Non-Classical Set Theories and Logics Associated With Them

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Sourav Tarafder

Department of Pure Mathematics

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To My Family

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Contents

List of papers included in the thesis

1. Löwe, B., & Tarafder, S. (2015). Generalized Algebra-Valued Models of Set Theory. *Review of Symbolic Logic*, **8**(1), pp. 192–205.

2. Tarafder, S. (2015). Ordinals in an Algebra-Valued Model of a Paraconsistent Set Theory. In Banerjee, M. and Krishna, S., editors. *Logic and Its Applications, 6th International Conference, ICLA 2015, Mumbai, India, January 8–10, 2015*, Lecture Notes in Computer Science, Vol. 8923. Berlin: Springer-Verlag, pp. 195–206.

3. Tarafder, S., & Chakraborty, M. K. (2016). A Paraconsistent Logic Obtained from an Algebra-Valued Model of Set Theory. Beziau, J. Y., Chakraborty, M. K. and Dutta, S., editors, *New Directions in Paraconsistent Logic, 5th WCP, Kolkata, India, February 2014*, Springer Proceedings in Mathematics & Statistics, Vol. 152. New Delhi: Springer, pp. 165–183.

Chapter 1

Introduction

Classical and Non-classical set theories

Classical set theory. We shall briefly discuss about classical set theory in the framework of *first order predicate calculus*. The language of set theory contains $=$ and $\in$ as two predicate symbols. The connectives are $\wedge$, $\vee$, $\rightarrow$, and $\neg$. The symbols $\forall$ and $\exists$ are the quantifiers. For reference, we list the forms of the axioms and axiom schemes that we use in this thesis (in the schemes, φ is a formula with $n+2$ free variables); the concrete formulations follow [4] very closely:

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y] \tag{Extensionality}$$

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y)) \tag{Pairing}$$

$$\exists x [\exists y (\forall z (z \in y \rightarrow \bot) \wedge y \in x) \wedge \forall w \in x \exists u \in x (w \in u)]. \tag{Infinity}$$

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x (z \in x)) \tag{Union}$$

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w \in z (w \in x)) \tag{Power Set}$$

$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z, p_0, \ldots, p_n)) \tag{Separation$_\varphi$}$$

$$\forall p_0 \cdots \forall p_{n-1} \forall x[\forall y \in x \exists! z \varphi(y, z, p_0, \ldots, p_{n-1}) \\ \to \exists w \forall v \in x \exists u \in w \ \varphi(v, u, p_0, \ldots, p_{n-1})] \qquad (\mathsf{Replacement}_{\varphi})$$

$$\forall p_0 \cdots \forall p_n \forall x[\forall y \in x \ \varphi(y, p_0, \ldots, p_n) \to \varphi(x, p_0, \ldots, p_n)] \\ \to \forall z \varphi(z, p_0, \ldots, p_n) \qquad (\mathsf{Foundation}_{\varphi})$$

The axioms Extensionality, Pairing, Infinity[1], Union, Power Set, and Separation are due to Ernst Zermelo; Replacement is independently due to Abraham Fraenkel and Thoralf Skolem; Foundation is due to John von Neumann. The theory with the above axioms is called Zermelo-Fraenkel axiomatic set theory (ZF). The following sentence is known as the *Axiom of Choice* (AC):

$$\forall y \exists f[\mathrm{Func}(f) \wedge \mathrm{dom}(f) = y \wedge \forall x(x \in y \wedge x \neq \varnothing \to f(x) \in x)],$$

where Func(f) and dom(f) are the abbreviations for "f is a function" and "domain of f", respectively. The set theory ZF with the Axiom of Choice is denoted by ZFC. The set theory having all the axioms of ZF excluding Foundation is represented by ZF^-. In ZFC if the axiom scheme of Replacement is replaced by the scheme $\mathsf{Collection}_{\varphi}$, expressed below, then the theory remains equivalent in strength.

$\mathsf{Collection}_{\varphi}$

$$\forall p_0 \cdots \forall p_{n-1} \forall x[\forall y \in x \exists z \varphi(y, z, p_0, \ldots, p_{n-1}) \to \exists w \forall v \in x \exists u \in w \ \varphi(v, u, p_0, \ldots, p_{n-1})],$$

where φ is a formula with $n+2$ free variables. From now onwards whenever we shall refer the systems ZFC, ZF, and ZF^- it will be assumed that we are using $\mathsf{Collection}_{\varphi}$ instead of $\mathsf{Replacement}_{\varphi}$.

[1] In the formulation of Infinity, we avoid the use of the $\neg$ symbol since we shall later require that the axioms are in the negation-free fragment. Of course, this formulation is classically equivalent to the formulation with $\notin$.

In classical set theory we deal with only one type of object, viz. *sets*. But for simplicity sometimes we refer to *classes* whose informal notion is given below:

If φ is a formula in $n+1$ free variables, $x_1, ..., x_n$ are sets, then $\{x\colon \varphi(x, x_1, ..., x_n)\}$ is called the class defined by φ and $x_1, ..., x_n$. The members of this class are those sets which satisfy $\varphi(x, x_1, ..., x_n)$. Two classes are said to be equal if they have same elements:

$$\{x\colon \varphi(x, x_1, \ldots, x_n)\} = \{x\colon \psi(x, y_1, \ldots, y_n)\}$$

if and only if for any x the following holds;

$$\varphi(x, x_1, \ldots, x_n) \leftrightarrow \psi(x, y_1, \ldots, y_n).$$

From now onwards we shall consider the *universal class*, $\mathbf{V} = \{x\colon x = x\}$ as the standard class model of ZF unless otherwise it is stated to be a class model of ZFC.

By a *non-classical set theory* we mean here a set theory whose underlying logic is non-classical. Examples of non-classical set theories are *intuitionistic Zermelo-Fraenkel set theory* (IZF), *constructive Zermelo-Fraenkel set theory* (CZF), *quantum set theory*, and *paraconsistent set theory.*

Intuitionistic and constructive set theory. Both of the set theories IZF and CZF were proposed by John Myhill in 1973 [19]. Constructive mathematics can be developed in the logical framework of constructive set theory. The underlying logic for both of CZF and IZF is *Intuitionistic logic.* But they do not have the same set theoretic axioms [1, Section 2]. The set theoretic axioms of IZF are exactly the same as the axioms of ZF in the form expressed above. On the other hand the set theoretic axioms for CZF are Extensionality, Pairing, Infinity, Union, Restricted Separation, Strong Collection, Subset Collection, and Foundation; where

Restricted Separation$_\varphi$:

$$\forall p_0 \dots \forall p_n \forall a \exists b \forall x[x \in b \leftrightarrow x \in a \wedge \varphi(x, p_0, \dots, p_n)]$$

for all restricted formula φ whose quantifiers are either of the form $\forall x \in y$ or $\exists x \in y$,

Strong Collection$_\varphi$:

$$\forall p_0 \dots \forall p_{n-1} \forall a[\forall x \in a \exists y\, \varphi(x, y, p_0, \dots, p_{n-1}) \rightarrow$$
$$\exists b(\forall x \in a \exists y \in b\, \varphi(x, y, p_0, \dots, p_{n-1}) \ \wedge\ \forall y \in b \exists x \in a\, \varphi(x, y, p_0, \dots, p_{n-1}))],$$

Subset Collection$_\varphi$:

$$\forall a \forall b \exists c[\forall u \forall x \in a \exists y \in b\, \varphi(x, y, u) \rightarrow$$
$$\exists z \in c(\forall x \in a \exists y \in z\, \varphi(x, y, u) \ \wedge\ \forall y \in z \exists x \in a\, \varphi(x, y, u))];$$

in Restricted Separation and Strong Collection, φ is a formula having $n + 2$ free variables.

Intuitionistic Zermelo-Fraenkel set theory IZF has rich collection of mathematical models. For example *Beth models, topological models, (pre)sheaf models, realizability toposes* etc. *Heyting-valued models* of IZF were introduced by R. J. Grayson in 1977 [20]. The construction is similar to the construction of the *Boolean-valued models* of ZF. In Chapter 2 we shall discuss the construction of Boolean-valued model $\mathbf{V}^{(\mathbb{B})}$ of ZF corresponding to a given *complete Boolean algebra* $\mathbb{B}$. For any *complete Heyting algebra* $\mathbb{H}$ it can be proved that $\mathbf{V}^{(\mathbb{H})}$ is an algebra-valued model of IZF. In this thesis we shall mainly emphasize on this kind of algebra-valued model constructions of *non-classical set theories*.

Quantum set theory. G. Takeuti first proposed quantum set theory in 1981 [30]. As the name indicates, the underlying logic of quantum set theory is *quantum logic*. Let $\mathbf{H}$ be a *Hilbert space* and $\mathbf{L}$ be the collection of all closed linear subspaces of $\mathbf{H}$. It can be proved that $\mathbf{L}$ will be a *complete orthomodular lattice*. Following the construction of the Boolean-valued models or Heyting-valued models, the set theory developed in the $\mathbf{L}$-valued universe $\mathbf{V}^{(\mathbf{L})}$ will be called a quantum set theory. In 1999, S. Titani generalised the idea to a *lattice-valued valued set theory*. By using any *complete lattice* $\mathscr{L}$ the $\mathscr{L}$-valued universe $\mathbf{V}^{(\mathscr{L})}$ was developed for the lattice valued set theory.

Paraconsistent set theory. In this thesis we have constructed an algebra-valued model of a paraconsistent set theory. Hence paraconsistent set theory will be our main area of interest. Before the discussions of paraconsistent set theories it is worthwhile to concentrate on the *paraconsistent logic* first.

History of paraconsistent logic

In the year 1948, Stanislaw Jaskowśki first constructed a formal system of paraconsistent propositional calculus. Independently of S. Jaskowśki, Newton C. A. da Costa started the general study of contradictory systems in 1958 [13, p. 657]. The adjective "paraconsistent" was first used by the Peruvian philosopher Francisco Miró Quesada Cantuarias in the *Simposio Latinoamericano de Lógica Matemática* held at the State University of Campinas, in 1976.

A set Γ of formulas is *inconsistent* if there is a formula φ in its language such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$. A set Γ of formulas is *trivial* or *explosive* if for any formula φ of its language, $\Gamma \vdash \varphi$. In the context of classical logic the above two notions are equivalent. A logic is said to be *paraconsistent* if there exists a set Γ of formulas such that Γ is inconsistent but not explosive. So we have the following definition of paraconsistent logic:

Definition. A logic is called paraconsistent if there exist formulas φ and ψ such that

$$\{\varphi, \neg\varphi\} \nvdash \psi. \tag{Par}$$

Many paraconsistent logics have been studied. These logics were developed with various motivations. For example: Jaskowśki's paraconsistent logic, Da Costa's paraconsistent logic systems C_n where $0 < n < \omega$, Priest's logic of paradox, also other paraconsistent logics defined by C. Mortensen, R. Brady, J. Marcos, W.A. Carnielli, A. Avron and many others.

In 1948 S. Jaśkowski proposed three conditions for a paraconsistent propositional logic the simplified versions of which are as follows (cf. [16, p. 81],[17, p. 53]):

Jas1. the logic does not satisfy the implicational law of overfilling:

$$\varphi \to (\neg\varphi \to \psi);$$

Jas2. the logic should be rich enough to enable practical inferences: it satisfies modus ponens and the following formulas,

$$\varphi \to \varphi,$$
$$(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \psi)),$$
$$(\varphi \to (\psi \to \gamma)) \to (\psi \to (\varphi \to \gamma));$$

Jas3. it should have an intuitive justification: restriction to $\{0, 1\}$ gives the classical valuation.

Driven by some different motivation in 1963 Newton da Costa wanted to characterise paraconsistency by proposing a whole hierarchy of paraconsistent propositional calculi, known as $\mathrm{C_n}$, for $0 < n < \omega$. The following four conditions are the basic requirements for these calculi (cf. [17, p. 53]):

NdaC1. the *law of non-contradiction*, $\neg(\varphi \wedge \neg\varphi)$, should not be a valid schema;

NdaC2. from the set of formulae, $\{\varphi, \neg\varphi\}$, not all formulas should be derived in general;

NdaC3. extension to the predicate calculi (with or without equality) of these propositional calculi are simple;

NdaC4. without violating **NdaC1**, the calculi should contain the most part of the schemata and rules of the classical propositional calculus.

We shall introduce a three-valued paraconsistent logic $\mathbb{L}\mathrm{PS}_3$ in Chapter 3, which will be the base logic for the paraconsistent set theory described in this thesis. The connection between paraconsistent logics [12, 14, 27] and three-valued matrices is now well established and a widely discussed issue, where an algebra with designated elements is usually called a *matrix* in logic-literature. There are several articles dealing with this relationship [3, 7, 17, 26, 29]. Characterization of three-valued matrices giving rise to paraconsistency is shown in [3, 2, 17]. The three-valued matrix obtained and dealt with in this thesis is naturally one of them. The particular choice, however, stems from a specific motivation which will be mentioned in Chapter 2 and Chapter 3. Although many paraconsistent logics have been discussed in terms of three-valued semantics, in some cases a sound and complete axiomatic system has not been obtained (e.g. Priest's *Logic of Paradox* [26]).

Paraconsistent set theory

Some paraconsistent set theories are already developed [5, 15, 21, 36]. In [36] the author used a paraconsistent logic as the underlying logic and the following axioms as set theoretic axioms.

Abstraction. $x \in \{z : \varphi(z)\} \leftrightarrow \varphi(x)$.

Extensionality. $\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$.

In this paraconsistent set theory the *comprehension principle* is a theorem, where the principle is as follows: for any formula $\varphi(x)$ having one free variable x,

Comprehension. $\exists y \forall x(x \in y \leftrightarrow \varphi(x))$.

Using the comprehension scheme one can get existence of the *Russell's set*, $\mathrm{R} = \{x : x \notin x\}$ in the paraconsistent set theory developed in [36]. After showing the existence of R it can be proved that $\mathrm{R} \neq \mathrm{R}$. Hence the formula $\exists x(x \neq x)$ is a theorem there. On the other hand it can also be proved that identity is an equivalence relation. Though in the classical set theory it seems contradictory, it is not explosive in Weber's set theory [36] as the background logic is paraconsistent. Moreover the comprehension principle produces the strength to have a *universal set*, the set of all ordinals etc. Hence *Russell's paradox, Burali-Forti paradox* are not paradoxes in this set theory. It should also be mentioned that all the axioms of ZF^- hold in Weber's set theory together with AC and *Zorn's Lemma*.

The comprehension axiom scheme is derived as a theorem in many of the other paraconsistent set theories developed till now. As an immediate effect the paradoxes discussed above do not remain paradoxes. Generally the paraconsistency enters in the corresponding set theory through this axiom scheme.

In [5] some new paraconsistent set theories are developed, such as ZFmbC and ZFCil. Both of these two set theories have the same language, where besides the two binary predicate symbols "=" (for equality) and "∈" (for membership) there is one more unary predicate symbol "C" (for consistency). The underlying logic for these set theories is named as QmbC, which is the predicate extension of the logic mbC. The propositional logic mbC is known as one of the basic *Logic of Formal Inconsistencies* (LFIs) [8, 6]. The paraconsistent set theory ZFmbC is a subsystem of ZFCil.

An overview of the research work done in this thesis

The idea of Boolean-valued model $\mathbf{V}^{(\mathbb{B})}$ where $\mathbf{V}$ is the standard class model of classical set theory and $\mathbb{B}$ is any arbitrary but fixed complete Boolean algebra was introduced in 1960s (cf. [4]). The motivation behind this construction was to understand in a different way Cohen's method of *forcing* [9] which is used to prove the results regarding consistency and the independence of set theoretic statements. We have already discussed some other algebra-valued models too.

Generally speaking an $\mathbb{A}$-valued model $\mathbf{V}^{(\mathbb{A})}$ validate all (or some) of the ZF-axioms where $\mathbb{A}$ is the algebraic model for the underlying logic of the corresponding set theory. For instance Boolean algebras are algebraic models of classical logic, Heyting algebras are algebraic models of *intuitionistic logic*, and orthomodular lattices are algebraic models for *quantum logic*. Accordingly, $\mathbf{V}^{(\mathbb{A})}$ is called a model for classical set theory or intuitionistic set theory or quantum set theory when $\mathbb{A}$ is a Boolean algebra or a Heyting algebra or an orthomodular lattice, respectively. In Chapter 2 we shall describe the construction of the Boolean-valued model $\mathbf{V}^{(\mathbb{B})}$ in brief. The construction of the generalised algebra-valued model $\mathbf{V}^{(\mathbb{A})}$ corresponding to some algebra $\mathbb{A}$ is same as the construction of the Boolean-valued model.

We have already said that our goal is to construct a paraconsistent set theory. As one step towards this, an algebra $\mathbb{A}$ is defined so that $\mathbf{V}^{(\mathbb{A})}$ becomes an algebra-valued model of some versions of ZF^{-}-axioms [22, Section 2.1] [2]. This algebra $\mathbb{A}$ is sufficiently general to give rise to some non-classical logics other than paraconsistent logics. In this project we have chosen a three-valued matrix PS_3 as an instance of the above mentioned algebra $\mathbb{A}$. The matrix PS_3 is shown to give a semantics of a paraconsistent logic $\mathbb{L}\mathrm{PS}_3$ [33]. We would

[2]We should like to mention that Joel Hamkins independently investigated the construction of general algebra-valued model of set theory and proved a result equivalent to our Theorem 3.5.12 (presented at the Workshop on Paraconsistent Set Theory in Storrs, CT in October 2013; personal communication).

like to mention some of the results obtained in [22] to justify the selection of the particular algebra PS_3 in Chapter 3. It should be focused that claiming an algebra to be the model of some logic means that the logic should be *sound* and *complete* with respect to the semantics given in the algebra. In the algebra some elements need to be designated. In the Boolean algebras and Heyting algebras the designated element is only the top element of the bounded lattice. But in PS_3 we will consider more than one element in the designated set.

The structure $\mathbf{V}^{(\mathrm{PS}_3)}$ becomes an algebra-valued model of the paraconsistent set theory mentioned above. Some properties of *ordinal-like elements* and in particular *natural number-like elements* will be investigated in this model, where the notion of ordinal-like elements will be defined in Chapter 4. For each ordinal number α in $\mathbf{V}$ the collection of all α-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ will form an equivalence class made by the identity relation $\sim$ in $\mathbf{V}^{(\mathbb{A})}$, defined in Section 2.2.3. Some properties of the ordinal-like elements will be studied in comparison with the corresponding properties of ordinals in classical set theories. In particular for any natural number n in $\mathbf{V}$ the n-like elements and the arithmetic of them will be proposed. It will be proved that the formula corresponding to *mathematical induction* is valid in this algebra-valued model.

In $\mathbf{V}^{(\mathrm{PS}_3)}$, though the identity relation $\sim$ is an equivalence relation still two elements from the same class may not agree on the same properties. It will be shown that any two elements from the same class will either both satisfy or both dissatisfy any *negation-free formula* (defined in Section 2.1.3). On the other hand an example of a formula (having negation) will be provided such that there are two elements from a same $\sim$-class, one of which will satisfy the formula where as the other will not (for reference see Section 4.4).

It was already discussed that, if $\mathbb{A}$ is a deductive reasonable implication algebra then $\mathbf{V}^{(\mathbb{A})}$ may become an algebra-valued model of set theory. The logic corresponding to $\mathbb{A}$ will act as the underlying logic of the set theory having the algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. It may happen that such an algebra $\mathbb{A}$ is generating none of classical logic, intuitionistic logic and paraconsistent logic, but some other non-classical logic. In that case a new non-classical set theory can be made in the same vein and not only that, an algebra-valued model viz. $\mathbf{V}^{(\mathbb{A})}$ of that set theory can be produced. Hence the research work found in this thesis has the strength to produce different non-classical set theories corresponding to different deductive reasonable implication algebras and a way to get the axiom systems of their underlying non-classical logics.

It is worthwhile to mention that Chapter 2, Chapter 3, and Chapter 4 are based on the papers [22], [33], and [32], respectively.

Comparison with previous paraconsistent set theories. The paraconsistent set theory developed in this thesis does not validate the comprehension axiom scheme. Not only that, it will be established that the statements "there is a Russell set", "there is a set of all ordinals", and "there is a set of all sets" are invalid in our set theory. Hence the statements of Russell's paradox, Burali-Forti's paradox, Cantor's Paradox remain paradoxical in this paraconsistent set theory. This is a notable difference with almost all of the existing paraconsistent set theories.

There is a notion of *inconsistent set* in [36, p. 77], but there is no such notion in our paraconsistent set theory. In Section 4.4 it will be shown that Leibniz's law of *indiscernible of identicals* will be violated here. Hence the property *substitution* is not in general valid in this set theory (but is valid if restricted to negation-free formulas), whereas the law of substitution is taken as a rule in the set theory developed in [36]. Unlike in classical set theory, in the paraconsistent set theory of Weber, *inaccessible cardinals* can be proved to exist [36, p. 90]. We shall develop some properties of ordinal numbers and in particular natural numbers in our paraconsistent set theory in Section 4, but the theory of cardinal numbers will not be investigated here.

One of the differences between the paraconsistent set theories ZFmbC and ZFCil, and the one developed in this thesis is that the Leibniz's axiom for equality,

$$x = y \to (\varphi \to \varphi(x/y))$$

(where $\varphi(x/y)$ denotes any formula obtained from φ by replacing some free occurrences of the variable x by the variable y provided that y remains free in those occurrences) is taken as a set theoretic axiom scheme in both of ZFmbC and ZFCil [5, p. 5]. But as mentioned above, in Section 4.4 we shall show that Leibniz's law of indiscernible of identicals is not valid in our paraconsistent set theory. It is proved that ZFmbC and ZFCil are non-explosive if ZF is consistent [5, Corollary 3.16], which is similar with the paraconsistent set theory we have developed in this thesis as it will be shown in Section 3.6.1.

Chapter 2

Algebra-valued models of set theories

On the basis of the standard textbook by J. L. Bell [4], we shall discuss briefly in the following steps how a Boolean valued model is constructed.

1. Let us take the standard class model $\mathbf{V}$ of classical set theory and a complete Boolean algebra, $\mathbb{B} = \langle \mathbf{B}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$.

2. For any ordinal $\alpha \in \mathbf{V}$ we define,

$$\mathbf{V}_\alpha^{(\mathbb{B})} = \{x : \mathrm{Func}(x) \wedge \mathrm{ran}(x) \subseteq \mathbf{B} \wedge \exists \xi < \alpha(\mathrm{dom}(x) \subseteq \mathbf{V}_\xi^{(\mathbb{B})})\}$$

   where as above, $\mathrm{Func}(x)$ is the abbreviation for the formula expressing "x is a function" and in this case, we use $\mathrm{dom}(x)$ for the domain of the function x and $\mathrm{ran}(x)$ for the range of the function x.

3. Using the above we get a Boolean valued model as,

$$\mathbf{V}^{(\mathbb{B})} = \{x : \exists \alpha(x \in \mathbf{V}_\alpha^{(\mathbb{B})})\}.$$

4. The language of classical ZFC is extended by adding a name corresponding to each element of $\mathbf{V}^{(\mathbb{B})}$, in it.

5. Each formula of the new language is associated by recursion on the complexity of the formula with a value of the complete Boolean algebra, $\mathbb{B}$. We start with the atomic formulas $u \in v$ and $u = v$, for any u, v in $\mathbf{V}^{(\mathbb{B})}$. Their values are determined by a simultaneous transfinite recursion,

$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!]), \text{ and}$$

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]).$$

Then for any sentences σ and τ of the new language we define,

$$\begin{aligned}
[\![\sigma \wedge \tau]\!] &= [\![\sigma]\!] \wedge [\![\tau]\!], \\
[\![\sigma \vee \tau]\!] &= [\![\sigma]\!] \vee [\![\tau]\!], \\
[\![\sigma \to \tau]\!] &= [\![\sigma]\!] \Rightarrow [\![\tau]\!], \\
[\![\neg\sigma]\!] &= [\![\sigma]\!]^*, \\
[\![\forall x \varphi(x)]\!] &= \bigwedge_{u \in V^{(\mathbb{B})}} [\![\varphi(u)]\!], \text{ and} \\
[\![\exists x \varphi(x)]\!] &= \bigvee_{u \in V^{(\mathbb{B})}} [\![\varphi(u)]\!].
\end{aligned}$$

6. A sentence σ will be called *valid* in $\mathbf{V}^{(\mathbb{B})}$ or $\mathbf{V}^{(\mathbb{B})}$ will be called an algebra-valued model of a sentence σ if $[\![\sigma]\!] = \mathbf{1}$. It will be denoted as $\mathbf{V}^{(\mathbb{B})} \models \sigma$.

7. It can be shown that all the axioms of ZFC are *valid* in $\mathbf{V}^{(\mathbb{B})}$ i.e., $\mathbf{V}^{(\mathbb{B})}$ is a Boolean-valued model of ZFC.

It is shown in [20] that if the same construction is done with a complete Heyting algebra $\mathbb{H}$ instead of the complete Boolean algebra then $\mathbf{V}^{(\mathbb{H})}$ becomes an algebra-valued model of IZF. This idea was further generalized in [31, 34, 35]; and [24, 25], replacing the Heyting algebra $\mathbb{H}$ by suitable lattices that produce algebra-valued models of quantum set theory (where the algebra is an algebra of truth-values in quantum logic) or fuzzy set theory.

Generalising this idea we have found the algebraic properties needed for making the axioms of set theories valid in the corresponding algebra-valued model.

2.1 Reasonable implication algebras

2.1.1 Implication algebras and implication-negation algebras

In this thesis, all structures $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ will be complete distributive lattices with smallest element $\mathbf{0}$ and largest element $\mathbf{1}$.

As usual, we abbreviate $x \wedge y = x$ as $x \leq y$. An expansion of this structure by an additional binary operation $\Rightarrow$ is called an *implication algebra* and an expansion with $\Rightarrow$ and another unary operation * is called an *implication-negation algebra* [22]. We emphasize that no requirements are made for $\Rightarrow$ and * at this point. In [28, p. 30] a notion of "implication algebra" is provided having some properties on the operators of the algebra. The property for $\Rightarrow$ in [28] is given by

$$(a \Rightarrow b) \Rightarrow a = a$$

for all a and b in the domain of the algebra. Since no property is assigned on $\Rightarrow$ in the implication algebra defined in this thesis, these two notions of implication algebra are completely different.

2.1.2 Interpreting propositional logic in algebras

By $\mathcal{L}_{\mathrm{Prop}}$ we denote the language of propositional logic without negation (with connectives $\wedge$, $\vee$, $\rightarrow$, and $\perp$ and countably many variables Var); we write $\mathcal{L}_{\mathrm{Prop},\neg}$ for the expansion of this language to include the negation symbol $\neg$. Let $\mathcal{L}$ be either $\mathcal{L}_{\mathrm{Prop}}$ or $\mathcal{L}_{\mathrm{Prop},\neg}$, and let $\mathbb{A}$ be either an implication algebra or an implication-negation algebra, respectively.

Any map ι from Var to A (called an *assignment*) allows us to interpret $\mathcal{L}$-formulas φ as elements $\iota(\varphi)$ of the algebra.

For an $\mathcal{L}$-formula φ and some $X \subseteq A$, we write $\varphi \in X$ to mean "for all assignments $\iota : \mathrm{Var} \rightarrow A$, we have that $\iota(\varphi) \in X$".

As usual, we call a set $D \subseteq A$ a *filter* if the following four conditions hold: (i) $\mathbf{1} \in D$, (ii) $\mathbf{0} \notin D$, (iii) if $x, y \in D$, then $x \wedge y \in D$, and (iv) if $x \in D$ and $x \leq y$, then $y \in D$; in this context, we call filters *designated sets of truth values*, since the algebra $\mathbb{A}$ and a filter D together determine a *logic* $\vdash_{\mathbb{A},D}$ by defining for every set Γ of $\mathcal{L}_{\mathrm{Prop}}$-formulas and every $\mathcal{L}_{\mathrm{Prop}}$-formula φ

$$\Gamma \vdash_{\mathbb{A},D} \varphi :\Longleftrightarrow \text{ if for all } \psi \in \Gamma, \text{ we have } \psi \in D, \text{ then } \varphi \in D.$$

We write $\mathrm{Pos}_{\mathbb{A}} := \{x \in A\,;\, x \neq \mathbf{0}\}$ for the set of positive elements in $\mathbb{A}$. In all of the examples considered in this project, this set will be a filter.

2.1.3 The negation-free fragment

If $\mathcal{L}$ is any first-order language including the connectives $\wedge$, $\vee$, $\perp$ and $\rightarrow$ and Λ any class of $\mathcal{L}$-formulas, we denote closure of Λ under $\wedge$, $\vee$, $\perp$, $\exists$, $\forall$, and $\rightarrow$ by $\mathrm{Cl}(\Lambda)$ and call it the *negation-free closure of* Λ. A class Λ of formulas is *negation-free closed* if $\mathrm{Cl}(\Lambda) = \Lambda$. By NFF we denote the negation-free closure of the atomic formulas; its elements are called the *negation-free formulas.*[1]

Obviously, if $\mathcal{L}$ does not contain any connectives beyond $\wedge$, $\vee$, $\perp$, and $\rightarrow$, then NFF $= \mathcal{L}$.

Similarly, if the logic we are working in, allows to define negation in terms of the other connectives (as is the case, e.g., in classical logic), then every formula is equivalent to one in NFF.

2.1.4 Reasonable implication algebras

We call an implication algebra $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow)$ *reasonable* if the operation $\Rightarrow$ satisfies the following axioms:

P1. $(x \wedge y) \leq z$ implies $x \leq (y \Rightarrow z)$,

P2. $y \leq z$ implies $(x \Rightarrow y) \leq (x \Rightarrow z)$, and

P3. $y \leq z$ implies $(z \Rightarrow x) \leq (y \Rightarrow x)$.

We say that a reasonable implication algebra is *deductive* if

$$((x \wedge y) \Rightarrow z) = (x \Rightarrow (y \Rightarrow z)). \qquad \textbf{(P4)}$$

Proposition 2.1.1 *In a reasonable implication algebra $\mathbb{A}$ for any two elements a and b if $a \leq b$ holds then $a \Rightarrow b = 1$ also holds.*

Proof. Let $a, b \in A$ (the domain of $\mathbb{A}$) be such that $a \leq b$. Then by the property of lattice, $1 \wedge a \leq b$. Now by **P1** we can write $1 \leq a \Rightarrow b$. Hence $a \Rightarrow b = 1$, since 1 is the top element of $\mathbb{A}$. $\square$

[1]In some contexts, our negation-free fragment is called the *positive fragment*; in other contexts, the *positive closure* is the closure under $\wedge$, $\vee$, $\perp$, $\exists$, and $\forall$ (not including $\rightarrow$). In order to avoid confusion with the latter contexts, we use the phrase "negation-free" rather than "positive".

Proposition 2.1.2 *In a deductive reasonable implication algebra $\mathbb{A}$ for any two elements a and b, $a \Rightarrow b \ = \ a \Rightarrow (a \wedge b)$ holds.*

Proof. For any two elements a and b in a reasonable implication algebra $\mathbb{A}$ using **P1**, **P2** and **P4** we have

$$\begin{aligned} a \wedge b &\leq a \wedge b \\ \text{i.e.,} \qquad b &\leq a \Rightarrow (a \wedge b) \\ \text{i.e.,}\ a \Rightarrow b &\leq a \Rightarrow (a \Rightarrow (a \wedge b)) \\ &\leq (a \wedge a) \Rightarrow (a \wedge b) \\ &\leq a \Rightarrow (a \wedge b) \end{aligned}$$

On the other hand since $a \wedge b \leq b$ by using **P2** we have $a \Rightarrow (a \wedge b) \leq a \Rightarrow b$. Hence combining the above results we have $a \Rightarrow b \ = \ a \Rightarrow (a \wedge b)$. □

Proposition 2.1.3 *Let $\mathbb{A}$ be a reasonable implication algebra having D as a designated set. If $d \in D$ be any element then $1 \Rightarrow d \in D$.*

Proof. For any $d \in D$ we have $1 \wedge d \leq d$. Then, by using **P1** we can write, $d \leq 1 \Rightarrow d$. Since $d \in D$ and D is a filter, by the condition (iv) of a filter given in Section 2.1.2 it can be said that $1 \Rightarrow d \in D$, and hence the proof is complete. □

One can immediately check that all Boolean algebras and Heyting algebras are reasonable and deductive implication algebras.

2.2 The model construction

2.2.1 Definitions and basic properties

Our construction follows very closely the Boolean-valued construction as it is shown in the beginning of this chapter.

By $\mathcal{L}_\in$, we denote the first-order language of set theory using the predicate symbols $=$ and $\in$; propositional connectives $\wedge$, $\vee$, $\perp$, and $\to$; quantifiers $\forall$ and $\exists$. We can now expand this language by adding all of the elements of $\mathbf{V}^{(\mathbb{A})}$ as constants; the expanded language will be called $\mathcal{L}_\mathbb{A}$. For convenience we shall identify the elements of $\mathbf{V}^{(\mathbb{A})}$ by their corresponding names in $\mathcal{L}_\mathbb{A}$.

As in the Boolean case (cf. [4, Induction Principle 1.7]), the *(meta-)induction principle* for $\mathbf{V}^{(\mathbb{A})}$ can be proved by a simple induction on the rank function: for every property Φ of names, if for all $x \in \mathbf{V}^{(\mathbb{A})}$, we have

$$\forall y \in \mathrm{dom}(x)(\Phi(y)) \text{ implies } \Phi(x),$$

then all names $x \in \mathbf{V}^{(\mathbb{A})}$ have the property Φ.

As in the Boolean-valued model construction, we define a map $[\![\cdot]\!]$ assigning to each negation-free formula in $\mathcal{L}_\mathbb{A}$ a truth value in A together with $[\![\perp]\!] = \mathbf{0}$.

We abbreviate $\exists x(x \in u \wedge \varphi(x))$ by $\exists x \in u\ \varphi(x)$ and $\forall x(x \in u \to \varphi(x))$ by $\forall x \in u\ \varphi(x)$ and call these *bounded quantifiers*. Bounded quantifiers will play a crucial role in this thesis.

If D is a filter on A and σ is a sentence of $\mathcal{L}_\mathbb{A}$, we say that σ is D-valid or valid in $\mathbf{V}^{(\mathbb{A})}$ if $[\![\sigma]\!] \in D$ and write $\mathbf{V}^{(\mathbb{A})} \models_D \sigma$, or simply $\mathbf{V}^{(\mathbb{A})} \models \sigma$.

Proposition 2.2.1 *If $\mathbb{A}$ is a reasonable implication algebra and $u \in \mathbf{V}^{(\mathbb{A})}$, then $[\![u = u]\!] = \mathbf{1}$ and for each $x \in \mathrm{dom}(u)$, $u(x) \leq [\![x \in u]\!]$.*

Proof. We will prove this theorem by induction principle for $\mathbf{V}^{(\mathbb{A})}$. Let us take any $u \in \mathbf{V}^{(\mathbb{A})}$ and assume $[\![x = x]\!] = 1$ for all $x \in \mathrm{dom}(u)$. Now,

$$\begin{aligned}
[\![u = u]\!] &= \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in u]\!]] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in u]\!]] \\
&= \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![x \in u]\!]]
\end{aligned}$$

Take any $x \in \mathrm{dom}(u)$, then we get,

$$\begin{aligned}
[\![x \in u]\!] &= \bigvee_{y \in \mathrm{dom}(u)} [u(y) \wedge [\![x = y]\!]] \\
&\geq u(x) \wedge [\![x = x]\!] \\
&= u(x) \wedge 1, \text{(by Induction hypothesis)} \\
&= u(x)
\end{aligned}$$

Therefore we have for any $x \in \mathrm{dom}(u)$, $[u(x) \Rightarrow [\![x \in u]\!]] = 1$, by the above theorem. Hence it can be immediately concluded that $[\![u = u]\!] = 1$.

For the proof of the second part let us take an $u \in \mathbf{V}^{(\mathbb{A})}$. Then for any $x \in \mathrm{dom}(u)$,

$$\begin{aligned}
[\![x \in u]\!] &= \bigvee_{y \in \mathrm{dom}(u)} [u(y) \wedge [\![x = y]\!]] \\
&\geq u(x) \wedge [\![x = x]\!] \\
&= u(x) \wedge 1, \ \text{(using the first part of the theorem)} \\
&= u(x)
\end{aligned}$$

Hence the proof is complete. $\square$

If $\mathbb{A}$ is a Boolean algebra or Heyting algebra then for all $u, v, w \in \mathbf{V}^{(\mathbb{A})}$ the following holds:

$$[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!], \tag{$\star$}$$

which is not true for any reasonable implication algebra $\mathbb{A}$.

Observation 2.2.2 *There is a reasonable implication algebra $\mathbb{A}$ and $u, v, w \in \mathbf{V}^{(\mathbb{A})}$ such that the property $\star$ does not hold.*

Proof. Let $Ł_3 = \langle \{1, 1/2, 0\}, \wedge, \vee, \Rightarrow, ^* \rangle$ be the three valued Łukasiewicz algebra having the following truth tables:

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

$*$	
1	0
1/2	1/2
0	1

and $\{1\}$ as the designated set. It can be verified that $\langle \{1, 1/2, 0\}, \wedge, \vee, \Rightarrow \rangle$ is a reasonable implication algebra.

Consider the algebra-valued model $\mathbf{V}^{(Ł_3)}$ and three elements (which are actually functions) p_0, $p_{1/2}$ and p_1 in $\mathbf{V}^{(Ł_3)}$ as $\mathrm{dom}(p_0) = \mathrm{dom}(p_{1/2}) = \mathrm{dom}(p_1) = \{\varnothing\}$ and $p_0(\varnothing) = 0$, $p_{1/2}(\varnothing) = 1/2$, and $p_1(\varnothing) = 1$, i.e., in the notation of naive set theory

$$\begin{aligned} p_{\mathbf{0}} &= \{\langle \varnothing, 0 \rangle\}, \\ p_{1/2} &= \{\langle \varnothing, 1/2 \rangle\}, \text{ and} \\ p_{\mathbf{1}} &= \{\langle \varnothing, 1 \rangle\}. \end{aligned}$$

By fixing $u = p_{\mathbf{0}}$, $v = p_{1/2}$ and $w = p_{\mathbf{1}}$ one get

$$[\![u = v]\!] = (p_{\mathbf{0}}(\varnothing) \Rightarrow [\![\varnothing \in p_{1/2}]\!]) \wedge (p_{1/2}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{0}}]\!])$$

$$
\begin{aligned}
&= (0 \Rightarrow [\![\varnothing \in p_{1/2}]\!]) \wedge ({}^1\!/\!_2 \Rightarrow (p_{\mathbf{0}}(\varnothing) \wedge [\![\varnothing = \varnothing]\!])) \\
&= 1 \wedge ({}^1\!/\!_2 \Rightarrow 0) \\
&= {}^1\!/\!_2,
\end{aligned}
$$

$$
\begin{aligned}
[\![v = w]\!] &= (p_{1/2}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{1}}]\!]) \wedge (p_{\mathbf{1}}(\varnothing) \Rightarrow [\![\varnothing \in p_{1/2}]\!]) \\
&= ({}^1\!/\!_2 \Rightarrow (p_{\mathbf{1}}(\varnothing) \wedge [\![\varnothing = \varnothing]\!])) \wedge (1 \Rightarrow (p_{1/2}(\varnothing) \wedge [\![\varnothing = \varnothing]\!])) \\
&= ({}^1\!/\!_2 \Rightarrow (1 \wedge 1)) \wedge (1 \Rightarrow ({}^1\!/\!_2 \wedge 1)) \\
&= ({}^1\!/\!_2 \Rightarrow 1) \wedge (1 \Rightarrow {}^1\!/\!_2) \\
&= 1 \wedge {}^1\!/\!_2 \\
&= {}^1\!/\!_2,
\end{aligned}
$$

$$
\begin{aligned}
[\![u = w]\!] &= (p_{\mathbf{0}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{1}}]\!]) \wedge (p_{\mathbf{1}}(\varnothing) \Rightarrow [\![\varnothing \in p_{\mathbf{0}}]\!]) \\
&= (0 \Rightarrow [\![\varnothing \in p_{\mathbf{1}}]\!])) \wedge (1 \Rightarrow (p_{\mathbf{0}}(\varnothing) \wedge [\![\varnothing = \varnothing]\!])) \\
&= 1 \wedge (1 \Rightarrow 0) \\
&= 0
\end{aligned}
$$

Hence there are $u, v, w \in \mathbf{V}^{(Ł_3)}$ so that the inequality $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$ does not hold. $\square$

The following proposition provides a class of reasonable implication algebras such that if $\mathbb{A}$ belongs to that class then there exist $u, v, w \in \mathbf{V}^{(\mathbb{A})}$ for which the property $(\star)$ of Observation 2.2.2 does not hold.

Proposition 2.2.3 *Let $\mathbb{A} = \langle A, \wedge, \vee, \Rightarrow, ^*, 0, 1\rangle$ be an algebra such that $\langle A, \wedge, \vee, \Rightarrow, 0, 1\rangle$ is a reasonable implication algebra and the unary operator * satisfies $1^* = 0$ and $a \Rightarrow b \leq a^* \vee b$*

for all $a, b \in A$. If there exist two elements p and q in $\mathbb{A}$ such that, $(p \wedge p^* \Rightarrow q) \neq 1$ then there exist $x, y, z \in \mathbf{V}^{(\mathbb{A})}$ such that, $[\![x = y]\!] \wedge [\![y = z]\!] \not\leq [\![x = z]\!]$.

Proof. Let us take those $p, q \in A$ such that $(p \wedge p^* \Rightarrow q) \neq 1$. By this property and the property **P2** of reasonable implication algebra we get,

$$(p \wedge p^* \Rightarrow 0) \leq (p \wedge p^* \Rightarrow q) < 1.$$

Since $a \leq b$ implies $a \Rightarrow b = 1$ for all $a, b \in A$ and 0 is the bottom element in $\mathbb{A}$, one can conclude that $p \wedge p^* > 0$. Let us define three elements in $\mathbf{V}^{(\mathbb{A})}$ as,

$$\begin{aligned} x_0 &= \{\langle \varnothing, 0 \rangle\}, \\ x_p &= \{\langle \varnothing, p \rangle\}, \text{ and} \\ x_1 &= \{\langle \varnothing, 1 \rangle\}. \end{aligned}$$

We shall show that,

$$[\![x_0 = x_p]\!] \wedge [\![x_p = x_1]\!] > [\![x_0 = x_1]\!].$$

Let us first find the value of $[\![x_0 = x_p]\!]$.

$$\begin{aligned} [\![x_0 = x_p]\!] &= \bigwedge_{u \in \mathrm{dom}(x_0)} (x_0(u) \Rightarrow [\![u \in x_p]\!]) \wedge \bigwedge_{v \in \mathrm{dom}(x_p)} (x_p(v) \Rightarrow [\![v \in x_0]\!]) \\ &= (x_0(\varnothing) \Rightarrow [\![\varnothing \in x_p]\!]) \wedge (x_p(\varnothing) \Rightarrow [\![\varnothing \in x_0]\!]) \\ &= (0 \Rightarrow p) \wedge (p \Rightarrow 0) \\ &= 1 \wedge (p \Rightarrow 0) \\ &\geq p^* \vee 0 \\ &= p^* \end{aligned}$$

Similarly,

$$\begin{aligned}
[\![x_p = x_1]\!] &= (x_p(\varnothing) \Rightarrow [\![\varnothing \in x_1]\!]) \wedge (x_1(\varnothing) \Rightarrow [\![\varnothing \in x_p]\!]) \\
&= (p \Rightarrow 1) \wedge (1 \Rightarrow p) \\
&\geq 1 \wedge (1^* \vee p) \\
&= p, \text{ [since } 1^* = 0].
\end{aligned}$$

Again we can prove that, $[\![x_0 = x_1]\!] = 0$.
Hence $[\![x_0 = x_p]\!] \wedge [\![x_p = x_1]\!] > [\![x_0 = x_1]\!]$ is proved. □

Therefore Observation 2.2.2 is a particular case of Proposition 2.2.3.

We are now interested in the formulas $\exists x(x \in u \wedge \varphi(x))$ and $\forall x(x \in u \rightarrow \varphi(x))$ for which we use the abbreviations as $\exists x \in u\varphi(x)$ and $\forall x \in u\varphi(x)$, respectively.

Proposition 2.2.4 *If $\mathbb{A}$ is a reasonable implication algebra, $\varphi(x)$ an $\mathcal{L}_{\mathbb{A}}$-formula with one free variable x, and $u \in \mathbf{V}^{(\mathbb{A})}$, then*

$$[\![\exists x \in u \ \varphi(x)]\!] \geq \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]).$$

Proof. By definition,

$$\begin{aligned}
[\![\exists x \in u\varphi(x)]\!] &= [\![\exists x(x \in u \wedge \varphi(x))]\!] \\
&= \bigvee_{y \in \mathbf{V}^{(\mathbb{A})}} ([\![y \in u]\!] \wedge [\![\varphi(y)]\!]) \\
&\geq \bigvee_{x \in \mathrm{dom}(u)} ([\![x \in u]\!] \wedge [\![\varphi(x)]\!]) \\
&\geq \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]), \quad \text{using proposition 2.2.1.}
\end{aligned}$$

Hence the proposition is proved. $\square$

In the Boolean case, the inequality proved in Proposition 2.2.4 is an equality [4, Corollary 1.18]:

$$[\![\exists x \in u\ \varphi(x)]\!] = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![\varphi(x)]\!]).$$

Similarly the following equality can be proved in Boolean case:

$$[\![\forall x \in u\ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

But this one breaks down for general reasonable implication algebras:

Proposition 2.2.5 *There exists a reasonable implication algebra $\mathbb{A}$ such that in the extended language of the set theory corresponding to $\mathbf{V}^{(\mathbb{A})}$ there exists a formula $\varphi(x)$ so that the following holds*

$$[\![\forall x \in u\ \varphi(x)]\!] < \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

Proof. We shall show that $Ł_3$ is such a reasonable implication algebra. Consider the Łukasiewicz-valued model $\mathbf{V}^{(Ł_3)}$ and the three elements $p_{\mathbf{0}}$, $p_{1/2}$, and $p_{\mathbf{1}}$ in $\mathbf{V}^{(Ł_3)}$ as

$$\begin{aligned} p_{\mathbf{0}} &= \{\langle \varnothing, 0\rangle\}, \\ p_{1/2} &= \{\langle \varnothing, 1/2\rangle\}, \text{ and} \\ p_{\mathbf{1}} &= \{\langle \varnothing, 1\rangle\}. \end{aligned}$$

Consider the formula $\varphi(x) := (x = p_{\mathbf{0}})$ as well as the name $u = \{\langle p_{1/2}, 1/2\rangle\}$. Hence

$$[\![\forall x \in u\varphi(x)]\!] = \bigwedge_{x \in \mathbf{V}^{(Ł_3)}} ([\![x \in u]\!] \Rightarrow [\![\varphi(x)]\!])$$

$$
\begin{aligned}
&\leq [\![p_1 \in u]\!] \Rightarrow [\![\varphi(p_1)]\!] \\
&= (u(p_{1/2}) \wedge [\![p_1 = p_{1/2}]\!]) \Rightarrow [\![p_1 = p_0]\!] \\
&= ({}^1\!/_2 \wedge {}^1\!/_2) \Rightarrow 0 \\
&= {}^1\!/_2,
\end{aligned}
$$

$$
\begin{aligned}
\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]) &= u(p_{1/2}) \Rightarrow [\![\varphi(p_{1/2})]\!]) \\
&= {}^1\!/_2 \Rightarrow [\![p_{1/2} = p_0]\!] \\
&= {}^1\!/_2 \Rightarrow {}^1\!/_2 \\
&= 1.
\end{aligned}
$$

As a conclusion the following is proved

$$
[\![\forall x \in u \varphi(x)]\!] < \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).
$$

□

This means that in the setting of reasonable implication algebras, the following equality

$$
[\![\forall x \in u \ \varphi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]) \qquad (\mathrm{BQ}_\varphi)
$$

becomes a new axiom, one whose validity depends on the choice of the formula φ and on $\mathbb{A}$ (and conceivably on the model of set theory $\mathbf{V}$).

If Λ is any class of formulas of the extended language, we say that $\mathbf{V}^{(\mathbb{A})}$ *satisfies the* Λ*-bounded quantification property*, if BQ_φ holds for every $\varphi \in \Lambda$.

2.2.2 Set theory

It can be observed that all axioms and axiom schemes have natural forms that do not include any negation symbols,[2] so unless we instantiate one of the schemes with a formula containing a negation symbol, we will always have formulas in NFF.

We write NFF-Separation and NFF-Collection for the axiom schemes where we only allow the instantiation by negation-free formulas, and NFF-ZF^- and NFF-ZF stand for *negation-free set theory* using these schemes.

We emphasize once more that in settings where negation can be defined in terms of negation-free formulas (such as in classical logic negation of a formula, $\neg\varphi$ can be defined as $\varphi \to \bot$), NFF-ZF coincides (up to provable equivalence) with standard Zermelo-Fraenkel set theory. But there are logics where $\varphi \to \bot$ do not define the formula $\neg\varphi$, e.g., the logic $\mathbb{L}\mathrm{PS}_3$, introduced in Chapter 3.

Theorems 2.2.6 and 2.2.7 are the core of this chapter, establishing validity of NFF-ZF^- in our $\mathbb{A}$-valued model.

Theorem 2.2.6 *Let $\mathbb{A}$ be a reasonable implication algebra such that $\mathbf{V}^{(\mathbb{A})}$ satisfies the* NFF-*bounded quantification property, and let D be any filter on A. Then Extensionality, Pairing, Infinity, Union and* NFF-*Collection are D-valid in $\mathbf{V}^{(\mathbb{A})}$; in fact, they all get the value $\mathbf{1}$.*

Proof. (i) Extensionality: We know that,
$[\![\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]]\!]$

$$= \bigwedge_{u \in \mathbf{V}^{(\mathbb{A})}} \bigwedge_{v \in \mathbf{V}^{(\mathbb{A})}} [\![\forall z(z \in u \leftrightarrow z \in v) \to u = v]\!]$$

[2]Note that this is only the case because we formulated the occurrence of the empty set in Infinity appropriately and because we used the axiom scheme of set induction (or $\in$-induction) instead of the usual formulation of Foundation; the latter is not negation-free.

$$= \bigwedge_{u\in \mathbf{V}^{(\mathbb{A})}} \bigwedge_{v\in \mathbf{V}^{(\mathbb{A})}} [(\llbracket \forall z \in u(z \in v)\rrbracket \wedge \llbracket \forall z \in v(z \in u)\rrbracket) \Rightarrow \llbracket u = v\rrbracket]$$

$$= \bigwedge_{u\in \mathbf{V}^{(\mathbb{A})}} \bigwedge_{v\in \mathbf{V}^{(\mathbb{A})}} [((\bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow \llbracket x \in v\rrbracket)) \wedge (\bigwedge_{y\in \mathrm{dom}(v)} (v(y) \Rightarrow \llbracket y \in u\rrbracket))) \Rightarrow \llbracket u = v\rrbracket]$$

$$= \bigwedge_{u\in \mathbf{V}^{(\mathbb{A})}} \bigwedge_{v\in \mathbf{V}^{(\mathbb{A})}} (\llbracket u = v\rrbracket \Rightarrow \llbracket u = v\rrbracket)$$

$$= 1$$

So we have shown that the Axiom of Extensionality is D-valid in $\mathbf{V}^{(\mathbb{A})}$.

(ii) Pairing: For proving the Paring Axiom is true in $\mathbf{V}^{(\mathbb{A})}$, it is to be shown,

$$\bigwedge_{u\in \mathbf{V}^{(\mathbb{A})}} \bigwedge_{v\in \mathbf{V}^{(\mathbb{A})}} \bigvee_{z\in \mathbf{V}^{(\mathbb{A})}} \llbracket \forall w(w \in z \leftrightarrow (w = u \vee w = v))\rrbracket = 1.$$

Let us take arbitrary $u, v \in \mathbf{V}^{(\mathbb{A})}$ and then define $z \in \mathbf{V}^{(\mathbb{A})}$ such that $\mathrm{dom}(z) = \{u, v\}$ and $\mathrm{ran}(z) = \{1\}$. Now,

$$\llbracket \forall w(w \in z \rightarrow (w = u \vee w = v))\rrbracket = \bigwedge_{x\in \mathrm{dom}(z)} (z(x) \Rightarrow \llbracket x = u\rrbracket \vee \llbracket x = v\rrbracket) = 1.$$

Similarly,

$$\begin{aligned}\llbracket \forall w((w = u \vee w = v) \rightarrow w \in z)\rrbracket &= \bigwedge_{w\in \mathbf{V}^{(\mathbb{A})}} (\llbracket w = u \vee w = v\rrbracket \Rightarrow \bigvee_{x\in \mathrm{dom}(z)} (z(x) \wedge \llbracket x = w\rrbracket)) \\ &= \bigwedge_{w\in \mathbf{V}^{(\mathbb{A})}} (\llbracket w = u \vee w = v\rrbracket \Rightarrow \llbracket w = u \vee w = v\rrbracket) \\ &= 1\end{aligned}$$

Hence we are done.

(iii) Infinity: Let us first define for each $x \in V$,

$$\hat{x} = \{\langle \hat{y}, 1\rangle : y \in x\}$$

where $\mathbf{V}$ is the standard model of classical set theory. Clearly $\hat{x} \in \mathbf{V}^{(\mathbb{A})}$ for every $x \in \mathbf{V}$. Particularly $\hat{\omega} \in \mathbf{V}^{(\mathbb{A})}$, where ω is the first infinite ordinal. We will show that,

$$[\![\forall z(z \in \varnothing \to \perp)]\!] \wedge [\![\varnothing \in \hat{\omega}]\!] \wedge [\![\forall x \in \hat{\omega} \exists y \in \hat{\omega}(x \in y)]\!] = 1.$$

Consider the first conjunct:

$$[\![\forall z(z \in \varnothing \to \perp)]\!] = \bigwedge_{z \in \mathbf{V}^{(\mathrm{PS}_3)}} (0 \Rightarrow 0) = 1.$$

The value of the second conjunct is,

$$[\![\varnothing \in \hat{\omega}]\!] = \bigvee_{x \in \mathrm{dom}(\hat{\omega})} (\hat{\omega}(x) \wedge [\![\varnothing = x]\!]) \geq (\hat{\omega}(\varnothing) \Rightarrow [\![\varnothing = \varnothing]\!]) = 1.$$

Now we will show that the value of the third conjunct is also 1.

$$\begin{aligned}
[\![\forall x \in \hat{\omega} \exists y \in \hat{\omega}(x \in y)]\!] &= \bigwedge_{x \in \mathrm{dom}(\hat{\omega})} (\hat{\omega}(x) \Rightarrow \bigvee_{y \in \mathrm{dom}(\hat{\omega})} (\hat{\omega}(y) \wedge [\![x \in y]\!])) \\
&= \bigwedge_{x \in \mathrm{dom}(\hat{\omega})} (1 \Rightarrow \bigvee_{y \in \mathrm{dom}(\hat{\omega})} [\![x \in y]\!])
\end{aligned}$$

Let us take any arbitrary $x \in \mathrm{dom}(\hat{\omega})$. By our construction there exists some $m \in \omega$ such that $x = \hat{m}$. Clearly, $\widehat{(m+1)} \in \mathrm{dom}(\hat{\omega})$, where " + " is taken as the standard addition operator on the ordinals of $\mathbf{V}$. Now,

$$\bigvee_{y\in\mathrm{dom}(\hat{\omega})} [\![x \in y]\!] = \bigvee_{y\in\mathrm{dom}(\hat{\omega})} [\![\hat{m} \in y]\!] \geq [\![\hat{m} \in (\widehat{m+1})]\!] \geq (\widehat{m+1})(\hat{m}) \wedge [\![\hat{m} = \hat{m}]\!] = 1.$$

Therefore we get,

$$\begin{aligned} [\![\forall x \in \hat{\omega} \exists y \in \hat{\omega}(x \in y)]\!] &= \bigwedge_{x\in\mathrm{dom}(\hat{\omega})} (1 \Rightarrow \bigvee_{y\in\mathrm{dom}(\hat{\omega})} [\![x \in y]\!]) \\ &= \bigwedge_{x\in\mathrm{dom}(\hat{\omega})} (1 \Rightarrow 1) \\ &= 1. \end{aligned}$$

Hence the axiom of infinity is valid in $\mathbf{V}^{(\mathbb{A})}$.

(iv) Union: We know that,

$$[\![\forall u \exists v \forall x(x \in v \leftrightarrow \exists y \in u(x \in y))]\!] = \bigwedge_{u\in\mathbf{V}^{(\mathbb{A})}} \bigvee_{v\in\mathbf{V}^{(\mathbb{A})}} [\![\forall x(x \in v \leftrightarrow \exists y \in u(x \in y))]\!].$$

Therefore, if for any $u \in \mathbf{V}^{(\mathbb{A})}$ we can find a $v \in \mathbf{V}^{(\mathbb{A})}$ for which,
$[\![\forall x(x \in v \leftrightarrow \exists y \in u(x \in y))]\!]$

$$= [\![\forall x \in v \exists y \in u(x \in y)]\!] \wedge [\![\forall x(\exists y \in u(x \in y) \to (x \in v))]\!] = 1$$

then Union will be valid in $\mathbf{V}^{(\mathbb{A})}$.

Let us take any $u \in \mathbf{V}^{(\mathbb{A})}$ and then define a $v \in \mathbf{V}^{(\mathbb{A})}$ so that,

$$\mathrm{dom}(v) = \bigcup\{\mathrm{dom}(y) \colon y \in \mathrm{dom}(u)\}$$

and for any $x \in \mathrm{dom}(v)$,

$$v(x) = [\![\exists y \in u(x \in y)]\!].$$

For this v we have,

$$[\![\forall x \in v \exists y \in u(x \in y)]\!] = \bigwedge_{x \in \mathrm{dom}(v)} ([\![\exists y \in u(x \in y)]\!] \Rightarrow [\![\exists y \in u(x \in y)]\!]) = 1.$$

Let us now check the value of the second conjunct.

$$\begin{aligned}
[\![\forall x(\exists y \in u(x \in y) \to (x \in v))]\!] &= \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} ([\![\exists y \in u(x \in y)]\!] \Rightarrow [\![x \in v]\!]) \\
&= \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} [[\![\exists y \in u(x \in y)]\!] \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} (v(z) \wedge [\![x = z]\!])] \\
&= \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} [(\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge [\![x \in y]\!])) \\
&\qquad \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} (\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge [\![z \in y]\!]) \wedge [\![x = z]\!])]
\end{aligned}$$

We claim, for any $x \in \mathbf{V}^{(\mathbb{A})}$,

$$\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge [\![x \in y]\!]) \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} (\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge [\![z \in y]\!]) \wedge [\![x = z]\!]) = 1$$

i.e.,

$$\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge [\![x \in y]\!]) \Rightarrow \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \bigvee_{z \in \mathrm{dom}(v)} ([\![z \in y]\!] \wedge [\![x = z]\!])) = 1.$$

It will be proved if we can show, for any $y \in \mathrm{dom}(u)$,

$$[\![x \in y]\!] \leq \bigvee_{z \in \mathrm{dom}(v)} ([\![z \in y]\!] \wedge [\![x = z]\!])).$$

Let us take an element $y \in \mathrm{dom}(u)$. Then,

$$\bigvee_{z\in \mathrm{dom}(v)} (\llbracket z \in y \rrbracket \wedge \llbracket x = z \rrbracket)) = \bigvee_{z\in \mathrm{dom}(v)} \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket).$$

By our construction, $w \in \mathrm{dom}(y)$ implies $w \in \mathrm{dom}(v)$. Hence for any $w \in \mathrm{dom}(y)$,

$$\begin{aligned} \bigvee_{z\in \mathrm{dom}(v)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket) &\geq (y(w) \wedge \llbracket w = w \rrbracket \wedge \llbracket x = w \rrbracket) \\ &= (y(w) \wedge \llbracket x = w \rrbracket). \end{aligned}$$

Hence we get,

$$\bigvee_{z\in \mathrm{dom}(v)} \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket) \geq \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket x = w \rrbracket),$$

i.e.,

$$\bigvee_{z\in \mathrm{dom}(v)} \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket) \geq \llbracket x \in y \rrbracket.$$

But we know that

$$\bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket) \wedge \llbracket x = z \rrbracket \geq \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket),$$

which implies

$$\bigvee_{z\in \mathrm{dom}(v)} (\bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket) \wedge \llbracket x = z \rrbracket) \geq \bigvee_{z\in \mathrm{dom}(v)} \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket),$$

i.e.,

$$\bigvee_{z\in \mathrm{dom}(v)} (\llbracket z \in y \rrbracket \wedge \llbracket x = z \rrbracket) \geq \bigvee_{z\in \mathrm{dom}(v)} \bigvee_{w\in \mathrm{dom}(y)} (y(w) \wedge \llbracket z = w \rrbracket \wedge \llbracket x = z \rrbracket),$$

and as a conclusion we get,

$$\bigvee_{z\in\mathrm{dom}(v)} ([\![z \in y]\!] \wedge [\![x = z]\!])) \geq [\![x \in y]\!].$$

(v) NFF-Collection: Let $\varphi(x, y)$ be a negation-free formula having two free variables. By definition,
$[\![\forall u[\forall x \in u \exists y \varphi(x, y) \to \exists v \forall x \in u \exists y \in v \varphi(x, y)]]\!]$

$$= \bigwedge_{u\in\mathbf{V}^{(\mathbb{A})}} [\![\forall x \in u \exists y \varphi(x, y) \to \exists v \forall x \in u \exists y \in v \varphi(x, y)]\!].$$

Now fix any $u \in \mathbf{V}^{(\mathbb{A})}$ and then consider,

$$\begin{aligned} [\![\forall x \in u \exists y \varphi(x, y)]\!] &= \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow [\![\exists y \varphi(x, y)]\!]] \\ &= \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow \bigvee_{y\in\mathbf{V}^{(\mathbb{A})}} [\![\varphi(x, y)]\!]]. \end{aligned}$$

We have taken A as a set. Therefore we may apply the Axiom of Replacement in $\mathbf{V}$ so that for any $x \in \mathrm{dom}(u)$ we get an ordinal α_x such that

$$\bigvee_{y\in\mathbf{V}^{(\mathbb{A})}} [\![\varphi(x, y)]\!] = \bigvee_{y\in\mathbf{V}^{(\mathbb{A})}_{\alpha_x}} [\![\varphi(x, y)]\!].$$

Let $\alpha = \bigcup\{\alpha_x \colon x \in \mathrm{dom}(u)\}$. So we get,

$$\bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow \bigvee_{y\in\mathbf{V}^{(\mathbb{A})}} [\![\varphi(x, y)]\!]] = \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow \bigvee_{y\in\mathbf{V}^{(\mathbb{A})}_{\alpha_x}} [\![\varphi(x, y)]\!]].$$

Since $\alpha_x \subseteq \alpha$, for each $x \in \mathrm{dom}(u)$,

$$\bigvee_{y \in \mathbf{V}_{\alpha_x}^{(\mathbb{A})}} [\![\varphi(x,y)]\!] \leq \bigvee_{y \in \mathbf{V}_{\alpha}^{(\mathbb{A})}} [\![\varphi(x,y)]\!].$$

Then, by the property **P2**, on page 17, for each $x \in \mathrm{dom}(u)$,

$$[u(x) \Rightarrow \bigvee_{y \in \mathbf{V}_{\alpha_x}^{(\mathbb{A})}} [\![\varphi(x,y)]\!]] \leq [u(x) \Rightarrow \bigvee_{y \in \mathbf{V}_{\alpha}^{(\mathbb{A})}} [\![\varphi(x,y)]\!]].$$

Hence we can easily conclude that,

$$\bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathbb{A})}} [\![\varphi(x,y)]\!]] \leq \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow \bigvee_{y \in \mathbf{V}_{\alpha}^{(\mathbb{A})}} [\![\varphi(x,y)]\!]].$$

Let us take, $v = \mathbf{V}_{\alpha}^{(\mathbb{A})} \times 1$; then clearly $v \in \mathbf{V}^{(\mathbb{A})}$ and

$$\begin{aligned}
[\![\exists y \in v\varphi(x,y)]\!] &= \bigvee_{y \in \mathrm{dom}(v)} [v(y) \wedge [\![\varphi(x,y)]\!]] \\
&= \bigvee_{y \in \mathbf{V}_{\alpha}^{(\mathbb{A})}} [\![\varphi(x,y)]\!], \text{ [since } v(y) = 1, \text{ for all } y \in \mathrm{dom}(v) = \mathbf{V}_{\alpha}^{(\mathbb{A})}].
\end{aligned}$$

Now by putting the above results together we get,

$$[\![\forall x \in u \exists y \varphi(x,y)]\!] \leq \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow [\![\exists y \in v\varphi(x,y)]\!]] = [\![\forall x \in u \exists y \in v\varphi(x,y)]\!].$$

Hence it is easy to prove, the NFF-Collection Axiom Schema is D-valid in $\mathbf{V}^{(\mathbb{A})}$. $\square$

Theorem 2.2.7 *Let $\mathbb{A}$ be a reasonable and deductive implication algebra such that $\mathbf{V}^{(\mathbb{A})}$ satisfies the* NFF*-bounded quantification property, and let D be any filter on A. Then Power Set and* NFF*-Separation are D-valid in $\mathbf{V}^{(\mathbb{A})}$; in fact, they get the value* **1**.

Proof. (i) Power Set: Let us take any $u \in \mathbf{V}^{(\mathbb{A})}$ and fix $v \in \mathbf{V}^{(\mathbb{A})}$ such that, $\mathrm{dom}(v) = A^{\mathrm{dom}(u)}$ and for each $x \in \mathrm{dom}(v)$,

$$v(x) = [\![x \subseteq u]\!] = [\![\forall y \in x(y \in u)]\!].$$

Here we have abbreviated $\forall y \in x(y \in u)$ as $x \subseteq u$. Now it is enough to prove,

$$[\![\forall x(x \in v \leftrightarrow \forall y \in x(y \in u))]\!] = [\![\forall x \in v(x \subseteq u)]\!] \wedge [\![\forall x(x \subseteq u \to x \in v)]\!] = 1.$$

The value of the first conjunct is clearly 1 by our definitions and the fact that $a \Rightarrow a = 1$ for all $a \in A$. So we are interested to prove that the value of the second conjunct is also 1, if $\mathbb{A}$ has the property, $a \Rightarrow b \leq a \Rightarrow (a \wedge b)$. The second conjunct is,

$$[\![\forall x(x \subseteq u \to x \in v)]\!] = \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} ([\![\forall y \in x(y \in u)]\!] \Rightarrow [\![x \in v]\!]).$$

Let us fix an arbitrary $x \in \mathbf{V}^{(\mathbb{A})}$. Then,
$[\![\forall y \in x(y \in u)]\!] \Rightarrow [\![x \in v]\!]$

$$\begin{aligned}
&= [\![\forall y \in x(y \in u)]\!] \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} (v(z) \wedge [\![x = z]\!]) \\
&= [\![\forall y \in x(y \in u)]\!] \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} ([\![\forall y \in z(y \in u)]\!] \wedge [\![x = z]\!]) \\
&= \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in u]\!]) \Rightarrow \bigvee_{z \in \mathrm{dom}(v)} [\bigwedge_{p \in \mathrm{dom}(z)} (z(p) \Rightarrow [\![p \in u]\!]) \wedge \\
&\qquad \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in z]\!]) \wedge \bigwedge_{p \in \mathrm{dom}(z)} (z(p) \Rightarrow [\![p \in x]\!])] \\
&\geq \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in u]\!]) \Rightarrow \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in z']\!])
\end{aligned}$$

[for some $z' \in \mathrm{dom}(v)$ such that, $z'(p) = u(p) \wedge [\![p \in x]\!]$, for all $p \in \mathrm{dom}(z')$].

Now let us take any $w \in \mathrm{dom}(x)$. Then,

$$\begin{aligned}
x(w) \Rightarrow [\![w \in z']\!] &= x(w) \Rightarrow \bigvee_{p \in \mathrm{dom}(z')} (z'(p) \wedge [\![p = w]\!]) \\
&= x(w) \Rightarrow \bigvee_{p \in \mathrm{dom}(z)} [u(p) \wedge (\bigvee_{w' \in \mathrm{dom}(x)} (x(w') \wedge [\![p = w']\!])) \wedge [\![p = w]\!]] \\
&\geq x(w) \Rightarrow \bigvee_{p \in \mathrm{dom}(u)} [u(p) \wedge (x(w) \wedge [\![p = w]\!]) \wedge [\![p = w]\!]] \\
&= x(w) \Rightarrow \bigvee_{p \in \mathrm{dom}(u)} (u(p) \wedge [\![p = w]\!]) \wedge x(w) \\
&= x(w) \Rightarrow [\![w \in u]\!] \wedge x(w) \\
&\geq x(w) \Rightarrow [\![w \in u]\!], \text{ (by our assumption).}
\end{aligned}$$

Hence the following holds,

$$\bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in u]\!]) \leq \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in z']\!]).$$

Therefore for any $x \in \mathbf{V}^{(\mathbb{A})}$,

$$\bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in u]\!]) \Rightarrow \bigwedge_{w \in \mathrm{dom}(x)} (x(w) \Rightarrow [\![w \in z']\!]) = 1$$

i.e.,

$$[\![\forall y \in x(y \in u)]\!] \Rightarrow [\![x \in v]\!] = 1.$$

Hence we get, the value of the second conjunct is also 1. Therefore the Power Set Axiom is true in $\mathbf{V}^{(\mathbb{A})}$.

(ii) NFF-Separation: Let $\varphi(x)$ be a negation-free formula having one free variable. We know that,

$$[\![\forall u \exists v \forall x (x \in v \leftrightarrow x \in u \wedge \varphi(x))]\!] = \bigwedge_{u \in \mathbf{V}^{(\mathbb{A})}} \bigvee_{v \in \mathbf{V}^{(\mathbb{A})}} [\![\forall x (x \in v \leftrightarrow x \in u \wedge \varphi(x))]\!].$$

Let us take any $u \in \mathbf{V}^{(\mathbb{A})}$ and define $v \in \mathbf{V}^{(\mathbb{A})}$ by $\mathrm{dom}(u) = \mathrm{dom}(v)$ and for each $x \in \mathrm{dom}(v)$,

$$v(x) = u(x) \wedge [\![\varphi(x)]\!].$$

Now,

$$[\![\forall x (x \in v \leftrightarrow x \in u \wedge \varphi(x))]\!] = [\![\forall x \in v (x \in u \wedge \varphi(x))]\!] \wedge [\![\forall x (x \in u \wedge \varphi(x) \to x \in v)]\!].$$

Let us first find the value of the first conjunct.

$$[\![\forall x \in v (x \in u \wedge \varphi(x))]\!] = \bigwedge_{x \in \mathrm{dom}(v)} (u(x) \wedge [\![\varphi(x)]\!] \Rightarrow [\![x \in u]\!] \wedge [\![\varphi(x)]\!]) = 1,$$

since, $\mathrm{dom}(v) = \mathrm{dom}(u)$ and $u(x) \leq [\![x \in u]\!]$ for each $x \in \mathrm{dom}(u)$.
The value of the second conjunct is,

$$\begin{aligned}
[\![\forall x (x \in u \wedge \varphi(x) \to x \in v)]\!] &= \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} ([\![x \in u]\!] \wedge [\![\varphi(x)]\!] \Rightarrow [\![x \in v]\!]) \\
&= \bigwedge_{x \in \mathbf{V}^{(\mathbb{A})}} ([\![x \in u]\!] \Rightarrow ([\![\varphi(x)]\!] \Rightarrow [\![x \in v]\!])) \\
&= [\![\forall x (x \in u \to (\varphi(x) \to x \in v)]\!] \\
&= [\![\forall x \in u (\varphi(x) \to x \in v)]\!] \\
&= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow ([\![\varphi(x)]\!] \Rightarrow [\![x \in v]\!])) \\
&= \bigwedge_{x \in \mathrm{dom}(u)} [(u(x) \wedge [\![\varphi(x)]\!]) \Rightarrow [\![x \in v]\!]]
\end{aligned}$$

$$= \bigwedge_{x \in \mathrm{dom}(v)} (v(x) \Rightarrow [\![x \in v]\!])$$
$$= 1.$$

Hence the theorem is proved. □

2.2.3 Notion of "set"

Usually, in the model theory of set theory, we build models (M, E) where M is a class of objects and E is a binary relation on M. We interpret these structures as $\mathcal{L}_\in$-structures where the symbol $\in$ is interpreted by E and the symbol $=$ is interpreted by equality of objects in M. So, the objects of M will be interpreted as "sets".

This is quite different in the setting of algebra-valued models: the elements of $\mathbf{V}^{(\mathbb{A})}$ are not interpreted as the objects of our model of set theory; in particular, equality between them is not the correct interpretation for the $=$ symbol in $\mathcal{L}_\in$. Instead, the elements of $\mathbf{V}^{(\mathbb{A})}$ are *names* for sets, and a given set can have many different names.

If $\mathbb{A}$ is a Boolean algebra, i.e., we are looking at Boolean-valued models, there is a way of transforming $\mathbf{V}^{(\mathbb{A})}$ into a model of set theory. This transformation links the theory of Boolean-valued models to the theory of forcing and is discussed in [4, Chapter 4]:

If G is an $\mathbb{A}$-*generic filter* over $\mathbb{V}$, then we define an equivalence relation $\sim_G$ on $\mathbf{V}^{(\mathbb{A})}$ that will respect the definition of the evaluation of $\in$ in the sense that if

$$\mathbf{V}^{(\mathbb{A})} \models \tau \in \tau^* \text{ and } \tau \sim_G \tau', \text{ then } \mathbf{V}^{(\mathbb{A})} \models \tau' \in \tau^*$$

for any names τ, τ', and τ^*. So, on the quotient $\mathbf{V}^{(\mathbb{A})}/{\sim_G}$, we can define a relation E_G by

$$[\tau]_{\sim_G} \, E_G \, [\tau']_{\sim_G} \text{ iff } \mathbf{V}^{(\mathbb{A})} \models \tau \in \tau'.$$

As a consequence, the structure $(\mathbf{V}^{(\mathbb{A})}/{\sim_G}, E_G)$ is now an actual $\mathcal{L}_\in$-structure and one can show that it is a wellfounded model of the axioms of set theory [4, Theorem 4.22].

The situation is different for more general algebras $\mathbb{A}$ where we do not have an analogue of the notion of a generic filter. A natural and naïve idea would be to think of the equivalence classes of the following relation:

$$\tau \sim \tau' \text{ iff } \mathbf{V}^{(\mathbb{A})} \models \tau = \tau'.$$

For Boolean algebras $\mathbb{A}$, this naïve idea does not give us a proper ontology: for instance, if $\mathbb{B}_4 = \{0, 1, l, r\}$ is the four element Boolean algebra, then the empty name $\varnothing$ is the name for the empty set. For $x \in \mathbb{B}_4$, let τ_x be the name with domain $\{\varnothing\}$ and $\tau_x(\varnothing) = x$. These four names are pairwise non-equivalent in $\sim$, but correspond in the set theory of $\mathbf{V}^{(\mathbb{B}_4)}$ to only two sets, viz. $\varnothing$ and $\{\varnothing\}$.

Moving from Boolean valued models to Heyting valued models, this phenomenon is exploited to provide models of intuitionistic set theory that have large power sets of the ordinal number 1.

For our algebra-valued models, we shall explore the properties of this relation in Chapter 4 in depth. Since we are working in a weak logic, it is not immediately obvious that $\sim$ is an equivalence relation. For example consider Proposition 2.2.3:

Let $\mathbb{A}$ be a reasonable-implication algebra which satisfies all the conditions of Proposition 2.2.3. If we observe the proof of this proposition, we get three elements x_0, x_p, and x_1 in $\mathbf{V}^{(\mathbb{A})}$ such that $[\![x_0 = x_p]\!] = p^*$, $[\![x_p = x_1]\!] = p$, and $[\![x_0 = x_1]\!] = 0$. Now if we fix the designated set $D_\mathbb{A}$ of $\mathbb{A}$ such a way that p and p^* both are in $D_\mathbb{A}$ then

$$\mathbf{V}^{(\mathbb{A})} \models (x_0 = x_p) \wedge (x_p = x_1) \text{ but } \mathbf{V}^{(\mathbb{A})} \not\models x_0 = x_1.$$

This shows that $\sim$ is not an equivalence relation in this algebra-valued model $\mathbf{V}^{(\mathbb{A})}$.

In Chapter 3 we shall introduce a reasonable implication algebra PS_3 which will give rise to an algebra-valued model of a paraconsistent set theory. From Lemma 3.5.2 (i), one can deduce that $\sim$ will be an equivalence relation in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Chapter 3

The three-valued matrix PS_3 and the paraconsistent logic $\mathbb{L}\mathrm{PS}_3$

In Chapter 2 (Theorems 2.2.6 and 2.2.7), we gave sufficient conditions on an algebra $\mathbb{A}$ such that the relevant axioms of set theory are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. This raises the immediate question whether there are any algebras $\mathbb{A}$ that are neither Boolean algebras nor Heyting algebras that satisfy these sufficient conditions. In this chapter, we shall introduce such an algebra PS_3. The logic $\mathbb{L}\mathrm{PS}_3$ is found which is sound and (weak)complete with respect to PS_3.

3.1 The three-valued matrix PS_3

We introduce a three-valued matrix $\mathrm{PS}_3 = \langle \{1, {}^1\!/_2, 0\}, \wedge, \vee, \Rightarrow, ^* \rangle$ having the following truth tables:

$\wedge$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	0
0	0	0	0

$\vee$	1	$1/2$	0
1	1	1	1
$1/2$	1	$1/2$	$1/2$
0	1	$1/2$	0

$\Rightarrow$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	1	1	1

$*$	
1	0
$1/2$	$1/2$
0	1

and $D = \{1, 1/2\}$ as the designated set.

The algebra PS_3 is a *complete distributive lattice* relative to $\wedge, \vee$. It can be checked that PS_3 is a deductive reasonable implication algebra. By asserting values $1/2$ to φ and 0 to ψ one can check that "Par" is satisfied. Later we will prove that BQ_φ holds in $\mathbf{V}^{(\mathrm{PS}_3)}$ for every negation-free formula φ (see Theorem 3.5.7).

This matrix is included in the collection of 2^{13} three-valued matrices of the *Logic of Formal Inconsistencies* (cf. [6]) after exclusion of the *inconsistency operator* "$\bullet$". For our purpose we do not need the operator for inconsistency which acts for internalising *inconsistency* within the *object language*. Now it is important to explain why we have chosen PS_3. First of all $(\{1, 1/2, 0\}, \wedge, \vee)$ has to be a complete distributive lattice for which $\wedge$ and $\vee$ have to be the operators *minimum* and *maximum* respectively. Secondly for satisfying properties **P1**, **P2**, **P3**, and **P4** the only possibilities for the implication are given below:

$\Rightarrow_1$	1	$1/2$	0
1	1	1	1
$1/2$	1	1	1
0	1	1	1

$\Rightarrow_2$	1	$1/2$	0
1	0	0	0
$1/2$	0	0	0
0	0	0	0

$\Rightarrow_3$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	1	0
0	1	1	1

$\Rightarrow_4$	1	$1/2$	0
1	1	1	0
$1/2$	1	1	0
0	1	1	1

The implications $\Rightarrow_1$ and $\Rightarrow_2$ cannot produce a reasonable logic as these two are degenerated. The implication $\Rightarrow_3$ satisfies both **P1** and its converse, though the converse of **P1** is not needed for being a reasonable implication algebra. Besides, $\Rightarrow_3$ together with the above mentioned operators $\wedge$ and $\vee$ produce the three-valued Heyting algebra. As a consequence we are interested in $\Rightarrow_4$ which is the implication of PS_3. Before fixing the truth table of * it should be noticed, since $1 \Rightarrow {}^1\!/_2 = 1$ in the chosen truth table for $\Rightarrow$, for getting *Modus Ponens* as a valid rule in our system the designated set has to be fixed as $\{1, {}^1\!/_2\}$.

Following are the requirements for fixing the truth table of *.

(*i*) For generating a paraconsistent logic, there should exist an element $d \in \{1, {}^1\!/_2\}$ such that $d^* \in \{1, {}^1\!/_2\}$ as well.

(*ii*) For getting a reasonable logic one can expect $0^* \in \{1, {}^1\!/_2\}$ and either $1^* = 0$ or ${}^1\!/_2{}^* = 0$.

Hence the following are only possibilities for the truth table of *.

$^*{}_1$		$^*{}_2$		$^*{}_3$		$^*{}_4$		$^*{}_5$		$^*{}_6$		$^*{}_7$		$^*{}_8$	
1	0	1	0	1	1/2	1	1	1	0	1	0	1	1/2	1	1
1/2	1/2	1/2	1	1/2	0	1/2	0	1/2	1/2	1/2	1	1/2	0	1/2	0
0	1	0	1	0	1	0	1	0	1/2	0	1/2	0	1/2	0	1/2

We are interested in taking $1^* = 0$ and $0^* = 1$ so that it does not violate the third criterion of Jaśkowski, **Jas3** for being a paraconsistent logic. Hence $^*{}_1$ and $^*{}_2$ are the only remaining possibilities. Since we want to have the rule of *double negation*, as in many of the other well known paraconsistent logics (shown in Section 3.3) the only choice for * is $^*{}_1$. However, it may be mentioned that in [23] a three-valued paraconsistent logic G'_3 having connectives $\wedge$, $\vee$, $\Rightarrow$ and * has been intensively investigated in which $\wedge$ and $\vee$ are same as PS_3 but $\Rightarrow$ and * are taken as $\Rightarrow_3$ and $^*{}_2$, respectively. Since $\Rightarrow_3$ is same as the implication operator of the three-valued Heyting algebra, and Heyting-valued models are already studied, we decide to work with the implication operator $\Rightarrow_4$, which is same as the operator $\Rightarrow$ of PS_3. Some comparisons between PS_3 and G'_3 are given in Section 3.3.2. It is to be noted that PS_3 is a fixed, particular algebra of type $(2, 2, 2, 1, 0, 0)$ which satisfies the conditions **P1**, **P2**, **P3**, **P4**, and Par.

3.2 The logic $\mathbb{L}\mathrm{PS}_3$

In this section we introduce an axiom system for the propositional logic $\mathbb{L}\mathrm{PS}_3$ having the matrix PS_3 as the three-valued semantics. The alphabet of $\mathbb{L}\mathrm{PS}_3$ consists of propositional letters $p_1, p_2, \ldots$; logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$. the well formed formulas are constructed in the usual way.

3.2.1 The axiom system for $\mathbb{L}\mathrm{PS}_3$

The following formulas are taken as the axioms for $\mathbb{L}\mathrm{PS}_3$:

(Ax1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(Ax2) $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$

(Ax3) $\varphi \wedge \psi \rightarrow \varphi$

(Ax4) $\varphi \wedge \psi \rightarrow \psi$

(Ax5) $\varphi \rightarrow \varphi \vee \psi$

(Ax6) $(\varphi \rightarrow \gamma) \wedge (\psi \rightarrow \gamma) \rightarrow (\varphi \vee \psi \rightarrow \gamma)$

(Ax7) $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \psi \wedge \gamma)$

(Ax8) $\varphi \leftrightarrow \neg\neg\varphi$

(Ax9) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

(Ax10) $(\varphi \wedge \neg\varphi) \rightarrow (\neg(\psi \rightarrow \varphi) \rightarrow \gamma)$

(Ax11) $(\varphi \to \psi) \to (\neg(\varphi \to \gamma) \to \psi)$

(Ax12) $(\neg\varphi \to \psi) \to (\neg(\gamma \to \varphi) \to \psi)$

(Ax13) $\perp \to \varphi$

(Ax14) $(\varphi \wedge (\psi \to \perp)) \to \neg(\varphi \to \psi)$

(Ax15) $(\varphi \wedge (\neg\varphi \to \perp)) \vee (\varphi \wedge \neg\varphi) \vee (\neg\varphi \wedge (\varphi \to \perp))$

where φ, ψ, γ are any well formed formulas and $\perp$ is the abbreviation for $\neg(\theta \to \theta)$ for any arbitrary formula θ.

The rules for $\mathbb{L}\mathrm{PS}_3$ are the following:

1. $\dfrac{\varphi,\ \psi}{\varphi \wedge \psi}$

2. $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

Let $\vdash$ be defined in the usual way and $\models$ be defined as follows: let Γ be a set of formulas and φ be a formula in $\mathbb{L}\mathrm{PS}_3$; if for any valuation v, $v(\psi) \in \{1, 1/2\}$ for all $\psi \in \Gamma$, implies $v(\varphi) \in \{1, 1/2\}$ then we say $\Gamma \models \varphi$. Below we shall show that the propositional axiom system is sound and (weak)complete with respect to PS_3.

3.2.2 Soundness

Theorem 3.2.1 *For any formula φ and a set of formulas Γ, if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.*

Proof. It is immediate that under any valuation the values of the axioms are either 1 or $1/2$ and all the rules are valid. Hence the value of any theorem will belong to the designated set, with respect to any valuation. So the theorem is proved. $\square$

3.2.3 Completeness

For the proof of *completeness* we need a few lemmas.

Lemma 3.2.2 *For any formula φ, $\vdash \varphi \to \varphi$ holds.*

Proof. Let us take an arbitrary formula φ. The following deduction will prove the theorem.

$\vdash$ 1. $[\varphi \to ((\varphi \to \varphi) \to \varphi)] \to [(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)]$	Axiom 2
2. $\varphi \to ((\varphi \to \varphi) \to \varphi)$	Axiom 1
3. $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	M.P. 1, 2
4. $\varphi \to (\varphi \to \varphi)$	Axiom 1
5. $\varphi \to \varphi$	M.P. 3, 4

Hence the proof is complete. □

Lemma 3.2.3 (Deduction Theorem). *If $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$.*

Proof. Let $x_1, x_2, \ldots, x_n$ be a derivation of ψ from $\Gamma \cup \{\varphi\}$. So by the definition of derivation we get, $x_n = \psi$. We will prove the theorem by showing, $\Gamma \vdash \varphi \to x_i$ for $i = 1, 2, \ldots, n$.
Case 1. Let x_1 be an axiom. The following derivation will prove $\Gamma \vdash \varphi \to x_1$.

$\Gamma \vdash$ 1. x_1	Axiom
2. $x_1 \to (\varphi \to x_1)$	Axiom 1
3. $\varphi \to x_1$	M.P. 1, 2

Case 2. Let x_1 be in Γ. Now $\Gamma \vdash \varphi \to x_1$ will be proved by following way.

$\Gamma \vdash$ 1. x_1	Assumption

2. $x_1 \rightarrow (\varphi \rightarrow x_1)$ Axiom 1

3. $\varphi \rightarrow x_1$ *M.P. 1, 2*

Case 3. Suppose x_1 is φ. By the previous theorem we know that $\vdash \varphi \rightarrow \varphi$ for any formula φ. So in this case we can also prove, $\Gamma \vdash \varphi \rightarrow \varphi$ i.e., $\Gamma \vdash \varphi \rightarrow x_1$ by monotonicity.

Case 4. Now suppose x_k $(k \leq n)$ is such that it is derived from x_i and x_j $(1 \leq i, j \leq k-1)$ where $\Gamma \vdash \varphi \rightarrow x_i$ and $\Gamma \vdash \varphi \rightarrow x_j$ either by Rule 1 or by M.P.

Subcase 4.1. Let x_k is derived from x_i and x_j by Rule 1, i.e., $x_k = x_i \wedge x_j$. We have to prove $\Gamma \vdash \varphi \rightarrow x_k$. The derivation for proving this is as follows:

$\Gamma \vdash$ 1. $\varphi \rightarrow x_i$ *Assumption*

2. $\varphi \rightarrow x_j$ *Assumption*

3. $(\varphi \rightarrow x_i) \wedge (\varphi \rightarrow x_j)$ *Rule 1 on 1, 2*

4. $(\varphi \rightarrow x_i) \wedge (\varphi \rightarrow x_j) \rightarrow (\varphi \rightarrow x_i \wedge x_j)$ Axiom 7

5. $\varphi \rightarrow x_i \wedge x_j$ M.P. 3, 4

So in this subcase it can be proved, $\Gamma \vdash \varphi \rightarrow x_k$.

Subcase 4.2. Let x_k is derived from x_i and x_j by M.P. Without loss of generality we can assume $x_j = x_i \rightarrow x_k$. Now for proving $\Gamma \vdash \varphi \rightarrow x_k$, the derivation is as follows:

$\Gamma \vdash$ 1. $\varphi \rightarrow x_i$ Assumption

2. $\varphi \rightarrow (x_i \rightarrow x_k)$ *Assumption, since* $x_j = x_i \rightarrow x_k$

3. $[\varphi \rightarrow (x_i \rightarrow x_k)] \rightarrow [(\varphi \rightarrow x_i) \rightarrow (\varphi \rightarrow x_k)]$ Axiom 2

4. $(\varphi \to x_i) \to (\varphi \to x_k)$ M.P. 2, 3

5. $\varphi \to x_k$ M.P. 1, 4

By all these cases together it has been shown, $\Gamma \vdash \varphi \to x_i$ for each $i = 1, 2, \ldots, n$. So in particular $\Gamma \vdash \varphi \to x_n$ i.e., $\Gamma \vdash \varphi \to \psi$. Hence the proof is completed. $\square$

Using the Deduction theorem one can also prove the following theorems.

Theorem 3.2.4 *For any formulas φ, ψ and γ the following formulas are theorems.*

(i) $(\varphi \to \psi) \to ((\varphi \wedge \gamma) \to \psi)$.

(ii) $(\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))$.

(iii) $(\varphi \to \psi) \wedge (\psi \to \gamma) \to (\varphi \to \gamma)$.

Lemma 3.2.5 is the most important step to prove the completeness theorem.

Lemma 3.2.5 *For any formula φ and a given valuation v with respect to* PS_3 *let φ' be defined by,*

$$\varphi' = \begin{cases} \varphi \wedge (\neg\varphi \to \bot) & \textit{if} \quad v(\varphi) = 1; \\ \varphi \wedge \neg\varphi & \textit{if} \quad v(\varphi) = {}^1\!/_2; \\ \neg\varphi \wedge (\varphi \to \bot) & \textit{if} \quad v(\varphi) = 0. \end{cases}$$

If $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ are the propositional letters in φ then $\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_k}\} \vdash \varphi'$.

Proof. As it was indicated that the proof will be by induction on the complexity of φ. Let, $\Gamma = \{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_k}\}$.
Base step: It is obvious when the complexity is 0.

Induction hypothesis: Assume the lemma holds well for formulas with complexity less than n.

Induction step: Let the complexity of φ be n.

Case 1: Let us consider $\varphi = \neg\psi$.

Clearly the complexity of ψ is less than n and the propositional letters in ψ are exactly same as the propositional letters in φ.

Subcase 1.1: If $v(\psi) = 1$ holds then we have $v(\varphi) = 0$. Hence by our construction,

$$\psi' = \psi \wedge (\neg\psi \to \bot) \text{ and } \varphi' = \neg\varphi \wedge (\varphi \to \bot).$$

Here we get,

$\Gamma \vdash$

1. ψ'	Induction hypothesis
2. ψ	Axiom 3 and M.P.
3. $\neg\psi \to \bot$	Axiom 4 and M.P.
4. $\psi \to \neg\neg\psi$	Axiom 8
5. $\neg\neg\psi$	M.P. 2, 4
6. $\neg\varphi \wedge (\varphi \to \bot)$	Rule 1 on 3 and 5

Hence in subcase1.1, $\Gamma \vdash \varphi'$.

Subcase 1.2: If $v(\psi) = 1/2$ holds then we get $v(\varphi) = 1/2$. So by the construction,

$$\psi' = \psi \wedge \neg\psi \text{ and } \varphi' = \varphi \wedge \neg\varphi.$$

So we have,

$\Gamma \vdash$

1. ψ'	Induction hypothesis

2. ψ Axiom 3 and M.P.

3. $\neg\psi$ Axiom 4 and M.P.

4. $\psi \to \neg\neg\psi$ Axiom 8

5. $\neg\neg\psi$ M.P. 2, 4

6. $\varphi \wedge \neg\varphi$ Rule 1 on 3 and 5

Hence $\Gamma \vdash \varphi'$ holds here.

Subcase 1.3: If we consider $v(\psi) = 0$ then $v(\varphi) = 1$ holds. Hence,

$$\psi' = \neg\psi \wedge (\psi \to \bot) \text{ and } \varphi' = \varphi \wedge (\neg\varphi \to \bot).$$

The following derivation can be made,

$\Gamma \vdash$

1. ψ' Induction hypothesis

2. $\neg\psi$ Axiom 3 and M.P.

3. $\psi \to \bot$ Axiom 4 and M.P.

4. $(\neg\neg\psi \to \psi) \to [(\psi \to \bot) \to (\neg\neg\psi \to \bot)]$ Theorem 3.2.4(ii)

5. $\neg\neg\psi \to \psi$ Axiom 8

6. $(\psi \to \bot) \to (\neg\neg\psi \to \bot)$ M.P. 4, 5

7. $\neg\neg\psi \to \bot$ M.P. 3, 6

8. φ' Rule 1 on 2, 7

Hence in *Case 1* we always get $\Gamma \vdash \varphi'$.

Case 2: Let us take $\varphi = \psi \wedge \gamma$.

Obviously both the complexities of ψ and γ are less than n and the sets of propositional letters in φ and ψ are proper subsets of $\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_k}\}$, the set of propositional letters in

φ. Hence clearly by the induction hypothesis and monotonicity property we get,

$$\Gamma \vdash \psi' \quad \text{and} \quad \Gamma \vdash \gamma'.$$

Subcase 2.1: If any one of $v(\psi)$ and $v(\gamma)$ is 0 then it can be proved $\Gamma \vdash \varphi'$. Without loss of generality, let $v(\psi) = 0$, then $v(\varphi) = 0$. Hence we get the following.

$$\psi' = \neg\psi \wedge (\psi \to \bot) \quad \text{and} \quad \varphi' = \neg\varphi \wedge (\varphi \to \bot).$$

Since $\Gamma \vdash \psi'$,

$\Gamma \vdash$

1. $\neg\psi$	Axiom 3 and M.P.
2. $\psi \to \bot$	Axiom 4 and M.P.
3. $\neg\psi \to (\neg\psi \vee \neg\gamma)$	Axiom 5
4. $\neg\psi \vee \neg\gamma$	M.P. 1, 3
5. $(\neg\psi \vee \neg\gamma) \to \neg(\psi \wedge \gamma)$	Axiom 9
6. $\neg(\psi \wedge \gamma)$	M.P. 4, 5
7. $(\psi \to \bot) \to (\psi \wedge \gamma \to \bot)$	Theorem 3.2.4(i)
8. $\psi \wedge \gamma \to \bot$	M.P. 2, 7
9. φ'	Rule 1 on 6, 8

Subcase 2.2: If $v(\psi) = 1/2$ and $v(\gamma) = 1/2$ hold together then $v(\varphi) = 1/2$ will also hold. So by the definition,

$$\psi' = \psi \wedge \neg\psi, \quad \gamma' = \gamma \wedge \neg\gamma \quad \text{and} \quad \varphi' = \varphi \wedge \neg\varphi.$$

Now for proving $\Gamma \vdash \varphi'$ i.e., $\Gamma \vdash (\psi \wedge \gamma) \wedge \neg(\psi \wedge \gamma)$ we go through the following derivation, by using $\Gamma \vdash \psi'$ and $\Gamma \vdash \gamma'$.

$\Gamma \vdash$

1. ψ Axiom 3 and M.P.
2. $\neg\psi$ Axiom 4 and M.P.
3. γ Axiom 3 and M.P.
4. $\psi \wedge \gamma$ Rule 1 on 1, 3
5. $\neg\psi \to (\neg\psi \vee \neg\gamma)$ Axiom 5
6. $(\neg\psi \vee \neg\gamma)$ M.P. 2, 5
7. $(\neg\psi \vee \neg\gamma) \to \neg(\psi \wedge \gamma)$ Axiom 9
8. $\neg(\psi \wedge \gamma)$ M.P. 6, 7
9. φ' Rule 1 on 4, 8

Subcase 2.3: If we have $v(\psi) = 1/2$ and $v(\gamma) = 1$ then $v(\varphi) = 1/2$ holds. Hence,

$$\psi' = \psi \wedge \neg\psi, \quad \gamma' = \gamma \wedge (\neg\gamma \to \bot) \quad \text{and} \quad \varphi' = \varphi \wedge \neg\varphi.$$

Since $\Gamma \vdash \psi'$ and $\Gamma \vdash \gamma'$, using Axiom 1, 2 and rule M.P. we get,

$$\Gamma \vdash \psi, \quad \Gamma \vdash \neg\psi \quad \text{and} \quad \Gamma \vdash \gamma.$$

Now following the same derivation as above we can prove $\Gamma \vdash \psi'$.

Subcase 2.4: If $v(\psi) = 1$ and $v(\gamma) = 1$ hold then $v(\varphi) = 1$ is true. Therefore by our construction,

$$\psi' = \psi \wedge (\neg\psi \to \bot), \quad \gamma' = \gamma \wedge (\neg\gamma \to \bot) \quad \text{and} \quad \varphi' = \varphi \wedge (\neg\varphi \to \bot).$$

So we have to prove $\Gamma \vdash \varphi'$, i.e., $\Gamma \vdash (\psi \wedge \gamma) \wedge [\neg(\psi \wedge \gamma) \to \bot]$. The derivation is as follows.

$\Gamma \vdash$

1. ψ Axiom 3 and M.P.
2. γ Axiom 3 and M.P.
3. $\neg\psi \to \bot$ Axiom 4 and M.P.
4. $\neg\gamma \to \bot$ Axiom 4 and M.P.
5. $\psi \wedge \gamma$ Rule 1 on 1 and 2
6. $(\neg\psi \to \bot) \wedge (\neg\gamma \to \bot)$ Rule 1 on 3 and 4
7. $(\neg\psi \to \bot) \wedge (\neg\gamma \to \bot) \to (\neg\psi \vee \neg\gamma \to \bot)$ Axiom 6
8. $\neg\psi \vee \neg\gamma \to \bot$ M.P. 6, 7
9. $\neg(\psi \wedge \gamma) \to (\neg\psi \vee \neg\gamma)$ Axiom 9
10. $[\neg(\psi \wedge \gamma) \to (\neg\psi \vee \neg\gamma)] \to$
    $[((\neg\psi \vee \neg\gamma) \to \bot) \to (\neg(\psi \wedge \gamma) \to \bot)]$ Theorem 3.2.4(ii)
11. $\neg(\psi \wedge \gamma) \to \bot$ M.P. repeatedly on 10, 9, 8
12. φ' Rule 1 on 5, 11

Subcase 2.5: If $v(\psi) = 1$ and $v(\gamma) = 1/2$ hold together then by the same derivation in Subcase 2.3 it can be proved $\Gamma \vdash \varphi'$.

Hence in *Case 2* we can always prove $\Gamma \vdash \varphi'$.

Case 3: Let us assume $\varphi = \psi \vee \gamma$.

Since $\varphi \vee \psi$ can be abbreviated as $\neg(\neg\varphi \wedge \neg\psi)$ therefore by using Case 1 and Case 2, $\Gamma \vdash \varphi'$ can also be proved in this case.

Case 4: Let us consider $\varphi = \psi \to \gamma$.

Obviously both the complexities of ψ and γ are less than n and the sets of propositional letters in φ and ψ are subsets of $\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_k}\}$, the set of propositional letters in φ. Hence clearly by the induction hypothesis and monotonicity property we get,

$$\Gamma \vdash \psi' \quad \text{and} \quad \Gamma \vdash \gamma'.$$

Subcase 4.1: If $v(\gamma) = 1$ is the case then no matter what $v(\psi)$ is, $v(\varphi) = 1$ is always true. So we get, $\gamma' = \gamma \wedge (\neg\gamma \to \bot)$ and $\varphi' = \varphi \wedge (\neg\varphi \to \bot) = (\psi \to \gamma) \wedge [\neg(\psi \to \gamma) \to \bot]$.

Now for proving $\Gamma \vdash \varphi'$ we go through the following derivation.

$\Gamma \vdash$

1. γ	Axiom 3 and M.P.
2. $\neg\gamma \to \bot$	Axiom 4 and M.P.
3. $\gamma \to (\psi \to \gamma)$	Axiom 1
4. $\psi \to \gamma$	M.P. 1, 3
5. $(\neg\gamma \to \bot) \to [\neg(\psi \to \gamma) \to \bot]$	Axiom 12
6. $\neg(\psi \to \gamma) \to \bot$	M.P. 2, 5
7. φ'	Rule 1 on 4, 6

Subcase 4.2: If $v(\gamma) = 1/2$ holds then always $v(\varphi) = 1$ will hold. Hence by the definition,

$$\gamma' = \gamma \wedge \neg\gamma \quad \text{and} \quad \varphi' = \varphi \wedge (\neg\varphi \to \bot) = (\psi \to \gamma) \wedge [\neg(\psi \to \gamma) \to \bot].$$

Hence we get the following,

$\Gamma \vdash$

1. $\gamma \wedge \neg\gamma$	Iinduction hypothesis
2. γ	Axiom 3 and M.P.
3. $\gamma \to (\psi \to \gamma)$	Axiom 1
4. $\psi \to \gamma$	M.P. 1, 2

5. $(\gamma \wedge \neg\gamma) \to [\neg(\psi \to \gamma) \to \bot]$ Axiom 10

6. $\neg(\psi \to \gamma) \to \bot$ M.P. 1, 5

7. φ' Rule 1 on 3, 6

Subcase 4.3: If $v(\gamma) = 1/2$ and $v(\psi) = 0$ are true then $v(\varphi) = 1$ holds. So by the construction,

$$\gamma' = \neg\gamma \wedge (\gamma \to \bot),\ \psi' = \neg\psi \wedge (\psi \to \bot) \text{ and}$$

$$\begin{aligned}\varphi' &= \varphi \wedge (\neg\varphi \to \bot)\\ &= (\psi \to \gamma) \wedge [\neg(\psi \to \gamma) \to \bot].\end{aligned}$$

Now the following derivation shows that $\Gamma \vdash \varphi'$ holds in this subcase also.

$\Gamma \vdash$

1. $\psi \to \bot$ Axiom 4 and M.P.

2. $\bot \to \gamma$ Axiom 13

3. $(\psi \to \bot) \to [(\bot \to \gamma) \to (\psi \to \gamma)]$ Theorem 3.2.4(ii)

4. $(\bot \to \gamma) \to (\psi \to \gamma)$ M.P. 1, 3

5. $\psi \to \gamma$ M.P. 2, 4

6. $(\psi \to \bot) \to [\neg(\psi \to \gamma) \to \bot]$ Axiom 11

7. $\neg(\psi \to \gamma) \to \bot$ M.P. 1, 6

8. φ' Rule 1 on 5, 7

Subcase 4.4: If we have $v(\gamma) = 0$ and $v(\psi) = 1$ then $v(\varphi) = 0$ is true. Therefore,

$$\gamma' = \neg\gamma \wedge (\gamma \to \bot),\ \psi' = \psi \wedge (\neg\psi \to \bot) \text{ and}$$

$$\varphi' = \neg\varphi \wedge (\varphi \to \bot)$$

$$= \neg(\psi \to \gamma) \wedge [(\psi \to \gamma) \to \bot].$$

The deduction theorem will be used here for proving $\Gamma \vdash \varphi'$. Since we know $\Gamma \vdash \psi'$ and $\Gamma \vdash \gamma'$

$\Gamma \cup \{\psi \to \gamma\} \vdash$

1. ψ'	Monotonicity
2. ψ	Axiom 3 and M.P. with 1
3. γ'	Monotonicity
4. $\gamma \to \bot$	Axiom 4 and M.P. with 3
5. $\psi \to \gamma$	Assumption
6. γ	M.P. 2, 5
7. $\bot$	M.P. 4, 6

Now applying the Deduction theorem we get, $\Gamma \vdash (\psi \to \gamma) \to \bot$.

Again for proving $\Gamma \vdash \neg(\psi \to \gamma)$ we do the following derivation.

$\Gamma \vdash$

1. ψ'	Induction hypothesis
2. ψ	Axiom 3 and M.P. with 1
3. γ'	Induction hypothesis
4. $\gamma \to \bot$	Axiom 4 and M.P. with 3
5. $\psi \wedge (\gamma \to \bot)$	Rule 1 on 2 and 4
6. $\psi \wedge (\gamma \to \bot) \to \neg(\psi \to \gamma)$	Axiom 14
7. $\neg(\psi \to \gamma)$	M.P. 3, 4

Hence, again by *Rule 1* it is derived $\Gamma \vdash \neg(\psi \to \gamma) \wedge [(\psi \to \gamma) \to \bot]$ i.e., $\Gamma \vdash \varphi'$.

Subcase 4.5: If both of $v(\gamma) = 0$ and $v(\psi) = 1/2$ hold then we have $v(\varphi) = 0$. Therefore by definition,

$$\gamma' = \neg\gamma \wedge (\gamma \to \bot),\ \psi' = \psi \wedge \neg\psi \text{ and}$$
$$\varphi' = \neg\varphi \wedge (\varphi \to \bot) = \neg(\psi \to \gamma) \wedge [(\psi \to \gamma) \to \bot].$$

Since $\Gamma \vdash \psi'$ and $\Gamma \vdash \gamma'$, by Axiom 1, 2 and using M.P. we get

$$\Gamma \vdash \psi \text{ and } \Gamma \vdash \gamma \to \bot.$$

Therefore in this subcase $\Gamma \vdash \varphi'$ can be proved by following the same steps used in Subcase 4.4.

Hence combining all the cases Lemma 3.2.5 is proved. □

Using Lemma 3.2.5 one can prove the weak completeness theorem:

Theorem 3.2.6 (Completeness). *For any formula φ, if $\models \varphi$ then $\vdash \varphi$.*

Proof. Let φ be a formula such that $\models \varphi$. Moreover let $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$ be n propositional letters in φ. By Lemma 3.2.5, for any arbitrarily fixed valuation we have, $\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_n}\} \vdash \varphi'$. Since, $\models \varphi$, by the definition, φ' is either $\varphi \wedge (\neg\varphi \to \bot)$ or $\varphi \wedge \neg\varphi$. So in any case $\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_n}\} \vdash \varphi$ can be derived by Axiom 3 and using M.P. Hence Deduction theorem gives,

$$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_{n-1}}\} \vdash p'_{i_n} \to \varphi.$$

Now, since the valuation was arbitrary, we get,

$$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_{n-1}}\} \vdash [p_{i_n} \wedge (\neg p_{i_n} \to \bot)] \to \varphi,$$
$$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_{n-1}}\} \vdash (p_{i_n} \wedge \neg p_{i_n}) \to \varphi, \text{ and}$$
$$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_{n-1}}\} \vdash [\neg p_{i_n} \wedge (p_{i_n} \to \bot)] \to \varphi.$$

Hence the following derivation can be established:

$\{p'_{i_1}, p'_{i_2}, \ldots, p'_{i_{n-1}}\} \vdash$

1. $[(p_{i_n} \wedge (\neg p_{i_n} \to \bot)) \to \varphi] \wedge [(p_{i_n} \wedge \neg p_{i_n}) \to \varphi] \wedge$
   $[(\neg p_{i_n} \wedge (p_{i_n} \to \bot)) \to \varphi]$ — Rule 1
2. $[(p_{i_n} \wedge (\neg p_{i_n} \to \bot)) \vee (p_{i_n} \wedge \neg p_{i_n}) \vee$
   $(\neg p_{i_n} \wedge (p_{i_n} \to \bot))] \to \varphi$ — Axiom 6
3. $(p_{i_n} \wedge (\neg p_{i_n} \to \bot)) \vee (p_{i_n} \wedge \neg p_{i_n}) \vee (\neg p_{i_n} \wedge (p_{i_n} \to \bot))$ — Axiom 15
4. φ. — M.P. 2, 3

Repeating this process for each of the remaining p'_{i_j}, where $j = n-1, n-2, \ldots, 1$ we get, $\vdash \varphi$. Hence the (weak) completeness theorem is proved. □

Strong completeness is however not yet investigated.

3.3 $\mathbb{L}\mathrm{PS}_3$ and other three-valued paraconsistent logics

In this section some important properties of $\mathbb{L}\mathrm{PS}_3$ will be discussed and comparisons between $\mathbb{L}\mathrm{PS}_3$ and some other well known three-valued paraconsistent logics will be pointed out with respect to some logical properties.

3.3.1 Maximality relative to classical propositional logic

Maximality is an important issue in the study of paraconsistent logics (cf. [2, 7]).

Definition. A logic $\mathbf{L}_1 = \langle \mathscr{L}, \vdash_1 \rangle$ is said to be *maximal relative to* a logic $\mathbf{L}_2 = \langle \mathscr{L}, \vdash_2 \rangle$ iff (*i*) $\vdash_1 \varphi$ implies $\vdash_2 \varphi$ for any φ, and

(*ii*) if $\nvdash_1 \varphi$, $\vdash_2 \varphi$ and $\vdash_1$ is enhanced to $\vdash_1'$ by adding φ to the theorem set of $\mathbf{L}_1$ then $\langle \mathscr{L}, \vdash_1' \rangle = \langle \mathscr{L}, \vdash_2 \rangle$.

Definition 3.3.1 is more demanding than what is there in [7]. The change is in the condition (*ii*) of the definition 3.3.1. In [7] the condition (*ii*) was taken as: if $\nvdash_1 \varphi$, $\vdash_2 \varphi$ and $\vdash_1$ is enhanced to $\vdash_1'$ by adding φ to the theorem set of $\mathbf{L}_1$ then the set of theorems in $\langle \mathscr{L}, \vdash_1' \rangle$ is same as the set of theorems in $\langle \mathscr{L}, \vdash_2 \rangle$.

The relationship between $\mathbb{L}\mathrm{PS}_3$ and the classical propositional logic will be explored now.

Lemma 3.3.1 *By adding any theorem of Classical Propositional Logic* (CPL) *which is not a theorem of* $\mathbb{L}\mathrm{PS}_3$*, as an axiom schema in* $\mathbb{L}\mathrm{PS}_3$*, all the theorems of* CPL *can be proved in the enhanced system of* $\mathbb{L}\mathrm{PS}_3$*.*

Proof. First we show that by adding any theorem of CPL which is not a theorem of $\mathbb{L}\mathrm{PS}_3$, as an axiom schema in $\mathbb{L}\mathrm{PS}_3$, all the theorems of CPL can be proved.

Let $\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$ be a theorem of CPL but not a theorem of $\mathbb{L}\mathrm{PS}_3$, where $p_{i_1}, p_{i_2}, \ldots, p_{i_n}$ are the propositional variables. Hence for any valuation v from the set of all formulas of $\mathbb{L}\mathrm{PS}_3$ to PS_3 for which

$$v(\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})) = 0$$

there must exist some p_{i_ℓ}, $1 \leq \ell \leq n$ such that $v(p_{i_\ell}) = 1/2$. Now using this fact without loss of generality we may assume that for any given valuation v we have $v(\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})) = 0$ iff $v(p_{i_\ell}) = 1/2$ for all $\ell \in \{1, \ldots, n\}$. It is guaranteed by the following fact: Suppose a formula $\psi(p_{r_1}, p_{r_2}, \ldots p_{r_{t+1}})$ is such that $v(p_{r_\ell}) = 1/2$ for all $\ell \in \{1, \ldots, t\}$ but $v(p_{r_{t+1}}) \neq 1/2$. We then replace the propositional variable $p_{r_{t+1}}$

(i) by $\neg(p_{r_1} \to p_{r_1})$ if $v(p_{r_{t+1}}) = 0$

(ii) by $(p_{r_1} \to p_{r_1})$ if $v(P_{r_{t+1}}) = 1$

in the formula $\psi(p_{r_1}, p_{r_2}, \ldots p_{r_{t+1}})$ and therefore after replacing, the formula will get the value 0 iff all its propositional variables take the value $1/2$. In this way we always get such a formula $\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$.

Let us now assume $\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$ be another arbitrarily chosen theorem of CPL which is not a theorem of $\mathbb{L}\mathrm{PS}_3$, where $p_{k_1}, p_{k_2}, \ldots, p_{k_m}$ are the propositional variables. It will be proved that $\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$ can be derived from the axiom system of $\mathbb{L}\mathrm{PS}_3$ if it is extended by the new axiom schema $\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$. Let $\varphi(p_{k_j})$ be the formula replacing each propositional variable of $\varphi(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$ by p_{k_j}, for all $j \in \{1, \ldots, m\}$.

Claim 3.3.2 *The formula* $\varphi(p_{k_1}) \wedge \varphi(p_{k_2}) \wedge \ldots \wedge \varphi(p_{k_m}) \to \sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$ *is a theorem of* $\mathbb{L}\mathrm{PS}_3$.

Proof. Let v be any valuation.

If $v(\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})) = 0$ and since $\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$ is a theorem of CPL there must exist p_{k_j} such that $v(p_{k_j}) = 1/2$ for some $j \in \{1, \ldots, m\}$. Hence $v(\varphi(p_{k_j})) = 0$ and therefore

$$v(\varphi(p_{k_1}) \wedge \varphi(p_{k_2}) \wedge \ldots \wedge \varphi(p_{k_m}) \to \sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})) = 1.$$

Again if $v(\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})) \neq 0$ then by the truth tables of PS_3

$$v(\varphi(p_{k_1}) \wedge \varphi(p_{k_2}) \wedge \ldots \wedge \varphi(p_{k_m}) \to \sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})) = 1.$$

Hence for any valuation the formula

$$\varphi(p_{k_1}) \wedge \varphi(p_{k_2}) \wedge \ldots \wedge \varphi(p_{k_m}) \to \sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$$

always get the value 1. So by the completeness theorem of $\mathbb{L}\mathrm{PS}_3$ the claim is proved. $\square$

Now let us extend the axiom system of $\mathbb{L}\mathrm{PS}_3$ by including $\varphi(p_{i_1}, \ldots, p_{i_n})$ as an axiom schema. Let this system be denoted by $\mathbb{L}'\mathrm{PS}_3$. Then by the Rule 1, $\varphi(p_{k_1}) \wedge \varphi(p_{k_2}) \wedge \ldots \wedge \varphi(p_{k_m})$ is a theorem of $\mathbb{L}'\mathrm{PS}_3$. Now by using M.P. we have $\sigma(p_{k_1}, p_{k_2}, \ldots, p_{k_m})$ as a theorem of the new system. Hence the lemma is proved. $\square$

From Lemma 3.3.1 it follows that for the enhanced system $\mathbb{L}'\mathrm{PS}_3$,

$$\Gamma \vdash_{\mathbb{L}'\mathrm{PS}_3} \varphi \text{ iff } \Gamma \vdash_{\mathrm{CPL}} \varphi.$$

Hence we get the following theorem.

Theorem 3.3.3 *The logic* $\mathbb{L}\mathrm{PS}_3$ *is maximal relative to the classical propositional logic* (CPL).

3.3.2 Comparison with other logics

The three-valued paraconsistent logics chosen for comparisons are G. Priest's *Logic of Paradox* (LP) [26], *Logic of Formal Inconsistency 1* (LFI1) and *Logic of Formal Inconsistency 2* (LFI2) by W. A. Carnielli, J. Marcos, and S. de Amo [7], I. M. L. D'Ottaviano's logic (J_3) [11], The Logic RM_3 by the Entailment school (cf. [16]), A. M. Sette's three-valued paraconsistent logic P_1 [29], the logic G'_3 by M. O. Galindo and J. L. C. Carranza [23], and C. Mortensen's paraconsistent logic $\mathrm{C}_{0,2}$ [18]. Moreover we will also make a comparison table with respect to S. Jaśkowski's and N. Da Costa's criteria for paraconsistent logics viz. **Jas1**, **Jas2**, **Jas3** and **NdaC1**, **NdaC2**, **NdaC3**, **NdaC4**, which are described in Chapter 1.

We will compare the following properties:

$$\vdash \neg\neg\varphi \leftrightarrow \varphi. \tag{DN}$$

$$\vdash \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi). \tag{DM1}$$

$$\vdash \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi). \tag{DM2}$$

$$\vdash \varphi \vee \neg\varphi. \tag{LEM}$$

$$\vdash (\neg\varphi \to \neg\psi) \to (\psi \to \varphi). \tag{C}$$

$$\vdash (\neg\varphi \to \psi) \to (\neg\psi \to \varphi). \tag{C1}$$

$$\vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi). \tag{C2}$$

$$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi). \tag{C3}$$

$$\vdash (\varphi \to \psi) \wedge (\psi \to \gamma) \to (\varphi \to \gamma). \tag{HS}$$

$$\varphi, \varphi \to \psi \vdash \psi \tag{MP}$$

We write DT for the Deduction Theorem.

The mark (✓) indicates that the property holds and the mark (X) indicates that the property does not hold in the corresponding logical system.

	DN	DM1	DM2	LEM	C	C1	C2	C3	HS	MP	DT
$\mathbb{L}PS_3$	✓	✓	✓	✓	X	X	X	X	✓	✓	✓
LP	✓	✓	✓	✓	✓	✓	✓	✓	X	X	✓
LFI1	✓	✓	✓	✓	X	X	X	X	✓	✓	✓
LFI2	✓	X	X	✓	X	X	X	X	✓	✓	✓
J_3	✓	✓	✓	✓	X	X	X	X	✓	✓	✓
RM_3	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	x
P_1	X	X	X	✓	X	X	X	X	✓	✓	✓
G'_3	X	✓	✓	✓	X	X	X	X	✓	✓	✓
$C_{0,2}$	✓	X	X	✓	X	X	X	X	✓	✓	✓

We also present the following comparison table with respect to **Jas1**, **Jas2**, **Jas3**, **NdaC1**, and **NdaC2**.

	Jas1	Jas2	Jas3	NdaC1	NdaC2
$\mathbb{L}PS_3$	✓	✓	✓	X	✓
LP	X	X	✓	X	✓
LFI1	✓	✓	✓	X	✓
LFI2	✓	✓	✓	X	✓
J_3	✓	✓	✓	X	✓
RM_3	✓	✓	✓	X	✓
P_1	✓	✓	✓	✓	✓
G'_3	✓	✓	✓	X	✓
$C_{0,2}$	✓	✓	✓	✓	✓

Note. All the observations for LP and $C_{0,2}$ are semantical.

In the literature, one can get several paraconsistent logics having algebraic semantics. Some of them are mentioned above. These logics and their algebraic semantics were developed from various motivations. Our motivation is to construct models of some paraconsistent set theories.

With respect to the algebraic properties discussed in Section 2.1.4 which are needed for making an algebra-valued model of a paraconsistent set theory the following comparison with othetr paraconsistent logics is made. The algebras of the three-valued semantics for LFI2, P^1 and $C_{0,2}$ are not lattices and hence we exclude these three logics from the following comparison table.

	P1	**P2**	**P3**	**P4**
$\mathbb{L}PS_3$	✓	✓	✓	✓
LP	X	✓	✓	✓
LFI1	X	✓	✓	✓
J_3	X	✓	✓	✓
RM_3	X	✓	✓	X
G'_3	✓	✓	✓	✓

Note. The comparisons are made with respect to the corresponding three-valued semantics of the respective logics.

Thus PS_3 differs from the other three-valued matrices mentioned here in forming an algebraic structure suitable to construct some model of set theory within **V** with an underlying paraconsistent logic. It is yet unknown whether other logics are suitable for this purpose.

3.4 Predicate logic for PS_3 and the equality

In [10] the authors used a three valued matrix $\langle\{1, 1/2, 0\}, \wedge, \vee, \Rightarrow, ^*, \circ, 0, 1\rangle$ where the unary operator $\circ$ is taken as the consistency operator and the structure $\langle\{1, 1/2, 0\}, \wedge, \vee, \Rightarrow, ^*, 0, 1\rangle$ is similar to PS_3. The operator $\circ$ can be defined by the other operators as $\circ\alpha = (\alpha \wedge \alpha^*) \Rightarrow (\alpha \Rightarrow \alpha)^*$. In [10] a propositional logic LPT is defined which is sound and complete with respect to the above mentioned three-valued matrix, PS_3. The axiom system for LPT is given below:

(Ax1) $\varphi \to (\psi \to \varphi)$

(Ax2) $(\varphi \to \psi) \to ((\varphi \to (\psi \to \gamma)) \to (\varphi \to \gamma))$

(Ax3) $\varphi \to (\psi \to (\varphi \wedge \psi))$

(Ax4) $\varphi \wedge \psi \to \varphi$

$$
\begin{array}{ll}
\text{(Ax5)} & \varphi \wedge \psi \to \psi \\
\text{(Ax6)} & \varphi \to \varphi \vee \psi \\
\text{(Ax7)} & \psi \to \varphi \vee \psi \\
\text{(Ax8)} & (\varphi \to \gamma) \to ((\psi \to \gamma) \to (\varphi \vee \psi \to \gamma)) \\
\text{(Ax9)} & \varphi \vee (\varphi \to \psi) \\
\text{(Ax10)} & \varphi \vee \neg\varphi \\
\text{(Ax11)} & \neg\neg\varphi \leftrightarrow \varphi \\
\text{(Ax12)} & \circ\,\varphi \to (\varphi \to (\neg\varphi \to \psi)) \\
\text{(Ax13)} & \neg\circ\varphi \to (\varphi \wedge \neg\varphi) \\
\text{(Ax14)} & \circ\,(\varphi \to \psi) \\
\text{(Ax15)} & (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi) \\
\text{(Ax16)} & (\varphi \wedge \neg\varphi \wedge \psi) \to (\neg(\varphi \wedge \psi) \wedge \neg(\psi \wedge \varphi))
\end{array}
$$

where φ, ψ, γ are any well formed formulas.

The rule of inference is Modus Ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$.

Then a predicate extension of LPT is developed whose soundness and completeness has been established with respect to the *pragmatic semantics of partial structures* whose definitions are given below [10, Section 3.2].

Definition. An n-ary partial relation R on a non-empty set D is an order triple $\langle R_+, R_-, R_u\rangle$ where $R_+, R_-, R_u \subseteq D^n$ are mutually disjoint and $R_+ \cup R_- \cup R_u = D^n$.

Definition. A partial structure for a first order language $\mathbf{L}$ is an order pair $\mathscr{S} = \langle D, (\cdot)^{\mathscr{S}}\rangle$ where D is non empty and the function $(\cdot)^{\mathscr{S}}$ defined on $\mathbf{L}$ is such that for every n-ary relation symbol R, $R^{\mathscr{S}} = (R_+^{\mathscr{S}}, R_-^{\mathscr{S}}, R_u^{\mathscr{S}})$ is an n-ary partial relation, constant symbols and function symbols are defined classically.

Definition. Let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables $x_1, \ldots, x_n$ and let $\mathscr{S}$ be a partial structure with domain D. Then the triple $\varphi^{\mathscr{S}} = (\varphi_+^{\mathscr{S}},\ \varphi_-^{\mathscr{S}},\ \varphi_u^{\mathscr{S}})$ is defined recursively as below:

(i) if $\varphi = P(t_1, \dots, t_n)$ is atomic then for $* \in \{+, -, u\}$

$$\varphi_*^{\mathscr{S}} = \{\overrightarrow{a} \in D^n : \langle t_1^{\mathscr{S}}[\overrightarrow{a}], \dots, t_n^{\mathscr{S}}[\overrightarrow{a}]\rangle \in P_*^{\mathscr{S}}\},$$

(ii) $(\neg\varphi)^{\mathscr{S}} = \langle \varphi_-^{\mathscr{S}},\ \varphi_+^{\mathscr{S}},\ \varphi_u^{\mathscr{S}}\rangle$,

(iii) $(\varphi \wedge \psi)^{\mathscr{S}} = \langle \varphi_+^{\mathscr{S}} \cap \psi_+^{\mathscr{S}},\ \varphi_-^{\mathscr{S}} \cup \psi_-^{\mathscr{S}},\ D^n - (\varphi_+^{\mathscr{S}} \cap \psi_+^{\mathscr{S}}) \cup (\varphi_-^{\mathscr{S}} \cup \psi_-^{\mathscr{S}})$, and

(iv) $(\varphi \to \psi)^{\mathscr{S}} = \langle \varphi_-^{\mathscr{S}} \cup (\psi_+^{\mathscr{S}} \cup \psi_-^{\mathscr{S}}),\ (\varphi_+^{\mathscr{S}} \cup \varphi_u^{\mathscr{S}}) \cap \psi_-^{\mathscr{S}},\ \varnothing\rangle$.

From the definition $(\varphi\wedge\neg\varphi)^{\mathscr{S}} = \langle \varnothing,\ \varphi_+^{\mathscr{S}} \cup \varphi_-^{\mathscr{S}},\ \varphi_u^{\mathscr{S}}\rangle$ and $(\varphi\vee\neg\varphi)^{\mathscr{S}} = \langle \varphi_+^{\mathscr{S}} \cup \varphi_-^{\mathscr{S}},\ \varnothing,\ \varphi_u^{\mathscr{S}}\rangle$ are proved. Let us consider a subset A of D^{n+1} and define $\forall(A), \exists(A) \subseteq D^n$ as follows:

$$\forall(A) = \{\overrightarrow{a} \in D^n : (b, \overrightarrow{a}) \in A \text{ for all } b \in D\} \text{ and}$$
$$\exists(A) = \{\overrightarrow{a} \in D^n : (b, \overrightarrow{a}) \in A \text{ for some } b \in D\}$$

Moreover if $A \subseteq D$ (i.e., $n = 0$) then,

$$\forall(A) = \begin{cases} 1 & \text{if } A = D; \\ 0 & \text{otherwise.} \end{cases}$$

$$\exists(A) = \begin{cases} 1 & \text{if } A \neq \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

Definition. Assume that $\varphi^{\mathscr{S}} = \langle \varphi_+^{\mathscr{S}},\ \varphi_-^{\mathscr{S}},\ \varphi_u^{\mathscr{S}}\rangle$ is defined on D^{n+1} for a formula $\varphi(x_0, \dots, x_n)$ where $n \geq 1$. Then

$$(\forall x_0\varphi)^{\mathscr{S}} = \langle \forall(\varphi_+^{\mathscr{S}}),\ \exists(\varphi_-^{\mathscr{S}}),\ D^n - [\forall(\varphi_+^{\mathscr{S}}) \cup \exists(\varphi_-^{\mathscr{S}})]\rangle$$

and $\exists x_0\varphi = \neg\forall x_0\neg\varphi$.

Below the definition of pragmatic satisfaction of a formula is given.

Definition. Let $\varphi(x_1,\ldots,x_n)$ be a formula, $\mathscr{S} = \langle D, (\cdot)^{\mathscr{S}}\rangle$ a partial structure and $\overrightarrow{a} \in D^n$. The sequence $\overrightarrow{a}$ *pragmatically satisfies* φ in $\mathscr{S}$, denoted by $\mathscr{S} \Vdash \varphi[\overrightarrow{a}]$ in the following cases.

(*i*) $\mathscr{S} \Vdash R(t_1,\ldots,t_n)[\overrightarrow{a}]$ iff $(t_1^{\mathscr{S}}[\overrightarrow{a}],\ldots,t_n^{\mathscr{S}}[\overrightarrow{a}]) \in R_+^{\mathscr{S}} \cup R_u^{\mathscr{S}}$;

(*ii*) $\mathscr{S} \Vdash \neg\varphi[\overrightarrow{a}]$ iff $\overrightarrow{a} \in \varphi_-^{\mathscr{S}} \cup \varphi_u^{\mathscr{S}}$;

(*iii*) $\mathscr{S} \Vdash (\varphi \wedge \psi)[\overrightarrow{a}]$ iff $\mathscr{S} \Vdash \varphi^{\mathscr{S}}[\overrightarrow{a}]$ and $\mathscr{S} \Vdash \psi^{\mathscr{S}}[\overrightarrow{a}]$;

(*iv*) $\mathscr{S} \Vdash (\varphi \to \psi)[\overrightarrow{a}]$ iff $\mathscr{S} \nVdash \varphi^{\mathscr{S}}[\overrightarrow{a}]$ or $\mathscr{S} \Vdash \psi^{\mathscr{S}}[\overrightarrow{a}]$;

(*v*) $\mathscr{S} \Vdash \forall x\varphi[\overrightarrow{a}]$ iff $\mathscr{S} \Vdash \varphi[b, \overrightarrow{a}]$ for all $b \in D$.

Definition. A formula $\varphi(x_1,\ldots,x_n)$ is said to be *pragmatically satisfied* by a partial structure $\mathscr{S}$ which is denoted by $\mathscr{S} \Vdash \varphi$, if for all sequences $\overrightarrow{a} \in D^n$, $\mathscr{S} \Vdash \varphi[\overrightarrow{a}]$.

Definition. Let $\Gamma \cup \{\varphi\}$ be a set of well formed formulas. The formula φ is said to be *pragmatic consequence* of the set of formulas Γ which is denoted by $\Gamma \Vdash \varphi$, if $\mathscr{S} \Vdash \varphi$ for every partial structure $\mathscr{S}$ such that $\mathscr{S} \Vdash \psi$ for all $\psi \in \Gamma$.

After defining all these definitions the authors extended the propositional logic LPT to a predicate logic system LPT1 in [10, Section 6]. The language of LPT1 is the usual first order language based on the connectives $\wedge$, $\to$, $\neg$ and the universal quantifier $\forall$. The axiom system for LPT1 is as follows.

All the axioms of LPT together with one more axiom

(Ax17) $\forall x\varphi \to \varphi[x/t]$, where t is a term free for x in φ.

The rules of inference for LPT1 are

1. The rule of modus ponens: $\dfrac{\varphi,\ \varphi \to \psi}{\psi}$

2. $\dfrac{\varphi \to \psi}{\varphi \to \forall x \psi}$

whenever x does not occur free in φ.

After that it is proved that LPT1 is sound and complete with respect to the semantics of the pragmatic satisfaction by partial structures [10, Section 6].

3.5 The algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$ of a paraconsistent set theory

In Section 3.1 we have proved that PS_3 satisfies the properties **P1**, **P2**, **P3**, and **P4**. If the property BQ_φ would hold in $\mathbf{V}^{(\mathrm{PS}_3)}$ for any formula φ then we could conclude that all the axioms of ZF^- are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. But we have the following theorem.

Theorem 3.5.1 *There is a formula φ such that the property BQ_φ does not hold in $\mathbf{V}^{(\mathrm{PS}_3)}$.*

Proof. Let $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any arbitrary element, then $u = \{\langle w, 1\rangle\}$ and $v = \{\langle w, 1/2\rangle\}$ be two elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. Define a formula $\varphi(x) := \neg(w \in x)$. Therefore $[\![\varphi(u)]\!] = 0$ and $[\![\varphi(v)]\!] = 1/2$. Let us now consider $k \in \mathbf{V}^{(\mathrm{PS}_3)}$ as $k = \{\langle v, 1\rangle\}$. Then the following can be derived.

$$\begin{aligned}
[\![\forall x(x \in k \to \varphi(x))]\!] &= \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![y \in k]\!] \Rightarrow [\![\varphi(y)]\!]) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} ((k(v) \wedge [\![v = y]\!]) \Rightarrow [\![\varphi(y)]\!])
\end{aligned}$$

$$\begin{aligned} &\leq (k(v) \wedge [\![v = u]\!]) \Rightarrow [\![\varphi(u)]\!] \\ &= (1 \wedge 1) \Rightarrow 0 \\ &= 0 \end{aligned}$$

On the other hand,

$$\bigwedge_{x \in \mathrm{dom}(k)} (k(x) \Rightarrow [\![\varphi(x)]\!]) \;=\; k(v) \Rightarrow [\![\varphi(v)]\!] \;=\; 1 \Rightarrow {}^1\!/_2 \;=\; 1.$$

Hence we get,

$$[\![\forall x(x \in k \to \varphi(x))]\!] < \bigwedge_{x \in \mathrm{dom}(k)} (k(x) \Rightarrow [\![\varphi(x)]\!])$$

and the theorem is proved. □

Though BQ_φ does not hold in $\mathbf{V}^{(\mathrm{PS}_3)}$ for the formula φ defined in the proof of Theorem 3.5.1, below we shall show the property BQ_φ is valid in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all negation-free formula φ for which the following lemmas and theorems are needed.

Lemma 3.5.2 *For any three elements $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$, we have the following:*

(i) $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$

(ii) $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$

Proof. Let $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any three elements.

(i) We will prove $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$ by induction. Suppose for any $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ and any $z \in \mathrm{dom}(w)$ we have the following:

$$[\![u = v]\!] \wedge [\![v = z]\!] \leq [\![u = z]\!].$$

By the truth tables of the operators in PS_3 it is clear that $[\![u = w]\!]$ cannot be $1/2$, so it is either 1 or 0. If it is 1 then we have nothing to prove. Now suppose $[\![u = w]\!] = 0$. So in this case our aim will be to prove either $[\![u = v]\!] = 0$ or $[\![v = w]\!] = 0$. By definition,

$$[\![u = w]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in w]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow [\![z \in u]\!]).$$

Case 1. Suppose $\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in w]\!]) = 0$. So, there exists $x_0 \in \mathrm{dom}(u)$ such that, $[u(x_0) \Rightarrow [\![x_0 \in w]\!]] = 0$; which implies

$$u(x_0) \neq 0 \text{ and } \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![x_0 = z]\!]) = 0. \tag{$\star$}$$

We know that,

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]),$$

and

$$[\![v = w]\!] = \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in w]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow [\![z \in v]\!]).$$

Since $u(x_0) \neq 0$, if $[\![x_0 \in v]\!] = 0$ then it can be concluded $[\![u = v]\!] = 0$. By definition,

$$[\![x_0 \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![x_0 = y]\!]).$$

Now if for any $y_0 \in \mathrm{dom}(v)$, $v(y_0) \neq 0$ then we will show, either $[\![y_0 \in w]\!] = 0$ or $[\![x_0 = y_0]\!] = 0$. If $[\![y_0 \in w]\!] = 0$ then we get $[\![v = w]\!] = 0$ and the proof is complete. On the other hand if $[\![y_0 \in w]\!] \neq 0$, i.e.,

$$\bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![y_0 = z]\!]) \neq 0$$

then there exists $z_0 \in \mathrm{dom}(w)$ such that, $w(z_0) \neq 0$ and $[\![y_0 = z_0]\!] \neq 0$ both. Since $w(z_0) \neq 0$, from $(\star)$ we get, $[\![x_0 = z_0]\!] = 0$. Now by induction hypothesis,

$$[\![x_0 = y_0]\!] \wedge [\![y_0 = z_0]\!] \leq [\![x_0 = z_0]\!].$$

Hence we get $[\![x_0 = y_0]\!] = 0$. Therefore if $v(y_0) \neq 0$ then either $[\![v = w]\!] = 0$ or $[\![x_0 = y_0]\!] = 0$, i.e., $[\![x_0 \in v]\!] = 0$. Now since $u(x_0) \neq 0$ we get $[\![u = v]\!] = 0$. As a conclusion of Case 1, if $[\![u = w]\!] = 0$ then either $[\![u = v]\!] = 0$ or $[\![v = w]\!] = 0$.

Case 2. Suppose $\bigwedge_{z \in \mathrm{dom}(w)}(w(z) \Rightarrow [\![z \in u]\!]) = 0$. So, there exists $z_0 \in \mathrm{dom}(w)$ such that, $[w(z_0) \Rightarrow [\![z_0 \in u]\!]] = 0$, which gives

$$w(z_0) \neq 0 \text{ and } \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![z_0 = x]\!]) = 0. \tag{$\dagger$}$$

We know that,

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]),$$

and

$$[\![v = w]\!] = \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in w]\!]) \wedge \bigwedge_{z \in \mathrm{dom}(w)} (w(z) \Rightarrow [\![z \in v]\!]).$$

Since $w(z_0) \neq 0$ if $[\![z_0 \in v]\!] = 0$ then it can be concluded $[\![v = w]\!] = 0$. By definition,

$$[\![z_0 \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![z_0 = y]\!]).$$

Now if for any $y_0 \in \mathrm{dom}(v)$, $v(y_0) \neq 0$ then we will show, either $[\![y_0 \in u]\!] = 0$ or $[\![z_0 = y_0]\!] = 0$. If $[\![y_0 \in u]\!] = 0$ then we get $[\![u = v]\!] = 0$ and the proof is complete. On the other hand if

$[\![y_0 \in u]\!] \neq 0$, i.e.,

$$\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y_0 = x]\!]) \neq 0$$

then there exists $x_0 \in \mathrm{dom}(u)$ such that, $u(x_0) \neq 0$ and $[\![y_0 = x_0]\!] \neq 0$ both hold. Since $u(x_0) \neq 0$, from (†) we get, $[\![x_0 = z_0]\!] = 0$. Now by induction hypothesis,

$$[\![x_0 = y_0]\!] \wedge [\![y_0 = z_0]\!] \leq [\![x_0 = z_0]\!].$$

Hence we get $[\![z_0 = y_0]\!] = 0$. Therefore if $v(y_0) \neq 0$ then either $[\![u = v]\!] = 0$ or $[\![z_0 = y_0]\!] = 0$, i.e., $[\![z_0 \in v]\!] = 0$. Now since $w(z_0) \neq 0$ we get $[\![v = w]\!] = 0$. Therefore in *Case 2* also, if $[\![u = w]\!] = 0$ then either $[\![u = v]\!] = 0$ or $[\![v = w]\!] = 0$.

By the induction principle now we can argue,

$$[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!],$$

for all $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$.

(ii) The proof of $[\![u = v]\!] \wedge [\![u \in w]\!] \leq [\![v \in w]\!]$ is as follows:

$$\begin{aligned} [\![u = v]\!] \wedge [\![u \in w]\!] &= [\![u = v]\!] \wedge \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![u = z]\!]) \\ &= \bigvee_{z \in \mathrm{dom}(w)} [w(z) \wedge ([\![u = z]\!] \wedge [\![u = v]\!])] \\ &\leq \bigvee_{z \in \mathrm{dom}(w)} (w(z) \wedge [\![v = z]\!]), \qquad \text{(using (1))} \\ &= [\![v \in w]\!]. \end{aligned}$$

Hence the proof is complete. □

Observation 3.5.3 *There exist $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that the inequality*

$$[\![u = v]\!] \wedge [\![w \in u]\!] \leq [\![w \in v]\!]$$

does not hold.

Proof. Let $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any arbitrary element and u, v be defined as,

$$\mathrm{dom}(u) = \mathrm{dom}(v) = \{w\} \text{ and } u(w) = 1, v(w) = 1/2.$$

Then clearly $[\![u = v]\!] = 1$, $[\![w \in u]\!] = 1$ and $[\![w \in v]\!] = 1/2$ and hence the inequality is not valid generally in $\mathbf{V}^{(\mathrm{PS}_3)}$. $\square$

Hence there exists a formula $\varphi(x)$ and $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ so that the inequality $[\![u = v]\!] \wedge [\![\varphi(u)]\!] \leq [\![\varphi(v)]\!]$ is not valid. But we can prove the following lemma.

Lemma 3.5.4 *For any three elements $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$, we have the following:*

(*i*) $[\![u = v]\!] \Rightarrow [\![v = w]\!] \;=\; [\![u = v]\!] \Rightarrow [\![u = w]\!]$.

(*ii*) $[\![u = v]\!] \Rightarrow [\![u \in w]\!] \;=\; [\![u = v]\!] \Rightarrow [\![v \in w]\!]$.

(*iii*) $[\![u = v]\!] \Rightarrow [\![w \in u]\!] \;=\; [\![u = v]\!] \Rightarrow [\![w \in v]\!]$.

Proof. (*i*) From Lemma 3.5.2, for any $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$, it is very easy to get:

$$[\![u = v]\!] \wedge [\![v = w]\!] \;\leq\; [\![u = v]\!] \wedge [\![u = w]\!] \;\leq\; [\![u = v]\!] \wedge [\![v = w]\!].$$

Hence it is clear that,

$$[\![u = v]\!] \wedge [\![v = w]\!] \;=\; [\![u = v]\!] \wedge [\![u = w]\!].$$

Since PS_3 is a reasonable implication algebra, using Proposition 2.1.2 we get,

$$[\![u = v]\!] \Rightarrow [\![v = w]\!] \ = \ [\![u = v]\!] \Rightarrow [\![u = w]\!].$$

The equality (ii) can be proved similarly as (1).

(iii) Here we have to prove, $[\![u = v]\!] \Rightarrow [\![w \in u]\!] \ = \ [\![u = v]\!] \Rightarrow [\![w \in v]\!]$.

By definition it is clear that, either $[\![u = v]\!] = 0$ or $[\![u = v]\!] = 1$. If $[\![u = v]\!] = 0$ then we are done.

Now suppose $[\![u = v]\!] = 1$. We will show, if $[\![w \in u]\!] = 0$ then $[\![w \in v]\!] = 0$ and vice versa. Let $[\![w \in u]\!] = 0$; i.e.,

$$\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![w = x]\!]) = 0. \tag{$\star$}$$

Again, by our assumption $[\![u = v]\!] = 1$; i.e.,

$$\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]) \ = \ 1.$$

Hence if for some $y_0 \in \mathrm{dom}(v)$, $v(y_0) \neq 0$ then $[\![y_0 \in u]\!] \neq 0$; i.e.,

$$\bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![y_0 = x]\!]) \neq 0.$$

So there exists $x_0 \in \mathrm{dom}(u)$ such that $u(x_0) \neq 0 \neq [\![y_0 = x_0]\!]$. Now since $u(x_0) \neq 0$, from $(\star)$ we get $[\![w = x_0]\!] = 0$. Again by previous theorem we have

$$[\![w = y_0]\!] \wedge [\![y_0 = x_0]\!] \leq [\![w = x_0]\!]$$

by which it is easily concluded that $[\![w = y_0]\!] = 0$. Hence we get if for some $y \in \mathrm{dom}(v)$ if $v(y) \neq 0$ then $[\![w = y]\!] = 0$; therefore $[\![w \in v]\!] = 0$.

The other part can also be proved similarly. This completes the proof. $\square$

Let us now consider the equality

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!]. \tag{\#}$$

In Lemma 3.5.4 it is shown that for any atomic well formed formula $\varphi(x)$ (having one free variable x) of the extended language of set theory corresponding to $\mathbf{V}^{(\mathrm{PS}_3)}$, and for any two elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ the equality (#) holds. If (#) could be proved for any formula $\varphi(x)$, having one free variable x, and for any two elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ then BQ_φ could also be proved in $\mathbf{V}^{(\mathrm{PS}_3)}$ for any formula φ, contradicting Theorem 3.5.1. But we have the following observation.

Observation 3.5.5 *There is a formula $\varphi(x)$ and $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that*

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!]$$

does not hold.

Let $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any arbitrary element, then $u = \{\langle w, 1\rangle\}$ and $v = \{\langle w, 1/2\rangle\}$ be two elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. Now define a formula $\varphi(x) := \neg(w \in x)$. Hence clearly we will get $[\![u = v]\!] = 1$, $[\![\varphi(u)]\!] = 0$ and $[\![\varphi(v)]\!] = 1/2$. Therefore,

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] = 0 \neq 1 = [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!].$$

If we notice carefully, the formulas used in Theorem 3.5.1 and Observation 3.5.5 are not negation-free formulas. For the negation-free formulas we get the following theorem.

Theorem 3.5.6 *For any two elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ and any negation-free formula $\varphi(x)$ the following holds:*

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] \ = \ [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!].$$

Proof. The theorem will be proved by the mathematical induction on the complexity of an arbitrarily chosen negation-free formula $\varphi(x)$.

Base step: If the complexity of $\varphi(x)$ is 0 the theorem will be proved by Lemma 3.5.4.

Induction hypothesis: Let the theorem hold for any negation-free formula with complexity less than a non zero natural number n.

Induction step: Let the complexity of $\varphi(x)$ be n. It is known that the value of $[\![u = v]\!]$ is either 0 or 1. If it is 0 then the theorem is proved immediately. Therefore through out in this proof we assume $[\![u = v]\!] = 1$. The following cases may occur.

Case 1. Let $\varphi(x) \ = \ \psi(x) \wedge \gamma(x)$.
Clearly the complexities of $\psi(x)$ and $\gamma(x)$ are less than n and therefore by the induction hypothesis we get,

1. $[\![u = v]\!] \Rightarrow [\![\psi(u)]\!] \ = \ [\![u = v]\!] \Rightarrow [\![\psi(v)]\!]$ and

2. $[\![u = v]\!] \Rightarrow [\![\gamma(u)]\!] \ = \ [\![u = v]\!] \Rightarrow [\![\gamma(v)]\!]$.

We have to prove:

$$[\![u = v]\!] \Rightarrow [\![\varphi(u)]\!] \ = \ [\![u = v]\!] \Rightarrow [\![\varphi(v)]\!]$$

i.e.,

$$[\![u = v]\!] \Rightarrow [\![\psi(u)]\!] \wedge [\![\gamma(u)]\!] \ = \ [\![u = v]\!] \Rightarrow [\![\psi(v)]\!] \wedge [\![\gamma(v)]\!].$$

If $[\![\psi(u)]\!] \wedge [\![\gamma(u)]\!] = 0$ then any one of $[\![\psi(u)]\!]$ and $[\![\gamma(u)]\!]$ is 0. If $[\![\psi(u)]\!] = 0$ then by (1), $[\![\psi(v)]\!] = 0$ and if $[\![\gamma(u)]\!] = 0$ then by (2), $[\![\gamma(v)]\!] = 0$. Hence in any case, if $[\![\psi(u)]\!] \wedge [\![\gamma(u)]\!] = 0$ then $[\![\psi(v)]\!] \wedge [\![\gamma(v)]\!] = 0$.

If $[\![\psi(u)]\!] \wedge [\![\gamma(u)]\!] \neq 0$ then clearly $[\![\psi(u)]\!] \neq 0$ and $[\![\gamma(u)]\!] \neq 0$. Hence from (1) and (2) it can be concluded that $[\![\psi(v)]\!] \neq 0$ and $[\![\gamma(v)]\!] \neq 0$ also. So, $[\![\psi(v)]\!] \wedge [\![\gamma(v)]\!] \neq 0$ and hence in this case we are done.

Case 2. Let $\varphi(x) \;=\; \psi(x) \vee \gamma(x)$.
Following the similar process as shown in *Case 1* the theorem can be proved in this case also.

Case 3. Let $\varphi(x) \;=\; \psi(x) \Rightarrow \gamma(x)$.
Clearly the complexities of both $\psi(x)$ and $\gamma(x)$ are less than n and similarly as the above cases by the induction hypothesis we get,

1. $[\![u = v]\!] \Rightarrow [\![\psi(u)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\psi(v)]\!]$ and

2. $[\![u = v]\!] \Rightarrow [\![\gamma(u)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\gamma(v)]\!]$

For proving the theorem, in this case the following has to be proved,

$$[\![u = v]\!] \Rightarrow ([\![\psi(u)]\!] \Rightarrow [\![\gamma(u)]\!]) \;=\; [\![u = v]\!] \Rightarrow ([\![\psi(v)]\!] \Rightarrow [\![\gamma(v)]\!]).$$

If $[\![\psi(u)]\!] \Rightarrow [\![\gamma(u)]\!] = 0$ then $[\![\psi(u)]\!] \neq 0$ and $[\![\gamma(u)]\!] = 0$. Hence by using (1) and (2) we get $[\![\psi(v)]\!] \neq 0$ and $[\![\gamma(v)]\!] = 0$. So, $[\![\psi(v)]\!] \Rightarrow [\![\gamma(v)]\!] = 0$ and the proof is complete.

If $[\![\psi(u)]\!] \Rightarrow [\![\gamma(u)]\!] \neq 0$ then either $[\![\gamma(u)]\!] \neq 0$, in which case $[\![\gamma(v)]\!] \neq 0$ also, by (2); or both of $[\![\psi(u)]\!]$ and $[\![\gamma(u)]\!]$ is 0, in which case again by (1) and (2) we have, $[\![\psi(v)]\!] = 0$ and $[\![\gamma(v)]\!] = 0$. In any of the possibilities it can be concluded that

$$[\![\psi(u)]\!] \Rightarrow [\![\gamma(u)]\!] \;=\; [\![\psi(v)]\!] \Rightarrow [\![\gamma(v)]\!].$$

Hence this case is also proved.

Case 4. Let $\varphi(x) = \forall y \psi(x, y)$.

So the complexity of $\psi(x, y)$ is less than n. Hence by the induction hypothesis,

$$[\![u = v]\!] \Rightarrow [\![\psi(u, y)]\!] = [\![u = v]\!] \Rightarrow [\![\psi(v, y)]\!]$$

for all $y \in \mathbf{V}^{(\mathrm{PS}_3)}$. We have to prove

$$[\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] = [\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!].$$

Now if

$$[\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] = 0$$

then there exists $y \in \mathbf{V}^{(\mathrm{PS}_3)}$ for which $[\![\psi(u, y)]\!] = 0$. Hence by the induction hypothesis $[\![\psi(v, y)]\!] = 0$; therefore $\bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] = 0$ and hence

$$[\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] = 0.$$

Similarly it can be proved, if

$$[\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] = 0$$

then

$$[\![u = v]\!] \Rightarrow \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] = 0.$$

Hence *Case 4* is immediately proved by the truth table of $\Rightarrow$.

Case 5. Let $\varphi(x) = \exists y \psi(x, y)$.

So the complexity of $\psi(x, y)$ is less than n and therefore by the induction hypothesis,

$$[\![u = v]\!] \Rightarrow [\![\psi(u, y)]\!] \;=\; [\![u = v]\!] \Rightarrow [\![\psi(v, y)]\!]$$

for all $y \in \mathbf{V}^{(\mathrm{PS}_3)}$. We have to prove

$$[\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] \;=\; [\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!].$$

If

$$[\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] \;=\; 0$$

then $[\![\psi(u, y)]\!] \;=\; 0$ for all $y \in \mathbf{V}^{(\mathrm{PS}_3)}$. Hence by the induction hypothesis $[\![\psi(v, y)]\!] \;=\; 0$ for all $y \in \mathbf{V}^{(\mathrm{PS}_3)}$; therefore $\bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] \;=\; 0$ and hence

$$[\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] \;=\; 0.$$

Similarly it can be proved, if

$$[\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(v, y)]\!] \;=\; 0$$

then

$$[\![u = v]\!] \Rightarrow \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![\psi(u, y)]\!] \;=\; 0.$$

Hence this case is also proved.

So, combining all the cases above the theorem is proved. □

By using Theorem 3.5.6 we can reach to our main goal:

Theorem 3.5.7 *The property* BQ_φ *holds in* $\mathbf{V}^{(\mathrm{PS}_3)}$ *for any negation-free formula* $\varphi(x)$ *having one free variable* x*.*

Proof. We have to prove, for any $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ and any negation-free formula $\varphi(x)$,

$$[\![\forall x(x \in u \to \varphi(x))]\!] \;=\; \bigwedge_{x\in\mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

The proof is done as follows.

$$\begin{aligned}
[\![\forall x(x \in u \to \varphi(x))]\!] &= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} ([\![y \in u]\!] \Rightarrow [\![\varphi(y)]\!]) \\
&= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} [\bigvee_{x\in\mathrm{dom}(u)} (u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(y)]\!]] \\
&= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in\mathrm{dom}(u)} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(y)]\!]], \\
&\qquad \text{since in } \mathrm{PS}_3,\ (\bigvee_i a_i \Rightarrow b) = \bigwedge_i (a_i \Rightarrow b) \\
&= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow ([\![y = x]\!] \Rightarrow [\![\varphi(y)]\!])], \\
&\qquad \text{since } \mathrm{PS}_3 \text{ is deductive} \\
&= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in\mathrm{dom}(u)} [u(x) \Rightarrow ([\![y = x]\!] \Rightarrow [\![\varphi(x)]\!])], \\
&\qquad \text{by using Theorem 3.5.6, as } \varphi(x) \text{ is a negation-free formula} \\
&= \bigwedge_{y\in\mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in\mathrm{dom}(u)} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]]
\end{aligned}$$

Now for any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$ and $x \in \mathrm{dom}(u)$,

$$u(x) \Rightarrow [\![\varphi(x)]\!] \;\le\; (u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]$$

hence,

$$\bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]) \;\leq\; \bigwedge_{x\in \mathrm{dom}(u)} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]]$$

hence,

$$\bigwedge_{y\in \mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]) \;\leq\; \bigwedge_{y\in \mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{x\in \mathrm{dom}(u)} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]].$$

Again for the reverse direction, for any $x \in \mathrm{dom}(u)$,

$$\begin{aligned}\bigwedge_{y\in \mathbf{V}^{(\mathrm{PS}_3)}} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]] &\leq\; (u(x) \wedge [\![x = x]\!]) \Rightarrow [\![\varphi(x)]\!] \\ &=\; u(x) \Rightarrow [\![\varphi(x)]\!]\end{aligned}$$

hence,

$$\bigwedge_{x\in \mathrm{dom}(u)} \bigwedge_{y\in \mathbf{V}^{(\mathrm{PS}_3)}} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]] \;\leq\; \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

So, by combining the results we have,

$$\bigwedge_{x\in \mathrm{dom}(u)} \bigwedge_{y\in \mathbf{V}^{(\mathrm{PS}_3)}} [(u(x) \wedge [\![y = x]\!]) \Rightarrow [\![\varphi(x)]\!]] \;=\; \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

i.e.,

$$[\![\forall x(x \in u \rightarrow \varphi(x))]\!] \;=\; \bigwedge_{x\in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]).$$

So the theorem is proved. □

Hence as a corollary of Theorems 2.2.6 and 2.2.7 we get:

Theorem 3.5.8 *The axioms and axiom schemas* Extensionality, Pairing, Infinity, Union, Power

Set, NFF-Separation and NFF-Collection are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

In addition we can also prove the following theorem in $\mathbf{V}^{(\mathrm{PS}_3)}$:

Theorem 3.5.9 NFF-*Foundation is valid in* $\mathbf{V}^{(\mathrm{PS}_3)}$.

Proof. Let $\varphi(x)$ be a negation-free formula having one free variable x. We have to prove

$$\mathbf{V}^{(\mathrm{PS}_3)} \models \forall x[\forall y \in x\ \varphi(y) \to \varphi(x)] \to \forall x \varphi(x).$$

Case 1. Suppose $\varphi(x)$ is such that $[\![\varphi(x)]\!] \neq 0$ for any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$. Hence in this case $[\![\forall x \varphi(x)]\!] \neq 0$ and therefore $[\![\forall x[\forall y \in x\ \varphi(y) \to \varphi(x)] \to \forall x \varphi(x)]\!] = 1$.

Case 2. Suppose there exist $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ for which $[\![\varphi(x)]\!] = 0$. Then take a minimal $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ satisfying this, i.e., $[\![\varphi(u)]\!] = 0$ but for any $y \in \mathrm{dom}(u)$; $[\![\varphi(y)]\!] \neq 0$. Then clearly $[\![\forall x \varphi(x)]\!] = 0$. Now since the property BQ_φ holds in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all negation-free formulas φ we get the following.

$$\begin{aligned} [\![\forall x[(\forall y \in x \varphi(y)) \to \varphi(x)]]\!] &\leq [\![(\forall y \in u \varphi(y)) \to \varphi(u)]\!] \\ &= \bigwedge_{y \in \mathrm{dom}(u)} (u(y) \Rightarrow [\![\varphi(y)]\!]) \Rightarrow [\![\varphi(u)]\!] \\ &= 0 \end{aligned}$$

Hence we get

$$[\![\forall x[\forall y \in x \varphi(y) \to \varphi(x)] \to \forall x \varphi(x)]\!] = 1.$$

Combining *Case 1* and *Case 2* we can proved that $\mathbf{V}^{(\mathrm{PS}_3)}$ is an algebra-valued model of NFF-Foundation. □

In ZF one can prove from Foundation that "there is no set containing itself". The following theorem shows that $\mathbf{V}^{(\mathrm{PS}_3)}$ also agrees with this fact.

Theorem 3.5.10 *For all $u \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![u \in u]\!] = 0$. So, in particular, $[\![\exists x(x \in x)]\!] = 0$.*

Proof. By meta-induction, if there is a counterexample to the claim, there is a minimal counterexample, i.e., a name u with $[\![u \in u]\!] \neq 0$, but for every $x \in \mathrm{dom}(u)$, we have that $[\![x \in x]\!] = 0$. The first claim means that there is some $x_0 \in \mathrm{dom}(u)$ with $u(x_0) \neq 0$ and $[\![u = x_0]\!] \neq 0$. Since $[\![u = x_0]\!]$ is defined in terms of a conjunction in which all expressions of the form $u(x) \Rightarrow [\![x \in x_0]\!]$ for $x \in \mathrm{dom}(u)$ occur, each of these must be non-zero. Take one of these and let $x = x_0$ in this expression; we obtain $u(x_0) \Rightarrow [\![x_0 \in x_0]\!]$. But we assumed that $u(x_0) \neq 0$ and $[\![x_0 \in x_0]\!] = 0$. Contradiction! □

In Chapter 1, we have said that the paraconsistent set theory developed in this thesis does not satisfy Comprehension. Below we shall prove this claim.

Theorem 3.5.11 *The formula $\exists x \forall y(y \in x \leftrightarrow y \notin y)$ is not valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.*

Proof. Again, assume towards a contradiction that u satisfies $[\![\forall y(y \in u \leftrightarrow y \notin y)]\!] \neq 0$. By Theorem 3.5.10,

$$[\![y \notin y]\!] = [\![y \in y]\!]^* = 0^* = 1$$

for all $y \in \mathbf{V}^{(\mathrm{PS}_3)}$ and $[\![u \in u]\!] = 0$. But then $[\![u \notin u \to u \in u]\!] = 0$. Contradiction! □

Since the formula $\exists x \forall y(y \in x \leftrightarrow y \notin y)$ is one instance of Comprehension, Theorem 3.5.11 shows that the axiom scheme of Comprehension is not valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. It also assures that there does not exist any name for Russell's set in $\mathbf{V}^{(\mathrm{PS}_3)}$. In addition, like classical set theory, we can prove the formula stating 'there exists an universal set' is not valid in this algebra-valued model.

Theorem 3.5.12 *The formula $\exists x \forall y(y \in x)$ is invalid in $\mathbf{V}^{(\mathrm{PS}_3)}$.*

Proof. This follows immediately from Theorem 3.5.10: if $[\![\exists x \forall y(y \in x)]\!] \neq 0$ and u is a name witnessing this (i.e., $[\![\forall y(y \in u)]\!] \neq 0$), then $[\![u \in u]\!] \neq 0$ in contradiction to Theorem 3.5.10.

3.6 Paraconsistency in the language of set theory

3.6.1 The logic corresponding to $\mathbf{V}^{(\mathrm{PS}_3)}$ is non-explosive

We now built an algebra PS_3 which is paraconsistent and the axioms of set theory are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. Does the paraconsistency of PS_3 transfer to the set theory in $\mathbf{V}^{(\mathrm{PS}_3)}$, though? Or, in other words, can we find a sentence φ in the language $\mathcal{L}_\in$ such that $\mathbf{V}^{(\mathrm{PS}_3)} \models \varphi \wedge \neg\varphi$? In this section, we give a positive answer to this question.

Theorem 3.6.1 *For the formula*

$$\exists z\, \exists x\, \exists y\, (x = y \wedge z \in x \wedge z \notin y), \tag{$\mathsf{Paracon}_\exists$}$$

$\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Paracon}_\exists \wedge \neg\mathsf{Paracon}_\exists$ *holds; i.e.,* $[\![\mathsf{Paracon}_\exists]\!] = {}^1\!/\!_2$, *where* $x \notin y$ *is an abbreviation for* $\neg(x \in y)$.

Proof. Let us take an arbitrary element $w \in \mathbf{V}^{(\mathrm{PS}_3)}$. Then $u = \{\langle w, 1\rangle\}$ and $v = \{\langle w, {}^1\!/\!_2\rangle\}$ are two elements of $\mathbf{V}^{(\mathrm{PS}_3)}$. For these u, v and w we get

$$[\![w \in u]\!] = u(w) \wedge [\![w = w]\!] = 1;$$

similarly $[\![w \in v]\!] = {}^1\!/\!_2$ and

$$\begin{aligned}[\![u = v]\!] &= (u(w) \Rightarrow [\![w \in v]\!]) \wedge (v(w) \Rightarrow [\![w \in u]\!]) \\ &= (1 \Rightarrow {}^1\!/\!_2) \wedge ({}^1\!/\!_2 \Rightarrow 1) \\ &= 1.\end{aligned}$$

Hence we get the following:

$$\begin{aligned}[\![\exists z\ \exists x\ \exists y\ (x = y \wedge z \in x \wedge z \notin y)]\!] &= \bigvee_{z \in \mathbf{V}^{(\mathrm{PS}_3)}} \bigvee_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} \bigvee_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![x = y]\!] \wedge \\ &\qquad [\![z \in x]\!] \wedge [\![z \in y]\!]^*) \\ &\geq [\![u = v]\!] \wedge [\![w \in u]\!] \wedge [\![w \in v]\!]^* \\ &= 1 \wedge 1 \wedge {}^1\!/\!_2{}^* \\ &= {}^1\!/\!_2.\end{aligned}$$

We shall now show that

$$[\![\exists x\ \exists y\ \exists z\ (x = y \wedge z \in x \wedge z \notin y)]\!] \neq 1.$$

It will be proved if we can show for any $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$,

$$[\![u = v\ \wedge\ w \in u\ \wedge\ w \notin v]\!] \neq 1. \tag{i}$$

Hence we have to prove that simultaneously $[\![u = v]\!] = 1$, $[\![w \in u]\!] = 1$, and $[\![w \in v]\!] = 0$ are not possible. By definitions we have,

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!]) \tag{ii}$$

$$[\![w \in u]\!] = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge [\![x = w]\!]) \tag{iii}$$

$$[\![w \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge [\![y = w]\!]) \tag{iv}$$

Let us assume $[\![u = v]\!] = 1$ and $[\![w \in v]\!] = 0$. Then we shall prove the value of $[\![w \in u]\!]$ must be equal to 0. Since $[\![w \in v]\!] = 0$ from (iv) we get, for any $y \in \mathrm{dom}(v)$

$$\text{either } v(y) = 0 \text{ or } [\![y = w]\!] = 0. \tag{v}$$

Now let for some $x \in \mathrm{dom}(u)$, $u(x) \neq 0$. Then from (ii) and the assumption $[\![u = v]\!] = 1$ we have $[\![x \in v]\!] \neq 0$. So there exists $y \in \mathrm{dom}(v)$ such that $v(y) \neq 0 \neq [\![x = y]\!]$. If $v(y) \neq 0$ then from (v) we get $[\![y = w]\!] = 0$. Using (i) of Lemma 3.5.2 it can be concluded that

$$[\![x = y]\!] \wedge [\![x = w]\!] \leq [\![y = w]\!].$$

Hence $[\![x = w]\!] = 0$. So we have derived, for any $x \in \mathrm{dom}(u)$ either $u(x) = 0$ or $[\![x = w]\!] = 0$. Then using (iii) we get $[\![w \in u]\!] = 0$. Hence (i) is proved.

Combining all the above results we get,

$$[\![\mathsf{Paracon}_\exists]\!] = [\![\exists z\, \exists x\, \exists y\, (x = y \wedge z \in x \wedge z \notin y)]\!] = 1/2.$$

Hence we can conclude $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Paracon}_\exists \wedge \neg\mathsf{Paracon}_\exists$. □

By slightly changing the formula $\mathsf{Paracon}_\exists$ we can provide another formula whose valuation will still be $1/2$. Details are given in the following theorem.

Theorem 3.6.2 *For the formula*

$$\forall z\, \exists x\, \exists y\, (x = y \wedge z \in x \wedge z \notin y), \tag{$\mathsf{Paracon}_\forall$}$$

$[\![\mathsf{Paracon}_\forall]\!] = 1/2$ *holds i.e.,* $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Paracon}_\forall \wedge \neg\mathsf{Paracon}_\forall$.

Proof. For any $z \in \mathbf{V}^{(\mathrm{PS}_3)}$ fix $x = \{\langle z, 1\rangle\}$ and $y = \{\langle z, 1/2\rangle\}$. Then by following the arguments given in Theorem 3.6.1 we get $[\![x = y]\!] = 1$, $[\![z \in x]\!] = 1$, and $[\![z \notin y]\!] = [\![z \in y]\!]^* = 1/2$. Hence $[\![\mathsf{Paracon}_\forall]\!] \geq 1/2$. Since the condition (i) in the proof of Theorem 3.6.1 holds in $\mathbf{V}^{(\mathrm{PS}_3)}$, we get $[\![\mathsf{Paracon}_\forall]\!] < 1$. Combining both the results it can be derived that $[\![\mathsf{Paracon}_\forall]\!] = 1/2$ i.e., $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Paracon}_\forall \wedge \neg\mathsf{Paracon}_\forall$. □

Let us now consider the formula $\psi := \forall x(x \in x)$ in the language of set theory. One can prove that $[\![\psi]\!] = 0$ by the following derivation:

$$[\![\forall x(x \in x)]\!] = \bigwedge_{u \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![u \in u]\!] \leq [\![\varnothing \in \varnothing]\!] = 0.$$

As a conclusion, suppose φ is any one of the formulas $\mathsf{Paracon}_\exists$ and $\mathsf{Paracon}_\forall$, we can prove that $\mathbf{V}^{(\mathrm{PS}_3)}$ being an algebra-valued model of $\{\varphi, \neg\varphi\}$ cannot validate ψ. Hence by the definition, the first order set theory having $\mathbf{V}^{(\mathrm{PS}_3)}$ as an algebra-valued model, is paraconsistent.

3.6.2 De-Paraconsistification of a formula

If we observe carefully it can be understood that the paraconsistency entered into the language of set theory through the formulas φ which are such that $[\![\varphi]\!] = 1/2$. Also we know in the composition table of $\Rightarrow$ in PS_3 the only entries are 1 and 0. We shall use this fact to get from a formula φ with $[\![\varphi]\!] = 1/2$ a new, classically equivalent, formula ψ such that $[\![\psi]\!] = 1$.

Definition. Let $\varphi \in \mathbb{L}\mathrm{PS}_3$ be an arbitrary formula and v be a valuation such that $v(\varphi) = 1/2$. A formula ψ is said to be a *v-de-paraconsistification* of φ if $v(\psi) = 1$ and $\models (\varphi \leftrightarrow \psi)$.

Let $\varphi(x)$ be a formula in the language of set theory having one free variable. Let us define another formula, $\varphi^*(x) := \forall t(t = x \to \varphi(t))$.

Theorem 3.6.3 *Let $\varphi(x)$ be a formula having one free variable. Then $[\![\forall x\varphi(x)]\!] \in D$ iff $[\![\forall x\varphi^*(x)]\!] = 1$.*

Proof. Let $[\![\forall x\varphi(x)]\!] \in D$ hold. Hence for any $u \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![\varphi(u)]\!] \in D$. Hence by the composition table of $\Rightarrow$ in PS_3 we can say that $[\![\varphi^*(u)]\!] = 1$. Since u was arbitrary $[\![\forall x\varphi^*(x)]\!] = 1$ also.

Conversely let $[\![\forall x\varphi^*(x)]\!] = 1$ i.e., for any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![\varphi^*(x)]\!] = 1$. If there exists $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $[\![\varphi(u)]\!] = 0$ then we would get

$$\begin{aligned}
[\![\varphi^*(u)]\!] &= [\![\forall t(t = u \to \varphi(t))]\!] \\
&\leq [\![u = u \to \varphi(u)]\!] \\
&= [\![u = u]\!] \Rightarrow [\![\varphi(u)]\!] \\
&= 0
\end{aligned}$$

which is not the case. Hence for any $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ we get $[\![\varphi(u)]\!] \in D$. $\square$

In particular if there exists a formula $\varphi(x)$ such that $[\![\forall x\varphi(x)]\!] = 1/2$ then $\forall x\varphi^*(x)$ is a $[\![\cdot]\!]$-de-paraconsistification of the formula $\forall x\varphi(x)$. The formula $\mathsf{Paracon}_\forall$ assures that such a formula $\varphi(x)$ exists.

Theorem 3.6.4 *Let $\varphi(x)$ be a negation-free formula having one free variable. Then $[\![\exists x\varphi(x)]\!] \in D$ iff $[\![\exists x\varphi^*(x)]\!] = 1$.*

Proof. Let $\varphi(x)$ be a negation-free formula such that $[\![\exists x\varphi(x)]\!] \in D$. Then there exists

$u \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $[\![\varphi(u)]\!] \in D$. Hence for any $v \in \mathbf{V}^{(\mathrm{PS}_3)}$ we get

$$[\![v = u \to \varphi(v)]\!] = [\![v = u]\!] \Rightarrow [\![\varphi(v)]\!] = [\![v = u]\!] \Rightarrow [\![\varphi(u)]\!] \in D$$

using Theorem 3.5.6. This implies $[\![\varphi^*(u)]\!] = 1$. As a conclusion it can be derived that $[\![\exists x \varphi^*(x)]\!] = 1$.

Conversely let $[\![\exists x \varphi^*(x)]\!] = 1$. Hence there exists $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $[\![\varphi^*(u)]\!] = 1$. Now we know that

$$[\![\varphi^*(u)]\!] = \bigwedge_{v \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![v = u]\!] \Rightarrow [\![\varphi(v)]\!]) \leq [\![u = u]\!] \Rightarrow [\![\varphi(u)]\!].$$

From this it can be concluded that $[\![\varphi(u)]\!] \in D$. Hence $[\![\exists x \varphi(x)]\!] \in D$. $\square$

One side of the proof, viz. $[\![\exists x \varphi^*(x)]\!] = 1$ implies $[\![\exists x \varphi(x)]\!] \in D$, does not require φ to be a negation-free formula. As above, in particular if we get a negation-free formula $\varphi(x)$ so that $[\![\exists x \varphi(x)]\!] = 1/2$ then $\exists x \varphi^*(x)$ is a $[\![\cdot]\!]$-de-paraconsistification of the formula $\exists x \varphi(x)$.

We should stress that the results about $[\![\cdot]\!]$-de-paraconsistification are results in $\mathbf{V}^{(\mathrm{PS}_3)}$, not results for arbitrary reasonable implication algebras: this is because we heavily rely on the fact that the truth table of $\Rightarrow$ does not contain the value $1/2$ in our arguments. It is conceivable that there are other reasonable implication algebras where $[\![\cdot]\!]$-de-paraconsistification does not work. For example consider a reasonable implication-negation algebra $\mathbb{A}$, other than PS_3, having $D_{\mathbb{A}}$ as the designated set, which satisfy the following properties

(i) there exists $d \in D_{\mathbb{A}}$ such that $d^* \in D_{\mathbb{A}}$ as well, and

(ii) all the truth tables of the operators in $\mathbb{A}$ contain d.

Then the same idea of $[\![\cdot]\!]$-de-paraconsistification of a formula may not work in the set theory for $\mathbf{V}^{(\mathbb{A})}$.

Chapter 4

Ordinals in the algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$ and violation of Leibniz's Law

In this chapter we shall discuss the validity of the sentences: 'there is an ordinal number', 'there is no set containing all ordinals' etc. Before that it is necessary to define which elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ will be marked as names for ordinals. We shall also emphasize on some properties of ordinals in the algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$.

4.1 Definitions and preliminaries

4.1.1 Some classical definitions

We develop the classical theory of *transitive sets*, *well-ordered sets*, and ordinal numbers in the setting of $\mathbf{V}^{(\mathbb{A})}$. The following definitions are reminders of the classical definitions for the benefit of the reader.

Definition. A set x is said to be transitive if every element of x is a subset of x, or equivalently, if $y \in z$ and $z \in x$ then $y \in x$.

Definition. A set A is said to be well-ordered by a relation R if R is a linear order on A and any non-empty subset of A has a least element with respect to R.

Definition. An ordinal number is a transitive set well-ordered by $\in$.

4.1.2 Definition of α-like Elements

As in the theory of Boolean-valued models, we can recursively define the notion of a *canonical name*:

Definition. For each $x \in \mathbf{V}$, $\hat{x}$ is defined as:

$$\begin{aligned} \hat{\varnothing} &= \varnothing, \\ \hat{x} &= \{\langle \hat{y}, 1 \rangle : y \in x\}. \end{aligned}$$

Let ORD refer to the class of all ordinal numbers in $\mathbf{V}$. One of the main goals of this chapter is to identify elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ which behave almost similar to the classical ordinal numbers. It will be shown that there are more than one such elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ corresponding to each $\alpha \in$ ORD which will be named as α-like elements. But the non-classical behaviour of these elements will be discussed in the section 4.4.

For each $\alpha \in$ ORD the α-like names in $\mathbf{V}^{(\mathrm{PS}_3)}$ are defined by transfinite recursion as follows.

Definition. An element $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ is called

i. 0-like if for every $y \in \mathrm{dom}(x)$, we have that $x(y) = 0$; and

ii. α-like if for each $\beta \in \alpha$ there exists $y \in \text{dom}(x)$ which is β-like and $x(y) \in \{1, 1/2\}$, and for any $z \in \text{dom}(x)$ if it is not β-like for any $\beta \in \alpha$ then $x(z) = 0$.

Clearly, the canonical name $\hat{\alpha}$ is an α-like name for every $\alpha \in \text{ORD}$.

4.2 Properties of α-like Elements

For each $\alpha \in \text{ORD}$, there are many α-like names as the following results show.

Lemma 4.2.1 *For any $x \in \mathbf{V}^{(\text{PS}_3)}$ and $\alpha \in \text{ORD}$, $[\![x = \hat{\alpha}]\!] = 1$ if and only if x is α-like.*

Proof. The proof will be done by (meta-)induction. We assume that we have shown the result for all elements in the domain of $\hat{\alpha}$. We know

$$[\![x = \hat{\alpha}]\!] = \bigwedge_{y \in \text{dom}(x)} (x(y) \Rightarrow [\![y \in \hat{\alpha}]\!]) \wedge \bigwedge_{\hat{\beta} \in \text{dom}(\hat{\alpha})} (1 \Rightarrow [\![\hat{\beta} \in x]\!]).$$

Hence $[\![x = \hat{\alpha}]\!] = 1$ if and only if both of the conjuncts are 1. The second conjunct is 1, i.e.,

$$\bigwedge_{\hat{\beta} \in \text{dom}(\hat{\alpha})} (1 \Rightarrow [\![\hat{\beta} \in x]\!]) = 1;$$

if and only if for each $\hat{\beta} \in \text{dom}(\hat{\alpha})$, $1 \Rightarrow [\![\hat{\beta} \in x]\!] = 1$ i.e.,

$$1 \Rightarrow \bigvee_{y \in \text{dom}(x)} (x(y) \wedge [\![y = \hat{\beta}]\!]) = 1;$$

if and only if for each $\hat{\beta} \in \text{dom}(\hat{\alpha})$ there exists $y \in \text{dom}(x)$ such that $x(y) \in \{1, 1/2\}$ and $[\![y = \hat{\beta}]\!] = 1$; if and only if for each $\hat{\beta} \in \text{dom}(\hat{\alpha})$ there exists $y \in \text{dom}(x)$ such that y is β-like (by the induction hypothesis) and $x(y) \in \{1, 1/2\}$.

Again, since the first conjunct is 1, we have,

$$\bigwedge_{y \in \mathrm{dom}(x)} (x(y) \Rightarrow [\![y \in \hat{\alpha}]\!]) = 1;$$

if and only if for each $y \in \mathrm{dom}(x)$, $(x(y) \Rightarrow [\![y \in \hat{\alpha}]\!]) = 1$, i.e.,

$$(x(y) \Rightarrow \bigvee_{\hat{\beta} \in \mathrm{dom}(\hat{\alpha})} [\![y = \hat{\beta}]\!]) = 1;$$

if and only if for each $y \in \mathrm{dom}(x)$, if y is not β-like for any $\beta \in \alpha$ then by induction hypothesis it can be derived that $x(y) = 0$.

Hence combining the above results we get $[\![x = \hat{\alpha}]\!] = 1$ if and only if x is α-like and hence by the (meta-)induction the proof is done. □

Lemma 4.2.2 *For any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ and $\alpha \in \mathrm{ORD}$, $[\![x \in \hat{\alpha}]\!] = 1$ if and only if x is β-like for some $\beta \in \alpha$.*

Proof. Using Lemma 4.2.1, the following three statements are equivalent:

1. $[\![x \in \hat{\alpha}]\!] = 1$;
2. $\bigvee_{\hat{u} \in \mathrm{dom}(\hat{\alpha})} [\![x = \hat{u}]\!] = 1$;
3. there exists $\hat{\beta} \in \mathrm{dom}(\hat{\alpha})$ such that $[\![x = \hat{\beta}]\!] = 1$; and
4. x is β-like for some $\beta \in \alpha$.

□

It is clear from the definition that for any $\alpha \in \mathrm{ORD}$, there are many α-like names in $\mathbf{V}^{(\mathrm{PS}_3)}$ in addition to $\hat{\alpha}$. We would desire that for any two α-like names, $\mathbf{V}^{(\mathrm{PS}_3)}$ validates the statement that they are equal, and if $\beta < \alpha$, for β-like names and α-like names, $\mathbf{V}^{(\mathrm{PS}_3)}$ validates the statement that the former are elements of the latter.

Of course, we would desire that α-like names are equal and that for $\beta < \alpha$, β-like names are elements of α-like names (in the formal sense of $\mathbf{V}^{(\mathbb{A})}$):

Theorem 4.2.3 *Let $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ be α-like for some $\alpha \in \mathrm{ORD}$. For any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![x = y]\!] = 1$ if and only if y is α-like.*

Proof. It is proved in 3.5.2(i) that for any $x, y, z \in \mathbf{V}^{(\mathrm{PS}_3)}$,

$$[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!].$$

Let x and y be two α-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. So $[\![x = \hat{\alpha}]\!] \wedge [\![\hat{\alpha} = y]\!] \leq [\![x = y]\!]$. By Lemma 4.2.1 we have $[\![x = \hat{\alpha}]\!] = 1 = [\![y = \hat{\alpha}]\!]$, which implies $[\![x = y]\!] = 1$.

Conversely let $[\![x = y]\!] = 1$. By a similar argument we can write, $[\![x = y]\!] \wedge [\![x = \hat{\alpha}]\!] \leq [\![y = \hat{\alpha}]\!]$ and hence $[\![y = \hat{\alpha}]\!] = 1$. Again by Lemma 4.2.1 it can be concluded that y is α-like. □

Theorem 4.2.4 *Let $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ be α-like for some non-zero $\alpha \in \mathrm{ORD}$. For any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![y \in x]\!] \in \{1, 1/2\}$ if and only if y is β-like for some $\beta \in \alpha$.*

Proof. Let y be β-like for some $\beta \in \alpha$. Now

$$\begin{aligned}
[\![y \in x]\!] &= \bigvee_{u \in \mathrm{dom}(x)} (x(u) \wedge [\![u = y]\!]) \\
&\geq x(v) \wedge [\![v = y]\!], \text{where } v \in \mathrm{dom}(x) \text{ is } \beta\text{-like and } x(v) \in \{1, 1/2\} \\
&\geq 1/2, \text{ by Theorem 4.2.3.}
\end{aligned}$$

Conversely, let $[\![y \in x]\!] \in \{1, 1/2\}$, i.e.,

$$\bigvee_{u \in \mathrm{dom}(x)} (x(u) \wedge [\![u = y]\!]) \in \{1, 1/2\}.$$

Hence there exists some β-like $v \in \mathrm{dom}(x)$ such that $x(v) \in \{1, 1/2\}$ and $[\![v = y]\!] = 1$, where $\beta \in \alpha$. So by Theorem 4.2.3, it follows that y is also β-like. $\square$

4.3 Ordinals in $\mathbf{V}^{(\mathrm{PS}_3)}$

We now rewrite the classical definitions given in section 4.1 in the language of set theory:

$$\mathrm{Trans}(x) = \forall y \forall z (z \in y \wedge y \in x \rightarrow z \in x)$$

$$\mathrm{LO}(x) = \forall y \forall z ((y \in x \wedge z \in x) \rightarrow (y \in z \vee y = z \vee z \in y))^{1}$$

$$\mathrm{WO}_{\in}(x) = \mathrm{LO}(x) \wedge \forall y (y \subseteq x \wedge (y \neq \varnothing) \rightarrow \exists z (z \in y \wedge z \cap y = \varnothing))$$

$$\mathrm{ORD}(x) = \mathrm{Trans}(x) \wedge \mathrm{WO}_{\in}(x)$$

where the following abbreviations are used in $\mathrm{WO}_{\in}(x)$:

$$y \subseteq x := \forall t (t \in y \rightarrow t \in x),$$

$$(y \neq \varnothing) := \exists z (z \in y), \text{ and}$$

$$(z \cap y = \varnothing) := \neg \exists w (w \in z \wedge w \in y).$$

Finally, we can connect the notion of α-like elements to the set theoretic notion of ordinals:

[1] $\mathrm{LO}(x)$ stands for the formula: x is a linear orderered set with respect to $\in$.

Lemma 4.3.1 *Let $\alpha \in \mathrm{ORD}$ and u be an α-like element in $\mathbf{V}^{(\mathrm{PS}_3)}$. Then the following hold:*

(i) $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Trans}(u)$.

(ii) $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{LO}(u)$.

(iii) $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{WO}_{\in}(u)$.

Proof. *(i)* We have to prove $[\![\forall y \forall z(z \in y \wedge y \in u \to z \in u)]\!] \in \{1, 1/2\}$. Since the truth table of $\Rightarrow$ in PS_3 does not contain $1/2$ it is sufficient to show $[\![\forall y \forall z(z \in y \wedge y \in u \to z \in u)]\!] = 1$.

Let us take any $z \in \mathbf{V}^{(\mathrm{PS}_3)}$. Then,

$$\begin{aligned}
[\![\forall y(y \in u \wedge z \in y \to z \in u)]\!] &= \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![y \in u]\!] \wedge [\![z \in y]\!] \Rightarrow [\![z \in u]\!]) \\
&= \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![y \in u]\!] \Rightarrow ([\![z \in y]\!] \Rightarrow [\![z \in u]\!])) \\
&= [\![\forall y \in u\ (z \in y \to z \in u)]\!] \\
&= \bigwedge_{y \in \mathrm{dom}(u)} (u(y) \Rightarrow [\![z \in y \to z \in u]\!]) \\
&= \bigwedge_{y \in \mathrm{dom}(u)} (u(y) \Rightarrow ([\![z \in y]\!] \Rightarrow [\![z \in u]\!]))
\end{aligned}$$

(since BQ_φ hold in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all negation-free formulas φ, and $(z \in y \to z \in u)$ is a negation-free formula.)

For any $y \in \mathrm{dom}(u)$ if $u(y) \neq 0$ then y is β-like for some non-zero $\beta \in \alpha$. Let for such an y, $[\![z \in y]\!] \in \{1, 1/2\}$. Therefore by Theorem 4.2.4, z is γ-like for some $\gamma \in \beta$. Clearly, $\gamma \in \alpha$. Therefore one more application of Theorem 4.2.4 provides $[\![z \in u]\!] \in \{1, 1/2\}$. Hence combining the above results we get

$$\bigwedge_{y \in \mathrm{dom}(u)} (u(y) \Rightarrow ([\![z \in y]\!] \Rightarrow [\![z \in u]\!])) = 1$$

for any $z \in \mathbf{V}^{(\mathrm{PS}_3)}$. This leads to the fact

$$[\![\forall y \forall z (y \in u \wedge z \in y \to z \in u)]\!] = 1, \text{ i.e., } \mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Trans}(u).$$

(*ii*) Since for any $\alpha, \beta \in \mathrm{ORD}$ exactly one of $\alpha \in \beta,\ \alpha = \beta$ and $\beta \in \alpha$ holds in $\mathbf{V}$, the proof can be derived easily by applying Theorems 4.2.3 and 4.2.4.

(*iii*) We already have $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{LO}(u)$ from (*ii*). So it is sufficient to prove that

$$[\![\forall y (y \subseteq u \wedge (y \neq \varnothing) \to \exists z (z \in y \wedge z \cap y = \varnothing))]\!] = 1,\text{[2]}$$

i.e., for any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$ if $[\![y \subseteq u \wedge (y \neq \varnothing)]\!] \in \{1, 1/2\}$ then $[\![\exists z (z \in y \wedge z \cap y = \varnothing)]\!] \in \{1, 1/2\}$. Now by definition and the fact that BQ_φ hold in $\mathbf{V}^{(\mathrm{PS}_3)}$ for all negation-free formulas φ,

$$[\![y \subseteq u]\!] = [\![\forall t (t \in y \to t \in u)]\!] = \bigwedge_{t \in \mathrm{dom}(y)} (y(t) \Rightarrow [\![t \in u]\!])$$

So, $[\![y \subseteq u]\!] \in \{1, 1/2\}$ iff for any $t \in \mathrm{dom}(y)$ if $y(t) \neq 0$ then $[\![t \in u]\!] \neq 0$, i.e., by Theorem 4.2.4 it can be concluded that t is β-like for some $\beta \in \alpha$. Again,

$$[\![(y \neq \varnothing)]\!] = [\![\exists z (z \in y)]\!] = \bigvee_{z \in \mathbf{V}^{(\mathrm{PS}_3)}} \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![z = t]\!]).$$

Therefore $[\![(y \neq \varnothing)]\!] \in \{1, 1/2\}$ iff there exists $t \in \mathrm{dom}(y)$ such that $y(t) \in \{1, 1/2\}$.

Hence $[\![y \subseteq u \wedge (y \neq \varnothing)]\!] \in \{1, 1/2\}$ iff there exists $t \in \mathrm{dom}(y)$ such that $y(t) \in \{1, 1/2\}$

[2]Since PS_3 satisfies the deductive principle: $((a \wedge b) \Rightarrow c) = (a \Rightarrow (b \Rightarrow c))$.

and for each $t \in \mathrm{dom}(y)$ if $y(t) \in \{1, 1/2\}$ then t is β-like for some $\beta \in \alpha$.

Let us now find the value of $[\![\exists z(z \in y \wedge z \cap y = \varnothing)]\!]$ assuming $[\![y \subseteq u \wedge (y \neq \varnothing)]\!] \in \{1, 1/2\}$. Let

$$\gamma = \min\{\beta \in \mathrm{ORD} \mid \text{there exists } t \in \mathrm{dom}(y) \text{ such that } y(t) \in \{1, 1/2\} \text{ and } t \text{ is } \beta\text{-like}\}.$$

So there exists $t' \in \mathrm{dom}(y)$ such that $y(t') \in \{1, 1/2\}$ and t' is γ-like.

$$\begin{aligned}
&[\![\exists z(z \in y \wedge z \cap y = \varnothing)]\!] \\
&= [\![\exists z(z \in y \wedge \neg\exists w(w \in z \wedge w \in y))]\!] \\
&\geq [\![t' \in y \wedge \neg\exists w(w \in t' \wedge w \in y))]\!] \\
&= \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![t = t']\!]) \wedge (\bigvee_{w \in \mathrm{dom}(t')} (t'(w) \wedge [\![w \in y]\!]))^* \\
&\geq (y(t') \wedge [\![t' = t']\!]) \wedge [\bigvee_{w \in \mathrm{dom}(t')} (t'(w) \wedge \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![w = t]\!]))]^* \\
&\geq 1/2 \wedge [\bigvee_{w \in \mathrm{dom}(t')} (t'(w) \wedge \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![w = t]\!]))]^*.
\end{aligned}$$

Claim 4.3.2 $[\bigvee_{w \in \mathrm{dom}(t')} (t'(w) \wedge \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![w = t]\!]))]^* = 1.$

Proof. It is sufficient to prove that

$$\bigvee_{w \in \mathrm{dom}(t')} (t'(w) \wedge \bigvee_{t \in \mathrm{dom}(y)} (y(t) \wedge [\![w = t]\!])) = 0.$$

If t' is 0-like then the claim is proved immediately. If not, then assume there exists $w \in \mathrm{dom}(t')$ such that $t'(w) \in \{1, 1/2\}$. If possible let there exist $t \in \mathrm{dom}(y)$ such that both $y(t), [\![w = t]\!] \in \{1, 1/2\}$. By our assumption $y(t) \in \{1, 1/2\}$ implies t is β-like for some $\beta \in \alpha$.

Since $[\![w = t]\!] \in \{1, 1/2\}$ by Theorem 4.2.3 we have w is β-like. Again since t' is γ like and $t'(w) \in \{1, 1/2\}$ therefore $\beta \in \gamma$. So combining the above results it can be concluded that there exists $t \in \mathrm{dom}(y)$ such that $y(t) \in \{1, 1/2\}$ and t is β-like where $\beta < \gamma$, which contradicts the minimality of γ. Hence the claim is proved. $\square$

Therefore $[\![\exists z(z \in y \wedge z \cap y = \varnothing)]\!] \geq 1/2 \wedge 1 = 1/2$. This leads to the fact that for any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, if $[\![y \subseteq u \wedge (y \neq \varnothing)]\!] \in \{1, 1/2\}$ then $[\![\exists z(z \in y \wedge z \cap y = \varnothing)]\!] \in \{1, 1/2\}$; i.e.,

$$[\![\forall y(y \subseteq u \wedge \neg(y = \varnothing) \to \exists z(z \in y \wedge z \cap y = \varnothing))]\!] = 1.$$

Hence we can conclude $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{WO}_{\in}(u)$. $\square$

Combining (i) and (iii) of Lemma 4.3.1 the following theorem can be derived.

Theorem 4.3.3 *Let $\alpha \in \mathrm{ORD}$ and u be an α-like element in $\mathbf{V}^{(\mathrm{PS}_3)}$. Then $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{ORD}(u)$.*

Theorem 4.3.3 shows that any α-like element satisfies the classical definition of ordinal number. It is proved in Theorem 3.5.11 that the general Comprehension axiom scheme is not valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. On the other hand the general Comprehension axiom is a theorem in Weber's paraconsistent set theory [36, Theorem 3.3], hence as a consequence, the collection of all ordinals becomes a set. This fact leads us to the important question, whether the collection of elements which make the first order formula $\mathrm{ORD}(x)$ valid is a name of set in $\mathbf{V}^{(\mathrm{PS}_3)}$. The following theorem assures the answer is negative.

Theorem 4.3.4 *There is no set of all ordinals:*

$$\mathbf{V}^{(\mathrm{PS}_3)} \not\models \exists O\, \forall x(\mathrm{ORD}(x) \to x \in O).$$

Proof. Let $O \in \mathbf{V}^{(\mathrm{PS}_3)}$ be arbitrarily chosen. Then by definition, $\mathrm{dom}(O)$ is a set in $\mathbf{V}$. By Theorem 4.2.3, if $\alpha \neq \beta$ for any $\alpha, \beta \in \mathrm{ORD}$ then for any α-like u and β-like v, $\mathbf{V}^{(\mathrm{PS}_3)} \not\models u = v$. Hence u and v are not equal as a set in $\mathbf{V}$. We conclude that if for each $\alpha \in \mathrm{ORD}$ there exists an α-like u in $\mathrm{dom}(O)$ then $\mathrm{dom}(O)$ cannot be a set in $\mathbf{V}$ as the collection of all ordinals is not a set in $\mathbf{V}$. Hence there exists an $\alpha' \in \mathrm{ORD}$ such that there is no α'-like element in $\mathrm{dom}(O)$. Let u be an α'-like element. Then by Theorem 4.3.3, $[\![\mathrm{ORD}(u)]\!] \in \{1, 1/2\}$ but

$$[\![u \in O]\!] = \bigvee_{x \in \mathrm{dom}(O)} (O(x) \wedge [\![x = u]\!]) = 0.$$

Hence $[\![\forall x(\mathrm{ORD}(x) \rightarrow x \in O)]\!] = 0$. Since O is arbitrary we have

$$[\![\exists O\, \forall x(\mathrm{ORD}(x) \rightarrow x \in O)]\!] = 0.$$

So the theorem is proved. $\square$

4.4 Violation of Leibniz's law: indiscernibility of identicals

Before going to the main points, for simplicity, let us assume that D stands for the set $\{1, 1/2\}$, the designated set of PS_3. From now onwards D will be used for this designated set, unless otherwise stated.

Leibniz's law of *indiscernibility of identicals* can be expressed as an axiom scheme which intuitively states the following:

Let x and y be any two objects. If $x = y$ then for any formula $\varphi(t)$ having one free variable t, $\varphi(x) \leftrightarrow \varphi(y)$ is satisfied.

If φ is a formula let LL_φ stands for the formula

$$\forall x \forall y(x = y \to (\varphi(x) \leftrightarrow \varphi(y))).$$

Let us call a formula φ *Leibnizian* if LL_φ is valid. This is equivalent to saying that the class $[\varphi] := \{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\varphi(u)]\!] \in D\}$ forms a union of equivalence classes of the relation $\sim$ defined in Section 2.2.3, by using the Lemma 3.5.2(i). Typically (and classically), we say that a formula $\varphi(x)$ defines a *unique object* if

$$\forall x \forall y((\varphi(x) \wedge \varphi(y)) \to x = y)$$

is valid. In $\mathbf{V}^{(\mathrm{PS}_3)}$, this is equivalent to saying that $[\varphi]$ is contained in one equivalence class of the relation $\sim$. So, together if a formula φ is Leibnizian and defines a unique object, this would just mean that $[\varphi]$ is an equivalence class of $\sim$ in $\mathbf{V}^{(\mathrm{PS}_3)}$.

All NFF-formulas are Leibnizian. But there are formulas that are not Leibnizian. Consider the formula

$$\mathsf{Empty}_\exists(x) := \neg\, \exists y(y \in x).$$

Let us choose any non-zero $\alpha \in \mathrm{ORD}$. Fix any two α-like elements u and v as $\mathrm{ran}(u) = \{1/2\}$ and $\mathrm{ran}(v) = \{1\}$. Then clearly $[\![\mathsf{Empty}_\exists(u)]\!] = 1/2$. But it is easy to calculate that $[\![\mathsf{Empty}_\exists(v)]\!] = 0$. Since u and v both are α-like by Theorem 4.2.3 $[\![u = v]\!] = 1$. Hence we get the following derivation:

$$\begin{aligned}
[\![\mathsf{LL}_{\mathsf{Empty}_\exists}]\!] &= \bigwedge_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} [\![x = y \to (\mathsf{Empty}_\exists(x) \leftrightarrow \mathsf{Empty}_\exists(y))]\!] \\
&\leq [\![u = v \to (\mathsf{Empty}_\exists(u) \leftrightarrow \mathsf{Empty}_\exists(v))]\!] \\
&= 0.
\end{aligned}$$

This establishes that $\mathbf{V}^{(\mathrm{PS}_3)} \not\models \mathsf{LL}_{\mathsf{Empty}_\exists}$ and hence the formula $\mathsf{Empty}_\exists$ is not Leibnizian. This shows that Leibniz's law of indiscernible of identicals fails to be satisfied in $\mathbf{V}^{(\mathrm{PS}_3)}$.

The idea of the proof showing the failure of Leibniz's law of indiscernible of identicals in $\mathbf{V}^{(\mathrm{PS}_3)}$ can be generalised for some reasonable implication algebras, other than PS_3, as well. Let $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow, ^*)$ be an algebra extended by a unary operator * from a reasonable implication algebra $(A, \wedge, \vee, \mathbf{0}, \mathbf{1}, \Rightarrow)$. As we defined in Section 2.1.1, $\mathbb{A}$ is a reasonable implication-negation algebra. Let $D_{\mathbb{A}}$ be a filter of $\mathbb{A}$ taken as a designated set. We say that the matrix $(\mathbb{A}, D_{\mathbb{A}})$ is paraconsistent if there are formulas φ and ψ in the language, $\mathcal{L}_{(\mathbb{A}, D_{\mathbb{A}})}$ (say) corresponding to $(\mathbb{A}, D_{\mathbb{A}})$ such that $v(\varphi) \in D_{\mathbb{A}}$, $v(\neg\varphi) \in D_{\mathbb{A}}$ and $v(\psi) \notin D_{\mathbb{A}}$ (where v stands for a valuation function of propositional formulas in A). It is worthwhile to say that for any formula φ of $\mathcal{L}_{(\mathbb{A}, D_{\mathbb{A}})}$ and any valuation function v we have $v(\neg\varphi) = v(\varphi)^*$.

Clearly, if $(\mathbb{A}, D_{\mathbb{A}})$ is paraconsistent, then there must be some $d \in D_{\mathbb{A}}$ such that $d^* \in D_{\mathbb{A}}$. We say that such a d witnesses the paraconsistency of $(\mathbb{A}, D_{\mathbb{A}})$.

Definition. The designated set $D_{\mathbb{A}}$ of an implication-negation algebra $(\mathbb{A}, D_{\mathbb{A}})$ is said to be a *reasonable designated set* if

(*i*) $1^* \notin D_{\mathbb{A}}$, and

(*ii*) if $a \in D_{\mathbb{A}}$ and $b \notin D_{\mathbb{A}}$ then $a \Rightarrow b \notin D_{\mathbb{A}}$.

Observation 4.4.1 *Let $(\mathbb{A}, D_{\mathbb{A}})$ be a reasonable implication-negation algebra where $D_{\mathbb{A}}$ satisfies the condition (ii) of a reasonable designated set. If there are two elements $u, v \in \mathbf{V}^{(\mathbb{A})}$, and a formula $\varphi(x)$ having one free variable x, in the language of $\mathbf{V}^{(\mathbb{A})}$ such that $[\![u = v]\!] = a$, and $([\![\varphi(u)]\!] \Rightarrow [\![\varphi(v)]\!]) = b$, where $a \in D_{\mathbb{A}}$ but $b \notin D_{\mathbb{A}}$, then LL_φ fails to be valid in $\mathbf{V}^{(\mathbb{A})}$.*

Theorem 4.4.2 *Let $(\mathbb{A}, D_{\mathbb{A}})$ be a reasonable implication-negation algebra where $D_{\mathbb{A}}$ is a reasonable designated set. If $(\mathbb{A}, D_{\mathbb{A}})$ is paraconsistent then there exists a formula φ such that LL_φ is not valid in $\mathbf{V}^{(\mathbb{A})}$.*

Proof. Let $(\mathbb{A}, D_{\mathbb{A}})$ be paraconsistent. Then, there exists $d \in D_{\mathbb{A}}$ which witnesses the paraconsistency of $(\mathbb{A}, D_{\mathbb{A}})$. Let $u_1, u_d \in \mathbf{V}^{(\mathbb{A})}$ be such that $\mathrm{dom}(u_1) = \mathrm{dom}(u_d)$ whereas $\mathrm{ran}(u_1) = \{1\}$ and $\mathrm{ran}(u_d) = \{d\}$. By definition we know that

$$[\![u_1 = u_d]\!] = \bigwedge_{x \in \mathrm{dom}(u_1)} (u_1(x) \Rightarrow [\![x \in u_d]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(u_d)} (u_d(y) \Rightarrow [\![y \in u_1]\!]).$$

Then from the first conjunct the following can be derived:

$$\begin{aligned}
\bigwedge_{x \in \mathrm{dom}(u_1)} (u_1(x) \Rightarrow [\![x \in u_d]\!]) &= \bigwedge_{x \in \mathrm{dom}(u_1)} (1 \Rightarrow \bigvee_{y \in \mathrm{dom}(u_d)} (u_d(y) \wedge [\![x = y]\!])) \\
&= \bigwedge_{x \in \mathrm{dom}(u_1)} (1 \Rightarrow d) \\
&= 1 \Rightarrow d.
\end{aligned}$$

Since $d \in D_{\mathbb{A}}$, by Proposition 2.1.3 we have $1 \Rightarrow d \in D_{\mathbb{A}}$. On the other hand

$$\begin{aligned}
\bigwedge_{y \in \mathrm{dom}(u_d)} (u_d(y) \Rightarrow [\![y \in u_1]\!]) &= \bigwedge_{y \in \mathrm{dom}(u_d)} (d \Rightarrow \bigvee_{x \in \mathrm{dom}(u_1)} (u_1(x) \wedge [\![y = x]\!]) \\
&= \bigwedge_{y \in \mathrm{dom}(u_d)} (d \Rightarrow 1) \\
&= d \Rightarrow 1 \\
&= 1
\end{aligned}$$

by Proposition 2.1.1. Hence we get $[\![u_1 = u_d]\!] \in D_{\mathbb{A}}$.

Now consider the formula $\mathsf{Empty}_{\exists}(x)$. Then

$$\begin{aligned}
[\![\mathsf{Empty}_{\exists}(u_1)]\!] &= [\![\neg \exists y (y \in u_1)]\!] \\
&= (\bigvee_{y \in \mathbf{V}^{(\mathbb{A})}} [\![y \in u_1]\!])^*
\end{aligned}$$

$$= (\bigvee_{y \in \mathbf{V}^{(\mathbb{A})}} \bigvee_{x \in \mathrm{dom}(u_1)} (u_1(x) \wedge [\![x = y]\!]))^*$$
$$= 1^*.$$

Similarly we can derive $[\![\mathsf{Empty}_\exists(u_d)]\!] = d^*$. Hence the following holds

$$[\![\mathsf{Empty}_\exists(u_1) \leftrightarrow \mathsf{Empty}_\exists(u_d)]\!] = (1^* \Rightarrow d^*) \wedge (d^* \Rightarrow 1^*).$$

Since $d \in D_\mathbb{A}$ witnesses paraconsistency, $d^* \in D_\mathbb{A}$ too. So, by the definition of a reasonable designated set, $d^* \Rightarrow 1^* \notin D_\mathbb{A}$. Since

$$(1^* \Rightarrow d^*) \wedge (d^* \Rightarrow 1^*) \leq d^* \Rightarrow 1^* \notin D_\mathbb{A}$$

it can be concluded that $(1^* \Rightarrow d^*) \wedge (d^* \Rightarrow 1^*) \notin D_\mathbb{A}$, otherwise $D_\mathbb{A}$ would not be a designated set.

Combining the above results we get

$$[\![u_1 = u_d \rightarrow (\mathsf{Empty}_\exists(u_1) \leftrightarrow \mathsf{Empty}_\exists(u_d))]\!] \notin D_\mathbb{A}$$

since $D_\mathbb{A}$ is a reasonable designated set. Hence it can be derived that $[\![\mathsf{LL}_{\mathsf{Empty}_\exists}]\!] \notin D_\mathbb{A}$ which means $\mathsf{LL}_{\mathsf{Empty}_\exists}$ is not valid in $\mathbf{V}^{(\mathbb{A})}$. □

In set theory, we define many objects by formulas that "define unique objects". In our paraconsistent set theory, this is problematic since if the formula is not Leibnizian, then there could be things that are equal in $\mathbf{V}^{(\mathrm{PS}_3)}$, but do not satisfy the formula. So, we need to make sure that our defining formulas are Leibnizian as well. We are now interested in defining natural number-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. In classical set theory ZFC, the formulas defining the natural numbers are formulas that define unique objects. In our paraconsistent set theory we need this property as well. But the same formula used in ZFC to define a natural number may not work in our paraconsistent set theory as it may not be Leibnizian and define unique objects. As an example we may consider the formula

$$\mathsf{Empty}_\forall(x) := \forall y \, \neg(y \in x).$$

We know that in ZFC, the formula $\mathsf{Empty}_\forall$ defines uniquely the natural number 0. But in Observation 4.4.3 it is proved that $\mathsf{Empty}_\forall$ does not define unique objects in $\mathbf{V}^{(\mathrm{PS}_3)}$, which formally says that there exist $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ so that $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Empty}_\forall(u) \wedge \mathsf{Empty}_\forall(v)$ but $\mathbf{V}^{(\mathrm{PS}_3)} \not\models u = v$. Before going to the proof of this, the following is an important issue to discuss. The two formulas $\mathsf{Empty}_\forall(x)$ and $\mathsf{Empty}_\exists(x)$ are equivalent in ZFC. Not only that, we can prove $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{Empty}_\forall(x) \leftrightarrow \mathsf{Empty}_\exists(x)$ as well. But there may exist a reasonable implication-negation algebra $\mathbb{A}$ so that $\mathsf{Empty}_\forall(x) \leftrightarrow \mathsf{Empty}_\exists(x)$ is not valid in $\mathbf{V}^{(\mathbb{A})}$.

Observation 4.4.3 *The collection* $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathsf{Empty}_\forall(u)]\!] \in \mathrm{D}\}$ *contains all 0-like elements together with some other elements which are not 0-like.*

Proof. Let us take any element $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ and then fix $v = \{(u, 1/2)\}$ and $w = \{(u, 0)\}$. Then $v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ and it can be derived that $[\![\mathsf{Empty}_\forall(v)]\!]$, $[\![\mathsf{Empty}_\forall(w)]\!] \in \mathrm{D}$. By definition w is a 0-like element whereas v is not and hence Theorem 4.2.3 says $\mathbf{V}^{(\mathrm{PS}_3)} \not\models v = w$. □

The closed formula $\mathsf{Empty}_\forall(v)$ containing the name corresponding to the element $v \in \mathbf{V}^{(\mathrm{PS}_3)}$ in the proof of Observation 4.4.3 is such that both $[\![\mathsf{Empty}_\forall(v)]\!]$ and $[\![\neg\mathsf{Empty}_\forall(v)]\!]$ are $1/2$. This is the reason why the formula $\mathsf{Empty}_\forall(x)$ having one free variable x is not defining unique object. We shall now use the idea of "de-paraconsistification" shown in Section 3.6.2. Consider the formula

$$\mathrm{Nat}_0(x) := \forall y \forall z (y = x \to z \notin y).$$

It is clear that for any $u \in \mathbf{V}^{(\mathrm{PS}_3)}$, if $[\![\mathsf{Empty}_\forall(u)]\!] = 1/2$ then $[\![\mathrm{Nat}_0(u)]\!] = 1$. Hence $\mathrm{Nat}_0(x)$

is a $[\![\cdot]\!]$-de-paraconsistification of the formula $\mathsf{Empty}_{\forall}(x)$. In Proposition 4.4.4 we will show that the formula $\mathrm{Nat}_0(x)$ is not only Leibnizian but also define unique objects.

Proposition 4.4.4 *The collection* $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Nat}_0(u)]\!] \in \mathrm{D}\}$ *consists of all 0-like elements only.*

Proof. Let u be a 0-like element i.e., $\mathrm{ran}(u) = \{0\}$. Hence

$$[\![\mathrm{Nat}_0(u)]\!] = \bigwedge_{y \in \mathbf{V}^{(\mathrm{PS}_3)}} \bigwedge_{z \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![y = u]\!] \Rightarrow [\![z \in y]\!]^*).$$

Take any $v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$. Since u is 0-like, $[\![v = u]\!] \in \mathrm{D}$ iff v is also 0-like. This ensures that $[\![w \in v]\!]^* = 1$. Hence $[\![v = u]\!] \Rightarrow [\![w \in v]\!]^* = 1 \in \mathrm{D}$. Again if $[\![v = u]\!] \notin \mathrm{D}$ then by the table of $\Rightarrow$ in PS_3 we directly get $[\![v = u]\!] \Rightarrow [\![w \in v]\!]^* = 1 \in \mathrm{D}$. Therefore in any case $[\![\mathrm{Nat}_0(u)]\!] = 1 \in \mathrm{D}$.

Conversely let $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ be such that $[\![\mathrm{Nat}_0(u)]\!] \in \mathrm{D}$. We shall prove that u is 0-like. If possible let u be not 0-like. Let us consider $v \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $\mathrm{dom}(v) = \mathrm{dom}(u)$ and if for some $x \in \mathrm{dom}(u)$, $u(x) \in \mathrm{D}$ then $v(x) = 1$. By our assumption there exists $w \in \mathrm{dom}(u)$ such that $u(w) \in \mathrm{D}$; so $v(w) = 1$ too. Clearly $[\![v = u]\!] = 1 = [\![w \in v]\!]$. Hence

$$[\![\mathrm{Nat}_0(u)]\!] \leq [\![v = u]\!] \Rightarrow [\![w \in v]\!]^* = 0$$

which is not the case. As a conclusion it can be said that u is a 0-like element. $\square$

4.5 Natural Numbers

4.5.1 Natural numbers and their properties

Inspired by Observation 4.4.3 and Proposition 4.4.4, in this section, we shall identify formulas which can uniquely define natural number-like elements.

Below we shall generalise the result of Proposition 4.4.4. Like $\mathrm{Nat}_0(x)$, for each $n \in \omega$ we shall recursively provide a formula $\mathrm{Nat}_n(x)$ whose instances are only n-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. Let us define

$\mathrm{Nat}_{n+1}(x) := \exists x_0\ \exists x_1 \ldots \exists x_n[(\mathrm{Nat}_0(x_0) \wedge\ \ldots\ \wedge \mathrm{Nat}_n(x_n)) \wedge$
$\quad (x_0 \in x\ \wedge\ \ldots\ \wedge\ x_n \in x)\ \wedge\ \forall y(y \in x \to (y = x_0\ \vee\ \ldots\ \vee\ y = x_n))]$

for each $n \in \omega$. The intuitive idea of defining the formula $\mathrm{Nat}_{n+1}(x)$ is that x is $n+1$-like iff there exist $x_0, x_1, \ldots, x_n$ such that (x_0 is 0-like, x_1 is 1-like, …, x_n is n-like) and ($x_0, x_1 \ldots, x_n \in x$) and (for any $y \in x$, one of $y = x_0$ or $y = x_1$ or … or $y = x_n$ holds).

Proposition 4.5.1 *For each $n \in \omega$, $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Nat}_{n+1}(u)]\!] \in \mathrm{D}\}$ is the collection of all $n+1$-like elements.*

Proof. We shall prove the proposition using mathematical induction.

Base step: Let us consider the case for $n = 0$. We have to prove that the 1-like elements are the only instances of the formula

$$\mathrm{Nat}_1(x) = \exists x_0\ [\mathrm{Nat}_0(x_0)\ \wedge\ x_0 \in x\ \wedge\ \forall y\ (y \in x \to y = x_0)].$$

Hence we have $[\![\mathrm{Nat}_1(x)]\!] \in \mathrm{D}$ iff there exists $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that all of $[\![\mathrm{Nat}_0(u)]\!]$, $[\![u \in x]\!]$, and $[\![\forall y\ (y \in x \to y = u)]\!]$ belong to D.

(i) Using Proposition 4.4.4 it can be said $[\![\mathrm{Nat}_0(u)]\!] \in \mathrm{D}$ iff u is 0-like.

(ii) Now $[\![u \in x]\!] \in \mathrm{D}$ i.e., $\bigvee_{t \in \mathrm{dom}(x)}(x(t) \wedge [\![t = u]\!]) \in \mathrm{D}$ iff there exists $t \in \mathrm{dom}(x)$ such that $x(t) \in \mathrm{D}$ and $[\![t = u]\!] \in \mathrm{D}$ iff $x(t) \in \mathrm{D}$ and t is 0-like, using theorem 4.2.3.

(iii) Lastly $[\![\forall y\ (y \in x \rightarrow y = u)]\!] \in \mathrm{D}$ i.e., $\bigwedge_{t \in \mathrm{dom}(x)}(x(t) \Rightarrow [\![t = u]\!]) \in \mathrm{D}$ iff for each $t \in \mathrm{dom}(x)$ if $x(t) \in \mathrm{D}$ then $[\![t = u]\!] \in \mathrm{D}$ also i.e., t is 0-like.

Hence combining (i), (ii), and (iii) it can be concluded that $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Nat}(1, u)]\!] \in \mathrm{D}\}$ is the collection of all 1-like elements.

Induction hypothesis: Let the proposition be true for all natural numbers less than $m \in \omega - \{0\}$.

Induction step: We shall prove the proposition for the natural number m. Hence we have to prove $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Nat}_{m+1}(u)]\!] \in \mathrm{D}\}$ is the collection of all $m+1$-like elements. Since

$$\begin{aligned}\mathrm{Nat}_{m+1}(x) := \exists x_0\ \exists x_1 \ldots \exists x_m[(\mathrm{Nat}_0(x_0) \wedge\ \ldots\ \wedge \mathrm{Nat}_m(x_m)) \wedge \\ (x_0 \in x\ \wedge\ \ldots\ \wedge\ x_m \in x)\ \wedge\ \forall y(y \in x \rightarrow (y = x_0\ \vee\ \ldots\ \vee\ y = x_m))],\end{aligned}$$

$[\![\mathrm{Nat}_{m+1}(x)]\!] \in \mathrm{D}$ iff there exist $u_0, u_1, \ldots, u_m \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that

(i) $[\![\mathrm{Nat}_0(u_0)]\!]$, $[\![\mathrm{Nat}_1(u_1)]\!]$, $\ldots$, $[\![\mathrm{Nat}_m(u_m)]\!] \in \mathrm{D}$,

(ii) $[\![u_0 \in x]\!]$, $[\![u_1 \in x]\!]$, $\ldots$, $[\![u_m \in x]\!] \in \mathrm{D}$, and

(iii) $[\![\forall y(y \in x \rightarrow (y = u_0\ \vee\ \ldots\ \vee\ y = u_m))]\!] \in \mathrm{D}$.

By induction hypothesis (i) assures that each u_k is a k-like element where $k \in \{0, 1, 2, \ldots, m\}$. From (ii) it can be concluded that for each $k \in \{0, 1, 2, \ldots, m\}$ there exists a k-like element, say v_k, in $\mathrm{dom}(x)$ such that $x(v_k) \in \mathrm{D}$. Condition (iii) describes for any $y \in \mathrm{dom}(x)$ if $x(y) \in \mathrm{D}$ then y is k-like for some $k \in \{0, 1, 2, \ldots, m\}$. Hence the proposition is true for m also. Therefore by the principle of mathematical induction the proposition is true for all $n \in \omega$. $\square$

Propositions 4.4.4 and 4.5.1 together produce the fact: for each $n \in \omega$, $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Nat}_n(u)]\!] \in \mathrm{D}\}$ is the collection of all n-like elements. Therefore, this collection represents an equivalence class with respect to $\sim$ in $\mathbf{V}^{(\mathrm{PS}_3)}$. Hence we can say for each $n \in \omega$ the formula $\mathrm{Nat}_n(x)$ is Leibnizian and define unique objects.

In the next few lemmas and theorems we shall discuss some properties about natural number-like elements and ω-like elements. It will be proved that if u is an ω-like element then $v \in u$ is true in $\mathbf{V}^{(\mathrm{PS}_3)}$ iff v is some natural number-like element.

Intuitively the following lemma says, if $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ satisfies the two conditions, (i) an element belongs to I implies its *successor* (in classical sense) also belongs to I and (ii) there does not exist any m-like element (where $m \in \omega - 0$) in I; then I does not contain any $(m-1)$-like element also.

Lemma 4.5.2 *Let $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ be an element which satisfies the following two conditions:*

(i) $[\![\forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)]\!] \in D$ and

(ii) for an arbitrary $m \in \omega - \{0\}$ there does not exist any m-like element $v \in \mathrm{dom}(I)$ such that $I(v) \in D$.

Then there does not exist any $(m-1)$-like element $u \in \mathrm{dom}(I)$ as well such that $I(u) \in D$.

Proof. If possible let u be an $(m-1)$-like element such that $u \in \mathrm{dom}(I)$ and $I(u) \in D$. We have
$[\![\forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)]\!]$
$\leq I(u) \Rightarrow [\![\exists s \forall y(y \in s \leftrightarrow y \in u \vee y = u) \wedge s \in I]\!]$.
Condition (i) and $I(u) \in D$ together implies

$$[\![\exists s \forall y(y \in s \leftrightarrow y \in u \vee y = u) \wedge s \in I]\!] \in D$$

i.e., there exists $v \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that

$$[\![\forall y(y \in v \leftrightarrow y \in u \vee y = u) \wedge v \in I]\!] \in D$$

which implies $[\![\forall y(y \in v \leftrightarrow y \in u \vee y = u)]\!] \wedge [\![v \in I]\!] \in D$. The first conjunct belongs to D means v is m-like (using Theorems 4.2.4 and 4.2.3). The second conjunct belongs to D implies $I(v) \in D$. This violates the condition (ii). Hence the proof is complete. □

Let us now consider the following three formulas

(i) $\mathrm{Ind}(I) := \exists e(\mathrm{Nat}_0(e) \wedge e \in I) \wedge \forall x(x \in I \rightarrow \exists s \forall y(y \in s \leftrightarrow y \in x \vee y = x) \wedge s \in I)$,

(ii) $\mathrm{Nat}(x) := \forall I(\mathrm{Ind}(I) \rightarrow x \in I)$, and

(iii) $\mathrm{SetNat}(w) := \forall x(x \in w \leftrightarrow \mathrm{Nat}(x))$.

Intuitively the formula $\mathrm{Ind}(I)$ says that I is *inductive*, $\mathrm{Nat}(x)$ states that x is a natural number which belongs to every inductive set, the formula $\mathrm{SetNat}(w)$ says that w contains of all natural numbers, provided all the three formulas are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Lemma 4.5.3 *If* $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Ind}(I)$ *for some* $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ *then for each natural number* n *there exists an* n*-like element* $u \in \mathrm{dom}(I)$ *such that* $I(u) \in D$.

Proof. Let us consider an $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Ind}(I)$ i.e., both the conjuncts of $\mathrm{Ind}(I)$ are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. Hence from the first conjunct we get there exists a 0-like element in $\mathrm{dom}(I)$ having image in D. Since the second conjunct is also valid, by Lemma 4.5.2 it can be concluded that for a natural number m in $\mathbf{V}$ if there exists an m-like element $u \in \mathrm{dom}(I)$ such that $I(u) \in D$ then there exists an $m+1$-like element $v \in \mathrm{dom}(I)$ such that $I(v) \in D$. Hence by the meta-induction on natural numbers in $\mathbf{V}$ the proof is complete. □

Theorem 4.5.4 *For any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$, $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{Nat}(x)$ iff x is n-like for some natural number n.*

Proof. Let u be an n-like element for some $n \in \omega$, where ω is the set of all natural numbers in $\mathbf{V}$. We get

$$\begin{aligned}
[\![\mathrm{Nat}(u)]\!] &= [\![\forall I(\mathrm{Ind}(I) \to u \in I)]\!] \\
&= \bigwedge_{I \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![\mathrm{Ind}(I)]\!] \Rightarrow [\![u \in I]\!]) \\
&\in D
\end{aligned}$$

since for each $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ if $[\![\mathrm{Ind}(I)]\!] \in D$ then by Lemma 4.5.3 we get $[\![u \in I]\!] \in D$ also.

Conversely let $[\![\mathrm{Nat}(x)]\!] \in D$. Hence for each $I \in \mathbf{V}^{(\mathrm{PS}_3)}$ if $[\![\mathrm{Ind}(I)]\!] \in D$ then $[\![x \in I]\!] \in D$. We shall show that x is a natural number-like element. Consider an ω-like element $u \in \mathbf{V}^{(\mathrm{PS}_3)}$. Then $[\![\mathrm{Nat}(u)]\!] \in D$ is immediate. But if x is not a natural number-like element using Lemma 4.5.3 we get $[\![x \in I]\!] \notin D$. Hence x should be some natural number-like element and the theorem is proved. $\square$

An immediate question should be what is the successor of an n-like element for some natural number n? In the next lemma we shall show that in $\mathbf{V}^{(\mathrm{PS}_3)}$ the successor of an n-like element will be an $(n+1)$-like element.

Lemma 4.5.5 *For any $n \in \omega$,*

$$\mathbf{V}^{(\mathrm{PS}_3)} \models \forall x(\mathrm{Nat}_n(x) \wedge \forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x) \to \mathrm{Nat}_{n+1}(y)).$$

Proof. Let us take an n-like element x for some $n \in \omega$. Now let

$$[\![\forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x)]\!] \in D.$$

From the first conjunct we get, $[\![\forall y \forall z(z \in y \to z \in x \vee z = x)]\!] \in D$ which implies for any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![\forall z(z \in y \to z \in x \vee z = x)]\!] \in D$ i.e.,

$$\bigwedge_{z \in \mathrm{dom}(y)} (y(z) \Rightarrow [\![z \in x]\!] \vee [\![z = x]\!]) \in D$$

i.e., for any $z \in \mathrm{dom}(y)$ if $y(z) \in D$ then either z is some m-like where $m < n$ or z is n-like.

The second conjunct gives $[\![\forall y \forall z(z \in x \vee z = x \to z \in y)]\!] \in D$. This implies for any $y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![\forall z(z \in x \vee z = x \to z \in y)]\!] \in D$ i.e.,

$$\bigwedge_{z \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![z \in x]\!] \vee [\![z = x]\!] \Rightarrow [\![z \in y]\!]) \in D,$$

i.e., for any $z \in \mathbf{V}^{(\mathrm{PS}_3)}$ which is either m-like for some $m < n$ or n-like, there exists $t \in \mathrm{dom}(y)$ which is either m-like or n-like, respectively, such that $y(t) \in D$.

Combining the above two derivations we can say if x is any n-like element and $[\![\forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x)]\!] \in D$ then by definition y is $(n+1)$-like. Hence

$$\bigwedge_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![\mathrm{Nat}_n(x)]\!] \wedge [\![\forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x)]\!] \Rightarrow [\![\mathrm{Nat}_{n+1}(y)]\!]) \in D$$

which implies

$$\mathbf{V}^{(\mathrm{PS}_3)} \models \forall x(\mathrm{Nat}_n(x) \wedge \forall y \forall z(z \in y \leftrightarrow z \in x \vee z = x) \to \mathrm{Nat}_{n+1}(y)).$$

$\square$

Theorem 4.5.6 *For any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$, $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{SetNat}(x)$ iff x is an ω-like element.*

Proof. Let u be an ω-like element. Then

$$\begin{aligned}
[\![\mathrm{SetNat}(u)]\!] &= [\![\forall x(x \in u \leftrightarrow \mathrm{Nat}(x))]\!] \\
&= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\mathrm{Nat}(x)]\!]) \wedge \bigwedge_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![\mathrm{Nat}(x)]\!] \Rightarrow [\![x \in u]\!]) \\
&\in D
\end{aligned}$$

using Theorems 4.5.4 and 4.2.4.

Conversely let $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathrm{SetNat}(u)$ i.e.,

$$\bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![\mathrm{Nat}(x)]\!]) \wedge \bigwedge_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![\mathrm{Nat}(x)]\!] \Rightarrow [\![x \in u]\!]) \in D.$$

We shall show u is ω-like. Let n be any natural number. For any n-like element x Theorem 4.5.4 says that $[\![\mathrm{Nat}(x)]\!] \in D$, hence from the second conjunct of our assumption $[\![x \in u]\!] \in D$ also. Using Theorem 4.2.3 it can be derived that $\mathrm{dom}(u)$ contains an n-like element x' (say) so that $u(x') \in D$. Again from the first conjunct of our assumption it can be said that, if there exists an element $y \in \mathrm{dom}(u)$ so that it is not a natural number-like element then by Theorem 4.5.4 $[\![\mathrm{Nat}(y)]\!] \notin D$ and hence $u(y) \notin D$ also. By definition u becomes an ω-like element. $\square$

Following the principle of *mathematical induction* in classical set theory we can intuitively think about the same principle in $\mathbf{V}^{(\mathrm{PS}_3)}$ as follows: for any two names $x, y \in \mathbf{V}^{(\mathrm{PS}_3)}$ if x is an ω-like element, y is a subset of x in $\mathbf{V}^{(\mathrm{PS}_3)}$, and y is inductive then $x = y$ holds in $\mathbf{V}^{(\mathrm{PS}_3)}$. Let us now consider the formula:

$$\forall x \forall y(\mathrm{SetNat}(x) \wedge y \subseteq x \wedge \mathrm{Ind}(y) \rightarrow x = y). \tag{MI}$$

In the following theorem we shall prove that mathematical induction holds in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Theorem 4.5.7 *The formula* MI *is valid in* $\mathbf{V}^{(\mathrm{PS}_3)}$ *i.e.,* $\mathbf{V}^{(\mathrm{PS}_3)} \models \mathsf{MI}$.

Proof. Consider any $x, y \in \mathbf{V}^{(\mathrm{PS}_3)}$ so that $[\![\mathrm{SetNat}(x) \wedge y \subseteq x \wedge \mathrm{Ind}(y)]\!] \in D$. We shall prove $[\![x = y]\!] \in D$ also. By our assumption $[\![\mathrm{SetNat}(x)]\!] \in D$, which shows that x is an ω-like element by Theorem 4.5.6. From the second conjunct we have $[\![y \subseteq x]\!] \in D$ which implies

$$[\![\forall t(t \in y \to t \in x)]\!] = \bigwedge_{t \in \mathrm{dom}(y)} (y(t) \Rightarrow [\![t \in x]\!]) \in D.$$

Since x is ω-like we get, if there exists $t \in \mathrm{dom}(y)$ which is not a natural number-like element then $u(t) \notin D$. From the third conjunct of our assumption $[\![\mathrm{Ind}(y)]\!] \in D$. Hence using Lemma 4.5.2 and above discussion we can say that y is also an ω-like element.

Hence by Theorem 4.2.3 we get $[\![x = y]\!] \in D$ and the theorem is proved. □

Once we have Theorem 4.5.7 the statement of Proposition 4.5.1 can be generalised for all successor ordinals less than $\omega + \omega$ as follows: let us define for each ordinal $\alpha < \omega + \omega$ the formula $\mathrm{Ord}_{\alpha+1}(u)$ as

$$\exists x(\mathrm{Ord}_\alpha(x) \wedge x \in u \wedge x \subseteq u \wedge \forall z(z \in u \to z = x \vee z \in x))$$

where $\mathrm{Ord}_0(x) = \mathrm{Nat}_0(x)$ and $\mathrm{Ord}_\omega(x) = \mathrm{SetNat}(x)$. Using this formula we get the following proposition:

Proposition 4.5.8 *For each ordinal* $\alpha < \omega + \omega$, $\{u \in \mathbf{V}^{(\mathrm{PS}_3)} : [\![\mathrm{Ord}_{\alpha+1}(u)]\!] \in \mathrm{D}\}$ *is the collection of all* $(\alpha + 1)$*-like elements.*

Proof. For any ordinal $\alpha < \omega + \omega$, $[\![\mathrm{Ord}_{\alpha+1}(u)]\!] \in \mathrm{D}$ iff there is $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that the following hold:

(*i*) $[\![\mathrm{Ord}_\alpha(x)]\!] \in \mathrm{D}$, (*ii*) $[\![x \in u]\!] \in \mathrm{D}$,

(*iii*) $[\![x \subseteq u]\!] \in \mathrm{D}$, and

(*iv*) $[\![\forall z(z \in u \to z = x \vee z \in x]\!] \in \mathrm{D}$.

From (*i*) we get x is α-like. Since x is α-like (*ii*) shows that there exists an element $x' \in \mathrm{dom}(u)$ which is α-like. Using the abbreviation for $x \subseteq u$, (*iii*) provides $[\![\forall y(y \in x \to y \in u)]\!] \in \mathrm{D}$, i.e.,

$$\bigwedge_{y \in \mathrm{dom}(x)} (x(y) \Rightarrow [\![y \in u]\!]) \in D.$$

Hence for each $\beta < \alpha$ if $y \in \mathrm{dom}(x)$ is such that y is β-like and $x(y) \in D$ then $[\![y \in u]\!] \in D$. It implies there exists some $y' \in \mathrm{dom}(u)$ such that y' is β-like and $u(y') \in D$. From (*iv*) we get

$$\bigwedge_{z \in \mathrm{dom}(u)} (u(z) \Rightarrow [\![z \in x]\!] \vee [\![z = x]\!]) \in D.$$

Hence if $z \in \mathrm{dom}(u)$ be such that it is not β-like for some $\beta \leq \alpha$ then $[\![z \in x]\!] \vee [\![z = x]\!] = 0$ which forces $u(z)$ to be 0.

Hence by definition u satisfies all the conditions of an $(\alpha + 1)$-like element and this completes the proof. $\square$

We have provided a theory of nice names for natural number objects in Section 4.5.1. If we try to generalise this to ordinals, we will need to deal with the limit step as well. For the case of the limit ordinal ω, the analysis of the property of being inductive provided the corresponding nice names (Theorem 4.5.6); this, in turn, allowed us to extend the inductive analysis beyond the first limit ordinal up to $\omega + \omega$ (Proposition 4.5.8). Similarly, one can formulate appropriate analyses for further limit ordinals and then continue beyond them with very similar proofs.

4.5.2 Addition in natural number-like elements

In the classical set theory we can think the addition $+$ between natural numbers as $\langle n, m, k\rangle \in +$ iff there are two subsets A and B of the natural number k such that $A \cup B = k$, $A \cap B = \varnothing$ and A is bijective with n, B is bijective with m. Following the above definition of $+$ between two natural numbers we shall define '$+$' on natural number-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ as a ternary relation which satisfies the condition: $\langle u, v, w\rangle \in +$ iff u is an m-like element, v is an n-like element and w is an $(m+n)$-like element. From now onwards whenever we shall talk about $n+m$ where $n, m \in \omega$ we shall mean the *usual addition* of two natural numbers in $\mathbf{V}$. Before the definition we have to understand what can be inferred from the validity of the formulas representing 'subset of an ordinal-like element', 'disjoint sets', 'partition of a set', and 'bijection between two sets' in $\mathbf{V}^{(\mathrm{PS}_3)}$. In some of the following observations and lemmas we shall discuss about it.

Let us consider an n-like element u where $n \in \omega$. Then $[\![x \subseteq u]\!] \in D$ implies $[\![\forall t(t \in x \to t \in u)]\!] \in D$ i.e., $\bigwedge_{t \in \mathrm{dom}(x)} (x(t) \Rightarrow [\![t \in u]\!]) \in D$, which implies for any $t \in \mathrm{dom}(x)$ if $x(t) \neq 0$ then $[\![t \in u]\!] \in D$. Hence we get

Observation 4.5.9 *If $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ is n-like for some $n \in \omega$ and $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ then $[\![x \subseteq u]\!] \in D$ iff for any $t \in \mathrm{dom}(x)$ if $x(t) \neq 0$ then t is m-like for some $m < n$.*

Definition. Let $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any two arbitrary names. Then u is a name for a proper subset of v in $\mathbf{V}^{(\mathrm{PS}_3)}$, if $\mathbf{V}^{(\mathrm{PS}_3)} \models (u \subseteq v) \wedge (v \nsubseteq u)$, which is denoted by $u \subsetneq_{\mathbf{V}^{(\mathrm{PS}_3)}} v$.

Observation 4.5.10 *For any $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$, $u \subsetneq_{\mathbf{V}^{(\mathrm{PS}_3)}} v$ if and only if*

(*i*) *for each $t \in \mathrm{dom}(u)$, if $u(t) \in D$ then there exists $t' \in \mathrm{dom}(v)$ such that both of $v(t')$, $[\![t = t']\!] \in D$, and* (*ii*) *there exists* $s \in \mathrm{dom}(v)$ *such that* $v(s) \in D$ *but for any* $s' \in dom(u)$ *either* $u(s') = 0$ *or* $[\![s = s']\!] = 0$.

Proof. Given that for $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$, $u \subsetneq_{\mathbf{V}^{(\mathrm{PS}_3)}} v$, which implies

$$\mathbf{V}^{(\mathrm{PS}_3)} \models \forall t(t \in u \to t \in v) \;\wedge\; \neg\forall s(s \in v \to s \in u).$$

By definition we can say that the condition (*i*) holds if and only if $\forall t(t \in u \to t \in v) \in D$. On the other hand

$$[\![\neg\forall s(s \in v \to s \in u)]\!] \in D \text{ if and only if } \bigwedge_{s\in\mathrm{dom}(v)} (v(s) \Rightarrow [\![s \in u]\!]) = 0,$$

since the truth table of $\Rightarrow$ does not contain $1/2$ in PS_3. This leads to the fact that condition (*ii*) holds if and only if $[\![\neg\forall s(s \in v \to s \in u)]\!] \in D$. $\square$

Next we shall discuss, if $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ then for which conditions on u and v, $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cap v = \varnothing$. The corresponding formula for $u \cap v = \varnothing$ is $\neg\exists x(x \in u \wedge x \in v)$. Now $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cap v = \varnothing$ means $[\![\neg\exists x(x \in u \wedge x \in v)]\!] \in D$ i.e., $[\![\exists x(x \in u \wedge x \in v)]\!]^* \in D$ which implies

$$\Big(\bigvee_{x\in\mathbf{V}^{(\mathrm{PS}_3)}} ([\![x \in u]\!] \wedge [\![x \in v]\!])\Big)^* \in D.$$

Observation 4.5.11 *In particular let* u, v *both are subsets of some* k*-like element in* $\mathbf{V}^{(\mathrm{PS}_3)}$*, where* $k \in \omega$*. Now from the above derivation it can be said that* $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cap v = \varnothing$ *iff for any* $x \in \mathbf{V}^{(\mathrm{PS}_3)}$*,* $[\![x \in u]\!] \wedge [\![x \in v]\!] < 1$ *iff either* x *is not* m*-like for any* $m < k$ *or if* x *is* m*-like for some* $m < k$ *then one of the following three cases holds:*

(*i*) *there do not exist any elements in* $\mathrm{dom}(u)$ *and* $\mathrm{dom}(v)$ *which are* m*-like,* (*ii*) *if there exists* $t \in \operatorname{dom}(u)$ *such that* t *is* m*-like and* $u(t) = 1$ *then either there does not exist any* m*-like element in* $\operatorname{dom}(v)$ *or if exists then its image under* v *belongs to* $\{0, 1/2\}$*. The same thing will happen if* u *and* v *interchange there positions, and*

(*iii*) *there exist* $t \in \operatorname{dom}(u)$ *and* $t' \in \operatorname{dom}(v)$ *such that both* t *and* t' *are* m*-like and* $u(t) = v(t') = 1/2$*.*

Let w be k-like for some $k \in \omega$ and $[\![u \subseteq w]\!], [\![v \subseteq w]\!] \in D$ then we shall discuss when we can say $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cup v = w$. By the definition we know that for any two elements $x, y \in \mathbf{V}^{(\mathrm{PS}_3)}$, $\mathbf{V}^{(\mathrm{PS}_3)} \models (x \subseteq y) \wedge (y \subseteq x)$ iff $\mathbf{V}^{(\mathrm{PS}_3)} \models x = y$. Hence we shall show when $\mathbf{V}^{(\mathrm{PS}_3)} \models (u \cup v \subseteq w) \wedge (w \subseteq u \cup v)$, i.e.,

$$[\![u \cup v \subseteq w]\!],\ [\![w \subseteq u \cup v]\!] \in D.$$

We already have,

$$\begin{aligned} [\![u \cup v \subseteq w]\!] &= [\![\forall x(x \in u \vee x \in v \rightarrow x \in w)]\!] \\ &= \bigwedge_{x \in \mathbf{V}^{(\mathrm{PS}_3)}} ([\![x \in u]\!] \vee [\![x \in v]\!] \Rightarrow [\![x \in w]\!]) \\ &\in D \end{aligned}$$

using Observation 4.5.9 and our assumption, i.e., $[\![u \cup v \subseteq w]\!] \in D$ is always true in this case. Therefore the required condition only depends on the fact $[\![w \subseteq u \cup v]\!] \in D$. Now $[\![w \subseteq u \cup v]\!] \in D$ implies

$$[\![\forall t(t \in w \rightarrow t \in u \vee t \in v)]\!] \in D,$$

i.e.,

$$\bigwedge_{t \in \operatorname{dom}(w)} (w(t) \Rightarrow [\![t \in u]\!] \vee [\![t \in v]\!]) \in D$$

which leads to the fact that for any $t \in \mathrm{dom}(w)$ if $w(t) \neq 0$ then either $[\![t \in u]\!] \in D$ or $[\![t \in v]\!] \in D$. Since for any $n < k$, there exists $t \in \mathrm{dom}(w)$ such that $w(t) \neq 0$ we get for each $n < k$ there exists an n-like element in $\mathrm{dom}(u)$ or in $\mathrm{dom}(v)$ whose corresponding image under u or v is in D. The converse also holds in this case. Hence we get the following observation.

Observation 4.5.12 *Let $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be such that w is k-like for some $k \in \omega$ and $\mathbf{V}^{(\mathrm{PS}_3)} \models (u \subseteq w) \wedge (v \subseteq w)$. Then $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cup v = w$ iff*

(i) for any $n < k$ there exists $x \in \mathrm{dom}(u) \cup \mathrm{dom}(v)$ such that x is n like and its corresponding image under u or v is in D and

(ii) for any $y \in \mathrm{dom}(u) \cup \mathrm{dom}(v)$ if y is m-like for some $m > k$ then its corresponding image under u or v is 0.

The property $\mathbf{V}^{(\mathrm{PS}_3)} \models (u \subseteq w) \wedge (v \subseteq w)$ is not necessary for Observation 4.5.12:

Observation 4.5.13 *Let $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ where w is k-like for some $k \in \omega$. Then $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cup v = w$ iff properties (i) and (ii) of Observation 4.5.12 are satisfied.*

Combining Observations 4.5.11 and 4.5.13 we get the following lemma.

Lemma 4.5.14 *Let $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ be k-like for some $k \in \omega$. Then for any $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$, $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cap v = \varnothing$ and $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cup v = w$ hold together iff*

(i) for each $n < k$ there exists $x \in \mathrm{dom}(u) \cup \mathrm{dom}(v)$ such that x is n-like and its corresponding image under u or v is in D,

(ii) if for any $n < k$ there exists $x \in \mathrm{dom}(u)$ such that x is n-like and $u(x) = 1$ then $v(x) \in \{0, 1/2\}$ (same will hold even if u and v interchange their places), and

(*iii*) *for any* $m > k$ *if there exists* $y \in \mathrm{dom}(u) \cup \mathrm{dom}(v)$ *such that* y *is* m*-like then its corresponding image under* u *or* v *is 0.*

In classical set theory for any three non-empty sets A, B, C the conditions $A = B \cup C$ and $B \cap C = \varnothing$ stands for the fact that A is partitioned by B and C. In particular if A is a natural number then B and C are in bijection with two natural numbers which are strictly less than A. Though the bijection is not defined yet, intuitively it does not hold in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Observation 4.5.15 *From Lemma 4.5.14 it can be observed, for any* $k \in \omega$ *there are* $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$ *such that all of them are* k*-like but still* $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cap v = \varnothing$ *and* $\mathbf{V}^{(\mathrm{PS}_3)} \models u \cup v = w$ *hold: just by setting* $\mathrm{ran}(u), \mathrm{ran}(v) \subseteq \{0, {}^1\!/_2\}$.

Now we shall define the notion when two names $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ represent *sets of same size*, where $\mathbf{V}$ is chosen to be a standard model of classical set theory including the *Axiom of Choice*. Let $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any arbitrary element. Then clearly $\mathrm{dom}(u)$ is a set in $\mathbf{V}$. We know that the identity relation $\sim$ is an equivalence relation in $\mathbf{V}^{(\mathrm{PS}_3)}$. Hence $\sim$ leads to a partition $\mathscr{S}$ in $\mathrm{dom}(u)$. Let $\mathscr{S}' \subseteq \mathscr{S}$ be such that for any $\bar{x} \in \mathscr{S}'$ there exists $a \in \bar{x}$ such that $u(a) \in D$ and for any $\bar{y} \in \mathscr{S} - \mathscr{S}'$ and any $b \in \bar{y}$, $u(b) = 0$. Since the axiom of choice holds in $\mathbf{V}$ the set $\mathscr{S}'$ can be well-ordered, and let α be the ordinal number having the same order type with $\mathscr{S}'$ in $\mathbf{V}$. Let us construct a set U in $\mathbf{V}$ such that for each $\beta < \alpha$ there exists exactly one β-like element in U and nothing else. Let $\mathrm{WO}_u \in \mathbf{V}^{(\mathrm{PS}_3)}$ be such that $\mathrm{dom}(\mathrm{WO}_u) = U$ and $\mathrm{ran}(\mathrm{WO}_u) = \{1\}$. Clearly WO_u is an α-like element.

Definition. Two elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ are said to have the same order type with respect to $\mathbf{V}^{(\mathrm{PS}_3)}$ if $[\![\mathrm{WO}_u = \mathrm{WO}_v]\!] \in D$.

Definition. Two elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ represent *sets of same size* if there exists a bijection between the two sets $\mathrm{dom}(\mathrm{WO}_u)$ and $\mathrm{dom}(\mathrm{WO}_v)$ in $\mathbf{V}$.

From the above two definitions we get the following important observation.

Observation 4.5.16 *Let $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be such that WO_u and WO_v be n-like and m-like elements respectively for some $n, m \in \omega$. Then u and v have the same order type with respect to $\mathbf{V}^{(\mathrm{PS}_3)}$ iff they represent sets of same size iff $n = m$ in $\mathbf{V}$.*

Let us now consider the following definition:

Definition. Let us take any $w \in \mathbf{V}^{(\mathrm{PS}_3)}$. Then w is called *partitioned by two elements u and v* in $\mathbf{V}^{(\mathrm{PS}_3)}$, if there exist $A, B \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that

(*i*) $\mathbf{V}^{(\mathrm{PS}_3)} \models (A \cap B = \varnothing) \wedge (A \cup B = w)$, and

(*ii*) A and B represent sets of same sizes as u and v, respectively.

From the condition (*i*) of the definition it is immediate that $\mathbf{V}^{(\mathrm{PS}_3)} \models (A \subseteq w) \wedge (B \subseteq w)$.

The definition of addition, $+$ between the natural number-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ should be such that if u and v are n-like and m-like elements respectively for some $n, m \in \omega$ then, $\langle u, v, w\rangle \in +$ iff w is an $(n+m)$-like element, where the $+$ between n and m is the usual addition in ω. But if we define the $+$ in between two natural number-like elements of $\mathbf{V}^{(\mathrm{PS}_3)}$ as follows: for two n-like and m-like elements $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$, where $n, m \in \omega$, $\langle u, v, w\rangle \in +$ iff w is partitioned by u and v in $\mathbf{V}^{(\mathrm{PS}_3)}$; then w may not be an $(n+m)$-like element.

Justification: Let $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be respectively 3-like and 5-like elements. If the definition of $+$ would be taken as above then we could prove that $\langle u, v, w\rangle \in +$ iff w is 5-like or 6-like or 7-like or 8-like element: let us consider $x_i, y_j \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that x_i is i-like and y_j is j-like elements where $0 \leq i \leq 7$ and $0 \leq j \leq 2$. Construct two subsets w_1 and w_2 of w in $\mathbf{V}^{(\mathrm{PS}_3)}$ so that the above claim hold.

Case I: Let $\mathrm{dom}(w_1) = \{y_0,\ y_1,\ y_2\}$, $\mathrm{dom}(w_2) = \{x_0,\ x_1, \ldots, x_4\}$ and $\mathrm{ran}(w_1) = \mathrm{ran}(w_2) = \{1/2\}$. Assume $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ is such that $\mathrm{dom}(w) = \mathrm{dom}(w_1) \cup \mathrm{dom}(w_2)$ and

$\text{ran}(w) = 1/2$. Then by Observation 4.5.9 and Lemma 4.5.14 we can derive

$$\mathbf{V}^{(\mathrm{PS}_3)} \models (w_1 \subseteq w) \wedge (w_2 \subseteq w) \wedge (w_1 \cap w_2 = \varnothing) \wedge (w_1 \cup w_2 = w).$$

By definition it can be said that w_1 and w_2 are of same size as u and v, respectively. Hence $\langle u, v, w\rangle$ would be a member of $+$. Note that in this example w is a 5-like element.

Case II: Every thing is as similar as Case I except the fact that here $\text{dom}(w_2) = \{x_1,\ x_2, \ldots, x_5\}$. Then by the same argument we would get $\langle u, v, w\rangle \in +$ but w is a 6-like element.

Case III & Case IV: These two cases are also same as Case I except $\text{dom}(w_2) = \{x_2,\ x_3, \ldots, x_6\}$ and $\text{dom}(w_2) = \{x_3,\ x_4, \ldots, x_7\}$ in the two cases respectively. Clearly in both the cases we could get $\langle u, v, w\rangle \in +$ where w is a 7-like element in Case III and a 8-like element in Case IV.

For avoiding the above problem we shall provide the definition of $+$ in between natural number-like elements having some extra condition.

Definition. Let $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any two natural number-like elements. Then $\langle u, v, w\rangle \in +$ where w is a k-like element for some $k \in \omega$ iff w is partitioned by u and v in $\mathbf{V}^{(\mathrm{PS}_3)}$ and if there exists $w' \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that w' can also be partitioned by u and v in $\mathbf{V}^{(\mathrm{PS}_3)}$ where w' is p-like element for some $p \in \omega$ then $p \leq k$.

The next theorem shows that the definition of $+$ fulfil our expectation.

Theorem 4.5.17 *If $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ are any two n-like and m-like elements respectively for some $n, m \in \omega$ then $\langle u, v, w\rangle \in +$ iff w is an $(n+m)$-like element.*

Proof. Let $u, v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be any two n-like and m-like elements where $n, m \in \omega$. Let w be an $(n+m)$-like element in $\mathbf{V}^{(\mathrm{PS}_3)}$. Now let us consider for each natural number i where

$0 \leq i \leq n+m-1$, an i-like element x_i in $\mathbf{V}^{(\mathrm{PS}_3)}$. Take two subsets A and B of w in $\mathbf{V}^{(\mathrm{PS}_3)}$ as $\mathrm{dom}(A) = \{x_0,\ x_1,\ \ldots,\ x_{n-1}\}$ and $\mathrm{dom}(B) = \{x_n,\ x_{n+1},\ \ldots,\ x_{n+m-1}\}$ and $\mathrm{ran}(A) = \mathrm{ran}(B) = \{1\}$. Clearly

$$\mathbf{V}^{(\mathrm{PS}_3)} \models (A \cap B = \varnothing) \wedge (A \cup B = w)$$

and A and B represent sets of same size as u and v, respectively. It is now sufficient to show that if w' is k-like for some natural number $k > n+m$ then $\langle u, v, w' \rangle \notin +$. It is immediate as for any two subsets A' and B' of w' in $\mathbf{V}^{(\mathrm{PS}_3)}$ we know that $\mathrm{dom}(A') \cup \mathrm{dom}(B')$ contains i-like elements for each $i = 0,\ 1,\ \ldots,\ k-1$. Then clearly A' is of same size as u and B' is of same size as v cannot hold simultaneously.

The converse direction can also be proved by the above arguments using the definition of $+$ in between two natural-number like elements. □

Observation 4.5.18 *From the definition it can be derived that if for three natural number-like elements $u, v, w \in \mathbf{V}^{(\mathrm{PS}_3)}$, $\langle u, v, w \rangle \in +$ then $\langle v, u, w \rangle \in +$ too.*

Next we shall discuss about the replica of *Cantor's theorem* in $\mathbf{V}^{(\mathrm{PS}_3)}$. In $\mathbf{V}$ Cantor's theorem is the following:
If $A \in \mathbf{V}$ and B is the power set of A then there does not exist any bijection between A and B but there exists a proper subset C of B so that there exist a bijection between A and C.
Since the axiom Power Set is valid in $\mathbf{V}^{(\mathrm{PS}_3)}$, for any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$ there exists $y \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that

$$[\![\forall t(t \subseteq x \leftrightarrow t \in y)]\!] \in D.$$

In this case we say y *is a name for the power set of* x. For the simplicity we shall write $x \subseteq_{\mathbf{V}} y$ to demonstrate that x is a subset of y in $\mathbf{V}$.

Lemma 4.5.19 *For an element $u \in \mathbf{V}^{(\mathrm{PS}_3)}$, if $v \in \mathbf{V}^{(\mathrm{PS}_3)}$ be such that v is a name for the power set of u then*

- (i) *corresponding to each subset (in classical sense) of* $\mathrm{dom}(\mathrm{WO}_u)$ *in* $\mathbf{V}$ *there exists an element in* $\mathrm{dom}(v)$ *whose image under v belongs to D i.e., if $A \subseteq_{\mathbf{V}}$* $\mathrm{dom}(\mathrm{WO}_u)$ *then there exists an element $x_A \in \mathrm{dom}(v)$ such that $v(x_A) \in D$, and*
- (ii) *if $x_A, y_B \in \mathrm{dom}(v)$ where $A, B \subseteq_{\mathbf{V}} \mathrm{dom}(\mathrm{WO}_u)$ and $A \neq B$ in* $\mathbf{V}$ *then* $[\![x_A = y_B]\!] = 0$.

Proof. Let $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ be an arbitrary element. Then by saying $[\![\forall x(x \subseteq u \leftrightarrow x \in v)]\!] \in D$ we mean that both of $[\![\forall x(x \subseteq u \rightarrow x \in v)]\!]$ and $[\![\forall x(x \in v \rightarrow x \subseteq u]\!]$ are in D, i.e.,

$$[\![\forall x(\forall t(t \in x \rightarrow t \in u) \rightarrow x \in v)]\!] \in D,$$

$$[\![\forall x(x \in v \rightarrow \forall t(t \in x \rightarrow t \in u))]\!] \in D.$$

From the first condition, for any $x \in \mathbf{V}^{(\mathrm{PS}_3)}$, $[\![\forall t(t \in x \rightarrow t \in u)]\!] \in D$ iff for any $t \in \mathrm{dom}(x)$ if $x(t) \in D$ then there exists some $t' \in \mathrm{dom}(u)$ such that $[\![t = t']\!] \in D$ and $u(t') \in D$. If the above holds then $[\![x \in v]\!] \in D$ i.e., there exists $x' \in \mathrm{dom}(v)$ such that both $[\![x = x']\!]$, $v(x') \in D$. From the second condition it can be said that

$$\bigwedge_{x \in \mathrm{dom}(v)} (v(x) \Rightarrow [\![x \subseteq u]\!]) \in D$$

i.e., for any $x \in \mathrm{dom}(v)$ if $v(x) \in D$ then for any $t \in \mathrm{dom}(x)$ if $x(t) \in D$ then there exists some $t' \in \mathrm{dom}(u)$ such that $[\![t = t']\!] \in D$ and $u(t') \in D$. Hence combining the above results we can say that the proof is complete. □

Using Lemma 4.5.19 we can get the replica of Cantor's theorem in $\mathbf{V}^{(\mathrm{PS}_3)}$.

Theorem 4.5.20 *If Cantor's theorem holds in* $\mathbf{V}$ *then, for any name* $u \in \mathbf{V}^{(\mathrm{PS}_3)}$ *if* v *is a name for the power set of* u*, then* u *and* v *do not represent sets of the same size. Furthermore, there is some name* $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ *where* $w \subsetneq_{\mathbf{V}^{(\mathrm{PS}_3)}} v$ *such that* u *and* w *represent sets of the same size.*

Proof. By Lemma 4.5.19 it can be concluded, if v is a name for a power set of u then there exists a bijection between $\mathrm{dom}(\mathrm{WO}_v)$ and the power set of $\mathrm{dom}(\mathrm{WO}_u)$ in $\mathbf{V}$. Hence by using Cantor's theorem of classical set theory it can be said that there is no bijection between $\mathrm{dom}(\mathrm{WO}_u)$ and $\mathrm{dom}(\mathrm{WO}_v)$ in $\mathbf{V}$. Therefore we can say that u and v do not represent sets of same size.

Let us now consider an element $w \in \mathbf{V}^{(\mathrm{PS}_3)}$ such that $\mathrm{dom}(w) = \{t_A \in \mathrm{dom}(v) : A \text{ is a singleton subset of } \mathrm{dom}(\mathrm{WO}_u) \text{ in } \mathbf{V} \text{ and } v(t_A) \in D\}$ and $\mathrm{ran}(w) = \{1\}$. By our construction, the condition (i) of Observation 4.5.10 is satisfied. Now let us take $B \subseteq_{\mathbf{V}} \mathrm{WO}_u$ where B is not singleton with respect to $\mathbf{V}$ and the corresponding element $t_B \in \mathrm{dom}(v)$ so that $v(t_B) \in D$. Since v is a name for the power set of u by definition $[\![t_B = t_A]\!] = 0$ for any $t_A \in \mathrm{dom}(w)$. Hence the condition (ii) of Observation 4.5.10 is also satisfied. As a conclusion we can say that $w \subsetneq_{\mathbf{V}^{(\mathrm{PS}_3)}} v$. Besides, by the construction we can say that u and w represent sets of same size. Hence the proof is complete. $\square$

Summary of the thesis and some issues which can be followed up as future work

The thesis consists of four chapters. The summary of the chapters are provided below.

Chapter 1

This chapter presents a brief overview of classical set theory, and some non-classical set theories. The definition of paraconsistent logic and its history is presented, and an overview of the difference between some of the existing paraconsistent set theories and the paraconsistent set theory discussed in this thesis, is delivered.

Chapter 2

The notion of implication algebra is defined in this chapter, followed by the two important notions reasonable implication algebra and deductive reasonable implication algebra. The construction of the Boolean-valued model for the classical ZFC is briefly stated. Then it is elaborately shown how the idea of Boolean-valued model construction can be generalised to a deductive reasonable implication-algebra valued model of the corresponding set theory. It is proved that some of the set theoretic results which are true in the classical ZFC may not be true for some given reasonable implication algebra. For example the transitivity of equality fails in the reasonable implication algebra $Ł_3$, the three-valued Łukasiewicz algebra. The notion of BQ_φ is introduced in this chapter which plays a very important role through out the thesis.

Chapter 3

A deductive reasonable implication algebra, PS_3 is defined in this chapter. It is mentioned that out of many other choices why PS_3 is chosen as a reasonable implication algebra. The propositional logic $\mathbb{L}\mathrm{PS}_3$ is developed which is proved to be sound and (weak) complete with respect to PS_3. The logic $\mathbb{L}\mathrm{PS}_3$ is shown to be maximal with respect to classical propositional logic (CPL). Then some comparisons between the logic $\mathbb{L}\mathrm{PS}_3$ and some existing paraconsistent logics are provided. It is mentioned that a predicate extension of $\mathbb{L}\mathrm{PS}_3$ is possible, and the details can be found in [10]. There exists formula φ for which BQ_φ does not hold in $\mathbf{V}^{(\mathrm{PS}_3)}$. But BQ_φ is valid in $\mathbf{V}^{(\mathrm{PS}_3)}$ for every negation free formula φ. Using this fact it is proved that the axioms and axiom schemas Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation, NFF-Collection, and NFF-Foundation are valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. In the algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$ we can prove that the Comprehension principle is not valid. There is no universal set in $\mathbf{V}^{(\mathrm{PS}_3)}$. It is proved that there are set theoretic formulas viz., $\mathsf{Paracon}_\exists$, $\mathsf{Paracon}_\forall$, so that $\mathbf{V}^{(\mathrm{PS}_3)} \models (\mathsf{Paracon}_\exists \wedge \neg\mathsf{Paracon}_\exists)$, and $\mathbf{V}^{(\mathrm{PS}_3)} \models (\mathsf{Paracon}_\forall \wedge \neg\mathsf{Paracon}_\forall)$, hold. The chapter ends with an idea of de-paraconsistification of a formula in $\mathbb{L}\mathrm{PS}_3$.

Chapter 4

In this chapter we have defined α-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ for each ordinal number α in $\mathbf{V}$. The ordinal-like elements satisfy the first order formula, which represents the ordinal numbers in ZFC. Some properties of ordinal numbers in ZFC are proved to be true for the ordinal-like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$ as well. It is proved that there is no set containing all ordinal like elements in $\mathbf{V}^{(\mathrm{PS}_3)}$. In $\mathbf{V}^{(\mathrm{PS}_3)}$ the Leibniz's law of indiscernibility of identicals is violated. Then we have investigated the natural numbers in $\mathbf{V}^{(\mathrm{PS}_3)}$ and defined addition. It is proved that the mathematical induction is valid in $\mathbf{V}^{(\mathrm{PS}_3)}$. The thesis ends with the proof of Cantor's theorem on Powersets.

Some issues to be investigated in future

Let us define a unary operator $\neg^c$ in $\mathbb{L}\mathrm{PS}_3$ as $\neg^c\varphi := (\varphi \to \bot)$. If $\neg^c$ is considered as the classical negation then it can be checked that one of the axioms

$$(\neg^c\psi \to \neg^c\varphi) \to (\varphi \to \psi)$$

of CPL is satisfied in the matrix PS_3. Hence the negation-free fragment of $\mathbb{L}\mathrm{PS}_3$ might be equivalent to CPL as the two axioms Ax1 and Ax2 of $\mathbb{L}\mathrm{PS}_3$ are also other two axioms of CPL. We have to resolve this issue. Though the set theory of $\mathbf{V}^{(\mathrm{PS}_3)}$ produces set theoretic formulas viz., $\mathsf{Paracon}_\exists$, $\mathsf{Paracon}_\forall$, so that $\mathbf{V}^{(\mathrm{PS}_3)} \models (\mathsf{Paracon}_\exists \wedge \neg\mathsf{Paracon}_\exists)$, and $\mathbf{V}^{(\mathrm{PS}_3)} \models (\mathsf{Paracon}_\forall \wedge \neg\mathsf{Paracon}_\forall)$; it will be better to look for some other reasonable implication algebras whose propositional logic are different from the CPL.

The addition + between two natural number-like elements are defined in the algebra-valued model $\mathbf{V}^{(\mathrm{PS}_3)}$ using the meta language. But it is not discussed in the thesis how it can be expressed in the first order language of the paraconsistent set theory corresponding to $\mathbf{V}^{(\mathrm{PS}_3)}$. This issue is taken as one of the further development of this set theory.

In the development of the research work in this thesis we have come up with the notion of deductive reasonable implication algebra, $\mathbb{A}$. But we did not investigate what are the properties of the complementation operator of $\mathbb{A}$ are necessary to make $\mathbf{V}^{(\mathbb{A})}$ a model of ZF instead of $\mathrm{NFF} - \mathrm{ZF}$. This should be one of the important investigations for finding the algebra-valued models of non-classical set theories.

The thesis does not shed any light on the relations between the logic of an algebra $\mathbb{A}$ and the logic of its algebra-valued model $\mathbf{V}^{(\mathbb{A})}$. We are already working on this issue.

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