter U. Let us show that (3) holds for λ. Take $M = \mathrm{Ult}(V, U)$ with the standard elementary map $j : V \to M$, $x \mapsto [c_x]_U$. Because U is a fine, normal, κ-complete ultrafilter, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. Left to show is that $M \models \lambda \in C^{(1\mathrm{Card})}$. It is standard to show that $X \in U$ iff $j``\lambda \in j(X)$ (cf., e.g., [Jec03, Lemma 20.13]). So because $\{s \in \mathcal{P}_\kappa\lambda \colon \mathrm{ot}(s) \in C^{(1\mathrm{Card})}\}$, we get that $M \models \lambda = \mathrm{ot}(j``\lambda) \in C^{(1\mathrm{Card})}$.

Finally assume (3) and let us show (1). Suppose $T = \bigcup_{s \in \mathcal{P}_\kappa\lambda} T_s$ is an increasing union of theories and $p(x) = \{\varphi_i(x) \colon i < \lambda\}$ a type both in the logic $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$ and over some vocabulary τ such that for all $s \in X$ for some club X in $\mathcal{P}_\kappa\lambda$, T_s has a model omitting $p_s = \{\varphi_i(x) \colon i \in s\}$. Take a $\gamma \in C^{(1\mathrm{Card})}$ with $\mathrm{cof}(\gamma) \geq \kappa$ such that $T, p \in V_\gamma$ and V_γ has models of T_s omitting p_s wherever possible. Let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$, $M^\gamma \subseteq M$ and $M \models \gamma \in C^{(1\mathrm{Card})}$. We may assume that j comes from an ultrafilter U over $\mathcal{P}_\kappa\gamma$ as in (2), and so for $Y \subseteq \mathcal{P}_\kappa\gamma$, $Y \in U$ iff $j``\gamma \in j(Y)$. It is standard to check that then

$$U^* = \{Y \subseteq \mathcal{P}_\kappa\lambda \colon \{s \in \mathcal{P}_\kappa\gamma \colon s \cap \lambda \in Y\} \in U\},$$

is a κ-complete, fine, normal ultrafilter over $\mathcal{P}_\kappa\lambda$. In particular, U^* extends the club filter over $\mathcal{P}_\kappa\lambda$, and therefore $X \in U^*$. Thus, $\{s \in \mathcal{P}_\kappa\gamma \colon s \cap \lambda \in X\} \in U$, which implies that $j``\lambda = j``\gamma \cap \lambda \in j(X)$, by definition of U.

Computing $j(T)$ and $j(p)$, we get that $j(T) = \bigcup_{s \in \mathcal{P}_{j(\kappa)} j(\lambda)} T^*_s$ and that $j(p)(x) = \{\varphi^*_i(x) \colon i < j(\lambda)\}$ for some theories T^*_s and formulas φ^*_i. By elementarity, for every $s \in j(X)$, M believes that T^*_s has a model omitting p^*_s. In particular, this is true for $s = j``\lambda \in j(X)$. We claim $j``T \subseteq T^*_{j``\lambda}$. If $\varphi \in T$, then $\varphi \in T_s$ for some $s \in \mathcal{P}_\kappa\lambda$. Now because $s \in \mathcal{P}_\kappa\lambda$, $s = \{\alpha_i \colon i < \beta\}$ for some $\beta < \kappa$ and $\alpha_i \in \lambda$. Therefore $j(s) = \{j(\alpha_i) \colon i < \beta\} \subseteq j``\lambda$. Then $j(\varphi) \in j(T_s) = T^*_{j(s)} \subseteq T^*_{j``\lambda}$. Further, $p^*_{j``\lambda} = \{\varphi^*_i(x) \colon i \in j``\lambda\} = j``p$.

Summarising, M has a model $\mathcal{A}^* \models j``T$ omitting $j``p$. Note that because M is closed under γ-sequences, it knows that $\mathcal{A}^* \restriction j``\tau$ can be renamed to a τ-structure $\mathcal{A}$ and it believes that $\mathcal{A}$ satisfies T and omits p. Further, also $\mathcal{A} \in V^M_{j(\gamma)}$. Because $V^M_{j(\gamma)}$ and M agree on $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$-satisfaction, $V^M_{j(\gamma)}$ thus satisfies the sentence

$$\exists \mathcal{B}(\mathcal{B} \models_{\mathcal{L}_{\kappa\kappa}(\mathsf{I})} T \text{ and } \mathcal{B} \text{ omits } p).$$

Because $\mathcal{L}_{\kappa\kappa}(\mathsf{I})$-satisfaction is $\Sigma_1(\mathrm{Card})$ definable using κ as a parameter, the above is a $\Sigma_1(\mathrm{Card})$ statement. By elementarity, $M \models j(\gamma) \in C^{(1\mathrm{Card})}$ and by assumption also $M \models \gamma \in C^{(1\mathrm{Card})}$. Therefore, $V_\gamma \prec_{\Sigma_1(\mathrm{Card})} V^M_{j(\gamma)}$. Hence, also V_γ satisfies the above statement and has a model $\mathcal{B} \in V_\gamma$ which it believes to satisfy T and to omit p. Clearly, V_γ is correct about this. □

Note that when in (2) of the above theorem we substitute $C^{(1\mathrm{Card})}$ for $C^{(1)}$, we get supercompact cardinals, while substituting for $C^{(2)}$ gives extendible cardinals by Bagaria's and Goldberg's Theorem 2.2.6, and so the notion (2) naturally lies in between supercompactness and extendibility. Moreover, property (1) above implies that in particular $\kappa = \mathrm{comp}(\mathcal{L}_{\kappa\kappa}(\mathsf{I}))$. By Theorem 3.5.13, then κ is cardinal correctly extendible. Thus, to answer Question 3.4.17 negatively, it would be sufficient to answer negatively:

Question 7.4.2. Can the smallest cardinal witnessing (2) of Theorem 7.4.1 be the smallest extendible cardinal?

A. Syntaxes of Logics

In this appendix, we carry out the formal definition of the syntax of the logics used throughout the thesis. For this, recall the coding of non-logical symbols from Definition 1.1.1. The syntax of our logics is coded similarly. Throughout, let us assume that $\kappa \geq \lambda$ are regular cardinals. First, let us fix (first- and second-order) variables.

Definition A.1. We define the following notions.

(i) For each set a and $n \in \omega$, we call $x = (4, (n, a))$ an *individual* or *first-order variable* (*of sort* n). We also write $s(x) = n$ to denote the sort of x.

(ii) For each set a and $n_1, \ldots, n_k \in \omega$, we call $X = (5, (n_1, \ldots, n_k, a))$ a *second-order* or *relation variable* (*of arity* k *between the sorts* $n_1, \ldots, n_k$). We also write $\operatorname{conf}(X) = (n_1, \ldots, n_k)$ to denote the configuration of X, and $s(X) = \{n_1, \ldots, n_k\}$ to denote the set of sorts appearing in X.

We also simply say *variables* to denote both first- and second-order variables.

First, let us fix the syntax of infinitary logics $\mathcal{L}_{\kappa\lambda}$. In particular, first-order logic results from considering $\kappa = \lambda = \omega$.

Definition A.2. Let τ be a vocabulary. The set of formulas $\mathcal{L}_{\kappa\lambda}$ is defined recursively in the following way.

(i) If $r \in \tau$ is a relation symbol with $\operatorname{conf}(r) = (n_1, \ldots, n_k)$ and $x_1, \ldots, x_k$ each are either individual variables in H_κ, or constant symbols in τ of sort $s(x_i) = n_i$, then $\varphi = (6, r, x_1, \ldots, x_k) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $r(x_1, \ldots, x_k)$.

(ii) If $f \in \tau$ is a function symbol with $\operatorname{conf}(f) = (n_1, \ldots, n_{k+1})$ and $x_1, \ldots, x_{k+1} \in H_\kappa$ each are either individual variables in H_κ, or constant symbols in τ of sort $s(x_i) = n_i$, then $\varphi = (7, f, n_1, \ldots, n_{k+1}) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $f(x_1, \ldots, x_k) = x_{n_{k+1}}$.

(iii) If x and y each are variable symbols in H_κ or constant symbols of a sort $s(x), s(y) \in s(\tau)$, then $\varphi = (8, x, y) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $x = y$.

(iv) If $\psi \in \mathcal{L}_{\kappa\lambda}[\tau]$, then $\varphi = (9, \varphi) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $\neg\psi$.

(v) If $T \subseteq \mathcal{L}_{\kappa\lambda}[\tau]$ and $T = \{\varphi_i \colon i < \gamma\}$ for some $\gamma < \kappa$, then $\varphi = (10, T) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $\bigwedge T$.

(vi) If $\psi \in \mathcal{L}_{\kappa\lambda}[\tau]$, $Z \in H_\lambda$, and $(x_i \colon i \in Z)$ is a sequence of first-order variables in H_κ each of a sort $s(x_i) \in s(\tau)$, then $\varphi = (11, (x_i \colon i \in Z), \psi) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $\exists(x_i \colon i \in Z)\psi$.

We further let $\mathcal{L}_{\infty\infty}[\tau] = \bigcup_{\kappa\in\text{Card}} \mathcal{L}_{\kappa\kappa}[\tau]$.

We may consider finite conjunctions $\wedge$ as special cases of (v), considering singletons T, and finite existential quantifications as special cases of (vi), considering singletons Z. We can further define the other boolean connectives $\vee$, $\to$, and $\leftrightarrow$, as well as infinite disjunctions $\bigvee$, and (finite or infinite) universal quantification $\forall$ in the usual way as abbreviations. Thus, the above develops the syntax of $\mathcal{L}_{\kappa\lambda}$ satisfactorily.

Note that our definition makes sure that for formulas of $\mathcal{L}_{\kappa\lambda}$ only variables in H_κ are used. In particular, if $\tau \in H_\kappa$, this makes sure that $\mathcal{L}_{\kappa\lambda}[\tau] \subseteq H_\kappa$, and thus $\text{dep}^*(\mathcal{L}_{\kappa\lambda}) = \kappa$.

For the other strong logics we consider, let us indicate how to adopt the above definition by adding additional conditions.

(1) For second-order logic $\mathcal{L}^2_{\kappa\lambda}$, use an analogous recursion as above, but add a condition dealing with atomic formulas given by relation variables, and one dealing with second-order quantification.

- (i*) If $X \in H_\kappa$ is a relation variable with configuration $\text{conf}(X) = (n_1, \dots, n_k)$ for $n_1, \dots, n_k \in s(\tau)$ and $x_1, \dots, x_k$ each are either individual variables in H_κ, or constant symbols in τ, of sort $s(x_i) = n_i$, then $\varphi = (6, 0, X, x_1, \dots, x_k) \in \mathcal{L}^2_{\kappa\lambda}[\tau]$. We also write φ as $X(x_1, \dots, x_k)$.
- (vi*) If $\psi \in \mathcal{L}^2_{\kappa\lambda}[\tau]$, $Z \in H_\lambda$, and $(X_i : i \in Z)$ is a sequence of second-order variables in H_κ each of a configuration $s(X_i) = (n^i_1, \dots, n^i_{k_i})$ such that each $n^i_j \in s(\tau)$, then $\varphi = (12, (X_i : i \in Z), \psi) \in \mathcal{L}^2_{\kappa\lambda}[\tau]$. We also write φ as $\exists(X_i : i \in Z)\psi$.

Further, $\mathcal{L}^2[\tau] = \mathcal{L}^2_{\omega\omega}[\tau]$.

(2) For $\mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\text{WF}})$, use an analogous recursion as above, but add as a condition:

- (vii) If $\psi \in \mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\text{WF}})[\tau]$ and $x, y \in H_\kappa$ are individual variables each of a sort $s(x), s(y) \in s(\tau)$, then $\varphi = (13, x, y, \psi) \in \mathcal{L}_{\kappa\lambda}(\mathsf{Q}^{\text{WF}})[\tau]$. We also write φ as $\mathsf{Q}^{\text{WF}}xy\psi$.

Then $\mathcal{L}(\mathsf{Q}^{\text{WF}})[\tau] = \mathcal{L}_{\omega\omega}(\mathsf{Q}^{\text{WF}})[\tau]$.

(3) For $\mathcal{L}_{\kappa\lambda}(\mathsf{I})$, use an analogous recursion as above, but add as a condition:

- (viii) If $\psi, \chi \in \mathcal{L}_{\kappa\lambda}(I)[\tau]$ and $x, y \in H_\kappa$ are individual variables each of a sort $s(x), s(y) \in s(\tau)$, then $\varphi = (14, x, y, \psi, \chi) \in \mathcal{L}_{\kappa\lambda}(\mathsf{I})[\tau]$. We also write φ as $\mathsf{I}xy\psi\chi$.

Then $\mathcal{L}(\mathsf{I})[\tau] = \mathcal{L}_{\omega\omega}(I)[\tau]$.

For sort logic, as we have extra conditions on where sort quantifiers can appear, let us state the full recursion separately. It is sufficient to define the class of formulas $\mathcal{L}^{\text{s}}_{\infty\omega}$, as this contains all formulas of $\mathcal{L}^{\text{s},n}_{\kappa\omega}$ for any n and any κ.

Definition A.3. Let τ be a vocabulary. The class of formulas of $\mathcal{L}^{\text{s}}_{\infty\omega}$ is defined recursively in the following way.

(i) If $r \in \tau$ is a relation symbol with $\mathrm{conf}(r) = (n_1, \ldots, n_k)$ and $x_1, \ldots, x_k$ each are either individual variables, or constant symbols in τ, of sort $s(x_i) = n_i$, then $\varphi = (6, r, x_1, \ldots, x_k) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ as $r(x_1, \ldots, x_k)$.

(ii) If X is a relation variable with configuration $\mathrm{conf}(X) = (n_1, \ldots, n_k)$ and $x_1, \ldots, x_k$ each are either individual variables, or constant symbols in τ, of sort $s(x_i) = n_i$, then $\varphi = (6, 0, X, x_1, \ldots, x_k) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ as $X(x_1, \ldots, x_k)$.

(iii) If $f \in \tau$ is a function symbol with $\mathrm{conf}(f) = (n_1, \ldots, n_{k+1})$ and $x_1, \ldots, x_{k+1} \in H_\kappa$ each are either individual variables in H_κ, or constant symbols in τ of sort $s(x_i) = n_i$, then $\varphi = (7, f, x_1, \ldots, x_{k+1}) \in \mathcal{L}_{\kappa\lambda}[\tau]$. We also write φ as $f(x_1, \ldots, x_k) = x_{n_{k+1}}$.

(iv) If x and y each are variable symbols or constant symbols of a sort $s(x), s(y) \in s(\tau)$, then $\varphi = (8, x, y) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ as $x = y$.

(v) If $\psi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$, then $\varphi = (9, \varphi) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ as $\neg\psi$.

(vi) If $T \subseteq \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ is a set, then $\varphi_0 = (10, 0, T), \varphi_1 = (10, 1, T) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ_0 as $\bigwedge T$ and φ_1 as $\bigvee T$.

(vii) If $\psi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ and ψ contains no sort quantifiers, then if x, y are individual variables, then $\varphi = (13, x, y, \psi) \in \mathcal{L}^{s}_{\kappa\omega}[\tau]$. We also write φ as $\mathsf{Q}^{\mathrm{WF}} xy\psi$.

(viii) If $\psi \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$ and ψ contains no sort quantifiers, then if x is an individual variable, then $\varphi_0 = (15, x, \psi), \varphi_1 = (16, x, \psi) \in \mathcal{L}^{\mathrm{s}}_{\kappa\omega}[\tau]$. We also write φ_0 as $\exists x\varphi$ and φ_1 as $\forall x\varphi$.

(ix) If $\psi \in \mathcal{L}^{s}_{\kappa\omega}[\tau]$ and X is a relation symbol such that $s(X) \cap s(\tau) = \emptyset$ and no free variable of φ involves a sort $n \in s(X)$, then $\varphi_0 = (17, X, \varphi), \varphi_1 = (18, X, \varphi) \in \mathcal{L}^{\mathrm{s}}_{\infty\omega}[\tau]$. We also write φ_0 as $\tilde{\exists} X\varphi$ and φ_1 as $\tilde{\forall} X\varphi$.

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English Summary

We study connections between *large cardinal axioms*, and statements that extensions of first-order logic, also known as *strong logics*, exhibit certain model-theoretic properties. The properties we consider can be separated into three types: compactness properties, upward Löwenheim-Skolem properties, and downward Löwenheim-Skolem properties. Each of these three types generalises an important theorem for first-order logic to stronger logics, namely the Compactness Theorem, and the upward and downward Löwenheim-Skolem Theorems.

An important large cardinal axiom intimately connected to model theory of strong logics is *Vopěnka's Principle* (VP). It was known before that VP stratifies naturally into a hierarchy of stronger and stronger assumptions along the $C^{(n)}$*-extendible cardinals*, and along compactness and downward Löwenheim-Skolem properties of logics.

In Chapter 2, we consider the notion of a (weak) Henkin model for an abstract logic $\mathcal{L}$ introduced in [BDGM24]. It was known that *Henkin chain compactness* (HCC) properties involving weak Henkin models characterise *strong* and *Woodin* cardinals [Bon20,BDGM24]. We introduce a natural strengthening of the notion of Henkin models considered in these characterisations called *strong Henkin models*. This comes with a natural strengthening of HCC properties to *strong Henkin compactness* (SHC) properties. We show that SHC properties of second-order logic can characterise stronger large cardinals, namely supercompact cardinals (Theorem 2.2.4). We further show that the known stratification of VP in terms of $C^{(n)}$-extendible cardinals has an entirely analogous one in terms of SHC properties of *sort logics* (Theorem 2.2.8). We get:

Corollary 2.2.12. The following are equivalent.

(1) VP.

(2) For every logic $\mathcal{L}$ and every natural number n, there is an n-SHC number.

We also continue the study of HCC properties and consider *weak Vopěnka's Principle* (WVP), a weakening of VP arising from category theory (cf. [ART88] and [AR94, Chapter 6]). It was known that WVP stratifies analogously to VP into a hierarchy of axioms along the Π_n*-strong cardinals*. Whether WVP also has a model-theoretic characterisation was open. We positively answer this question by showing that HCC properties of sort logics characterise the Π_n-strong cardinals (Theorem 2.3.6).[1] We get:

Corollary 2.3.11. The following are equivalent:

(1) WVP.

(2) For every logic $\mathcal{L}$ and every natural number n, there is an n-HCC number of $\mathcal{L}$.

[1]Note that Holy, Lücke, and Müller provide an independent positive answer (cf. [HLM24]). The model-theoretic properties they consider are entirely different from ours.

We further characterise other large cardinals, namely *jointly Π_n-strong and strongly compact*, and *superstrong cardinals* by HCC properties of logics

Chapter 3 introduces a new large cardinal notion called *cardinal correctly extendible* cardinals. They are a natural weakening of extendible cardinals, and we show that they arrive naturally from compactness properties for the logic $\mathcal{L}(\mathsf{I})$. Further, we study their relations to other large cardinal properties. We show that they are strongly compact, and separate them from supercompact cardinals. We prove a theorem that under certain assumptions on the relation of the universe of sets V to the inner model HOD, extendible cardinals preserve their cardinal correct extendibility in HOD:

Theorem 3.4.10. Suppose that there is a smallest extendible cardinal δ, that the HOD Hypothesis holds, and that every HOD-cardinal is a cardinal. Then δ is cardinal correctly extendible in HOD.

This separates the notion from extendibility (Corollary 3.4.15).

Chapter 4 studies upward Löwenheim-Skolem-Tarski (ULST) numbers of strong logics. We answer a question from [Gal19, GKV20] positively whether the ULST number of second-order logic is the smallest extendible cardinal (Theorem 4.5.1). We strengthen the notion and consider *strong* ULST numbers. We show that VP has yet another stratification by the ULST and strong ULST numbers of sort logics (Theorem 4.6.1). We get:

Corollary 4.6.4. The following are equivalent:

(1) VP.

(2) Every logic has a ULST number

(3) Every logic has a strong ULST number.

We determine the ULST and strong ULST numbers of several other logics, including the well-foundedness logic, the equicardinality logic, and infinitary logics. Their existence is, in turn, equivalent to that of measurable cardinals, variations of cardinal correctly extendible cardinals, and variations of tall cardinals. We show that for some logics, the ULST and strong ULST numbers are necessarily the same, while for others they may be separated. For an abstractly given logic $\mathcal{L}$, we introduce *$\mathcal{L}$-extendible cardinals* and show that their existence is equivalent to the existence of ULST numbers and strong ULST numbers of $\mathcal{L}$ for a large class of logics.

Chapter 5 considers *class logics*, which are logics that have a proper class of sentences over a mere set of non-logical symbols. We determine compactness numbers of class extensions of first-order logics, infinitary logics, second-order logic, and sort logics. A new compactness property of second-order class logics gives the first known model-theoretic characterisation of *Shelah cardinals* (Theorem 5.5.1). We show that certain downward Löwenheim-Skolem properties of class extensions of sort logics characterise Π_n-strong cardinals (Theorem 5.4.3). We therefore get a second stratification of WVP by entirely different model-theoretic properties:

Corollary 5.4.6. The following are equivalent.

(1) WVP.

(2) Every class logic $\mathcal{L}$ with $\mathcal{L} \leq \mathcal{L}^{s,n}_{\kappa\omega}(\vee^\infty, \forall^\infty)$ for some κ has an LS^ω number.

Chapter 6 continues some work from the author's Master's thesis [OP24]. Bagaria and Väänänen [BV16], and then Galeotti, Khomskii, and Väänänen [Gal19, GKV20] studied equivalences between model-theoretic properties of logics and reflection principles mediated by *symbiosis* between a logic and a set-theoretic predicate. They considered downward and upward Löwenheim-Skolem properties, respectively. In [Gal19, GKV20], the question was asked whether there is an analogous equivalence between compactness properties and reflection properties. We show that this is the case and formulate the reflection principle $\mathrm{EEP}^\lambda_\kappa(R)^+$ for classes of partial orders defined via some set-theoretic predicate R. Our result is:

Theorem 6.3.12. Let $\mathcal{L}$ be a logic and R a set-theoretic predicate. Assume $\mathcal{L} \geq \mathcal{L}_{\lambda\omega}$, $\mathrm{dep}^*(\mathcal{L}) = \lambda$ and that $\mathcal{L}$ and R are p-symbiotic. Then the following are equivalent for a regular cardinal κ:

(1) $\mathcal{L}$ is κ-compact.

(2) $\mathrm{EEP}^\lambda_\kappa(R)^+$.

Further, we give proofs of some results stated by Bagaria in [Bag23] how weak downward Löwenheim-Skolem properties of logics are equivalent to weak reflection principles, mediated by symbiosis and some additional assumption introduced by us about $\mathcal{L}$'s ability to define R (Theorem 6.4.13).

Finally, Chapter 7 considers *compactness for type omission* properties, which were first introduced by Benda [Ben78] and then recently studied by Boney [Bon20]. Wilson in [Wil22a] asked the question whether it is possible to have a property P of a finitary logic $\mathcal{L}$ such that the statement that $\mathcal{L}$ has P exceeds VP in consistency strength. It was first proven by Boney [Bon20] and Wilson [Wil22a] that there are such properties at all, but the logics they considered were infinitary. We prove a theorem showing that a property of a finitary logic can achieve this:

Theorem 7.3.2. Let κ and λ be cardinals such that $\lambda > \kappa$. Then κ is huge with target λ iff $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ is $[\lambda]^\kappa_*$-κ-compact for type omission.

We also show how a large cardinal notion which naturally sits in between supercompactness and extendibility can be characterised by compactness for type omission of infinitary versions of $\mathcal{L}(\mathsf{I})$ (Theorem 7.4.1).

Deutsche Zusammenfassung

Diese Dissertation untersucht Verbindungen zwischen *großen Kardinalzahlaxiomen* und Aussagen über modelltheoretische Eigenschaften *starker Logiken*, d.h., von Erweiterungen der Prädikatenlogik erster Stufe. Wir untersuchen drei Arten von Eigenschaften: Kompaktheitseigenschaften, sowie aufwärts- und abwärtsgerichtete Löwenheim-Skolem Eigenschaften. Jeder dieser drei Typen von Eigenschaften verallgemeinert einen wichtigen Satz über die Prädikatenlogik erster Stufe: den Kompaktheitssatz, sowie den aufwärts- und den abwärtsgerichteten Satz von Löwenheim-Skolem.

Ein wichtiges großes Kardinalzahlaxiom mit engen Verbindungen zur Modelltheorie starker Logiken ist *Vopěnkas Prinzip* (VP). Es war bekannt, dass VP eine natürliche Stratifizierung durch immer stärker werdende Annahmen durch die $C^{(n)}$*-erweiterbaren* Kardinalzahlen aufweist, sowie analoge Stratifizierungen durch Kompaktheits- und abwärtsgerichtete Löwenheim-Skolem Eigenschaften von Logiken.

In Kapitel 2 untersuchen wir den Begriff eines (schwachen) Henkin-Modells für eine abstrakte Logik $\mathcal{L}$ aus [BDGM24]. Es war bekannt, dass *Henkin-Ketten-Kompaktheitseigenschaften* (HKK-Eigenschaften) mit schwachen Henkin-Modellen *starke* und *Woodin* Kardinalzahlen charakterisieren [Bon20, BDGM24]. Wir führen mit *starken Henkin-Modellen* eine natürliche Verstärkung der schwachen Henkin-Modelle ein. Diese führen mit dem Begriff der *starken Henkin-Kompaktheit* (SHK) zu einer natürlichen Verstärkung von HKK-Eigenschaften. Wir zeigen, dass SHK-Eigenschaften der Prädikatenlogik zweiter Stufe superkompakte, und damit stärkere große Kardinalzahlen charakterisieren (Satz 2.2.4). Weiterhin führen SHK-Eigenschaften von *Sortenlogiken* zu den $C^{(n)}$-erweiterbaren Kardinalzahlen (Satz 2.2.8). Wir erhalten das folgende Resultat:

Korollar 2.2.12. Die folgenden Aussagen sind äquivalent.

(1) VP.

(2) Jede Logik hat für jede natürliche Zahl n eine n-SHK-Zahl.

Weiterhin untersuchen wir das *schwache Vopěnka-Prinzip* (SVP), eine natürliche Abschwächung von VP, die durch Kategorientheorie motiviert ist (s. [AR94, Kapitel 6]). Es war bekannt, dass SVP eine zu VP analoge Stratifizierung durch große Kardinalzahlen aufweist, die sogenannten Π_n*-starken Kardinalzahlen.* Ob SVP ebenfalls durch modelltheoretische Eigenschaften charakterisiert werden kann, war offen. Wir beantworten dies positiv und zeigen, dass HKK-Eigenschaften von Sortenlogiken Π_n-starke Kardinalzahlen charakterisieren.[2] Es folgt:

Satz 2.3.11. Die folgenden Aussagen sind äquivalent:

(1) SVP.

(2) Jede Logik hat für jede natürliche Zahl n eine n-HKK-Zahl.

[2] Holy, Lücke und Müller haben diese Frage unabhängig durch andere modelltheoretische Eigenschaften beantwortet (s. [HLM24]).

Wir charakterisieren außerdem *gleichzeitig Π_n-stark und stark kompakte*, sowie *superstarke* Kardinalzahlen durch HKK-Eigenschaften von Logiken.

Kapitel 3 führt einen neuen großen Kardinalzahlbegriff ein: *kardinalzahlkorrekt-erweiterbare Kardinalzahlen.* Diese stellen eine natürliche Abschwächung von erweiterbaren Kardinalzahlen dar, und wir zeigen, dass sie auf natürliche Weise aus Kompaktheitseigenschaften der Logik $\mathcal{L}(\mathsf{I})$ hervorgehen. Wir zeigen weiterhin, dass sie stark kompakt sind und separieren sie von superkompakten Kardinalzahlen. Wir zeigen, dass unter bestimmten Annahmen über das Verhältnis des Universums V zum inneren Modell HOD, die kardinalzahlkorrekte Erweiterbarkeit von erweiterbaren Kardinalzahlen in HOD erhalten bleibt:

Satz 3.4.10. Sei δ die kleinste erweiterbare Kardinalzahl. Wenn die HOD-Hypothese gilt und jede HOD-Kardinalzahl eine Kardinalzahl ist, dann ist δ eine kardinalzahlkorrekt-erweiterbare Kardinalzahl in HOD.

Dies separiert unseren Begriff vom Begriff der Erweiterbarkeit (Korollar 3.4.15).

Kapitel 4 untersucht *aufwärtsgerichtete Löwenheim-Skolem-Tarski-Zahlen* (ALST-Zahlen) starker Logiken. Wir geben eine positive Antwort auf eine Frage aus [Gal19, GKV20], ob die ALST-Zahl der Prädikatenlogik zweiter Stufe die erste erweiterbare Kardinalzahl ist (Satz 4.5.1). Wir führen außerdem den Begriff der *starken* ALST-Zahl ein. Wir zeigen, dass ALST- und starke ALST-Zahlen von Sortenlogiken eine weitere Stratifizierung von VP ergeben (Satz 4.6.1). Es folgt:

Satz 2.3.11. Die folgenden Aussagen sind äquivalent:

(1) VP.

(2) Jede Logik hat eine ALST-Zahl.

(3) Jede Logik hat eine starke ALST-Zahl.

Wir bestimmen die ALST- und starken ALST-Zahlen von weiteren Logiken, nämlich der Fundiertheitslogik $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$, der Härtig-Logik $\mathcal{L}(\mathsf{I})$, und von infinitären Logiken. Ihre Existenz ist, der Reihe nach, äquivalent zu der von messbaren Kardinalzahlen, Abwandlungen von kardinalzahlkorrekt-erweiterbaren Kardinalzahlen, sowie Abwandlungen von *hohen* Kardinalzahlen. Wir zeigen, dass für einige Logiken die Existenz von ALST- und starken ALST-Zahlen äquivalent ist, während diese für andere Logiken voneinander separiert werden können. Für eine beliebige abstrakte Logik $\mathcal{L}$ führen wir den Begriff der *$\mathcal{L}$-erweiterbaren Kardinalzahl* ein und zeigen, dass ihre Existenz für eine große Klasse von Logiken zur Existenz von ALST-Zahlen von $\mathcal{L}$ äquivalent ist.

Kapitel 5 untersucht *Klassenlogiken.* Wir bestimmen Kompaktheitszahlen von Klassenerweiterungen der Prädikatenlogik erster und zweiter Stufe, sowie von infinitären Logiken und Sortenlogiken. Wir beweisen, dass *Shelah* Kardinalzahlen eine modelltheoretische Charakterisierung durch eine Kompaktheitseigenschaft von Klassenerweiterungen der Prädikatenlogik zweiter Stufe aufweisen (Satz 5.5.1). Wir zeigen weiterhin, dass Π_n-starke Kardinalzahlen durch bestimmte abwärtsgerichtete Löwenheim-Skolem Eigenschaften von Sortenlogiken charakterisiert werden (Satz 5.4.3). Wir erhalten eine zweite Stratifizierung von SVP durch andere modelltheoretische Eigenschaften:

Theorem 5.4.6. Die folgenden Aussagen sind äquivalent.

(1) SVP.

(2) Jede Klassenlogik $\mathcal{L}$, sodass $\mathcal{L} \leq \mathcal{L}^{s,n}_{\kappa\omega}(\bigvee^{\infty}, \forall^{\infty})$ für ein κ, hat eine LS^{ω}-Zahl.

Kapitel 6 führt die Ergebnisse der Masterarbeit [Osi21] des Autors fort. Bagaria und Väänänen [BV16], sowie Galeotti, Khomskii und Väänänen [Gal19, GKV20] haben Äquivalenzen zwischen modelltheoretischen Eigenschaften von Logiken und Reflexionsprinzipien untersucht, die durch das Phänomen der *Symbiose* zwischen einer Logik und einem mengentheoretischen Prädikat vermittelt werden. Ihre Ergebnisse betrafen abwärts- und aufwärtsgerichtete Löwenheim-Skolem Eigenschaften. In [Gal19, GKV20] wurde die Frage gestellt, ob eine analoge Äquivalenz zwischen Kompaktheitseigenschaften und Reflexionsprinzipien besteht. Wir zeigen, dass dies der Fall ist. Für Klassen partieller Ordnungen, die mittels eines Prädikates R definiert sind, führen wir das Reflexionsprinzip $\mathrm{EEP}^{\lambda}_{\kappa}(R)^{+}$ ein. Wir zeigen:

Satz 6.3.12. Sei $\mathcal{L}$ eine Logik und R ein mengentheoretisches Prädikat, die p-symbiotisch zueinander sind, und sodass $\mathcal{L} \geq \mathcal{L}_{\lambda\omega}$ und $\mathrm{dep}^{*}(\mathcal{L}) = \lambda$. Dann sind die folgenden Aussagen für eine reguläre Kardinalzahl κ äquivalent:

(1) $\mathcal{L}$ ist κ-kompakt.

(2) $\mathrm{EEP}^{\lambda}_{\kappa}(R)^{+}$.

Weiterhin führen wir Beweise für einige von Bagaria [Bag23] erwähnte Resultate, dass schwache abwärtsgerichtete Löwenheim-Skolem Eigenschaften von Logiken äquivalent zu schwachen Reflexionsprinzipien sind, wiederum unter der Annahme von Symbiose, sowie einer zusätzlichen Annahme darüber, dass R in einem starken Sinne durch $\mathcal{L}$ definierbar ist (Satz 6.4.13).

Kapitel 7 betrachtet *Kompaktheitseigenschaften für Typenauslassung*, die zuerst von Benda [Ben78], sowie jüngst von Boney [Bon20] untersucht wurden. Wilson hat in [Wil22a] die Frage gestellt, ob Eigenschaften endlicher Logiken eine Konsistenzstärke aufweisen können, die die von Vopěnkas Prinzip übersteigt. Zunächst hatten Boney [Bon20] und Wilson [Wil22a] gezeigt, dass es überhaupt solche Eigenschaften von Logiken gibt, jedoch untersuchten beide infinitäre Logiken. Wir zeigen, dass es auch für endliche Logiken solch starke Resultate gibt:

Satz 7.3.2. Seien κ und λ Kardinalzahlen, sodass $\lambda > \kappa$. Dann ist κ genau dann riesig mit Ziel λ, wenn $\mathcal{L}(\mathsf{Q}^{\mathrm{WF}})$ $[\lambda]^{\kappa}_{*}$-κ-kompakt für Typenauslassung ist.

Wir zeigen außerdem, dass eine in natürlicher Weise zwischen Superkompaktheit und Erweiterbarkeit liegende Kardinalzahleigenschaft durch Kompaktheitseigenschaften für Typenauslassung von infinitären Versionen von $\mathcal{L}(\mathsf{I})$ charakterisiert wird (Satz 7.4.1).