Aspects of Forcing in Descriptive Set Theory and Computability Theory

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Chapter 1

Introduction

The method of forcing was first introduced by Cohen in [Coh63, Coh64] to show that the continuum hypothesis is independent from ZFC. Now, sixty years later, forcing is one of the most important tools in set theory and is used in almost all areas of set theory; one area particularly affected by forcing is set theory of the reals: the study of set theoretic properties of the real number line and its definable subsets. In this work, we shall study three different aspects of forcing in this area with connections to descriptive set theory and computability theory.

First, in Chapter 2, entitled “Regularity properties for forcing notions not living on the reals”, we shall investigate regularity properties that are defined from forcing notions. Such regularity properties have been a major object of study in descriptive set theory (cf., e.g., [Sol70, IS89, BL99, BHL05, Ike10, BL11, Kho12]). However, most of the studied regularity properties are defined on the real line, or more specifically on Cantor space and Baire space. We shall investigate three regularity properties that are defined on other Polish spaces.

Second, in Chapter 3, entitled “Descriptive choice principles”, we shall construct symmetric submodels to separate descriptive fragments of the axiom of choice from each other. The construction of a symmetric submodel is essentially a forcing construction, but with an extra step of permuting the names. This extra step allows us to construct models in which the axiom of choice can fail. Symmetric submodels were also first introduced by Cohen in [Coh63, Coh64].

Third, in Chapter 4, entitled “Set-theoretic forcing notions in computability theory”, we shall leave the area of descriptive set theory and study forcing in the closely related area of computability theory. Forcing in computability theory was first introduced by Feferman in [Fef64], shortly after Cohen’s work was published. We shall investigate the relationships between some set-theoretic forcing notions in computability theory.

1.1 Summary of results

In this section, we shall describe how this thesis is organized and list the main results of each chapter. Note that most of the terms and notations we shall use in this section have not yet been introduced. The reader is not expected to understand them at this point. They will be defined later in this thesis and in each case, a reference is given to the definition.

In Chapter 2, we shall define a general framework for regularity properties. We shall then use this framework to study three different regularity properties. First, we shall study amoeba regularity,[1] which was first introduced by Judah and Repický in [JR95]. Our main result for amoeba regularity is the following corollary.

Corollary 2.3.20. *The following are equivalent:*

(a) *every* $\mathbf{\Sigma}^1_2(\mathbf{R})$ *set*[2] *is amoeba regular,*

(b) *for every* $r \in \omega^\omega$*, the set* $\{P \in \mathbf{R} : P$ *is not an amoeba real*[3] *over* $\mathrm{L}[r]\}$ *is* $\mathcal{C}_{\mathbb{A}}$*-meager,*[4] *and*

(c) *for every* $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Second, we shall define a topological space for amoeba forcing for category[5] and study its Baire property. Finally, we shall do the same for localization forcing.[6] The following corollary and theorem are our main results for these regularity properties.

Corollary 2.4.11. *The following are equivalent:*

(a) *every* $\mathbf{\Sigma}^1_2(\mathbf{U})$ *set has the Baire property in the* $\mathbb{UM}$*-topology,*[7]

(b) *for every* $r \in \omega^\omega$*, the set* $\{x \in \mathbb{UM} : x$ *is not a* $\mathbb{UM}$*-generic real over* $\mathrm{L}[r]\}$ *is meager in the* $\mathbb{UM}$*-topology, and*

(c) *for every* $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Theorem 2.5.10. *The following are equivalent:*

(a) *every* $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ *set has the Baire property in the localizing topology,*[8]

(b) *for every real* $r \in \omega^\omega$*, the set* $\{f \in \mathbf{Loc} : f$ *is not a* $\mathbb{LOC}$*-generic real over* $\mathrm{L}[r]\}$ *is meager in the localizing topology,*

(c) *for every real* $r \in \omega^\omega$*, the set* $\{f \in \mathbf{Loc} : f$ *is not a localizing real*[9] *over* $\mathrm{L}[r]\}$ *is meager in the localizing topology, and*

(d) *for every real* $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

In Chapter 3, we shall compare the consistency strength of descriptive choice principles, i.e., fragments of the axiom of choice which are defined using descriptive pointclasses. Such choice principles have already been studied by Kanovei in [Kan79]. Our main result is the following theorem which generalizes a separation theorem of Kanovei.

Theorem 3.2.10. *For every* $n \geq 1$*, there is a model of* $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.[10]

[1] For the definition, cf. p. 56.
[2] For the definition, cf. p. 56.
[3] For the definition, cf. Definition 2.3.1 & p. 56.
[4] For the definition, cf. p. 56.
[5] For the definition, cf. Definition 2.4.1.
[6] For the definition, cf. Definition 2.5.1.
[7] For the definition, cf. p. 68.
[8] For the definition, cf. p. 73.
[9] For the definition, cf. p. 72.
[10] For the definition, cf. Definitions 3.2.1, 3.2.4, & 3.2.6.

Moreover, using a compactness argument, we get the following corollary.

Corollary 3.2.11. *There is a model of* $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.

In Chapter 4, we shall study set-theoretic forcing notions in computability theory and compare their n-generic reals.[11] Although much can be transferred from set theory, there are also differences. For example Cholak, Dzhafarov, Hirst, and Slaman proved in [CDHS14] that Mathias n-generic reals compute Cohen n-generic reals. Miller proved in [Mil81] that the analogue does not hold in set theory. We shall generalize their result. In particular, we shall show that Laver n-generic reals compute Cohen n-generic reals.

Corollary 4.2.40. *Let* $n \geq 3$. *Then every Laver* n*-generic real computes a Cohen* n*-generic real.*

Remarks on co-authorship. *Parts of Chapter 2 are co-authored with Raiean Banerjee; Chapter 3 is co-authored with Ned Wontner; Chapter 4 is the sole work of the author. Details about the contribution to these chapters are given at the very beginning of each chapter.*

1.2 Preliminaries

1.2.1 Set theory

Our basic theory is Zermelo-Fraenkel set theory together with the axiom of choice ZFC. In some instances, we shall drop the axiom of choice and work in Zermelo-Fraenkel set theory ZF. More specifically, we shall work in ZFC in Chapters 1, 2, and 4 and in ZF in Chapter 3. Moreover, in Chapter 3, we shall also deal with models of set theory in which the power set axiom fails. We write ZF^- for ZF without the power set axiom and the collection scheme instead of the replacement scheme and ZFC^- for ZF^- together with the axiom of choice. For a discussion of set theory without the power set axiom, we refer the reader to [GHJ16].

We assume that the reader is familiar with basic concepts of set theory, such as functions, relations, ordinals, cardinals, the universe of all sets, etc. In addition, we assume a basic knowledge of elementary set-theoretic topology and mathematical logic and its role in the formalization of set theory. Our notation follows classical textbooks such as [Jec03] and [Kan03].

Let us fix some basic notation. We denote the universe of all sets by V and the class of all ordinals by Ord. A *sequence* is a function whose domain is an ordinal. We often call the domain of a sequence s its *length* and denote it by $\mathrm{lh}(s)$. Let $\langle A_\beta : \beta < \alpha\rangle$ and $\langle A'_\beta : \beta < \alpha'\rangle$ be sequences. We denote the *concatenation* by $\langle A_\beta : \beta < \alpha\rangle^\frown\langle A'_\beta : \beta < \alpha'\rangle$. If A is a set, then we set $\langle A_\beta : \beta < \alpha\rangle^\frown A := \langle A_\beta : \beta < \alpha\rangle^\frown\langle A\rangle$. Let S be a non-empty set and let α be an ordinal. We write S^α for the set of all sequences from S with length α and $S^{<\alpha}$ for the set of all sequences from S with length $< \alpha$. Moreover, we denote the set of all countably infinite subsets of S by $[S]^\omega$ and the set of all finite subsets of S by $[S]^{<\omega}$. The quantifiers $\exists^\infty$ and $\forall^\infty$ are short for "there are infinitely many" and "for all but finitely many", respectively.

Let S be a non-empty set. A set $I \subseteq \mathcal{P}(S)$ of subsets of S is an *ideal on* S if I is non-empty and is closed under subsets and finite unions. An ideal is a σ*-ideal* if it is additionally closed under countable unions. We say that an ideal is *proper* if it contains all singletons, but not S itself. Let I be an ideal on S and let $A \subseteq S$. We call A I*-small* if $A \in I$ and I*-positive* otherwise. A set $F \subseteq \mathcal{P}(S)$ of subsets of S is a *filter on* S if F is non-empty and is closed under supersets and

[11] For the definition, cf. Definition 4.1.1.

finite intersections. If F is a filter on S, then $\{S \setminus A : A \in F\}$ is an ideal on S and the other way around. A set $\mathcal{S} \subseteq \mathcal{P}(S)$ of subsets of S is an *algebra on S* if $\mathcal{S}$ is non-empty and is closed under complements and finite unions. An algebra is a *σ-algebra* if it is additionally closed under countable unions.

1.2.2 Choice principles

The axiom of choice AC states that every family $\mathscr{F}$ of non-empty sets has a choice function, i.e., a function $f : \mathscr{F} \to \bigcup \mathscr{F}$ such that for every $A \in \mathscr{F}$, $f(A) \in A$. In set theory, we often work with fragments of the axiom of choice so called *choice principles*. More precisely, a choice principle is a statement Φ such that $\mathsf{ZFC} \vdash \Phi$, but $\mathsf{ZF} \nvdash \Phi$. There are many different choice principles in the literature (cf. [HR98]). In the following, we introduce a few choice principles which will be important throughout this thesis.

The axiom of choice can be stratified into fragments by requiring the existence of a choice function only for certain families. Let X and Y be non-empty sets. We write $\mathsf{AC}_X(Y)$ for the statement "every family $\{A_x : x \in X\}$ of non-empty subsets of Y has a choice function". Then AC holds if and only if $\mathsf{AC}_X(Y)$ holds for every non-empty X and Y. The axiom of countable choice AC_ω states that $\mathsf{AC}_\omega(Y)$ holds for every non-empty Y.

Another well-known choice principle is the axiom of dependent choice DC. It states that for every non-empty set X and every total relation $R \subseteq X \times X$ (i.e., for every $x \in X$, there is a $y \in X$ such that $x \mathrel{R} y$), there is a sequence $\langle x_k : k \in \omega \rangle \in X^\omega$ such that for every $k \in \omega$, $x_k \mathrel{R} x_{k+1}$. Just as the axiom of choice, the axiom of dependent choice can be stratified into fragments. Let X be a non-empty set. We write $\mathsf{DC}(X)$ for the statement "for every total relation $R \subseteq X \times X$, there is a sequence $\langle x_k : k \in \omega \rangle \in X^\omega$ such that for every $k \in \omega$, $x_k \mathrel{R} x_{k+1}$". Then it is clear that DC holds if and only if $\mathsf{DC}(X)$ holds for every non-empty X. It is well-known that DC implies AC_ω.

Proposition 1.2.1 (ZF, Folklore)**.** *For every non-empty set X, $\mathsf{DC}(X)$ implies $\mathsf{AC}_\omega(X)$.*

Proof. Let X be a non-empty set and let $\{A_k : k \in \omega\}$ be a family of non-empty subsets of X. We define $X_0 := X \setminus \bigcup_{k\in\omega} A_k$, $X_{k+1} := A_k$ for every $k \in \omega$, and $R := \bigcup_{k\in\omega} X_k \times X_{k+1}$. Then R is a total relation on X. By assumption, there is a sequence $\langle x_k : k \in \omega \rangle \in X^\omega$ such that for every $k \in \omega$, $x_k \mathrel{R} x_{k+1}$. Without loss of generality, $x_0 \in X_0$. Then $f := \{(A_k, x_{k+1}) : k \in \omega\}$ is the desired choice function. $\square$

In this thesis, we follow the convention that all results are ZFC-results unless the theorem is specifically marked with a different axiom system in which case, they are theorems in this axiom system.

1.2.3 Real numbers and Polish spaces

The real numbers have always been of special interest in mathematics and set theory is no exception. There are many different approaches to define the real numbers. However, up to isomorphism, they all produce the same field. The two most common approaches use either equivalence classes of Cauchy sequences or Dedekind cuts to define the real numbers from the rationals. We call the set of all real numbers equipped with the topology generated by the open intervals with rational endpoints the *real line* and denote it by $\mathbb{R}$. Mathias forcing will be also denoted by $\mathbb{R}$, but it will always be clear from the context whether we are talking about the real line or Mathias forcing.

Compared to the natural or rational numbers, the real numbers are rather complicated objects when considered as sets. Here, it does not really matter whether we use equivalence classes of Cauchy sequences, Dedekind cuts, or something else to define the real line. For this reason set theorists often work with different spaces that share most properties with the real line, but whose elements are much less complicated as sets. Let X be a non-empty set. For every $s \in X^{<\omega}$ we define $[s] := \{x \in X^\omega : s \subseteq x\}$. Then $\{[s] : s \in X^{<\omega}\}$ is a topology base on X^ω. We call the topology which is generated by this set the *standard topology on* X^ω. Unless otherwise specified, we always assume that X^ω is equipped with the standard topology. The spaces ω^ω and 2^ω are called *Baire space* and *Cantor space*, respectively. They are not homeomorphic to each other, but both share many relevant and important structural and topological properties with the real line; e.g., they have the same cardinality and they have a countable topology base (see also Theorem 1.2.4). It should be noted, however, that neither the Baire space nor the Cantor space is homeomorphic to the real line. Nevertheless, it is common practice in set theory to work with the Baire and Cantor space instead of the real line since their elements are very simple objects. Following set-theoretic conventions, we shall call the elements of the Baire and Cantor space *reals*.

Most of the time in this thesis we shall be working with the reals. In Chapter 2, we shall also consider a more general class of topological space called *Polish spaces*. Before we define them, we need a couple of definitions. A set is *dense* in a topological space if it meets every non-empty open set. We say that a topological space is *separable* if it contains a countable dense set. A topological space X is *completely metrizable* if there is a metric d on X such that (X, d) is a complete metric space which has the same open sets as X. We say that a topological space is a *Polish space* if it is separable and completely metrizable. The Baire space, the Cantor space, and the real line are all Polish spaces. In the rest of this section, we state some basic facts about Polish spaces. For further information, we refer the reader to [Kec95].

Proposition 1.2.2.

(a) The (possibly countably infinite) Cartesian product of Polish spaces with the product topology is Polish.

(b) A subset of a Polish space with the induced subset topology is a Polish space if and only if it is a countable intersection of open sets.

Proof. Cf., e.g., [Kec95, Proposition 3.3 & Theorem 3.11]. $\square$

Being Polish is a very restrictive property. One can show that every Polish space is either countable or has the cardinality of the continuum. Moreover, two Polish spaces of the same cardinality are isomorphic in the following sense.

Definition 1.2.3. Let X and Y be Polish spaces. A function $f : X \to Y$ is called *Borel* if the preimage of every Borel set in Y (cf. Section 1.2.5) is Borel in X. A function f is a *Borel isomorphism* if f is bijective and both f and f^{-1} are Borel.

Theorem 1.2.4. *Two Polish spaces are Borel isomorphic if and only if they have the same cardinality. Moreover, any two uncountable Polish spaces are Borel isomorphic.*

Proof. Cf., e.g., [Kec95, Theorem 15.6]. $\square$

1.2.4 Trees

Trees play an important role in set theory. Let X be a non-empty set. A *tree* on X is a non-empty set $T \subseteq X^{<\omega}$ which is closed under initial segments, i.e., for every $t \in T$ and every $n < \mathrm{lh}(t)$, $t{\restriction}n \in T$. We call the elements of a tree *nodes*. A *branch* through a tree $T \subseteq X^{<\omega}$ is an $x \in X^\omega$ such that for every $n \in \omega$, $x{\restriction}n \in T$. We denote the set of all branches through T by $[T]$. Let T be a tree on X and let $t \in T$. We write T_t for the subtree $\{s \in T : s \subseteq t \text{ or } t \subseteq s\}$. An *immediate successor of* t *in* T is a node $t' \in T$ such that there is some $x \in X$ with $t' = t^\frown x$. We say that t is *splitting in* T if t has more than one immediate successor in T and *non-splitting in* T otherwise. We denote the set of all immediate successors of t in T by $\mathrm{succ}_T(t)$. The *stem* of T is the unique node of the smallest length which is splitting. We denote it by $\mathrm{stem}(T)$.

In this work we shall only consider trees on 2 or ω. The following kind of trees will play a more important role.

Definition 1.2.5.

(a) A tree T on 2 or ω is *pruned* if every node in T has an immediate successor in T.

(b) A tree T on 2 is *uniform* if for every $t, t' \in T$ with $\mathrm{lh}(t) = \mathrm{lh}(t')$ and every $i \in 2$, $t^\frown i \in T$ if and only if $t'^\frown i \in T$.

(c) A tree T on 2 or ω is *perfect* if every node in T has a successor in T which is splitting.

(d) A tree T on ω is *super-perfect* if every node in T has a successor in T which has infinitely many immediate successors.

Especially important for the Baire and Cantor space are pruned trees.

Proposition 1.2.6. *Let X be a non-empty set. A set $A \subseteq X^\omega$ is closed if and only if there is a pruned tree T on X such that $[T] = A$. Moreover, for every closed set, the pruned tree is unique.*

Proof. Cf., e.g., [Kec95, Proposition 2.4]. □

1.2.5 Borel sets and projective sets

Classical descriptive set theory is the study of well-definable subsets of the reals or more general topological spaces. It is based on the work of Borel, Lebesgue, Luzin, and Suslin in the early 20th century. The following is intended to provide a brief introduction. For a more detailed introduction, we refer the reader to [Kan03], [Kec95], or [Mos09]. Most of basic descriptive set theory does not require the full axiom of choice and can be carried out in $\mathsf{ZF} + \mathsf{DC}$ or even in $\mathsf{ZF} + \mathsf{AC}_\omega$. However, even some of the most basic theorems of descriptive set theory require some amount of the axiom of choice and are not provable in ZF. In Section 3.1, we shall point out which of the results in this section are still true in ZF and which are not. To avoid defining everything again in ZF, we make sure that all definitions in this section are still well-defined in ZF. For this purpose, we highlight each use of the axiom of choice in this section.

In a topological space, the most easily definable sets are the open sets. The open sets are closed under unions and finite intersections. But in general they are not closed under complements and countable intersections. By closing the open sets under complements and countable unions/intersections, we obtain the Borel sets. More formally, for a topological space X, the collection of *Borel sets in* X is the smallest σ-algebra containing the open sets in X. We denote the set of Borel sets in X by $\mathcal{B}(X)$. If X is clear from the context, then we sometimes omit X and write $\mathcal{B}$ instead of $\mathcal{B}(X)$. Every Borel set can be built from the open sets by repeatedly taking complements and countable unions. For every Borel set, we can count the minimal number of steps it takes to construct it. This defines a hierarchy on the Borel sets we call the *Borel hierarchy.*

Definition 1.2.7. Let X be a topological space. For an ordinal $\xi \geq 1$, we recursively define the classes $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$, and $\mathbf{\Delta}^0_\xi(X)$:

$$\begin{aligned}
\mathbf{\Sigma}^0_1(X) &:= \{O \subseteq X : O \text{ is open in } X\},\\
\mathbf{\Pi}^0_\xi(X) &:= \{X \setminus A : A \in \mathbf{\Sigma}^0_\xi(X)\},\\
\mathbf{\Sigma}^0_\xi(X) &:= \{\bigcup_{n\in\omega} A_n : \forall n \exists \xi_n < \xi (A_n \in \mathbf{\Pi}^0_{\xi_n}(X))\} \text{ if } \xi > 1, \text{ and}\\
\mathbf{\Delta}^0_\xi(X) &:= \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X).
\end{aligned}$$

We say that a subset of X is a $\mathbf{\Sigma}^0_\xi(X)$ *set,* $\mathbf{\Pi}^0_\xi(X)$ *set*, and $\mathbf{\Delta}^0_\xi(X)$ *set* if is an element of $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$, and $\mathbf{\Delta}^0_\xi(X)$, respectively. If X is clear from the context, then we sometimes omit X and write $\mathbf{\Sigma}^0_\xi$ instead of $\mathbf{\Sigma}^0_\xi(X)$ and similarly for $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$.

It is clear that a set $B \subseteq X$ is Borel in X if and only if there is some ordinal ξ such that B is $\mathbf{\Sigma}^0_\xi(X)$. In $\mathsf{ZF} + \mathsf{AC}_\omega$, one can show that $\mathcal{B}(X) = \mathbf{\Sigma}^0_{\omega_1}(X) = \mathbf{\Pi}^0_{\omega_1}(X)$. Moreover, if $Y \subseteq X$ is a subspace, then for every $\xi \geq 1$, $\mathbf{\Sigma}^0_\xi(Y) = \{B \cap Y : B \in \mathbf{\Sigma}^0_\xi(X)\}$ and similarly for $\mathbf{\Pi}^0_\xi(Y)$.

Theorem 1.2.8 ($\mathsf{ZF} + \mathsf{AC}_\omega$)**.** *Let X be an uncountable Polish space. For every $1 \leq \xi < \omega_1$, we have that $\mathbf{\Sigma}^0_\xi(X) \neq \mathbf{\Pi}^0_\xi(X)$ and $\mathbf{\Delta}^0_\xi(X) \subsetneq \mathbf{\Sigma}^0_\xi(X) \cup \mathbf{\Pi}^0_\xi(X) \subsetneq \mathbf{\Delta}^0_{\xi+1}(X)$.*

Proof. Cf., e.g., [Kec95, Theorem 22.4]. □

Proposition 1.2.9 ($\mathsf{ZF} + \mathsf{AC}_\omega$)**.** *Let X be a topological space. For every $\xi \geq 1$, the classes $\mathbf{\Sigma}^0_\xi(X)$, $\mathbf{\Pi}^0_\xi(X)$, and $\mathbf{\Delta}^0_\xi(X)$ are closed under finite unions and intersections and continuous preimages. Moreover, $\mathbf{\Sigma}^0_\xi(X)$ is closed under countable unions, $\mathbf{\Pi}^0_\xi(X)$ under countable intersections, and $\mathbf{\Delta}^0_\xi(X)$ under complements.*

Proof. Cf., e.g., [Kec95, Proposition 22.1]. □

The set of all Borel sets is closed under countable unions and intersections, complements, and continuous preimages. In $\mathsf{ZF} + \mathsf{AC}_\omega$, however, the Borel sets are not closed under continuous images. In fact, there are even continuous images of the Baire space which are not Borel.

Definition 1.2.10. Let X be a Polish space. A set $A \subseteq X$ is *analytic in X* if either A is empty or there is a continuous function $f : \omega^\omega \to X$ such that $f[\omega^\omega] = A$.

Theorem 1.2.11 ($\mathsf{ZF} + \mathsf{AC}_\omega$, Suslin)**.** *Let X be an uncountable Polish space. Then every Borel set in X is analytic in X, but not vice versa.*

Proof. Cf., e.g., [Kec95, Theorem 14.2]. □

There are many different equivalent ways to define analytic sets. One of the most common ones uses projections. Let X and Y be sets. The *projection onto X* is the function $\mathrm{proj}_X : X \to Y$ with $\mathrm{proj}_X(x, y) := x$. One can easily check that if X and Y are topological spaces, then proj_X is continuous. The *projection of a set $A \subseteq X \times Y$ onto X* is the set $\mathrm{proj}_X(A) := \{x \in X : \exists y \in Y (x, y) \in A\}$.

Proposition 1.2.12 ($\mathsf{ZF} + \mathsf{AC}_\omega$)**.** *Let X be a Polish space and let $A \subseteq X$. The following are equivalent:*

(a) A is analytic,

(b) there is a Polish space Y, a Borel set B in Y, and a continuous function $f : Y \to X$ such that $f[B] = A$,

(c) there is a closed set $F \subseteq X \times \omega^\omega$ such that $\mathrm{proj}_X(F) = A$, and

(d) there is a Polish space Y and a Borel set B in $X \times Y$ such that $\mathrm{proj}_X(B) = A$.

Proof. Cf., e.g., [Jec03, Lemma 11.6]. □

The analytic sets are not closed under complements. We call the complements of analytic sets *co-analytic*.

Theorem 1.2.13 ($\mathsf{ZF} + \mathsf{AC}_\omega$, Suslin)**.** *Let X be an uncountable Polish space. Then a set is Borel in X if and only if it is analytic and co-analytic in X.*

Proof. Cf., e.g., [Kec95, Theorem 14.11]. □

There is no reason to stop here. We can define a second hierarchy similar to the Borel hierarchy, using projections instead of countable unions. This hierarchy is called the *projective hierarchy*.

Definition 1.2.14. Let X be a Polish space. For a natural number $n \geq 1$, we recursively define the classes $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X)$:

$$\begin{aligned}
\mathbf{\Sigma}^1_1(X) &:= \{A \subseteq X : A \text{ is analytic in } X\},\\
\mathbf{\Pi}^1_n(X) &:= \{X \setminus A : A \in \mathbf{\Sigma}^1_n(X)\},\\
\mathbf{\Sigma}^1_{n+1}(X) &:= \{\mathrm{proj}_X(A) : A \in \mathbf{\Pi}^1_n(X \times \omega^\omega)\}, \text{ and}\\
\mathbf{\Delta}^1_n(X) &:= \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X).
\end{aligned}$$

We say that a set $A \subseteq X$ is a $\mathbf{\Sigma}^1_n(X)$ *set*, $\mathbf{\Pi}^1_n(X)$ *set*, and $\mathbf{\Delta}^1_n(X)$ *set* if A is an element of $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X)$, respectively. Moreover, we call A *projective* if A is $\mathbf{\Sigma}^1_n(X)$ for some $n \geq 1$. If X is clear from the context, then we sometimes omit X and write $\mathbf{\Sigma}^1_n$ instead of $\mathbf{\Sigma}^1_n(X)$ and similarly for $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$.

Note that if $Y \subseteq X$ is a Polish subspace, then for every $n \geq 1$, $\mathbf{\Sigma}^1_n(Y) = \{A \cap Y : A \in \mathbf{\Sigma}^1_n(X)\}$ and similarly for $\mathbf{\Pi}^1_n(Y)$.

Theorem 1.2.15 ($\mathsf{ZF} + \mathsf{AC}_\omega$)**.** *Let X be an uncountable Polish space. For every $n \geq 1$, we have that $\mathbf{\Sigma}^1_n(X) \neq \mathbf{\Pi}^1_n(X)$ and $\mathbf{\Delta}^1_n(X) \subsetneq \mathbf{\Sigma}^1_n(X) \cup \mathbf{\Pi}^1_n(X) \subsetneq \mathbf{\Delta}^1_{n+1}(X)$.*

Proof. Cf., e.g., [Kec95, Theorem 37.7]. □

Proposition 1.2.16 ($\mathsf{ZF} + \mathsf{AC}_\omega$)**.** *Let X be a Polish space. For every $n \geq 1$, the classes $\mathbf{\Sigma}^1_n(X)$, $\mathbf{\Pi}^1_n(X)$, and $\mathbf{\Delta}^1_n(X)$ are closed under countable unions and intersection and preimages by Borel functions. Furthermore, $\mathbf{\Sigma}^1_n(X)$ is closed under images by Borel functions and $\mathbf{\Delta}^1_n(X)$ under complements.*

Proof. Cf., e.g., Proposition [Kec95, Proposition 37.1 & Exercise 37.3]. $\square$

So far, we only looked at easily definable sets from a topological point of view. In the rest of this section, we give an alternative, more syntactic approach. Let $\mathcal{A}^2 := (\omega, \omega^\omega, \mathrm{ap}, +, \cdot, <, 0, 1)$ be the structure of *second-order arithmetic*, where $\mathrm{ap} : \omega^\omega \times \omega \to \omega$ is the function with $\mathrm{ap}(x, n) := x(n)$ and $+$, $\cdot$, and $<$ are the addition, multiplication, and order on the natural numbers, respectively. We denote the first-order quantifiers and second-order quantifiers by $\forall^0, \exists^0$ and $\forall^1, \exists^1$, respectively. We say that a formula is *bounded* if contains no second-order quantifiers and every first-order quantifier is bounded by a term. The formulas in the language of second-order arithmetic are classified according to the number of alternating quantifiers.

Definition 1.2.17. Let $n \in \omega$. We say that a formula in the language of second-order arithmetic is

(a) Σ^0_0 if it is bounded,

(b) Π^0_n if it is of the form $\neg\varphi$ for some Σ^0_n formula φ,

(c) Σ^0_{n+1} if it is of the form $\exists^0 k\varphi$ for some Π^0_n formula φ,

(d) *arithmetical* if it is Σ^0_n or Π^0_n for some $n \in \omega$,

(e) Σ^1_1 if it is of the form $\exists^1 x\varphi$ for some arithmetical formula φ,

(f) Π^1_n if it is of the form $\neg\varphi$ for some Σ^1_n formula φ, and

(g) Σ^1_{n+1} if it is of the form $\exists^1 x\varphi$ for some Π^1_n formula φ.

That means a formula φ in the language of second-order arithmetic is Σ^0_n (or Π^0_n) if and only if there is a bounded formula ψ such that φ is of the form

$$\exists^0 k_0 \forall^0 k_1 \ldots Q^0 k_n \varphi \text{ (or } \forall^0 k_0 \exists^0 k_1 \ldots Q^0 k_n \varphi),$$

where $Q^0 k_n$ is $\exists^0$ if n is even (or odd) and $\forall^0$ otherwise. The same is true for Σ^1_n and Π^1_n formulas if we replace bounded, $\forall^0$, and $\exists^0$, with arithmetical, $\forall^1$, and $\exists^1$, respectively. Note that in $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ every formula in the language of second-order arithmetic is logically equivalent to some formula of this form. In the following, if we say that a statement is Σ^i_n (or Π^i_n), we mean that it is logically equivalent to a Σ^i_n (or Π^i_n) formula.

Each formula in the language of second-order arithmetic with at least one free second-order variable defines a subset of the reals. We can use this to classify subsets of the reals into another hierarchy.

Definition 1.2.18. Let $k, \ell \in \omega$ such that $\max\{k, \ell\} > 0$, let $n \in \omega$, and let $i \leq 1$.

(a) We say that a set $A \subseteq \omega^k \times (\omega^\omega)^\ell$ is Σ^i_n (or Π^i_n) if there is a Σ^i_n (or Π^i_n) formula φ with k free first-order variables and ℓ free second-order variables such that $A = \{w \in \omega^k \times (\omega^\omega)^\ell : \mathcal{A}^2 \models \varphi(w)\}$. Furthermore, a set A is Δ^i_n, if A is Σ^i_n and Π^i_n. We call a set *recursive* if it is Δ^0_1 and *arithmetical* if is definable by an arithmetical formula.

(b) Additionally, let $r \in \omega^\omega$ be a real number. We say that a set $A \subseteq \omega^k \times (\omega^\omega)^\ell$ is $\Sigma^i_n(r)$ (or $\Pi^i_n(r)$) if there is a Σ^i_n (or Π^i_n) formula φ with k free first-order variables and $\ell + 1$ free second-order variables such that $A = \{w \in \omega^k \times (\omega^\omega)^\ell : \mathcal{A}^2 \models \varphi(w, r)\}$. We define $\Delta^i_n(r)$, *recursive in* r, and *arithmetical in* r analogously to the case without parameters.

The hierarchies from Definition 1.2.18 are also called the *lightface* hierarchies and the Borel and projective hierarchies are called the *boldface* hierarchies. This comes from the typographical convention that lightface letters are usually used for the lightface hierarchy and boldface letters for the boldface hierarchy. The following theorem shows that the lightface and boldface hierarchies are talking about the same sets.

Theorem 1.2.19 ($\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$)**.** *Let $k, \ell \in \omega$ such that $\max\{k, \ell\} > 0$, let $n > 0$, and let $i \leq 1$. A set $A \subseteq \omega^k \times (\omega^\omega)^\ell$ is $\mathbf{\Sigma}^i_n$ (or $\mathbf{\Pi}^i_n$) if and only if there is some $r \in \omega^\omega$ such that A is $\Sigma^i_n(r)$ (or $\Pi^i_n(r)$).*

Proof. Cf., e.g., [Kan03, Proposition 12.6]. □

1.2.6 The Baire property and Lebesgue measurability

The Baire property and Lebesgue measurability are two of the best-known regularity properties. They were introduced by Baire and Lebesgue at the beginning of the last century in [Bai99] and [Leb02], respectively. Regularity properties in general separate the well-behaving sets from those which are not. It is well-known that every Borel set has the Baire property and is Lebesgue measurable. However, there are also sets which neither have the Baire property nor are Lebesgue measurable; e.g., Vitali [Vit05] and Bernstein sets [Ber08]. Both Vitali and Bernstein sets are constructed with explicit use of the axiom of choice and so do not have simple definitions. We shall talk about this in more detail in Section 1.2.14. Here, we give the definitions of the Baire property and Lebesgue measurability and discuss some basic properties of these. For further details, we refer the reader to classical textbooks such as [Kec95] or [Oxt80]. We begin with the Baire property.

Definition 1.2.20. Let X be a topological space.

(a) A set $A \subseteq X$ is *nowhere dense in X* if its closure $\overline{A}$ has empty interior. A set which is a countable union of nowhere dense sets in X is called *meager in X*. We denote the σ-ideal of meager sets in X by $\mathcal{M}_X$. For the reals, we omit the X and just write $\mathcal{M}$.

(b) A set $A \subseteq X$ has the *Baire property in X* if there is an open set $O \subseteq X$ such that $O \triangle A := (O \setminus A) \cup (A \setminus O)$ is meager in X.

By definition, every nowhere dense set in X is contained in a closed nowhere dense set in X and every meager set in X is contained in a meager $\mathbf{\Sigma}^0_2(X)$ set. Hence, $\mathcal{M}_X$ is *Borel generated*, in the sense that every meager set in X is contained in a meager Borel set in X. In general, we say that an ideal I on a topological space X is *Borel generated* if for every I-small set $A \subseteq X$, there is an I-small Borel set B in X such that $A \subseteq B$.

Proposition 1.2.21. *Let X be a topological space. The collection of all sets with the Baire property in X is the smallest σ-algebra containing all Borel and meager sets in X. Moreover, if X is Polish, then every analytic set has the Baire property.*

Proof. Cf., e.g., [Kec95, Proposition 8.21 & Theorem 29.5]. □

In descriptive set theory, one often works with an equivalent definition of the Baire property and nowhere dense sets.

Proposition 1.2.22 (Folklore)**.** *Let X be a topological space.*

(a) Then a set $A \subseteq X$ is nowhere dense in X if and only if for every non-empty open set $O \subseteq X$, there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$.

(b) If every family of pairwise disjoint open sets in X is countable, then a set $A \subseteq X$ has the Baire property in X if and only if for every non-empty open set $O \subseteq X$, there is a non-empty open set $O' \subseteq O$ such that $O' \setminus A$ or $O' \cap A$ is meager in X.

Proof. First, we prove (a) and start with the forward direction. Let $A \subseteq X$ be nowhere dense in X and let $O \subseteq X$ be a non-empty open set. Since $\overline{A}$ is nowhere dense in X, $\overline{A}$ contains no non-empty open set. Hence, $O' := O \cap (X \setminus \overline{A}) \subseteq O$ is a non-empty open set and $O' \cap A = \emptyset$.

Next, we prove the backward direction. Let $A \subseteq X$ be such that for every non-empty open set $O \subseteq X$, there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$. We suppose for a contradiction that the closure $\overline{A}$ contains a non-empty open set O. By assumption, there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$. Then $O' \subseteq \overline{A} \setminus A$ and so $A \subseteq \overline{A} \cap (X \setminus O') \subsetneq \overline{A}$. But this is a contradiction.

Now we prove (b) and start with the forward direction again. Let $A \subseteq X$ be a set with the Baire property and let $O \subseteq X$ be a non-empty open set. Since A has the Baire property in X, there is some open set $U \subseteq X$ such that $U \triangle A$ is meager in X. We make a case-distinction:

Case 1: $O \cap U$ is non-empty. Then $O' := O \cap U$ is a non-empty open set and $O' \setminus A \subseteq U \setminus A \subseteq U \triangle A$. Therefore, $O' \setminus A$ is meager in X.

Case 2: $O \cap U$ is empty. Then $O \cap A \subseteq (O \cap U) \cup (A \setminus U) \subseteq U \triangle A$. Therefore, $O \cap A$ is meager in X.

Finally, we prove the backward direction. Let $A \subseteq X$ such that the assumption holds and let $D := \{O \subseteq X : O \text{ is non-empty open and } O \setminus A \text{ or } O \cap A \text{ is meager}\}$. Then D is dense in the set of all non-empty open sets ordered by inclusion. Hence, there is a maximal antichain $\mathcal{A} \subseteq D$. By assumption, $\mathcal{A}$ is countable. Let $U := \bigcup\{O \in \mathcal{A} : O \setminus A \text{ is meager}\}$. Then U is open and $U \setminus A$ is meager in X. It remains to show that $A \setminus U$ is meager in X. Let $D' := \{O \subseteq X : O \text{ is non-empty open and } O \cap (A \setminus U) \text{ is meager}\}$ and let $O \subseteq X$ be a non-empty open set. Then there is some $O' \in \mathcal{A}$ such that $O \cap O'$ is non-empty. Since $O' \setminus A$ or $O' \cap A$ is meager, we have $O \cap O' \subseteq U$ or $(O \cap O') \cap A$ is meager. In both cases, $(O \cap O') \cap (A \setminus U)$ is meager and so $O \cap O' \in D'$. Therefore, D' is dense in the set of all non-empty open sets ordered by inclusion. Let $\mathcal{A}' \subseteq D'$ be a maximal antichain and let $A' := \bigcup_{O \in \mathcal{A}'} O \cap (A \setminus U)$. By assumption, $\mathcal{A}$ is countable and so A' is meager. It is enough to show that $(A \setminus U) \setminus A'$ is nowhere dense. Let $O \subseteq X$ be a non-empty open set. Then there is some $O' \in \mathcal{A}'$ such that $O \cap O' \neq \emptyset$. Since $O \cap O' \subseteq O'$, $(O \cap O') \cap (A \setminus U) \subseteq A'$. Therefore, $(O \cap O') \cap ((A \setminus U) \setminus A') = \emptyset$. By (a), $(A \setminus U) \setminus A'$ is nowhere dense. □

Proposition 1.2.22 is also true for general topological spaces. However, the proof is somewhat more involved. In the following, we shall only consider topological spaces which satisfy the additional assumption and therefore omit the proof.

Next, we introduce the Lebesgue measure on the real line. We define it in the standard way by first defining it for open intervals and then extend it to more sets via the outer measure.

Definition 1.2.23. We say that an open $(a, b) \subseteq \mathbb{R}$ has *Lebesgue measure* $\mu(a, b) := b - a$. For a set $A \subseteq \mathbb{R}$, we define the *outer measure*

$$\mu^*(A) := \inf\{\sum_{n \in \omega} \mu(I_n) : \langle I_n : n \in \omega \rangle \text{ is a sequence of open intervals with } A \subseteq \bigcup_{n \in \omega} I_n\}.$$

A set $A \subseteq \mathbb{R}$ is called *Lebesgue measurable* if for every set $A' \subseteq \mathbb{R}$, $\mu^*(A') = \mu^*(A' \cap A) + \mu^*(A' \cap (\mathbb{R} \setminus A))$. In this case, we set $\mu(A) := \mu^*(A)$. A Lebesgue measurable set $A \subseteq \mathbb{R}$ is called *Lebesgue null* if $\mu(A) = 0$.

The Lebesgue null sets and the Lebesgue measurable sets form a σ-ideal and σ-algebra, respectively. Moreover, it is well-known that every analytic set is Lebesgue measurable. Nowadays, it is common in descriptive set theory to work with the Lebesgue measure on Baire and Cantor space instead of on the real line. In this work, we focus on the Cantor space.

Definition 1.2.24. For $s \in 2^{<\omega}$, we say that the basic open set $[s]$ has *Lebesgue measure* $\mu([s]) := 2^{-\mathrm{lh}(s)}$. Using standard methods of measure theory, μ can be extended to all Borel sets (cf., e.g., [Kec95, p. 103]). We say that a set $A \subseteq 2^\omega$ is *Lebesgue null* if there is a Borel set $B \subseteq 2^\omega$ such that $A \subseteq B$ and $\mu(B) = 0$ and *Lebesgue measurable* if there is a Borel set $B \subseteq 2^\omega$ such that $B \triangle A$ is Lebesgue null. In this case, we set $\mu(A) := \mu(B)$.

As before, the Lebesgue null sets and the Lebesgue measurable sets form a σ-ideal and σ-algebra, respectively. Moreover, every analytic set is Lebesgue measurable. We denote the Lebesgue null ideal by $\mathcal{N}$. Clearly, $\mathcal{N}$ is Borel generated.

Definition 1.2.25. Let (X, μ) be a measure space and let $\mathcal{I}$ be an index set. We say that a family $\{A_i : i \in \mathcal{I}\}$ of sets of reals is *independent* if for every finite set $\mathcal{J} \subseteq \mathcal{I}$, $\mu(\bigcap_{j \in \mathcal{J}} A_j) = \prod_{j \in \mathcal{J}} \mu(A_j)$.

Definition 1.2.25 defines independent families for general measure spaces. However, in this thesis (X, μ) will always be either the Cantor space or the real line together with the Lebesgue measure. Independent families will be important in Section 2.3. In the rest of this section, we prove two lemmas which will be helpful later.

Lemma 1.2.26. *Let (X, μ) be a measure space and let $\{A^n_k : n, k \in \omega\}$ be a family of open independent sets of reals with $\mu(A^n_k) = 2^{-(n+1)}$ and let $x \in \omega^\omega$. For every $k, m \in \omega$, if $k \neq x(m)$, then $\mu(A^m_k \setminus \bigcup_{n \in \omega} A^n_{x(n)}) > 0$.*

Proof. Let $k, m \in \omega$ such that $k \neq x(m)$. Then

$$\begin{aligned}
\mu(A^m_k \cap \bigcup_{n \in \omega} A^n_{x(n)}) &= \mu(\bigcup_{n \in \omega} (A^m_k \cap A^n_{x(n)})) \\
&\leq \mu(\bigcup_{n \in 2} (A^m_k \cap A^n_{x(n)})) + \mu(\bigcup_{n > 1} (A^m_k \cap A^n_{x(n)})) \\
&\leq \sum_{n \in 2} \mu(A^m_k \cap A^n_{x(n)}) - \mu(A^m_k \cap A^0_{x(0)} \cap A^1_{x(1)}) + \sum_{n > 1} \mu(A^m_k \cap A^n_{x(n)}) \\
&= \sum_{n \in \omega} \mu(A^m_k \cap A^n_{x(n)}) - \frac{1}{2^{m+4}} = \frac{1}{2^{m+1}} - \frac{1}{2^{m+4}} < \frac{1}{2^{m+1}}.
\end{aligned}$$

Hence, $\mu(A^m_k \setminus \bigcup_{n \in \omega} A^n_{x(n)}) > 0$. □

In Section 2.3, we shall need a recursive family $\{A^n_k : k, n \in \omega\}$ of open independent subsets of 2^ω with certain measures. By a recursive family, we mean that the statement "$x \in A^n_k$" is recursive. Let $\{P^k_n : k, n \in \omega\}$ be a disjoint partition of ω such that $|P^n_k| = n + 1$ for every $k, n \in \omega$. We define $A^n_k := \{x \in 2^\omega : \forall i \in P^n_k (x(i) = 1)\}$. Then $\{A^n_k : k, n \in \omega\}$ is a recursive family of open independent sets such that for every $k, n \in \omega$, $\mu(A^n_k) = 2^{-(n+1)}$.

Lemma 1.2.27. *Let $A \subseteq 2^\omega$ be measurable. Then for every $n \in \omega$, $\lim_{k\to\infty} \mu(A_k^n \cap A) = 2^{-(n+1)}\mu(A)$.*

Proof. We first prove the lemma for basic open sets. Let $n \in \omega$ and let $s \in 2^{<\omega}$. Then there are only finitely many $k \in \omega$ such that $\mathrm{dom}(s) \cap P_n^k \neq \emptyset$. Let $m \in \omega$ such that for every $k \geq m$, $\mathrm{lh}(s) \cap P_n^k = \emptyset$. Hence, for every $k \geq m$,

$$\mu(A_k^n \cap [s]) = \frac{1}{2^{n+1}} \frac{1}{2^{\mathrm{lh}(s)}} = \frac{1}{2^{n+1}} \mu([s])$$

and so $\lim_{k\to\infty} \mu(A_k^n \cap [s]) = \frac{1}{2^{n+1}}\mu([s])$.

Now let $O \subseteq 2^\omega$ be open. Then O is a disjoint union of basic open sets $O_m \subseteq 2^\omega$. Without loss of generality, this union is infinite. Then

$$\begin{aligned}\lim_{k\to\infty} \mu(A_k^n \cap O) &= \lim_{k\to\infty} \mu(\bigcup_{m\in\omega}(A_k^n \cap O_m)) = \lim_{k\to\infty} \sum_{m\in\omega} \mu(A_k^n \cap O_m) = \sum_{m\in\omega} \lim_{k\to\infty} \mu(A_k^n \cap O_m)\\ &= \sum_{m\in\omega} \frac{1}{2^{n+1}}\mu(O_m) = \frac{1}{2^{n+1}} \sum_{m\in\omega} \mu(O_m) = \frac{1}{2^{n+1}}\mu(O).\end{aligned}$$

Finally, we prove the assertion for A. Since A is measurable, for every $\varepsilon > 0$, there is an open set $O \subseteq 2^\omega$ such that $A \subseteq O$ and $\mu(O \setminus A) < \varepsilon$. Hence, we can find a decreasing sequence $\langle O_m : m \in \omega\rangle$ of open sets such that for every $m \in \omega$, $A \subseteq O_m$ and $\mu(A) = \mu(\bigcap_{m\in\omega} O_m)$. Then

$$\lim_{k\to\infty} \mu(A_k^n \cap A) = \lim_{m\to\infty} \lim_{k\to\infty} \mu(A_k^n \cap O_m) = \lim_{m\to\infty} \frac{1}{2^{n+1}}\mu(O_m) = \frac{1}{2^{n+1}}\mu(A). \qquad \square$$

1.2.7 Constructible sets

The *constructible universe* L was introduced by Gödel in [Göd38]. It is an inner model of ZFC, i.e., a transitive model of ZFC containing all ordinals. Moreover, the generalized continuum hypothesis holds (GCH) in L. The constructible universe is constructed similarly to V, but in successor steps we do not take the entire power set, but only definable subsets.

Definition 1.2.28. A set $X \subseteq M$ is *definable over a set* M if there is a formula φ in the language of set theory and some $r_0, \dots, r_n \in M$ such that $X := \{x \in M : (M, \in) \models \varphi(x, r_0, \dots, r_n)\}$. We denote the collection of all definable sets over M by $\mathrm{def}(M)$.

Since M is a set, the satisfaction relation for M can be defined in V. Hence, $\mathrm{def}(M)$ is a set. Moreover, $M \in \mathrm{def}(M)$ and $\mathrm{def}(M) \subseteq \mathcal{P}(M)$. Using the definable sets, we can now define Gödel's constructible universe.

Definition 1.2.29. We define by transfinite recursion:

$$\begin{aligned}\mathrm{L}_0 &:= \emptyset,\\ \mathrm{L}_{\alpha+1} &:= \mathrm{def}(\mathrm{L}_\alpha),\\ \mathrm{L}_\lambda &:= \bigcup_{\alpha<\delta} \mathrm{L}_\alpha \text{ if } \delta \text{ is a limit, and}\\ \mathrm{L} &:= \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha.\end{aligned}$$

The class L is called *the constructible universe* and the elements of L are called *constructible*. The statement "every set is constructible" is the *axiom of constructibility* and is denoted usually by V=L.

Theorem 1.2.30 (Gödel)**.** *The constructible universe* L *is a transitive model of* $\mathsf{ZFC}+\mathsf{GCH}+\mathrm{V}=\mathrm{L}$*. Moreover,* L *is an inner model and every inner model of* ZF *contains* L*.*

Proof. Cf., e.g., [Jec03, Theorems 13.3, 13.16, 13.18, & 13.20]. □

The construction of the constructible universe can be relativized to a given set. Let A be a set and let $\mathcal{L}'$ be the language of set theory augmented with an additionally unary relation symbol. A set $X \subseteq M$ is *definable over a set M relative to A* if there is a formula φ in the language $\mathcal{L}'$ and some $r_0, \ldots, r_n \in M$ such that $X := \{x \in M : (M, \in, A \cap M) \models \varphi(x, r_0, \ldots, r_n)\}$. We denote the collection of all definable sets over M by $\mathrm{def}_A(M)$. In analogy to L, we define by transfinite recursion:

$$\begin{aligned}
\mathrm{L}_0[A] &:= \emptyset, \\
\mathrm{L}_{\alpha+1}[A] &:= \mathrm{def}_A(\mathrm{L}_\alpha[A]), \\
\mathrm{L}_\lambda[A] &:= \bigcup_{\alpha<\delta} \mathrm{L}_\alpha[A] \text{ if } \delta \text{ is a limit, and} \\
\mathrm{L}[A] &:= \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha[A].
\end{aligned}$$

The elements of $\mathrm{L}[A]$ are called *constructible from A*.

The class $\mathrm{L}[A]$ shares many properties with L, e.g., $\mathrm{L}[A]$ is a transitive model of ZFC and if M is an inner model of ZF such that $A \cap M \in M$, then M contains $\mathrm{L}[A]$. For more information about this or L and $\mathrm{L}[A]$ in general, we refer the reader to [Dev17], [Jec03], [Kan03], or [Mos09]. In the remainder of this section, we highlight a few properties of L and $\mathrm{L}[A]$ that will be particularly useful later. An important property of L is that one can construct a well-ordering of the whole constructible universe. The rough idea is to build this well-ordering recursively using the structure of L: assume that we already have constructed a well-ordering $<_{\mathrm{L}_\alpha}$ of L_α. Since there are only countably many formulas, they can be well-ordered. Then we can use $<_{\mathrm{L}_\alpha}$ and a well-ordering on the formulas to define a well-ordering $<_{\mathrm{L}_\alpha+1}$ of $\mathrm{L}_{\alpha+1}$. Finally, we set $<_{\mathrm{L}} := \bigcup_{\alpha\in\mathrm{Ord}} <_{\mathrm{L}_\alpha}$. Gödel showed that $<_{\mathrm{L}}$ is definable.

Theorem 1.2.31 (Gödel)**.**

(a) There is a sentence σ_0 in the language of set theory such that for every transitive class M, $(M, \in) \models \sigma_0$ if and only if either $N = \mathrm{L}$ or $N = \mathrm{L}_\delta$ for some limit ordinal $\delta > \omega$.

(b) There is a formula φ_0 in the language of set theory such that for every limit ordinal $\delta > \omega$ and every $x, y \in \mathrm{L}_\delta$, $x <_{\mathrm{L}} y$ if and only if $\mathrm{L}_\delta \models \varphi_0(x, y)$.

Proof. Cf., e.g., [Kan03, Theorem 3.3]. □

Just as for L, one can also construct a definable well-ordering of $\mathrm{L}[r]$ for every $r \in \omega^\omega$. Using these well-orderings, one can then show that for every $r \in \omega^\omega$, the set of reals in $\mathrm{L}[r]$ is $\Sigma^1_2(r)$.

Theorem 1.2.32 (Gödel)**.**

(a) The relation $\{(x, r) \in \omega^\omega \times \omega^\omega : x \in \mathrm{L}[r]\}$ *is a* Σ^1_2 *set of reals.*

(b) For every $r \in \omega^\omega$*, in* $\mathrm{L}[r]$*, there is a* $\boldsymbol{\Delta}^1_2(\omega^\omega \times \omega^\omega)$ *well-ordering of the reals.*

Proof. Cf., e.g., [Mos09, 8F.23 & 8F.24]. □

1.2.8 Absoluteness

Let M be a model of set theory and let N be a submodel of M. A formula φ is *absolute between* M *and* N if for every $x_1, \ldots, x_n \in N$, $M \models \varphi(x_1, \ldots, x_n)$ if and only if $N \models \varphi(x_1, \ldots, x_n)$. It is well-known that every formula in the language of set theory without unbounded quantifiers is absolute between any transitive models of set theory (cf., e.g., [Jec03, Lemma 12.9]). However, we are more interested in the absoluteness of formulas in the language of second-order arithmetic. Since arithmetical formulas have no unbounded quantifiers, any arithmetical formula is absolute between transitive models of set theory. But there are also formulas in the language of second-order arithmetic which are not absolute. Let φ be a Σ^1_2 formula such that $\varphi(x)$ is true if and only if $x \in \mathrm{L} \cap \omega^\omega$ and let $r \in \omega^\omega$ be not in L. Such a formula exists by Theorem 1.2.32. Then $\exists^1 x \neg\varphi(x)$ is false in L, but it is true in $\mathrm{L}[r]$. Therefore, there are Σ^1_3 formulas which are not absolute. But Σ^1_1 (and Π^1_1) formulas are.

Theorem 1.2.33 (Analytic absoluteness)**.** *Every* Σ^1_1 *(and* Π^1_1*) formula is absolute between transitive models of* $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$*.*

Proof. Cf., e.g., [Jec03, Theorem 25.4]. □

The analogous result does not hold for Σ^1_2 (or Π^1_2) formulas: suppose it does. Let Ψ be a finite fragment of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ which proves it. By reflection, there are countable transitive models of Ψ. Let M be such a model of minimal height. Then the statement "there is a transitive model of Ψ" is true in V but not in M. It remains to check that the statement is equivalent to a Σ^1_2 formula. Let $\pi : \omega^2 \to \omega$ be the canonical bijection. For every $x \in 2^\omega$, we define a relation E_x by $(n, m) \in E_x$ if and only if $x(\pi(n, m)) = 1$. By Mostowski's collapsing Lemma, there is a transitive model of Ψ if and only if there is a real $x \in 2^\omega$ such that (ω, E_x) is well-founded and $(\omega, E_x) \models \Psi$. Hence, "there is a transitive model of Ψ" is equivalent to a Σ^1_2 formula.

However, by Theorem 1.2.33, every Σ^1_2 formula is upwards absolute and every Π^1_2 formula is downwards absolute between transitive models of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$. In addition, Shoenfield has proven that they are absolute between sufficiently large models.

Theorem 1.2.34 (Shoenfield)**.** *Every* Σ^1_2 *(and* Π^1_2*) formula is absolute between inner models of* $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$*.*

Proof. Cf., e.g., [Jec03, Theorem 25.20]. □

1.2.9 Borel codes

The concept of coding Borel sets as reals was first formalized by Solovay. In [Sol70], he coded every Borel set as a real by giving a surjection from the reals onto the Borel sets. There are many different ways to define such a coding function. The explicit choice does not really matter as long as the decoding process is not too complex. In the following, we introduce a coding function which can be defined in ZF. Our codes are well-founded trees (which can be coded as reals). A tree T is *well-founded* if $(T, \supseteq)$ is well-founded. Then a tree T on ω is well-founded if and only if $[T] = \emptyset$. This is even true without the axiom of choice since $\omega^{<\omega}$ is well-orderable in ZF. For every well-founded tree T on ω, there is a unique rank function r_T. If ω_1 is regular, then $\sup(\mathrm{ran}(r_T)) < \omega_1$. However, it is not provable in ZF that ω_1 is regular. Still, one can show that $\sup(\mathrm{ran}(r_T)) < \omega_1$ (cf. [Fre08, 562A]). A *Borel code* is a real $c \in 2^\omega$ such that $T_c := \{s_n : c(n) = 1\}$ is a well-founded tree on ω. We denote the set of all Borel codes by BC.

It remains to define the decoding procedure. We do this for the Baire space. But this can be done for any other second countable topological space in the same way. Let $\{s_n : n \in \omega\}$ be the canonical enumeration of $\omega^{<\omega}$. For every Borel code $c \in \mathsf{BC}$, we *decode* c by recursion on r_{T_c}: we define for every $t \in T_c$,

$$B_t := \begin{cases} \emptyset & \text{if } r_{T_c}(t) = 0 \text{ and } t = \emptyset, \\ [s_n] & \text{if } r_{T_c}(t) = 0 \text{ and } t(\mathrm{lh}(t) - 1) = n, \\ X \setminus B_{t'} & \text{if } r_{T_c}(t) > 0 \text{ and } \mathrm{succ}_{T_c}(t) = \{t'\}, \\ \bigcup_{t' \in \mathrm{succ}_{T_c}(t)} B_{t'} & \text{otherwise.} \end{cases}$$

Finally, we set $B_c := B_\emptyset$. A set $B \subseteq \omega^\omega$ is *codable Borel* if there is a Borel code coding it. We denote the set of all codable Borel sets by $\mathcal{B}^*(\omega^\omega)$. Since $r_T(\emptyset) < \omega_1$, one can show inductively that every codable Borel set is Borel. The converse is also true in $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$. However, it is not provable in ZF. We shall take a closer look at this in Section 3.1.

In descriptive set theory, Borel codes are often used when working with more than one model. The advantage of Borel codes over Borel sets is that being a Borel code is absolute.

Lemma 1.2.35 (Solovay)**.** *The set BC is Π^1_1 and the set $\{(x, c) \in \omega^\omega \times \mathsf{BC} : x \in B_c\}$ is Δ^1_1. Therefore, for every $c \in \mathsf{BC}$ and every $x \in \omega^\omega$, the statements "c is a Borel code", "$x \in B_c$", and "$x \notin B_c$" are absolute between transitive models of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$.*

Proof. Cf., e.g., [Jec03, Lemmas 25.44 & 25.45]. □

1.2.10 Hereditarily countable sets

Let κ be a cardinal. We denote the collection of all sets whose transitive closure has cardinality $< \kappa$ by $\mathcal{H}_\kappa$. Then it is clear that $\mathcal{H}_\kappa$ is transitive for every cardinal κ. Moreover, if θ is an infinite regular cardinal, then $\mathcal{H}_\theta$ is a transitive model of ZFC^- (cf., e.g., [Kun11, Theorem II.2.1]). We are mostly interested in the case where $\theta = \aleph_1$. If $\theta = \aleph_1$, then we write HC instead of $\mathcal{H}_{\aleph_1}$ and call the elements of HC *hereditarily countable*. In the following, we highlight a few properties of HC which will be import later. For more information about HC and $\mathcal{H}_\kappa$, we refer the reader to [Kun11].

The reason we are mainly interested in HC is that it can be used to characterize the projective sets on the reals. To do this, we have to define the *Lévy hierarchy* first. As with the language of second-order arithmetic, the formulas in the language of set theory can be classified by the number of alternating quantifiers they contain. A formula in the language of set theory is called *bounded* if every quantifier is bounded by a variable.

Definition 1.2.36. Let $n \in \omega$. We say that a formula in the language of set theory is

(a) Σ_0 if it is bounded,

(b) Π_n if it is of the form $\neg\varphi$ for some Σ_n formula φ, and

(c) Σ_{n+1} if it is of the form $\exists x\varphi$ for some Π_n formula φ.

Now we can use the Lévy hierarchy to define a hierarchy of sets. However, we do not define this hierarchy in V, but in HC.

Definition 1.2.37. Let $n \in \omega$.

(a) A set $A \subseteq \mathrm{HC}$ is Σ_n^{HC} (or Π_n^{HC}) if there is a Σ_n (or Π_n) formula φ with only one free variable such that $A = \{x \in \mathrm{HC} : \mathrm{HC} \models \varphi(x)\}$. We say that A is Δ_n^{HC} if it is Σ_n^{HC} and Π_n^{HC}.

(b) A set $A \subseteq \mathrm{HC}$ is $\boldsymbol{\Sigma}_n^{\mathrm{HC}}$ (or $\boldsymbol{\Pi}_n^{\mathrm{HC}}$) if there is a Σ_n (or Π_n) formula φ with at least one free variable and $r_0, \dots, r_k \in \mathrm{HC}$ such that $A = \{x \in \mathrm{HC} : \mathrm{HC} \models \varphi(x, r_0, \dots, r_k)\}$. We say that A is $\boldsymbol{\Delta}_n^{\mathrm{HC}}$ if it is $\boldsymbol{\Sigma}_n^{\mathrm{HC}}$ and $\boldsymbol{\Pi}_n^{\mathrm{HC}}$.

The following well-known theorem connects the Lévy hierarchy in HC to the lightface projective hierarchy.

Theorem 1.2.38. *Let $n \geq 1$. A set of reals is Σ_{n+1}^1 if and only if it is Σ_n^{HC}.*

Proof. Cf., e.g., [Jec03, Lemma 25.25]. □

We conclude this section with a fact about HC in L. It is well-known that $\mathrm{HC} = \mathrm{L}_{\omega_1}$ in L (cf., e.g., [Kun11, Theorem II.6.23]). Moreover, in L, the restriction of $<_{\mathrm{L}}$ to HC is Δ_1^{HC}. The proof is essentially the same as the proof of (b) of Theorem 1.2.32.

Lemma 1.2.39 (Folklore)**.** *The set $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ is Δ_1^{HC}.*

Proof. We first show that $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ is Σ_1^{HC}. By Theorem 1.2.31, there is a formula φ_0 in the language of set theory such that for every limit ordinal $\delta > \omega$ and every $x, y \in \mathrm{L}_\delta$, $x <_{\mathrm{L}} y$ if and only if $\mathrm{L}_\delta \models \varphi_0(x, y)$. Let $x, y \in \mathrm{HC}$. Then $x <_{\mathrm{L}} y$ if and only if there is a limit ordinal $\omega < \delta < \omega_1$ such that $x, y \in \mathrm{L}_\delta$ and $\mathrm{L}_\delta \models \varphi_0(x, y)$. By Theorem 1.2.31, there is a sentence σ_0 in the language of set theory such that for every transitive class M, $(M, \in) \models \sigma_0$ if and only if either $N = \mathrm{L}$ or $N = \mathrm{L}_\delta$ for some limit ordinal $\delta > \omega$. Then $x <_{\mathrm{L}} y$ if and only if there is a pair (M, E) such that E is well-founded on M, $(M, E) \models \sigma_0$, and there are $a, b \in M$ such that $\pi(a) = x$, $\pi(b) = y$, and $(M, E) \models \varphi_0(a, b)$, where π is the transitive collapsing function. The statement "E is well-founded on M" is Δ_1 (cf., e.g., [Jec03, Lemma 13.11]). Hence, the statement "$x <_{\mathrm{L}} y$" is Σ_1. Therefore, the set $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ is Σ_1^{HC}.

It remains to show that $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ is Π_1^{HC}. Let $x, y \in \mathrm{HC}$. Then $x <_{\mathrm{L}} y$ if and only if for every limit ordinal $\omega < \delta < \omega_1$, if $x, y \in \mathrm{L}_\delta$, then $\mathrm{L}_\delta \models \varphi_0(x, y)$. With a similar argument as before, the latter statement is Π_1 and so $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ is Π_1^{HC}. □

1.2.11 Forcing and symmetric submodels

The method of *forcing* was first introduced by Cohen in [Coh63, Coh64] to produce a model of ZFC in which the continuum hypothesis does not hold. 15 years earlier, Gödel had already constructed a model of ZFC in which the continuum hypothesis holds (cf. Section 1.2.7). Thus the continuum hypothesis can neither be proved nor disproved in ZFC, which means that it is independent from ZFC. This solved one of Hilbert's millennium problems. Cohen also showed that the axiom of choice cannot be proved in ZF. Combined with Gödel's work, this means that the axiom of choice is independent from ZF. After Cohen's discovery, many others started to use forcing. They extended Cohen's work and created a more general framework. Today, forcing is one of the most important tools for set theorists and is used in almost all areas.

Thus, of course, forcing plays an important role in this thesis. We assume that the reader is familiar with the basic concept and method of forcing. For a detailed introduction, we refer the reader to [Kun11]. Let us fix some notation. Let M be a transitive model of ZF^- and let $\mathbb{P}$ be a partial order in M. Then we call $\mathbb{P}$ a *forcing notion* and the elements of $\mathbb{P}$ *forcing conditions*. We say that a forcing condition p is *stronger* than another q if $p \leq q$. Two conditions $p, q \in \mathbb{P}$ are *compatible* if there is a condition which is stronger than p and q and *incompatible* otherwise. We say that $\mathbb{P}$ satisfies the *countable chain condition (c.c.c.)* if every antichain in $\mathbb{P}$ is countable. Let G be a $\mathbb{P}$-generic filter over M. We denote the forcing extension of M by G by $M[G]$ and for a $\mathbb{P}$-name σ, we denote the interpretation of σ by G by σ_G. For $x \in M$, we denote the standard $\mathbb{P}$-name for x by $\check{x}$.

We shall also work with products and iterations of forcing notions. Let $\mathcal{I}$ be an index set and let $\langle \mathbb{P}_i : i \in \mathcal{I} \rangle$ be a family of forcing notions in M. The *$\mathcal{I}$-product of $\langle \mathbb{P}_i : i \in \mathcal{I} \rangle$ with finite (or countable) support* is the set of all partial functions p such that $\mathrm{dom}(p) \subseteq \mathcal{I}$ is finite (or countable) and for every $i \in \mathrm{dom}(p)$, $p(i) \in \mathbb{P}_i$ ordered by

$$q \leq p :\iff \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge \forall i \in \mathrm{dom}(p)(q(i) \leq p(i)).$$

Let $\mathbb{P}$ be the $\mathcal{I}$-product of $\langle \mathbb{P}_i : i \in \mathcal{I} \rangle$ with finite (or countable) support. We say that $\mathcal{I}$ is the *domain* of $\mathbb{P}$. Let G be a $\mathbb{P}$-generic filter over M. Then for every $i \in \mathcal{I}$, $\{p(i) : p \in G \wedge i \in \mathrm{dom}(p)\}$ is a $\mathbb{P}_i$-generic filter over M. We denote it by G_i. Let $\mathcal{J} \subseteq \mathcal{I}$. We define $\mathbb{P}{\restriction}\mathcal{J}$ to be the $\mathcal{J}$-product of $\langle \mathbb{P}_j : j \in \mathcal{J} \rangle$ with finite (or countable) support. Then $G{\restriction}\mathcal{J} := G \cap \mathbb{P}{\restriction}\mathcal{J}$ is a $\mathbb{P}{\restriction}\mathcal{J}$-generic filter over M. Iterations of forcing notions are a generalization of products, where the index set is an ordinal. The main difference between iterations and products is that with iterations, the forcing notions do not necessarily have to be in the ground model. The idea is that we recursively define a sequence of models by successively forcing with the forcing notions. But in order to handle limit steps we define the iteration as a single forcing notion in the ground model. For a formal definition and more details on products and iterations, we refer the reader to [Kun11, Chapter V].

Formally, forcing requires a countable transitive model as the ground model. For example, it is easy to show that there are no generic filters over V for non-atomic forcing notions. Nevertheless, it is common practice to talk about generic extensions of the whole universe V or other transitive class-sized models. The idea is that we pretend to be in the same situation as the people who live in a countable transitive model M. From their point of view there is nothing outside of M and so generic filters over M cannot exist. But from our perspective there are such generic filters. Now to force over V, we imagine that there is a model much bigger than V in which V is a countable transitive model. For a formalization of this argument and other approaches, we refer the reader to [Kun11, Section IV.5].

Forcing notions satisfying the c.c.c. have the nice property that they preserve $\aleph_1$, i.e., $\aleph_1^M = \aleph_1^{M[G]}$ for every generic filter G over M. But the c.c.c. is a rather strong property and many of the standard forcing notions do not satisfy it. However, there is another class of forcing notions which preserve $\aleph_1$. Let $\mathbb{P}$ be a forcing notion, let θ be a regular cardinal, and let M be a countable elementary substructure of $\mathcal{H}_\theta$ containing $\mathbb{P}$. A condition $q \in \mathbb{P}$ is an *$(M, \mathbb{P})$-master condition* if for every maximal antichain $\mathcal{A} \subseteq \mathbb{P}$ in M, the set $\mathcal{A} \cap M$ is predense below q. We say that a $\mathbb{P}$ is *proper* if for every sufficiently large regular cardinal θ, every countable elementary substructure M of $\mathcal{H}_\theta$

containing $\mathbb{P}$, and every $p \in \mathbb{P} \cap M$, there is an $(M, \mathbb{P})$-master condition below p. Every forcing notion that satisfies the c.c.c. is proper. In this thesis, the most forcing notion will be proper.

To construct a model of ZF in which the axiom of choice fails, Cohen started with a model M of ZFC, performed a forcing extension $M[G]$, and then carefully chose an inner model $M \subseteq N \subseteq M[G]$. The rough idea is not to evaluate all names, but only those that are symmetric with respect to a group of automorphisms. Therefore, such models are called *symmetric submodels*. The advantage of symmetric submodels over normal generic extensions is that they do not necessarily preserve the axiom of choice. In the following, we give a brief introduction to symmetric submodels. For more details, we refer the reader to [Jec08, Section 5.2]. Let M be a transitive model of ZFC and let $\mathbb{P}$ be a forcing notion in M. We say that $\pi : \mathbb{P} \to \mathbb{P}$ is an *automorphism* if $\mathbb{P}$ is bijective and order-preserving. Note that we can extend every automorphism π to the set of $\mathbb{P}$-names. We recursively define $\pi(\emptyset) := \emptyset$ and $\pi(\sigma) := \{(\pi(\tau), \pi(p)) : (\tau, p) \in \sigma\}$. Then $\pi(\sigma)$ is a $\mathbb{P}$-name and since $\mathbb{P}$ is a dense embedding, for every sentence φ in the forcing language and every condition $p \in \mathbb{P}$, $p \Vdash \varphi$ if and only if $\pi(p) \Vdash \pi(\varphi)$, where $\pi(\varphi)$ is the formula we get by replacing every $\mathbb{P}$-name σ in φ by $\pi(\sigma)$. Let $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$ in M. A filter $\mathcal{F}$ on the subgroups of $\mathcal{G}$ is *normal* if for every $\pi \in \mathcal{G}$ and every $K \in \mathcal{F}$, $\pi K \pi^{-1} \in \mathcal{F}$. We fix a normal filter $\mathcal{F}$. A $\mathbb{P}$-name σ is called *symmetric* if $\mathrm{sym}(\sigma) := \{\pi \in \mathcal{G} : \pi(\sigma) = \sigma\} \in \mathcal{F}$ and *hereditarily symmetric* if it itself and every $\mathbb{P}$-name in it is symmetric. Let G be a $\mathbb{P}$-generic filter over M. The symmetric submodel N associated to $\mathcal{F}$ and G is defined by $N := \{\sigma_G : \sigma \text{ is a hereditarily symmetric } \mathbb{P}\text{-name in } M\}$. In general, N might not be a model of ZFC, but still will be a model of ZF.

Theorem 1.2.40. *Let M be a transitive model of* ZFC*, let $\mathbb{P}$ be a forcing notion in M, let $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$ in M, let $\mathcal{F} \subseteq \mathcal{G}$ be a normal filter in M, and let G be a $\mathbb{P}$-generic filter over M. Then the symmetric submodel N associated to $\mathcal{F}$ and G is a transitive model of* ZF *and $M \subseteq N \subseteq M[G]$.*

Proof. Cf., e.g., [Jec08, Theorem 5.14]. ☐

We conclude this section with a list of standard forcing notions. We assume that the reader is somewhat familiar with these. For more details, we refer the reader to [BJ95] and [Hal17].

Definition 1.2.41.

(a) *Cohen forcing*, denoted by $\mathbb{C}$, is the set $2^{<\omega}$ (or $\omega^{<\omega}$) ordered by reverse inclusions. For every transitive model M of ZFC and every $\mathbb{C}$-generic filter G over M, $x_G := \bigcup G$ is a real in $M[G]$. We call such reals *Cohen reals over M*. Conversely, $G = \{s \in \mathbb{C} : s \subseteq x_G\}$ and so every $\mathbb{C}$-generic filter over M constructs a unique Cohen real over M and vice versa.

(b) *Random forcing*, denoted by $\mathbb{B}$, is the partial order of all Lebesgue positive Borel sets of reals ordered by inclusion. For every transitive model M of ZFC and every $\mathbb{B}$-generic filter G over M, there is a unique real x_G such that for every Borel set coded in M, $x_G \in B$ if and only if $B \in G$. Here by "$x_G \in B$" and "$B \in G$" we mean that if we take a Borel code $c \in M$ for B, then $x_G \in B_c^{M[G]}$ and $B_c^{M[G]} \in G$. We call such reals *random reals over M*. Then $x_G = \bigcap G$ and $G = \{B \in \mathbb{B} : x_G \in G\}$ and so every $\mathbb{B}$-generic filter over M constructs a unique random real over M and vice versa.

(c) *Hechler forcing*, denoted by $\mathbb{D}$, is the partial order of all pairs $(n, f) \in \omega \times \omega^\omega$ ordered by

$$(n, f) \leq (m, g) \iff n \geq m,\ f{\restriction}m = g{\restriction}m, \text{ and } \forall k \geq m(f(k) \geq g(k)).$$

For every transitive model M of ZFC and every $\mathbb{D}$-generic filter G over M, $x_G := \bigcup\{f \restriction n : (n, f) \in G\}$ is a real in $M[G]$. We call such reals *Hechler reals over* M. Conversely, $G = \{(n, f) \in \mathbb{D} : f \restriction n \subseteq x_G \text{ and } \forall k \geq n(f(k) \leq x_G(k))\}$ and so every $\mathbb{D}$-generic filter over M constructs a unique Hechler real over M and vice versa.

(d) *Eventually different forcing*, denoted by $\mathbb{E}$, is the partial order of all pairs $(s, F) \in \omega^{<\omega} \times [\omega^\omega]^{<\omega}$ ordered by

$$(s, F) \leq (t, E) \iff s \supseteq t,\ F \supseteq E, \text{ and } \forall f \in F \forall n \in \mathrm{dom}(s \setminus t)\ (s(n) \neq f(n)).$$

For every transitive model M of ZFC and every $\mathbb{E}$-generic filter G over M, $x_G := \bigcup\{s : \exists F \in [\omega^\omega]^{<\omega}((s, F) \in G)\}$ is a real in $M[G]$. We call such reals $\mathbb{E}$-*generic reals over* M. Conversely, $G = \{(s, F) \in \mathbb{E} : s \subseteq x_G \text{ and } \forall f \in F \forall k \geq \mathrm{lh}(s)(f(k) \neq x_G(k))\}$ and so every $\mathbb{E}$-generic filter over M constructs a unique $\mathbb{E}$-generic real over M and vice versa.

(e) *Sacks forcing*, denoted by $\mathbb{S}$, is the partial order of all perfect trees on 2 ordered by inclusion. For every transitive model M of ZFC and every $\mathbb{S}$-generic filter G over M, $x_G := \bigcup\{\mathrm{stem}(T) : T \in G\}$ is a real in $M[G]$. We call such reals *Sacks reals over* M. Conversely, $G = \{T \in \mathbb{S} : x_G \in [T]\}$ and so every $\mathbb{S}$-generic filter over M constructs a unique Sacks real over M and vice versa.

(f) *Miller forcing*, denoted by $\mathbb{M}$, is the partial order of all super-perfect trees on ω ordered by inclusion. For every transitive model M of ZFC and every $\mathbb{M}$-generic filter G over M, $x_G := \bigcup\{\mathrm{stem}(T) : T \in G\}$ is a real in $M[G]$. We call such reals *Miller reals over* M. Conversely, $G = \{T \in \mathbb{M} : x_G \in [T]\}$ and so every $\mathbb{S}$-generic filter over M constructs a unique Sacks real over M and vice versa.

(g) *Laver forcing*, denoted by $\mathbb{L}$, is the partial order of all perfect trees on ω such that every node above the stem splits infinitely often ordered by inclusion. For every transitive model M of ZFC and every $\mathbb{L}$-generic filter G over M, $x_G := \bigcup\{\mathrm{stem}(T) : T \in G\}$ is a real in $M[G]$. We call such reals *Laver reals over* M. Conversely, $G = \{T \in \mathbb{L} : x_G \in [T]\}$ and so every $\mathbb{L}$-generic filter over M constructs a unique Sacks real over M and vice versa.

(h) *Silver forcing*, denoted by $\mathbb{V}$, is the partial order of all perfect, uniform trees on 2 ordered by inclusion. For every transitive model M of ZFC and every $\mathbb{V}$-generic filter G over M, $x_G := \bigcup\{\mathrm{stem}(T) : T \in G\}$ is a real in $M[G]$. We call such reals *Silver reals over* M. Conversely, $G = \{T \in \mathbb{V} : x_G \in [T]\}$ and so every $\mathbb{V}$-generic filter over M constructs a unique Silver real over M and vice versa.

(i) *Mathias forcing*, denoted by $\mathbb{R}$, is the partial order of all pairs $(F, E) \in [\omega]^{<\omega} \times [\omega]^\omega$ such that $\max(F) < \min(E)$ ordered by

$$(F', E') \leq (F, E) \iff F' \cap (\max(F) + 1) = F,\ E' \subseteq E, \text{ and } F' \setminus F \subseteq E'.$$

For every transitive model M of ZFC and every $\mathbb{R}$-generic filter G over M, $x_G := \bigcup\{F : \exists E \in [\omega]^\omega((F, E) \in G)\}$ is an infinite subset of the naturals in $M[G]$. We call such objects *Mathias reals over* M. Conversely, $G = \{(F, E) \in \mathbb{R} : F \subseteq x_G \text{ and } x_G \subseteq F \cup E\}$ and so every $\mathbb{R}$-generic filter over M constructs a unique Mathias real over M and vice versa.

Cohen, random, Hechler, and eventually different forcing all satisfy the c.c.c. and the other five do not. However, all of them are proper.

1.2.12 Idealized forcing notions

The theory of idealized forcing was developed by Zapletal. In [Zap04] and [Zap08], he defined a forcing notion for every σ-ideal on a Polish space and studied the properties of these forcing notions. Moreover, he showed that many of the classical forcing notions are forcing equivalent to a forcing notion which is defined from a σ-ideal. In the following, we give the basic definition and state a few results. For a complete introduction, we refer the reader to Zapletal's work.

Definition 1.2.42. Let X be an uncountable Polish space and let I be a proper σ-ideal on X. We write $\mathbb{P}_I$ for the partial order of all I-positive Borel sets in X ordered by inclusion.

The partial orders of Definition 1.2.42 are called *idealized forcing notions*. It is clear that random forcing is an idealized forcing notion. Moreover, every forcing notion of Definition 1.2.41 is forcing equivalent to an idealized forcing notion.

Example 1.2.43.

(a) Cohen forcing is forcing equivalent to $\mathcal{B}(2^\omega) \setminus \mathcal{M}$, where $\mathcal{M}$ is the meager ideal in the Cantor space.

(b) Hechler forcing is forcing equivalent to $\mathcal{B}(\omega^\omega) \setminus \mathcal{M}_\mathbb{D}$, where $\mathcal{M}_\mathbb{D}$ is the meager ideal in the dominating topology. The dominating topology is the topology generated by $\{[n, f] : (n, f) \in \mathbb{D}\}$, where $[n, f] := \{x \in \omega^\omega : f{\restriction}n \subseteq x \text{ and } \forall m \geq n(f(m) \leq x(m))\}$. For more details about the dominating topology, we refer the reader to [ŁR95].

(c) Eventually different forcing is forcing equivalent to $\mathcal{B}(\omega^\omega) \setminus \mathcal{M}_\mathbb{E}$, where $\mathcal{M}_\mathbb{E}$ is the meager ideal in the eventually different topology. The eventually different topology is the topology generated by $\{[s, F] : (s, F) \in \mathbb{D}\}$, where $[s, F] := \{x \in \omega^\omega : s \subseteq x \text{ and } \forall f \in F \forall n \geq \mathrm{lh}(s)(f(m) \neq x(m))\}$. For more details about the eventually different topology, we refer the reader to [Łab96].

(d) Sacks forcing is forcing equivalent to $\mathcal{B}(2^\omega) \setminus \mathbf{ctbl}$, where $\mathbf{ctbl}$ is the ideal of countable sets of reals.

Idealized versions of Miller, Laver, Silver, and Mathias forcing can be found in [Zap04]. We shall prove in Proposition 2.2.26 that a wide class of forcing notions that includes every forcing notion of Definition 1.2.41 are forcing equivalent to an idealized forcing notion. To do so, we shall need the following proposition.

Proposition 1.2.44 (Folklore). *Let X be an uncountable Polish space and let I be a proper σ-ideal on X. Then there is a dense embedding from $\mathbb{P}_I$ to $(\mathcal{B}(X)/I)^+$.*

Proof. Let $j : \mathbb{P}_I \to (\mathcal{B}(X)/I)^+$ be defined by $j(B) := [B]$, where $[B]$ is the equivalence class of B in $\mathcal{B}(X)/I$. We show that j is a dense embedding. Let B and B' be I-positive Borel sets in X. It is clear that if $B \subseteq B'$, then $j(B) \leq j(B')$. If B and B' are compatible, then $C := B \cap B'$ is I-positive and so $j(C) \leq j(B), j(B')$. Hence, $j(B)$ and $j(B')$ are compatible. If $j(B)$ and $j(B')$ are compatible, then there is some I-positive Borel set C in X such that $B \setminus C$ and $B' \setminus C$ are I-small. Then $C \cap B \cap B'$ is I-positive and so in particular, $B \cap B'$ is an I-positive Borel set in X. Hence, B and B' are compatible. Moreover, $\mathrm{ran}(j)$ is dense in $(\mathcal{B}(X)/I)^+$. Therefore, j is a dense embedding. $\square$

Similar to random forcing, each generic filter of an idealized forcing notion is uniquely determined by a single element: let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X. We say that an element $x \in X$ is $\mathbb{P}_I$*-generic* if there is a $\mathbb{P}_I$-generic filter G over V such that for every Borel set B in X coded in V, $x_G \in B$ if and only if $B \in G$. Again, by "$x_G \in B$" and "$B \in G$" we mean that if we take a Borel code $c \in \mathrm{V}$ for B, then $x_G \in B_c^{\mathrm{V}[G]}$ and $B_c^{\mathrm{V}[G]} \in G$. Zapletal showed that for every $\mathbb{P}_I$-generic filter there is a unique $\mathbb{P}_I$-generic element.

Theorem 1.2.45 (Zapletal)**.** *Let X be an uncountable Polish subspace of the Baire space, let I be a proper σ-ideal on X, and let G be a $\mathbb{P}_I$-generic filter over* V*. Then in* $\mathrm{V}[G]$*, there is a unique element $x_G \in X$ such that for every Borel set B in X coded in* V*, $x_G \in B$ if and only if $B \in G$.*

Proof. Cf., [Zap08, Proposition 2.1.2]. □

Let G be a $\mathbb{P}_I$-generic filter and let x_G be the corresponding $\mathbb{P}_I$-generic element. By Theorem 1.2.45, $\{x_G\} = \bigcap G$ and conversely $G = \{B \in \mathbb{P}_I : x_G \in B\}$. Hence, $x_G \in \mathrm{V}[G]$ and $G \in \mathrm{V}[x_G]$ and so $\mathrm{V}[G] = \mathrm{V}[x_G]$. Therefore, forcing with an idealized forcing notion is the same as adding an element to X.

1.2.13 Cardinal characteristics of the continuum

Cardinal characteristics of the continuum are cardinal numbers which typically measure the size of certain sets related to the reals. Usually, cardinal characteristics of the continuum have values between $\aleph_1$ and $2^{\aleph_0}$ and their actual size may be different in different models of ZFC. The best known cardinal characteristic is the size of the continuum. In the following, we introduce a few cardinal characteristics which will be important later. For more details, we refer the reader to [Bla10].

Definition 1.2.46. Let I be a proper σ-ideal on a Polish space X.

(a) The *additivity number of* I, denoted by $\mathrm{add}(I)$, is the least size of a family $\mathscr{F} \subseteq I$ such that $\bigcup \mathscr{F}$ is I-positive.

(b) The *uniformity number of* I, denoted by $\mathrm{non}(I)$, is the least cardinality of an I-positive set.

(c) The *covering number of* I, denoted by $\mathrm{cov}(I)$, is the least size of a family $\mathscr{F} \subseteq I$ such that $\bigcup \mathscr{F} = X$.

(d) The *cofinality number of* I, denoted by $\mathrm{cof}(I)$, is the least size of a family $\mathscr{F} \subseteq I$ such that every I-small set is contained in some element of $\mathscr{F}$.

It is clear that $\aleph_1 \leq \mathrm{add}(I) \leq \mathrm{non}(I), \mathrm{cov}(I) \leq \mathrm{cof}(I)$ for every proper σ-ideal I on a Polish space X. Moreover, if I is also Borel generated, then $\mathrm{cof}(I) \leq 2^{\aleph_0}$.

Definition 1.2.47.

(a) Let $f, g \in \omega^\omega$. We say that f *dominates* g if for all but finitely many $n \in \omega$, $g(n) < f(n)$.

(b) A family $\mathscr{F} \subseteq \omega^\omega$ is called *dominating* if every real is dominated by some element of $\mathscr{F}$. The *dominating number*, denoted by $\mathfrak{d}$, is the least size of a dominating family.

(c) A family $\mathscr{F} \subseteq \omega^\omega$ is called *unbounded* if there is no real which dominates all elements of $\mathscr{F}$. The *unbounded number*, denoted by $\mathfrak{b}$, is the least size of an unbounded family.

Note that every dominating family is unbounded. Hence, $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\aleph_0}$. The relationships between the dominating and unbounded numbers and the cardinal characteristics for the meager and Lebesgue null ideal are well studied in descriptive set theory. For more details, we refer the reader to [BJ95, Chapter 2].

There are many more cardinal characteristics defined in a similar way to $\mathfrak{d}$ and $\mathfrak{b}$ using other kinds of families of reals. However, they are not important for this thesis. Nevertheless, some of these families of reals are. We list two of them which play a crucial role later.

Definition 1.2.48.

(a) A family $\mathscr{F} \subseteq [\omega]^\omega$ is called *almost disjoint* if for every $x, y \in \mathscr{F}$, $x \cap y$ is finite. We say that $\mathscr{F}$ is a *maximal almost disjoint* family if $\mathscr{F}$ is additionally infinite and there is no $\mathscr{F}' \supsetneq \mathscr{F}$ which is almost disjoint.

(b) We say that $f, g \in \omega^\omega$ are *eventually different* if for all but finitely many $n \in \omega$, $f(n) \neq g(n)$. A family $\mathscr{F} \subseteq \omega^\omega$ is called *pairwise eventually different* if every $f \neq g \in \mathscr{F}$ are eventually different.

Next, instead of dominating and unbounded families, we consider reals with similar properties. It is clear that there cannot be a single real which dominates all reals or which is not dominated by any real. However, if M is a model of set theory, then there may be a larger model containing a real that dominates all reals in M. For example, if M is a countable transitive model of ZFC, then V contains a real which dominates all reals in M. Note that this real is not in M.

Definition 1.2.49. Let M be a transitive model of ZFC.

(a) A real $f \in \omega^\omega$ is called *dominating over M* if f dominates all reals in $\omega^\omega \cap M$.

(b) A real $f \in \omega^\omega$ is called *unbounded over M* if there is no real in $\omega^\omega \cap M$ which dominates f.

(c) We say that a real $x \in [\omega]^\omega$ *splits* another real $y \in [\omega]^\omega$ if both $y \cap x$ and $y \setminus x$ are infinite. A real $x \in [\omega]^\omega$ is called *splitting over M* if it splits all reals in $[\omega]^\omega \cap M$.

It is clear that every dominating real over M is also unbounded over M. Moreover, it is well-known that if there is a dominating real over M, then there is also a splitting real over M (cf., e.g., [Hal17, Fact 21.1]).

1.2.14 Regularity properties and Ikegami's Theorem

In descriptive set theory, forcing notions are often used either to characterize already known regularity properties or to define new ones. In the following, we consider regularity properties associated with our standard forcing notions from Definition 1.2.41.

Example 1.2.50.

(a) Cohen forcing can be used to characterize the Baire property in the Cantor space: a set $A \subseteq 2^\omega$ has the Baire property if and only if for every $s \in \mathbb{C}$, there is some $s' \leq s$ such that either $[s'] \setminus A$ or $[s'] \cap A$ is meager in the Cantor space (cf. Proposition 1.2.22).

(b) Random forcing can be used to characterize Lebesgue measurability: a set $A \subseteq 2^\omega$ is Lebesgue measurable if and only if for every $B \in \mathbb{B}$, there is some $B' \leq B$ such that either $B' \subseteq A$ or $B' \cap A = \emptyset$.

(c) Hechler forcing can be used to characterize the Baire property in the dominating topology: a set $A \subseteq \omega^\omega$ has the Baire property in the dominating topology if and only if for every $(n, f) \in \mathbb{D}$, there is some $(n', f') \leq (n, f)$ such that either $[n', f'] \setminus A$ or $[n', f'] \cap A$ is meager in the dominating topology.

(d) Eventually different forcing can be used to characterize the Baire property in the eventually different topology: a set $A \subseteq \omega^\omega$ has the Baire property in the eventually different topology if and only if for every $(s, F) \in \mathbb{E}$, there is some $(s', F') \leq (s, F)$ such that either $[s', F'] \setminus A$ or $[s', F'] \cap A$ is meager in the eventually different topology.

Next, we introduce a class of regularity properties which are defined from forcing notions whose conditions are trees.

Definition 1.2.51. A forcing notion $\mathbb{P}$ is called *arboreal* if its conditions are perfect trees on 2 (or ω) ordered by inclusion and for every $T \in \mathbb{P}$ and every $t \in T$ there is a $T' \leq T$ such that $t \subseteq \mathrm{stem}(T')$.

Clearly, Sacks, Miller, Laver, and Silver forcing are arboreal forcing notions. Recall that generic filters for these forcing notions are uniquely determined be reals. The same is true for arboreal forcing notions: let $\mathbb{P}$ be an arboreal forcing notion and let G be a $\mathbb{P}$-generic filter over V. Then $x_G := \bigcup\{\mathrm{stem}(T) : T \in G\}$ is a real in $\mathrm{V}[G]$. Conversely, $G = \{T \in \mathbb{P} : x_G \in [T]\}$ and so $\mathrm{V}[G] = \mathrm{V}[x_G]$. We call such a real a $\mathbb{P}$-*generic real over* V.

Definition 1.2.52. Let $\mathbb{P}$ be an arboreal forcing notion.

(a) A set $A \subseteq \omega^\omega$ is $\mathbb{P}$-*null* if for every $T \in \mathbb{P}$, there is some $S \leq T$ such that $[T] \cap A = \emptyset$. We denote the set of all $\mathbb{P}$-null sets by $\mathcal{N}_\mathbb{P}$ and the σ-ideal generated by the $\mathbb{P}$-null sets by $I_\mathbb{P}$.

(b) A set $A \subseteq \omega^\omega$ is in $I^*_\mathbb{P}$ if for every $T \in \mathbb{P}$, there is some $S \leq T$ such that $[T] \cap A$ is $I_\mathbb{P}$-small.

(c) A set $A \subseteq \omega^\omega$ is $\mathbb{P}$-*measurable* if for every $T \in \mathbb{P}$, there is some $S \leq T$ such that either $[T] \setminus A$ or $[T] \cap A$ is $I_\mathbb{P}$-small.

We have already seen in Section 1.2.6 that every analytic and co-analytic set of reals has the Baire property and is Lebesgue measurable. Both Vitali and Bernstein sets are constructed with explicit use of the axiom of choice and so do not have simple definitions. However, Gödel showed in [Göd38] that, in L, there is a $\mathbf{\Sigma}^1_2$ set of reals which neither has the Baire property nor is Lebesgue measurable. Hence, in ZFC it is neither provable that every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property nor that every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable. In the 1960s, Solovay constructed a model of ZFC in which every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property and is Lebesgue measurable. Therefore, the statements "every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property" and "every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable" are independent from ZFC. The same is true for most other regularity properties. Since these statements are independent of ZFC, one can ask how strong they are viewed as set-theoretic axioms. Solovay proved the following characterization.

Theorem 1.2.53 (Solovay)**.**

(a) *Every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property if and only if for every real $r \in \omega^\omega$, the set $\{x \in 2^\omega : x$ is not a Cohen real over $\mathrm{L}[r]\}$ is meager.*

(b) Every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable if and only if for every real $r \in \omega^\omega$, the set $\{x \in 2^\omega : x$ is not a random real over $\mathrm{L}[r]\}$ is Lebesgue null.

Proof. Cf., e.g., [Sol70, Section III]. $\square$

Later, Judah and Shelah proved a similar characterization result for $\mathbf{\Delta}^1_2$ sets of reals.

Theorem 1.2.54 (Judah-Shelah)**.**

(a) Every $\mathbf{\Delta}^1_2$ set of reals has the Baire property if and only if for every real $r \in \omega^\omega$, there is a Cohen real over $\mathrm{L}[r]$.

(b) Every $\mathbf{\Delta}^1_2$ set of reals is Lebesgue measurable if and only if for every real $r \in \omega^\omega$, there is a random real over $\mathrm{L}[r]$.

Proof. Cf., e.g., [IS89, Theorem 3.1]. $\square$

There were many *Solovay-* and *Judah-Shelah-style* characterization results for other regularity properties and forcing notions; see, e.g., [BL99,BHL05,BL11], and it had been suggested that there must be an underlying general result that can be proved abstractly. This result was finally obtained by Ikegami in [Ike10] where he proved a general Solovay- and Judah-Shelah-style characterization for arboreal forcing notions. This is nowadays called *Ikegami's Theorem.* To prove his characterization result, Ikegami generalized a concept which was first introduced by Brendle, Halbeisen, and Löwe for Silver forcing in [BHL05].

Definition 1.2.55. Let X be an uncountable Polish space and let I be a proper σ-ideal on X, and let M be a transitive model of ZFC. An element $x \in X$ is *I-quasi-generic over M* if for every I-small Borel set B in X whose code is in M, $x \notin B$.

Ikegami proved that every $\mathbb{P}$-generic real over M is $I^*_{\mathbb{P}}$-quasi-generic over M and that both terms coincide if $\mathbb{P}$ satisfies the c.c.c. We shall talk about quasi-generics in more detail in Chapter 2. Table 1.1 lists the quasi-generics for our standard forcing notions.

Forcing notion	Regularity property	Quasi-generics over $L[r]$
Cohen	Baire property	Cohen reals
random	Lebesgue measurable	random reals
Hechler	Baire property dom. topology	Hechler reals
eventually different	Baire property ev. diff. topology	$\mathbb{E}$-generic reals
Sacks	$\mathbb{S}$-measurable	reals not in $L[r]$
Miller	$\mathbb{M}$-measurable	unbounded reals
Laver	$\mathbb{L}$-measurable	dominating reals
Silver	doughnut property	$I_{\mathbb{V}}$-quasi-generic reals
Mathias	Ramsey property [Jec03, pp. 201ff.]	?

Table 1.1: Forcing notions and associated regularity properties

Theorem 1.2.56 (Ikegami)**.** *Let $\mathbb{P}$ be a proper arboreal forcing notion such that $\{c \in \mathsf{BC} : B_c \in I^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$.*

(a) Every $\mathbf{\Delta}^1_2(\omega^\omega)$ set of reals is $\mathbb{P}$-measurable if and only if for every real $r \in \omega^\omega$ and every $T \in \mathbb{P}$, there is an $I^*_{\mathbb{P}}$-quasi-generic real over $\mathrm{L}[r]$.

(b) If $\mathcal{N}_{\mathbb{P}} = I_{\mathbb{P}}$ or $I_{\mathbb{P}}$ is Borel generated, then every $\mathbf{\Sigma}^1_2(\omega^\omega)$ set of reals is $\mathbb{P}$-measurable if and only if for every real $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x \text{ is not } I^*_{\mathbb{P}}\text{-quasi-generic real over } \mathrm{L}[r]\}$ is $I^*_{\mathbb{P}}$-small.

Proof. Cf., [Ike10, Theorems 4.3 & 4.4]. □

Theorem 1.2.56 can be used to obtain Solovay- and Judah-Shelah-style characterizations for all of the forcing notions in Table 1.1 except Mathias forcing. In fact, it is not known whether the Ramsey property for the second level of the projective hierarchy can be characterized using Mathias-quasi-generics. A reason for this is that for the Ramsey-null ideal I_{RN}, the set $\{c \in \mathsf{BC} : B_c \in I_{RN}\}$ is not $\mathbf{\Sigma}^1_2(2^\omega)$ (cf. [Sab12]). Therefore, we cannot apply Ikegami's Theorem. However, Judah and Shelah had already proved in the 1980s a characterization for the Ramsey property using different kinds of reals (cf. [IS89]).

Let $\mathbb{P}$ be a forcing notion from Table 1.1 and let Γ be a projective pointclass, i.e., Γ is $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, or $\mathbf{\Delta}^1_n$ for some $n \geq 1$. We write $\Gamma(\mathbb{P})$ for the statement "every Γ set has the associated regularity property". For example $\mathbf{\Delta}^1_2(\mathbb{C})$ stands for the statement "every $\mathbf{\Delta}^1_2(2^\omega)$ set has the Baire property". The statements $\Gamma(\mathbb{P})$ for the first and second level of the projective hierarchy are well studied. It is known that $\mathbf{\Sigma}^1_1(\mathbb{P})$ and $\mathbf{\Pi}^1_1(\mathbb{P})$ are provable in ZFC and that $\mathbf{\Delta}^1_2(\mathbb{P})$ and $\mathbf{\Sigma}^1_2(\mathbb{P})$ are independent from ZFC. Figure 1.1 summarizes what is known about their consistency strengths for the second level. The diagram is complete in the sense that if there is no arrow from a statement to another, then the implication does not hold in ZFC.[12]

Another well-known regularity property is the perfect set property. A set of reals has the *perfect set property* if it is either countable or contains a perfect set, i.e., a closed set which has no isolated points. Unlike the other regularity properties we have considered so far, the perfect set property is an asymmetric property. By this, we mean that it is not provable in ZFC that the sets which have the perfect set property are closed under complements. Nevertheless, it is provable in ZFC that every analytic set has the perfect set property. However, the question whether all co-analytic sets of reals have the perfect set property cannot be answered in ZFC.

Theorem 1.2.57. *The following are equivalent:*

(a) every co-analytic set of reals has the perfect set property,

(b) every $\mathbf{\Sigma}^1_2$ set of reals has the perfect set property, and

(c) for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Proof. Cf. [Jec03, Theorem 25.38]. □

Let Γ be a projective pointclass. We write $\Gamma(\mathrm{PSP})$ for the statement "every $\Gamma(\omega^\omega)$ set has the perfect set property". By Theorem 1.2.57, $\mathbf{\Delta}^1_2(\mathrm{PSP})$ and $\mathbf{\Sigma}^1_2(\mathrm{PSP})$ are equivalent. Moreover, they imply the existence of an inaccessible cardinal in L.

Theorem 1.2.58. *If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then for every $r \in \omega^\omega$, $\aleph_1$ is inaccessible in $\mathrm{L}[r]$.*

[12] For a long time it was not known if $\mathbf{\Delta}^1_2(\mathbb{L})$ implies $\mathbf{\Delta}^1_2(\mathbb{V})$. This was recently solved in the negative by Banerjee and Gaspar [BG22].

$$
\begin{tikzcd}
 & \boldsymbol{\Sigma}^1_2(\mathbb{D}) = \boldsymbol{\Sigma}^1_2(\mathbb{E}) \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & \boldsymbol{\Sigma}^1_2(\mathbb{B}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\boldsymbol{\Sigma}^1_2(\mathbb{R}) = \boldsymbol{\Delta}^1_2(\mathbb{R}) \arrow[d, Rightarrow] \arrow[ddr, Rightarrow] & \boldsymbol{\Sigma}^1_2(\mathbb{C}) = \boldsymbol{\Delta}^1_2(\mathbb{D}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \boldsymbol{\Delta}^1_2(\mathbb{E}) \arrow[dl, Rightarrow] & \boldsymbol{\Delta}^1_2(\mathbb{B}) \arrow[dddll, Rightarrow, bend left] \\
\boldsymbol{\Sigma}^1_2(\mathbb{L}) = \boldsymbol{\Delta}^1_2(\mathbb{L}) \arrow[dd, Rightarrow] & \boldsymbol{\Delta}^1_2(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & \boldsymbol{\Sigma}^1_2(\mathbb{V}) \arrow[d, Rightarrow] & & \\
\boldsymbol{\Sigma}^1_2(\mathbb{M}) = \boldsymbol{\Delta}^1_2(\mathbb{M}) \arrow[d, Rightarrow] & \boldsymbol{\Delta}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] & & \\
\boldsymbol{\Sigma}^1_2(\mathbb{S}) = \boldsymbol{\Delta}^1_2(\mathbb{S}) & & &
\end{tikzcd}
$$

Figure 1.1: Regularity properties for $\boldsymbol{\Delta}^1_2$ and $\boldsymbol{\Sigma}^1_2$ sets

Proof. Cf. [Kan03, Proposition 11.5]. $\square$

We shall see later that for every forcing notion $\mathbb{P}$ from Table 1.1, if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ holds. In fact, we shall show that this is true for most regularity properties which can be defined using forcing notions (cf. Proposition 2.1.6). Hence, the statement "for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$" is consistency strength wise an upper bound for statements of the form $\boldsymbol{\Delta}^1_2(\mathbb{P})$ and $\boldsymbol{\Sigma}^1_2(\mathbb{P})$. The converse is not true for most forcing notions $\mathbb{P}$ from Table 1.1. Only $\boldsymbol{\Sigma}^1_2(\mathbb{D})$ and $\boldsymbol{\Sigma}^1_2(\mathbb{E})$ imply that for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. In particular, for every forcing notion $\mathbb{P}$ from Table 1.1, $\boldsymbol{\Delta}^1_2(\mathbb{P})$ does not imply that for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. This is also true for regularity properties for which we have Judah-Shelah-style characterizations in the sense of Theorem 1.2.56. The idea is to extend L using an ω_1-iteration of $\mathbb{P}$ with countable support. Then by the Judah-Shelah-style characterization, $\boldsymbol{\Delta}^1_2(\mathbb{P})$ is true the extension. Hence, we can use forcing to produce a model of $\boldsymbol{\Delta}^1_2(\mathbb{P})$. Since the existence of an inaccessible cardinal in L cannot be forced, $\boldsymbol{\Delta}^1_2(\mathbb{P})$ does not imply that for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. Note that this means that there is no Judah-Shelah-style characterization for the perfect set property.

1.2.15 Computable sets

We assume that the reader is familiar with basic concepts of computability theory. For a detailed introduction, we refer the reader to [Soa16]. Let us fix some notation.[13] We fix a universal oracle Turing machine. Let $e, n, m, \sigma \in \omega$ and let $s \in 2^{<\omega}$. We write $\Phi^s_{e,\sigma}(n)\downarrow = m$ if $e, n, m < \sigma$ and the universal oracle Turing machine running the eth program with oracle s and input n halts after $< \sigma$ many steps and outputs m. Moreover, we write $\Phi^s_{e,\sigma}(n)\downarrow$ if there is some $m \in \omega$ such that $\Phi^s_{e,\sigma}(n)\downarrow = m$ and $\Phi^s_{e,\sigma}(n)\uparrow$ otherwise. Let $A \subseteq \omega$ and let $f \in 2^\omega$ be the characteristic function of A. Then we write $\Phi^A_{e,\sigma}$ for $\Phi^{f\restriction\sigma}_{e,\sigma}$ and write $\Phi^A_e(n)\downarrow = m$ if there is some $\sigma \in \omega$ such that $\Phi^A_{e,\sigma}(n)\downarrow = m$. Moreover, we write $\Phi^A_e(n)\downarrow$ if there is some $m \in \omega$ such that $\Phi^A_e(n)\downarrow = m$ and $\Phi^A_e(n)\uparrow$ otherwise. If $A = \emptyset$, then we usually omit it. Then Φ^A_e is a partial function from ω to ω. We say that a partial function p is *computable in* A if there is some $e \in \omega$ such that $p = \Phi^A_e$. If $A = \emptyset$, then we call p just *computable.* A set $B \subseteq \omega$ is called *computable (in A)* if its characteristic function is computable (in A). In this case, we write $B \leq_{\mathrm{T}} A$ and $A \equiv_{\mathrm{T}} B$ if $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$. Then $\equiv_{\mathrm{T}}$ is an equivalence relation. We call the equivalence classes *Turing degrees.*

For $A, B \subseteq \omega$, we write $A \oplus B$ for the set $\{2n : n \in A\} \cup \{2n+1 : n \in B\}$. Let $A \subseteq \omega$. A set $B \subseteq \omega$ is called *computably enumerable (in A)* if B is the domain of a partial function which is computable (in A). For $e \in \omega$, we define $W^A_e := \mathrm{dom}(\Phi^A_e) = \{n \in \omega : \Phi^A_e(n)\downarrow\}$. Then a set $B \subseteq \omega$ is computably enumerable in A if and only if there is some $e \in \omega$ such that $B = W^A_e$. Moreover, the computably enumerable sets can be approximated by computable sets. Let $e, \sigma \in \omega$. We define $W^A_{e,\sigma} := \{n < \sigma : \Phi^A_{e,\sigma}(n)\downarrow\}$. Then $W^A_{e,\sigma}$ is computable in A and for every $n \in \omega$ there is some $\sigma \in \omega$ such that $W^A_e \cap n = W^A_{e,\sigma} \cap n$. The *jump of* A is defined by

$$A' := \{e \in \omega : \Phi^A_e(e)\downarrow\}.$$

Let $n \in \omega$. We recursively define the *nth jump of* A, denoted by $A^{(n)}$, by $A^{(0)} = A$ and $A^{(n+1)} := (A^{(n)})'$. Note that $A^{(1)} = A'$.

We conclude this section with a theorem about the connection between descriptive set theory and computability theory.

Theorem 1.2.59 (Post's Hierarchy Theorem)**.** *Let $n \in \omega$, let $x \in \omega^\omega$ and let $A \subseteq \omega$. Then*

(a) A is computable in x if and only if it is recursive in x,

(b) A is computable in $x^{(n)}$ if and only if it is $\Delta^0_{n+1}(x)$, and

(c) A is computably enumerable in $x^{(n)}$ if and only if it is $\Sigma^0_{n+1}(x)$.

Proof. Cf., e.g., [Soa16, Theorem 4.2.2]. □

[13] Note that we introduce most notation only for sets of natural numbers. However, it can be easily generalized to the reals by identifying reals with sets of natural numbers.

Chapter 2

Regularity properties for forcing notions not living on the reals

Remarks on co-authorship. *The results of this chapter are partly due to a collaboration between Raiean Banerjee and the author. More specifically, all results in Sections 2.2.6, 2.3.2, 2.4, and 2.5 are, unless otherwise stated, joint work with Raiean Banerjee. In the listed sections, both authors contributed equally. The results in the other sections are, unless otherwise stated, solely due to the author. In particular, the results of Sections 2.1 and 2.3.1 and the ideas of Sections 2.2.2, 2.2.3, and 2.2.4 can be found in the author's Master's thesis [Wan19].*

Regularity properties on the reals are a well-studied field of descriptive set theory. However, there are also regularity properties which are not defined on the reals. One of the better known examples is amoeba regularity. Amoeba regularity was first introduced by Judah and Repický in [JR95]. They used amoeba forcing to define a regularity property on a Polish space whose elements are pruned trees $T \subseteq 2^{<\omega}$ such that $\mu([T]) = \frac{1}{2}$. Similar to the most other regularity properties, the amoeba regular sets form a σ-algebra containing all analytic sets and the statement "every $\mathbf{\Delta}^1_2$ set is amoeba regular" is independent from ZFC.

In this chapter, we shall study amoeba regularity and two other regularity properties which do not live on the reals, but on uncountable Polish subspaces of the Baire space. More precisely, in Section 2.1, we shall prove a variant of Ikegami's Theorem for uncountable Polish subspaces of the Baire space. In Section 2.2, we shall use a generalization of category bases to define a general framework for regularity properties. In particular, we shall prove a variant of Ikegami's Theorem for this framework in Section 2.2.3. Then in Section 2.3, we shall use this framework to investigate the consistency strength of the statements "every $\mathbf{\Delta}^1_2$ set is amoeba regular" and "every $\mathbf{\Sigma}^1_2$ set is amoeba regular". In Sections 2.4 and 2.5, we shall define a regularity property for amoeba forcing for category and localization forcing, respectively. Moreover, we shall investigate the consistency strength of the statements "every $\mathbf{\Delta}^1_2$ set is regular" and "every $\mathbf{\Sigma}^1_2$ set is regular" for both regularity properties. Figure 2.1 illustrates our main results from Sections 2.3, 2.4, and 2.5 by adding amoeba regularity and the regularity properties for amoeba forcing for category and localization forcing to Figure 1.1.

In Figure 2.1, the letters $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, $\mathbb{L}$, $\mathbb{LOC}$, $\mathbb{M}$, $\mathbb{R}$, $\mathbb{UM}$, and $\mathbb{V}$ stand for amoeba, random, Cohen, Hechler, eventually different, Laver, localization, Miller, Mathias, amoeba for category, and Silver forcing, respectively. Moreover, $\mathbf{\Delta}^1_2(\mathbb{P})$ is short for the statement "every $\mathbf{\Delta}^1_2$ satisfies the regularity property which is associated to $\mathbb{P}$" and analogously for $\mathbf{\Sigma}^1_2(\mathbb{P})$ (cf. Table 1.1). The diagram is complete in the sense that if there is no arrow from a statement to another, then the implication does not hold in ZFC. It should be noted that all implications and non-implications not involving $\mathbb{A}$, $\mathbb{LOC}$, or $\mathbb{UM}$ were already known before this work (cf. Figure 1.1).

$$
\begin{tikzcd}
 & \forall r \in \omega^\omega (\aleph_1^{\mathrm{L}[r]} < \aleph_1) \arrow[d, Leftrightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{A}) = \mathbf{\Sigma}^1_2(\mathbb{D}) = \mathbf{\Sigma}^1_2(\mathbb{E}) = \mathbf{\Sigma}^1_2(\mathbb{LOC}) = \mathbf{\Sigma}^1_2(\mathbb{UM}) \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & \mathbf{\Delta}^1_2(\mathbb{A}) = \mathbf{\Sigma}^1_2(\mathbb{B}) = \mathbf{\Delta}^1_2(\mathbb{LOC}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{\Sigma}^1_2(\mathbb{R}) = \mathbf{\Delta}^1_2(\mathbb{R}) \arrow[d, Rightarrow] \arrow[ddr, Rightarrow] & \mathbf{\Sigma}^1_2(\mathbb{C}) = \mathbf{\Delta}^1_2(\mathbb{D}) = \mathbf{\Delta}^1_2(\mathbb{UM}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{E}) \arrow[dl, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{B}) \arrow[dddll, Rightarrow, bend left] \\
\mathbf{\Sigma}^1_2(\mathbb{L}) = \mathbf{\Delta}^1_2(\mathbb{L}) \arrow[dd, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & & \\
\mathbf{\Sigma}^1_2(\mathbb{M}) = \mathbf{\Delta}^1_2(\mathbb{M}) \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] & & \\
\mathbf{\Sigma}^1_2(\mathbb{S}) = \mathbf{\Delta}^1_2(\mathbb{S}) & & &
\end{tikzcd}
$$

Figure 2.1: Regularity properties for $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets

2.1 Ikegami's Theorem for general Polish spaces

2.1.1 Regularity properties for ideals

The goal of Section 2.1 is to generalize Ikegami's Theorem to uncountable Polish spaces. However, we shall not generalize Theorem 1.2.56 directly, but a version of Ikegami's Theorem for regularity properties which are defined from σ-ideals.

Definition 2.1.1. Let X be an uncountable Polish space and let I be a proper σ-ideal on X. A set $A \subseteq X$ is called I*-regular* if for every I-positive Borel set B in X, there is an I-positive Borel set $B' \subseteq B$ such that either $B' \subseteq A$ or $B' \cap A = \emptyset$.

The notion of I-regularity was first introduced by Khomskii in [Kho12]. In his work, Khomskii focused only on σ-ideals on the reals. We shall investigate I-regularity for σ-ideals on general uncountable Polish spaces in Section 2.1. Here, we discuss a few of Khomskii's results, which will be helpful later.

Proposition 2.1.2 (Khomskii)**.** *Let I be a proper σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper.*

(a) The I-regular sets form a σ-algebra on ω^ω containing all analytic and co-analytic sets.

(b) In L*, there is a $\boldsymbol{\Delta}^1_2(\omega^\omega)$ set which is not I-regular.*

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\boldsymbol{\Sigma}^1_2(\omega^\omega)$ sets are I-regular.

Proof. Cf. [Kho12, Lemma 2.2.2, Propositions 2.2.3 & 2.2.4, and Corollary 2.2.7]. □

Item (b) of Proposition 2.1.2 utilizes the fact that one can use the $\boldsymbol{\Delta}^1_2(\omega^\omega \times \omega^\omega)$ well-ordering of the reals to construct a $\boldsymbol{\Delta}^1_2(\omega^\omega)$ Bernstein set, i.e., a set $A \subseteq \omega^\omega$ such neither A nor $\omega^\omega \setminus A$ contains a non-empty perfect set. Such a set never can be I-regular: let I be a proper σ-ideal on ω^ω, let $A \subseteq \omega^\omega$ be a Bernstein set, and let B be an I-positive Borel set. Since every Borel set has the perfect set property, B contains a perfect set. Therefore, B meets both A and $\omega^\omega \setminus A$ and so A cannot be I-regular.

A highlight of Khomskii's work was a version of Ikegami's Theorem for idealized forcing notions. For this, Khomskii defined a second ideal which coincides with I on Borel sets.

Definition 2.1.3. Let X be an uncountable Polish space and let I be a proper σ-ideal on X. A set $A \subseteq \omega^\omega$ is called *I-null* if for every I-positive Borel set B in X, there is an I-positive Borel set $B' \subseteq B$ such that $B' \cap A = \emptyset$. We denote the collection of all I-null sets by $\mathcal{N}_I$.

Proposition 2.1.4 (Khomskii)**.** *Let I be a proper σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper.*

(a) Then $\mathcal{N}_I$ is a σ-ideal which coincides with I on Borel sets.

(b) If I is Borel generated and $\mathbb{P}_I$ satisfies the c.c.c., then $\mathcal{N}_I = I$.

Proof. Cf. [Kho12, Lemma 2.1.10]. □

Theorem 2.1.5 (Khomskii's version of Ikegami's Theorem)**.** *Let I be a proper σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper and $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2(2^\omega)$.*

(a) Every $\boldsymbol{\Delta}^1_2(\omega^\omega)$ set of reals is I-regular if and only if for every $r \in \omega^\omega$ and every I-positive Borel set B, there is an I-quasi-generic real over L$[r]$ *in B.*

(b) Every $\boldsymbol{\Sigma}^1_2(\omega^\omega)$ set of reals is I-regular if and only if for every $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x$ is not I-quasi-generic over L$[r]\}$ *is I-null.*

Proof. Cf. [Kho12, Theorem 2.3.7 & Corollary 2.3.8]. □

2.1.2 Ikegami's Theorem for σ-ideals on general Polish spaces

In this section, we generalize Khomskii's version of Ikegami's Theorem (Theorem 2.1.5) to uncountable Polish spaces. This was done by the author in his Master's thesis [Wan19]: a proof of Theorem 2.1.12 can be found there. While the proof from [Wan19] was a modification of Khomskii's proof of Ikegami's Theorem, we provide an alternative, much simpler proof by reducing Theorem 2.1.12 to Khomskii's version of Ikegami's Theorem.

Before we can generalize Khomskii's version of Ikegami's Theorem, we have to first investigate I-regularity for σ-ideals on general uncountable Polish spaces and establish some basic properties.

The idea is to define a second σ-ideal on the reals and then to derive the desired properties for our original σ-ideal from it. More precisely, let X be an uncountable Polish space. By Theorem 1.2.4, there is a Borel isomorphism $f : X \to \omega^\omega$. Let I be a σ-ideal on X. We define $I_f := \{f[A] : A \in I\}$. Then I_f is a proper σ-ideal and f induces an isomorphism between $\mathbb{P}_I$ and $\mathbb{P}_{I_f}$. Hence, $\mathbb{P}_I$ and $\mathbb{P}_{I_f}$ are forcing equivalent and a set $A \subseteq X$ is I-regular (or I-null) if and only if $f[A]$ is I_f-regular (or I_f-null). Let Γ be a projective pointclass. Since f and f^{-1} are Borel, every $\Gamma(X)$ set is I-regular if and only if every $\Gamma(\omega^\omega)$ set is I_f-regular. Using this fact, we obtain the following proposition.

Proposition 2.1.6. *Let X be an uncountable Polish space and let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper.*

(a) The I-regular sets form a σ-algebra on X containing all analytic and co-analytic sets in X.

(b) In L, there is a $\mathbf{\Delta}^1_2(X)$ set which is not I-regular.

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\mathbf{\Sigma}^1_2(X)$ sets are I-regular.

(d) The set $\mathcal{N}_I$ is a σ-ideal on X which coincides with I on Borel sets in X.

(e) If I is Borel generated and $\mathbb{P}_I$ satisfies the c.c.c., then $\mathcal{N}_I = I$.

Proof. Follows directly from Propositions 2.1.2 and 2.1.4. □

Moreover, we obtain Solovay- and Judah-Shelah-style characterizations for σ-ideals on uncountable Polish spaces.

Proposition 2.1.7. *Let X be an uncountable Polish space, let $f : X \to \omega^\omega$ be a Borel isomorphism, and let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper and $\{c \in \mathsf{BC} : B_c \in I_f\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$.*

(a) Every $\mathbf{\Delta}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$ and every I_f-positive Borel set B in ω^ω, there is an I_f-quasi-generic real over $\mathrm{L}[r]$ in B.

(b) Every $\mathbf{\Sigma}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x$ is not I_f-quasi-generic over $\mathrm{L}[r]\}$ is I_f-null.

Proof. Follows directly from Khomskii's version of Ikegami's Theorem (Theorem 2.1.5) and the definition of I_f. □

Proposition 2.1.7 links I-regularity to I_f-quasi-genericity. However, we would prefer Solovay- and Judah-Shelah-style characterizations that link I-regularity directly to I-quasi-genericity. This inconvenience cannot simply be fixed by replacing I_f-quasi-genericity with I-quasi-genericity since we do not know whether f maps I-quasi-generics to I_f-quasi-generics and vice versa. Recall that an element from X is I-quasi-generic over an inner model M of ZFC if it omits all I-small Borel sets coded in M and similarly for I_f. Clearly, f maps I-small sets to I_f-small sets, but still there is no reason why f should map Borel sets coded in M to Borel sets coded in M. Hence, it might be possible that f does not preserve quasi-genericity.

To fix this problem, from now on we only consider σ-ideals on uncountable Polish subspaces of the Baire space. Let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X. We can extend I to a σ-ideal $\hat{I}$ on ω^ω by putting everything which does not meet X in $\hat{I}$. Formally, we define $\hat{I} := \{A \subseteq \omega^\omega : A \cap X \in I\}$. Then $\hat{I}$ is a proper σ-ideal on ω^ω which coincides with I on subsets of X and for every $\hat{I}$-positive Borel set $B \subseteq \omega^\omega$, $B \cap X$ is still an $\hat{I}$-positive Borel set. Hence, $\mathbb{P}_I$ is a dense subset of $\mathbb{P}_{\hat{I}}$ and so they are forcing equivalent. Moreover, even their generics coincide.

Proposition 2.1.8. *Let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X. A real is $\mathbb{P}_I$-generic over* V *if and only if it is $\mathbb{P}_{\hat{I}}$-generic over* V.

Proof. We start with the forward direction. Let $x \in \omega^\omega$ be $\mathbb{P}_I$-generic over V. Then there is a $\mathbb{P}_I$-generic filter G over V such that for every Borel set $B \subseteq X$ coded in V, $B \in G$ if and only if $x \in B$. Let $H := \{B \in \mathbb{P}_{\hat{I}} : \exists B' \in G(B' \subseteq B)\}$. Since $\mathbb{P}_I$ is a dense subset of $\mathbb{P}_{\hat{I}}$, H is a $\mathbb{P}_{\hat{I}}$-generic filter over V. It remains to check that for every Borel set of reals coded in V, $B \in H$ if and only if $x \in B$. Let $B \subseteq \omega^\omega$ be a Borel set coded in V. If $B \in H$, then there is some $B' \in G$ such that $B' \subseteq B$. Since $B' \in G$, $x \in B' \subseteq B$. On the other hand, if $x \in B$, then $x \in B \cap X$. Thus, $B \cap X \in G$ and so $B \in H$.

We prove the backward direction. Let $x \in \omega^\omega$ be $\mathbb{P}_{\hat{I}}$-generic over V. Then there is a $\mathbb{P}_{\hat{I}}$-generic filter H over V such that for every Borel set $B \subseteq \omega^\omega$ coded in V, $B \in H$ if and only if $x \in B$. Let $G := H \cap \mathbb{P}_I$. Then G is a $\mathbb{P}_I$-generic filter over V and for every Borel set $B \subseteq X$ coded in V, $B \in G$ if and only if $x \in B$. □

The fact that $\mathbb{P}_I$ is a dense subset of $\mathbb{P}_{\hat{I}}$ also tells us that I-regularity and $\hat{I}$-regularity coincide on subsets of X. In fact, we can prove even more.

Proposition 2.1.9. *Let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X. A set of reals $A \subseteq \omega^\omega$ is $\hat{I}$-regular if and only if $A \cap X$ is I-regular.*

Proof. Let $A \subseteq \omega^\omega$ be a set of reals. We start with the forward direction. Let B be an I-positive Borel set in X. Then B is also Borel in ω^ω and by definition $\hat{I}$-positive. Since A is $\hat{I}$-regular, there is an $\hat{I}$-positive Borel set $B' \subseteq B$ such that either $B' \subseteq A$ or $B' \cap A = \emptyset$. Then $B' \subseteq B \subseteq X$ and so B' is an I-positive Borel set in X such that either $B' \subseteq A \cap X$ or $B' \cap (A \cap X) = \emptyset$.

We prove the backward direction. Let B be an $\hat{I}$-positive Borel set in ω^ω. Then $B' := B \cap X$ is an I-positive Borel set in X. Since $A \cap X$ is I-regular, there is an I-positive Borel set $B'' \subseteq B'$ such that either $B'' \subseteq A \cap X$ or $B'' \cap (A \cap X) = \emptyset$. Then B'' is an $\hat{I}$-positive Borel set in ω^ω such that either $B'' \subseteq A$ or $B'' \cap A = \emptyset$. □

Since X is a Polish subspace of the Baire space, X is a $\mathbf{\Pi}^0_2(\omega^\omega)$ set of reals. Let Γ be a projective pointclass. Then A is in $\Gamma(\omega^\omega)$ if and only if $A \cap X$ is in $\Gamma(X)$. This means that every $\Gamma(X)$ set is I-regular if and only if every $\Gamma(\omega^\omega)$ set is $\hat{I}$-regular. As before, we get Solovay- and Judah-Shelah-style characterizations for I-regularity. Again, these do not really depend on I, but on $\hat{I}$. To change this, we need to investigate the connection between I-quasi-genericity and $\hat{I}$-quasi-genericity.

Proposition 2.1.10. *Let X be an uncountable Polish subspace of the Baire space, let I be a proper σ-ideal on X, and let M be an inner model of* ZFC *such that X is coded in M. Then a real is I-quasi-generic over M if it is $\hat{I}$-quasi-generic over M.*

Proof. By definition, $\omega^\omega \setminus X$ is $\hat{I}$-small. Hence, every $\hat{I}$-quasi-generic real over M is in X. Furthermore, a subset of X is I-small if and only if it is $\hat{I}$-small. Therefore, a real is I-quasi-generic over M if and only if it is $\hat{I}$-quasi-generic over M. □

Before we can prove a variant of Ikegami's Theorem for σ-ideals on uncountable Polish subspaces of the Baire space which only depends on I, we need one last lemma.

Lemma 2.1.11. *Let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X such that $\{c \in \mathsf{BC} : B_c \in I\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$. Then $\{c \in \mathsf{BC} : B_c \in \hat{I}\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$ as well.*

Proof. Let $c \in \mathsf{BC}$ be a Borel code. By definition, B_c is $\hat{I}$-small if and only if $B_c \cap X$ is I-small. Hence,

$$B_c \in \hat{I} \iff \exists c' \in \mathsf{BC}(B_{c'} \in I \wedge \forall x \in X(x \in B_c \leftrightarrow x \in B_{c'} \cap X)).$$

By Lemma 1.2.35, $\{c \in \mathsf{BC} : B_c \in \hat{I}\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$. □

Now we can put everything together and get a variant of Ikegami's Theorem for uncountable subspaces of the Baire space which only depends on I.

Theorem 2.1.12. *Let X be an uncountable Polish subspace of the Baire space and let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper and $\{c \in \mathsf{BC} : B_c \in I\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$.*

(a) Every $\mathbf{\Delta}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$ such that X is coded in L$[r]$ *and every I-positive Borel set B in X, there is an I-quasi-generic element over* L$[r]$ *in B.*

(b) Every $\mathbf{\Sigma}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$ such that X is coded in L$[r]$*, the set $\{x \in X : x$ is not I-quasi-generic over* L$[r]\}$ *is I-null.*

Proof. We start with proving (a). By Khomskii's version of Ikegami's Theorem (Theorem 2.1.5), we only have to check that for every $r \in \omega^\omega$ such that X is coded in L$[r]$ and every I-positive Borel set B in X, there is an I-quasi-generic element over L$[r]$ in B if and only if for every $r \in \omega^\omega$ and every $\hat{I}$-positive Borel set B in ω^ω, there is an $\hat{I}$-quasi-generic element over L$[r]$ in B. The backward direction follows directly from Proposition 2.1.10. Hence, we only have to show the forward direction. Let $r \in \omega^\omega$ and let B be an $\hat{I}$-positive Borel set in ω^ω. Without loss of generality, we can assume that X is not coded in L$[r]$ since otherwise we can just use Proposition 2.1.10. Let $a \in \omega^\omega$ such that X is coded in L$[a]$. By assumption, there is an I-quasi-generic element $x \in B$ over L$[r, a]$. Then x is also I-quasi-generic over L$[r]$. The proof of (b) is analogous. □

2.1.3 The special case for idealized forcing notions satisfying the c.c.c.

In this section, we consider Theorem 2.1.12 in the special case that $\mathbb{P}_I$ satisfies the c.c.c. Recall that Ikegami has shown that for forcing notions satisfying the c.c.c. generics and quasi-generics coincide. However, before we can replace I-quasi-genericity with $\mathbb{P}_I$-genericity in Theorem 2.1.12, we have to make sure that for inner models, $\mathbb{P}_I$-genericity is well-defined. Let X be an uncountable Polish subspace of the Baire space. Since X is a Polish subspace of ω^ω, X is $\mathbf{\Pi}^0_2(\omega^\omega)$. Let φ be a Π^0_2 formula with parameter $a \in \omega^\omega$ which defines X and let M be an inner model of ZFC containing a. Then $X \cap M = \{x : M \models \varphi(x, a)\}$ is $\mathbf{\Pi}^0_2(\omega^\omega)$ in M. Hence, $X \cap M$ is an uncountable Polish subspace of the reals in M. Next, let I be a proper σ-ideal on X such that $I_{\mathsf{BC}} := \{c \in \mathsf{BC} : B_c \in I\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$ and let ψ be a Σ^1_2 formula with parameter $a' \in \omega^\omega$ defining I_{BC}. By Shoenfield absoluteness, if $a' \in M$, then $I_{\mathsf{BC}} \cap M = \{c : M \models \psi(c, a')\}$ is in M. Let I^M be the ideal on X, in M, generated by $\{B_c : c \in I_{\mathsf{BC}} \cap M\}$. Then I^M is a Borel generated, proper σ-ideal on X in M. We write $\mathbb{P}^M_I$ for the idealized forcing notion defined from I^M in M. Then for every $c \in \mathsf{BC}$, $B^M_c \in \mathbb{P}^M_I$ if and only if $c \in M$ and $B^{\mathrm{V}}_c \in \mathbb{P}_I$. In the following, we often omit the M in $\mathbb{P}^M_I$ if it is clear from the context, e.g., we write $\mathbb{P}_I$-generic over M if we mean $\mathbb{P}^M_I$-generic over M.

Proposition 2.1.13 (Ikegami)**.** *Let X be an uncountable Polish subspace of the Baire space, let I be a Borel generated, proper σ-ideal on X such that $\{c \in \mathsf{BC} : B_c \in I\}$ is $\mathbf{\Sigma}^1_2$, let $a \in \omega^\omega$ such that X and I_{BC} are $\Pi^0_2(a)$ and $\Sigma^1_2(a)$, respectively, and let M be an inner model of* ZFC *such that*

$a \in M$. If $\mathbb{P}_I$ satisfies the c.c.c. in M, then an element of X is $\mathbb{P}_I$-generic over M if and only if it is I-quasi-generic over M.

Proof. By definition, every $\mathbb{P}_I$-generic over M is I-quasi-generic over M. Let $x \in X$ be an I-quasi-generic element over M and let $G_x := \{B \in \mathbb{P}_I : x \in B\}$. We have to show that G_x is a $\mathbb{P}_I$-generic filter over M. It is clear that G_x is a filter. Let $D \subseteq \mathbb{P}_I$ be dense in M. Then there is a maximal antichain $\mathcal{A} \subseteq D$. Since $\mathbb{P}_I$ satisfies the c.c.c. in M, $B := \bigcup \mathcal{A}$ is a Borel set coded in M. Moreover, $X \setminus B$ is I-null in M. By Proposition 2.1.6, $X \setminus B$ is I-small in M. Since $\{c \in \mathsf{BC} : B_c \in I\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$, $X \setminus B$ is I-small in V by Shoefield absoluteness. Hence, $x \notin X \setminus B$. Then $x \in B$ and so there is some $B' \in \mathcal{A}$ such that $x \in B'$. Therefore, B' witnesses that G meets D. □

Proposition 2.1.13 is not necessarily true if $\mathbb{P}_I$ does not satisfy the c.c.c.: note that every real which is not in M is **ctbl**-quasi-generic over M. Since every Borel set has the perfect set property, we can find a dense embedding from Sacks forcing into $\mathbb{P}_{\mathbf{ctbl}}$. Hence, $\mathbb{S}$ and $\mathbb{P}_{\mathbf{ctbl}}$ are forcing equivalent. Let x be a Cohen real over M. It is well-known that $M[x]$ does not contain any Sacks reals over M. Therefore, x is **ctbl**-quasi-generic over M but not $\mathbb{P}_{\mathbf{ctbl}}$-generic over M.

Using Proposition 2.1.13, we can replace I-quasi-genericity with $\mathbb{P}_I$-genericity in Theorem 2.1.12. We can simplify Theorem 2.1.12 even more if we also assume that I is Borel generated. Recall that if I is Borel generated and $\mathbb{P}_I$ satisfies the c.c.c., then by Proposition 2.1.6, $\mathcal{N}_I = I$. Hence, in this case, we can replace I-null with I-small in the Solovay-style characterization. To simplify the Judah-Shelah-style characterization, we use another result of Khomskii.

Proposition 2.1.14 (Khomskii)**.** *Let I be a proper, Borel generated σ-ideal on X such that $\{c \in \mathsf{BC} : B_c \in I\}$ is $\Sigma^1_2(a)$. If for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, there is a $\mathbb{P}_I$-generic real over $\mathrm{L}[r]$, then every $\mathbf{\Delta}^1_2(\omega^\omega)$ set of reals is I-regular.*

Proof. Cf. [Kho12, Proposition 2.2.5]. □

By Proposition 2.1.14, we can formulate the Judah-Shelah-style characterization without quantifying over I-positive Borel sets. Putting it all together, we get the following version of Ikegami's Theorem.

Corollary 2.1.15. *Let X be an uncountable Polish subspace of the Baire space, let $a \in \omega^\omega$ such that X is $\Pi^0_2(a)$, and let I be a proper, Borel generated σ-ideal on X such that $\{c \in \mathsf{BC} : B_c \in I\}$ is $\Sigma^1_2(a)$ and $\mathbb{P}_I$ satisfies the c.c.c. in every inner model of ZFC containing a.*

(a) Every $\mathbf{\Delta}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, there is an $\mathbb{P}_I$-generic element over $\mathrm{L}[r]$.

(b) Every $\mathbf{\Sigma}^1_2(X)$ set is I-regular if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, the set $\{x \in X : x$ is not $\mathbb{P}_I$-generic over $\mathrm{L}[r]\}$ is I-small.

Proof. Follows directly from Propositions 2.1.6, 2.1.13, 2.1.14 and Theorem 2.1.12. □

2.2 A general framework for regularity properties

2.2.1 Category bases

The goal of Section 2.2 is to provide a general framework for regularity properties. A good candidate for such a framework are category bases, which were first introduced by Morgan in [Mor77] to generalize the Baire property. They were also used by Judah and Repický in [JR95] to define amoeba regularity. In this section, we shall give a brief introduction to category bases and discuss some of Morgan's results. For a complete introduction, we refer the reader to [Mor90].

Definition 2.2.1. Let X be a set and let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a set of non-empty subsets of X. We say that $(X, \mathcal{C})$ is a *category base* if

(a) $X = \bigcup \mathcal{C}$ and

(b) for every $A \in \mathcal{C}$ and every non-empty disjoint family $\mathcal{C}' \subseteq \mathcal{C}$ with $|\mathcal{C}'| < |\mathcal{C}|$,

 (i) if $A \cap \bigcup \mathcal{C}'$ contains some element of $\mathcal{C}$, then there is a $C \in \mathcal{C}'$ such that $A \cap C$ contains some element of $\mathcal{C}$, and

 (ii) if $A \cap \bigcup \mathcal{C}'$ does not contain an element of $\mathcal{C}$, then there is some $A' \in \mathcal{C}$ such that $A' \subseteq A$ and $A' \cap \bigcup \mathcal{C}' = \emptyset$.

We call the elements of $\mathcal{C}$ *regions.*

Category bases are a generalization of topological spaces. In fact, every topological space induces a category base: let X be a topological space and let $\mathcal{C}$ be the set of non-empty open sets in X. It is clear that $\bigcup \mathcal{C} = X$. Let $O \in \mathcal{C}$ be a non-empty open set and let $\mathcal{C}' \subseteq \mathcal{C}$ be a family of disjoint open sets. If $O \cap \bigcup \mathcal{C}'$ contains a non-empty open set, then there is some open set $U \in \mathcal{C}'$ such that $O \cap U \neq \emptyset$. If $A \cap \bigcup \mathcal{C}'$ does not contain a non-empty open set, then $A \cap \bigcup \mathcal{C}' = \emptyset$. Therefore, $(X, \mathcal{C})$ is a category base. Note that if $\mathcal{C}' \subseteq \mathcal{C}$ is a basis for $\mathcal{C}$, then $(X, \mathcal{C}')$ is a category base as well.

Morgan defined for every category base a regularity property which coincides with the Baire property for topological spaces. For this purpose, he first generalized the concept of meager sets.

Definition 2.2.2. Let $(X, \mathcal{C})$ be a category base. A set $A \subseteq X$ is $\mathcal{C}$*-singular* if for every region $C \in \mathcal{C}$, there is a region $C' \subseteq C$ such that $C' \cap A = \emptyset$. A set is $\mathcal{C}$*-meager* if it is a countable union of $\mathcal{C}$-singular sets. We denote the σ-ideal of all $\mathcal{C}$-meager sets by $I_{\mathcal{C}}$.

Note that for topological spaces this is the alternative definition of nowhere dense sets from Proposition 1.2.22. Therefore, a set is nowhere dense if and only if it is singular and so the meager sets coincide as well. But there is a big difference between topological spaces and category bases. Unlike open sets, the intersection of two regions is in general not necessarily either region or empty (cf. Theorem 2.3.4). However, Morgan proved something slightly weaker. Let $(X, \mathcal{C})$ be a category base. Two sets are *essentially disjoint in* $(X, \mathcal{C})$ if their intersection is $\mathcal{C}$-meager.

Theorem 2.2.3 (Morgan)**.** *Let $(X, \mathcal{C})$ be a category base. The intersection of two regions either contains a region or is essentially disjoint in $(X, \mathcal{C})$. In fact, it either contains a region or is $\mathcal{C}$-singular.*

Proof. Cf. [Mor90, Chaper 1, Section II, Theorem 2]. □

In what follows, we will often use category bases ordered by inclusion as forcing notions. Let $(X, \mathcal{C})$ be a category base such that every region is not $\mathcal{C}$-meager. By Theorem 2.2.3, two regions are incompatible in $(\mathcal{C}, \subseteq)$ if and only if they are essentially disjoint in $(X, \mathcal{C})$. Therefore, a family of regions is an antichain in $(\mathcal{C}, \subseteq)$ if and only if its elements are pairwise essentially disjoint in $(X, \mathcal{C})$. This leads us to the following definition.

Definition 2.2.4. Let $(X, \mathcal{C})$ be a category base. We say that $(X, \mathcal{C})$ satisfies the *countable chain condition (c.c.c.)* if every family of pairwise essentially disjoint regions is countable.

Note that Morgan defined a category base to satisfy the c.c.c. if every disjoint family of regions is countable. Morgan's definition may seem a bit weaker at first sight, but he showed that it is equivalent to Definition 2.2.4 if every region is non-meager.

Proposition 2.2.5 (Morgan)**.** *Let $(X, \mathcal{C})$ be a category base such that every region is non-$\mathcal{C}$-meager. Then $(X, \mathcal{C})$ satisfies the c.c.c. if and only if every disjoint family of regions is countable.*

Proof. The forward direction is clear. We prove the backward direction. Let $\mathcal{C}' \subseteq \mathcal{C}$ be a pairwise essential disjoint family of regions. We assume for a contradiction that $\mathcal{C}'$ is uncountable. Without loss of generality, we can assume that $|\mathcal{C}'| = \aleph_1$. Let $\{C_\alpha : \alpha < \omega_1\}$ be an enumeration. We define a disjoint family $\{C'_\alpha : \alpha < \omega_1\} \subseteq \mathcal{C}$ such that for every $\alpha < \omega_1$, C'_α is a subregion of C_α by recursion. We set $C'_0 := C_0$. If C'_β is already defined for every $\beta < \alpha$, then $\{C'_\beta : \beta < \alpha\}$ is a countable disjoint family. We make a case-distinction:

Case 1: $C_\alpha \cap \bigcup\{C'_\beta : \beta < \alpha\}$ contains a region. Then there is a $\beta < \alpha$, such that $C_\alpha \cap C'_\beta$ contains a region. But this is not possible since C_α and C_β are essentially disjoint.

Case 2: $C_\alpha \cap \bigcup\{C'_\beta : \beta < \alpha\}$ contains no region. Then there is a subregion $C \subseteq C_\alpha$ such that for every $\beta < \alpha$, $C \cap C_\beta = \emptyset$. We set $C'_\beta := C$.

Then $\{C'_\alpha : \alpha < \omega_1\}$ is an uncountable disjoint family of regions. But this is not possible by assumption. Therefore, $(X, \mathcal{C})$ satisfies the c.c.c. $\square$

Most of the time we are only interested in category bases whose regions are non-meager. Therefore, it does not really matter which definition we choose for category bases. The reason we use the "stronger" version as our definition is that in Section 2.2.2 we deal with a generalization of category bases for which the author does not know whether Proposition 2.2.5 holds.

Note that a category base $(X, \mathcal{C})$ whose regions are non-$\mathcal{C}$-meager satisfies the c.c.c. as a category base if and only if $(\mathcal{C}, \subseteq)$ satisfies the c.c.c. as a forcing notion. Other non-trivial examples of category bases satisfying the c.c.c. are Polish spaces.

Proposition 2.2.6. *Let X be a Polish topological space and let $\mathcal{C}$ be the set of non-empty open sets. Then $(X, \mathcal{C})$ satisfies the c.c.c.*

Proof. Let $\mathcal{C}' \subseteq \mathcal{C}$ be an essentially disjoint family of non-empty open sets. In a Polish space every non-empty open set is non-meager. Hence, the elements of $\mathcal{C}'$ are pairwise disjoint. Let $D \subseteq X$ be a countable dense set and for every $C \in \mathcal{C}$, let $x_C \in C \cap D$. Then for every $C \neq C' \in \mathcal{C}'$, $x_C \neq x_{C'}$. Therefore, $\mathcal{C}'$ is countable. $\square$

Recall that Morgan introduced category bases as a more general carrier for the Baire property. Recall that in a topological space, a set has the Baire property if its symmetric difference with some open set is meager. If we apply this definition one-to-one to category bases, then the sets with the Baire property may not form a σ-algebra: let X be an infinite set and let $\mathcal{C}$ be the set of all singletons of X. Then $(X, \mathcal{C})$ is a category base and only the empty set is $\mathcal{C}$-meager. Hence, a subset of X has the Baire property if and only if it is a singleton. Then the collection of sets having the Baire property is neither closed under complements nor under unions. However, there are many other equivalent definitions of the Baire property (cf., e.g., [Kur66, §11.IV]). Morgan used one of them to define the Baire property for category bases.

Definition 2.2.7. Let $(X, \mathcal{C})$ be a category base. A set $A \subseteq X$ is *$\mathcal{C}$-Baire* if for every region $C \in \mathcal{C}$, there is a region $C' \subseteq C$ such that $C' \setminus A$ or $C' \cap A$ is $\mathcal{C}$-meager. We denote the collection of all $\mathcal{C}$-Baire sets by $\mathfrak{B}(\mathcal{C})$.

Note that this is the alternative definition of the Baire property from Proposition 1.2.22. Therefore, a subset of a Polish space has the Baire property if and only if it is Baire and so the Baire sets form a σ-algebra. Morgan showed that the latter is even true for general category bases.

Theorem 2.2.8 (Morgan)**.** *Let $(X, \mathcal{C})$ be a category base. Then $\mathfrak{B}(\mathcal{C})$ is a σ-algebra containing all regions. Moreover, if $(X, \mathcal{C})$ satisfies the c.c.c., then $\mathfrak{B}(\mathcal{C})$ is the smallest σ-algebra containing all regions and $\mathcal{C}$-meager sets.*

Proof. Cf. [Mor90, Chaper 1, Section III, Theorems 6 & 8]. □

2.2.2 Weak category bases

In this section, we introduce a generalization of category bases, which we then use as the basis of our framework. The reason why we use a generalization of category bases rather than category bases themselves is that we want to include I-regularity and $\mathbb{P}$-measurability in our framework and it is not known whether they can always be expressed in terms of category bases. By this, we mean that it is not known whether for every proper σ-ideal I on a topological space X, there is always a category base whose Baire sets are the I-regular sets and similarly for arboreal forcing notions $\mathbb{P}$ on 2 (or ω). Natural candidates for such category bases would be $(X, \mathbb{P}_I)$ or $(2^\omega, \{[T] : T \in \mathbb{P}\})$ (or $(\omega^\omega, \{[T] : T \in \mathbb{P}\})$). However, it is not known whether these are always category bases (cf. Question 2.6.1).

Definition 2.2.9. Let X be a set and let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a set of non-empty subsets of X. We say that $(X, \mathcal{C})$ is a *weak category base* if

(a) $X = \bigcup \mathcal{C}$ and

(b) for every $A, A' \in \mathcal{C}$, $A \cap A'$ contains an element of $\mathcal{C}$ or for every $C \in \mathcal{C}$, there is some $C' \in \mathcal{C}$ such that $C' \subseteq C$ and $C' \cap (A \cap A') = \emptyset$.

We adopt the terms *region*, *$\mathcal{C}$-singular*, *$\mathcal{C}$-meager*, *$\mathcal{C}$-Baire*, and *c.c.c.* from category bases.

In other words, (b) of Definition 2.2.9 says that the intersection of two regions must contain a region or be $\mathcal{C}$-singular. Thus, by Theorem 2.2.3, every category base is a weak category base. Next, we show that all regularity properties considered so far, except the perfect set property, can be characterized using weak category bases.

Proposition 2.2.10.

(a) Let X be a topological space and let $\mathcal{C}$ be the set of non-empty open sets or a basis for the topology of X. Then $(X, \mathcal{C})$ is a weak category base. Moreover, if $(X, \mathcal{C})$ satisfies the c.c.c., then a set $A \subseteq X$ has the Baire property if and only if it is $\mathcal{C}$-Baire.

(b) Every category base is a weak category base.

(c) Let I be a proper σ-ideal on a topological space X such that $\mathbb{P}_I$ is proper. Then $(X, \mathbb{P}_I)$ is a weak category base and a set $A \subseteq X$ is I-regular if and only if it is $\mathbb{P}_I$-Baire.

(d) *Let $\mathbb{P}$ be an arboreal forcing notion on 2 (or ω) and let $\mathcal{C}_{\mathbb{P}} := \{[T] : T \in \mathbb{P}\}$. Then $(2^\omega, \mathcal{C}_{\mathbb{P}})$ (or $(\omega^\omega, \mathcal{C}_{\mathbb{P}})$) is a weak category base and a set $A \subseteq X$ is $\mathbb{P}$-measurable if and only if it is $\mathcal{C}_{\mathbb{P}}$-Baire.*

Proof. Item (a) follows directly from (b) and Proposition 1.2.22 and (b) follows directly from Theorem 2.2.3. So we only have to prove (c) and (d). We start with proving (c). Let B, B' be I-positive Borel sets such that $B \cap B'$ is I-small. Since $\mathbb{P}_I$ is proper, $B \cap B'$ is I-null. Hence, $(X, \mathbb{P}_I)$ is a weak category base. It remains to show that a subset of X is I-regular if and only if it is $\mathbb{P}_I$-Baire. The forward direction is clear. Let $A \subseteq X$ be $\mathbb{P}_I$-Baire and let B be an I-positive Borel set in X. Then there is an I-positive Borel set $B' \subseteq B$ such that either $B' \setminus A$ or $B' \cap A$ is $\mathbb{P}_I$-meager. Note that a set is $\mathbb{P}_I$-singular if and only if it is I-null. Since $\mathbb{P}_I$ is proper, $\mathcal{N}_I$ is a σ-ideal. Hence, a set is $\mathbb{P}_I$-meager if and only if it is I-null and so either $B' \setminus A$ or $B' \cap A$ is I-null. Then there is an I-positive Borel set $B'' \subseteq B'$ such that either $B'' \cap (B' \setminus A)$ or $B'' \cap (B' \cap A)$ is empty. In the former case $B'' \subseteq A$ and in the latter case $B'' \cap A = \emptyset$. Therefore, A is I-regular.

It remains to prove (d). Let $T, T' \in \mathbb{P}$ be incompatible and let $S \in \mathbb{P}$. Since T and T' are incompatible, there is some $s \in S \setminus (T \cap T')$. Let $S' \in \mathbb{P}$ such that $S' \leq S$ and $s \subseteq \mathrm{stem}(S')$. Then $[S'] \cap ([T] \cap [T']) = \emptyset$ and so $(2^\omega, \mathcal{C}_{\mathbb{P}})$ (or $(\omega^\omega, \mathcal{C}_{\mathbb{P}})$) is a weak category base. The second part follows directly from the definitions. □

We shall investigate the converse of (c) of Proposition 2.2.10 in Section 2.2.3. Here, we show that the converse of (b) is not true, i.e., we show that there are weak category bases which are not category bases: let $X = \omega_2$ and let $\{A_\alpha : \alpha \leq \omega_2\}$ be a partition of X in ω_2 many disjoint sets with cardinality ω_1 and let $\{\gamma_\alpha : \alpha < \omega_1\}$ be an enumeration of A_{ω_2}. For every $\alpha < \omega_1$, we define $A'_\alpha := A_\alpha \cup \{\gamma_\alpha\}$. Let

$$\mathcal{C} := \{A_\alpha : \alpha \leq \omega_2\} \cup \{A'_\alpha : \alpha < \omega_1\} \cup \{A \subseteq A_{\omega_2} : A_{\omega_2} \setminus A \text{ is finite}\}.$$

We show that $(X, \mathcal{C})$ is a weak category base. Let $C, C' \in \mathcal{C}$ be regions such that $C \cap C' \neq \emptyset$. Then C or C' is a subset of A_{ω_2}. Without loss of generality, $C \subseteq A_{\omega_2}$. If $C' \subseteq A_{\omega_2}$ as well, then $C \cap C'$ is co-finite in A_{ω_2} and so $C \cap C' \in \mathcal{C}$. Hence, we can assume that $C' \nsubseteq A_{\omega_2}$. Then there is some $\alpha < \omega_1$ such that $C' = A'_\alpha$. Thus, $|C \cap C'| \leq 1$. By definition, every finite set $F \subseteq A_{\omega_2}$ is $\mathcal{C}$-singular. Therefore, $(X, \mathcal{C})$ is a weak category base. Now let $\mathcal{C}' := \{A'_\alpha : \alpha < \omega_1\}$. Then $\mathcal{C}'$ is a pairwise disjoint family of regions with $|\mathcal{C}'| < |\mathcal{C}|$, Moreover, $A_{\omega_2} \cap \bigcup\{A'_\alpha : 0 < \alpha < \omega_1\}$ contains a region, namely A_{ω_2}. However, there is no $\alpha < \omega_1$ such that $A_{\omega_2} \cap A'_\alpha$ contains a region. Therefore, $(X, \mathcal{C})$ is a weak category base but not a category base.

In the following, we prove some basic properties of weak category bases. It should be noted that although the concept of weak category bases is new, most of the proofs in this section and Sections 2.2.3 and 2.2.4 are not. They can be found in a less general setting in the author's Master's thesis [Wan19].

It is clear that the $\mathcal{C}$-singular sets and the $\mathcal{C}$-meager sets form an ideal and a σ-ideal, respectively. More complicated is the question of whether the $\mathcal{C}$-Baire sets always form a σ-algebra. To answer it, we use a concept which was also used by Ikegami for arboreal forcing notions in [Ike10].

Definition 2.2.11. Let $(X, \mathcal{C})$ be a weak category base. A subset of X is *$\mathcal{C}$-abundant* if it is not $\mathcal{C}$-meager. Let $C \in \mathcal{C}$ be a region. We say that a set $A \subseteq X$ is *$\mathcal{C}$-abundant in a region* C if $C \cap A$ is abundant and that A is *$\mathcal{C}$-abundant everywhere in* C if A is $\mathcal{C}$-abundant in every subregion of C. We define $I^*_{\mathcal{C}}$ as the set of all $A \subseteq X$ such that there is no region in which A is $\mathcal{C}$-abundant everywhere.

In other words, a set $A \subseteq X$ is in $I^*_{\mathcal{C}}$ if for every region $C \in \mathcal{C}$, there is a subregion $C' \subseteq C$ such that $C' \cap A$ is $\mathcal{C}$-meager. Therefore, $I^*_{\mathcal{C}}$ is an ideal containing all $\mathcal{C}$-meager sets. We can use $I^*_{\mathcal{C}}$ to obtain an alternative characterization of the $\mathcal{C}$-Baire sets.

Lemma 2.2.12. *Let $(X, \mathcal{C})$ be a weak category base. Then a set $A \subseteq X$ is $\mathcal{C}$-Baire if and only if for every region $C \in \mathcal{C}$ there is a subregion $C' \subseteq C$ such that either $C' \setminus A$ or $C' \cap A$ is $I^*_{\mathcal{C}}$-small.*

Proof. The forward direction is clear. We prove the backward direction. Let $A \subseteq X$ and assume that for every region $C \in \mathcal{C}$ there is a subregion $C' \subseteq C$ such that either $C' \setminus A$ or $C' \cap A$ is $I^*_{\mathcal{C}}$-small. Then there is a subregion $C' \subseteq C$ such that either $C' \setminus A = C' \cap (C \setminus A)$ or $C' \cap A = C' \cap (C \cap A)$ is $\mathcal{C}$-meager. □

Lemma 2.2.13. *Let $(X, \mathcal{C})$ be a weak category base such that $I^*_{\mathcal{C}}$ is a σ-ideal. Then $\mathfrak{B}(\mathcal{C})$ is a σ-algebra.*

Proof. We only have to check that $\mathfrak{B}(\mathcal{C})$ is closed under countable unions. Let $A_n \subseteq X$ be $\mathcal{C}$-Baire, let $A := \bigcup_{n \in \omega} A_n$, and let $C \in \mathcal{C}$ be a region. Without loss of generality, $C \cap A$ is not $I^*_{\mathcal{C}}$-small. Then there is some $n \in \omega$ such that $C \cap A_n$ is not $I^*_{\mathcal{C}}$-small. Hence, there is a region $B \in \mathcal{C}$ such that $C \cap A_n$ is $\mathcal{C}$-abundant everywhere in B. In particular, $B \cap C$ is $\mathcal{C}$-abundant. Thus, there is a region $C' \subseteq B \cap C$. Since A_n is $\mathcal{C}$-Baire, there is a subregion $C'' \subseteq C'$ such that either $C'' \setminus A_n$ or $C'' \cap A_n$ is $\mathcal{C}$-meager. The latter is not possible because C'' is a subregion of B and so $C'' \cap A_n = C'' \cap (C \cap A_n)$ is $\mathcal{C}$-abundant. Therefore, $C'' \setminus A_n$ is $\mathcal{C}$-meager. □

Lemma 2.2.13 reduces the question of whether the $\mathcal{C}$-Baire sets form a σ-algebra to the question of whether $I^*_{\mathcal{C}}$ is a σ-ideal.

Question 2.2.14. *Let $(X, \mathcal{C})$ be a weak category base. Is $I^*_{\mathcal{C}}$ is always a σ-ideal?*

In the rest of this section, we give partial answers to Question 2.2.14. We start with category bases. The following theorem shows that for every category base $(X, \mathcal{C})$, $I_{\mathcal{C}} = I^*_{\mathcal{C}}$. Therefore, Question 2.2.14 can be answered positively for category bases.

Theorem 2.2.15 (Morgan)**.** *Let $(X, \mathcal{C})$ be a category base. Then every $\mathcal{C}$-abundant set is $\mathcal{C}$-abundant everywhere in a region.*

Proof. Cf. [Mor90, Chapter 1, Section II, Fundamental Theorem]. □

Theorem 2.2.15 does not hold for weak category bases: let $X = \omega_1$ and let $\{A_\alpha : \alpha < \omega_1\}$ be a partition of X in ω_1 many disjoint sets with cardinality $\aleph_1$. We define

$$\begin{aligned}
\mathcal{C}_\alpha :=& \{A \subseteq A_\alpha : |A_\alpha \setminus A| < \aleph_0\} \text{ for } \alpha < \omega_1,\\
\mathcal{C}' :=& \{A \subseteq X : \forall \alpha < \omega_1 (|A \cap A_\alpha| \leq 1) \wedge |\{\alpha < \omega_1 : A \cap A_\alpha \neq \emptyset\}| = \aleph_1\}, \text{ and}\\
\mathcal{C} :=& \bigcup_{\alpha < \omega_1} \mathcal{C}_\alpha \cup \mathcal{C}'.
\end{aligned}$$

Then $(X, \mathcal{C})$ is a weak category base. We shall show that a subset of X is $\mathcal{C}$-meager if and only if it is countable. Then every set $A \subseteq X$ which meets every A_α in exactly $\aleph_0$ many elements is $\mathcal{C}$-abundant and $I^*_{\mathcal{C}}$-small. It is clear that every finite set is $\mathcal{C}$-singular. Hence, every countable set is $\mathcal{C}$-meager. Let $A \subseteq X$ be uncountable. Then $|\{\alpha < \omega_1 : A \cap A_\alpha \neq \emptyset\}| = \aleph_1$ or there is some

$\alpha < \omega_1$ such that $|A \cap A_\alpha| = \aleph_1$. In both cases there is a region with no subregion which is disjoint from A. Hence, A cannot be $\mathcal{C}$-singular and so every $\mathcal{C}$-meager set is countable.

However, Theorem 2.2.15 holds for large classes of weak category bases including those satisfying the c.c.c. To show this, we first prove a lemma characterizing the singular sets for weak category bases satisfying the c.c.c.

Lemma 2.2.16. *Let $(X, \mathcal{C})$ be a weak category base such that every region is $\mathcal{C}$-abundant and let $A \subseteq X$. Then A is $\mathcal{C}$-singular if and only if there is a maximal family $\mathcal{A} \subseteq \mathcal{C}$ of pairwise essentially disjoint regions such that $A \subseteq X \setminus \bigcup_{C \in \mathcal{A}} C$.*

Proof. We start with the forward direction. Since A is $\mathcal{C}$-singular, the set $D := \{C \in \mathcal{C} : C \cap A = \emptyset\}$ is dense in $(\mathcal{C}, \subseteq)$. Let $\mathcal{A} \subseteq D$ be a maximal antichain. Then $A \subseteq X \setminus \bigcup_{C \in \mathcal{A}} C$ and every $C \neq C' \in \mathcal{A}$ are essentially disjoint.

We prove the backward direction. Let $\mathcal{A} \subseteq \mathcal{C}$ be a maximal family of pairwise essentially disjoint regions such that $A \subseteq X \setminus \bigcup_{C \in \mathcal{A}} C$ and let $C \in \mathcal{C}$ be a region. Since $\mathcal{A}$ is maximal, C is compatible with some element $B \in \mathcal{A}$. Let $C' \subseteq B \cap C$ be a witness. Then $C' \cap A = \emptyset$. Therefore, A is $\mathcal{C}$-singular. $\square$

Note that in the statement of Lemma 2.2.16 we can exchange maximal family of pairwise essentially disjoint regions with maximal antichain in $(\mathcal{C}, \subseteq)$. In fact, if every region is $\mathcal{C}$-abundant, then both terms coincide. In the following, we shall use both interchangeably.

Proposition 2.2.17. *Let $(X, \mathcal{C})$ be a weak category base such that every region is $\mathcal{C}$-abundant. If $(X, \mathcal{C})$ satisfies the c.c.c. or every $\mathcal{C}$-meager set is $\mathcal{C}$-singular, then $I_\mathcal{C} = I^*_\mathcal{C}$.*

Proof. Let $A \subseteq X$ be $I^*_\mathcal{C}$-small. First, we assume that $(X, \mathcal{C})$ satisfies the c.c.c. Let $D := \{C \in \mathcal{C} : C \cap A \text{ is } \mathcal{C}\text{-meager}\}$. Then D is dense in $(\mathcal{C}, \subseteq)$. Let $\mathcal{A} \subseteq D$ be a maximal antichain. Since $(X, \mathcal{C})$ satisfies the c.c.c., $\mathcal{A}$ is countable and so $M := \bigcup_{C \in \mathcal{A}} C \cap A$ is $\mathcal{C}$-meager. We shall show that $A \setminus M$ is also $\mathcal{C}$-meager. By Lemma 2.2.16, $X \setminus \bigcup_{C \in \mathcal{A}} C$ is $\mathcal{C}$-singular. Since $A \setminus M = A \setminus \bigcup_{C \in \mathcal{A}} C$, $A \setminus M$ is also $\mathcal{C}$-singular and so A is $\mathcal{C}$-meager.

Now we assume that every $\mathcal{C}$-meager set is $\mathcal{C}$-singular. Let $C \in \mathcal{C}$ be a region. Since A is $I^*_\mathcal{C}$-small, there is a subregion $C' \subseteq C$ such that $C' \cap A$ is $\mathcal{C}$-meager. By assumption, $C' \cap A$ is $\mathcal{C}$-singular. Hence, there is a subregion $C'' \subseteq C'$ such that $C'' \cap A = C'' \cap (C' \cap A) = \emptyset$. Therefore, A is $\mathcal{C}$-singular. $\square$

Next, we introduce a class of weak category bases for which we can show that $I^*_\mathcal{C}$ is a σ-ideal.

Definition 2.2.18. A weak category base $(X, \mathcal{C})$ is *proper* if $(\mathcal{C}, \subseteq)$ is a proper forcing notion, $I_\mathcal{C}$ is a proper σ-ideal, and every region is $\mathcal{C}$-abundant.

Note that for every Polish space X, X together with the set of non-empty open sets forms a proper weak category base. Moreover, every proper arboreal and every proper idealized forcing notion defines a proper weak category base.

Proposition 2.2.19. *Let $(X, \mathcal{C})$ be a proper weak category base. Then $I^*_\mathcal{C}$ is a proper σ-ideal and every region is $\mathcal{C}$-abundant everywhere in a region, i.e., every region is $I^*_\mathcal{C}$-positive.*

Proof. First, we show that $I^*_\mathcal{C}$ is a σ-ideal. It is enough to check that $I^*_\mathcal{C}$ is closed under countable unions. Let $A_n \subseteq X$ be $I^*_\mathcal{C}$-small, let $A := \bigcup_{n \in \omega} A_n$, and let $C \in \mathcal{C}$ be a region. Then for every $n \in \omega$, the set $D_n := \{B \in \mathcal{C} : B \cap A \text{ is } \mathcal{C}\text{-meager}\}$ is dense in $(\mathcal{C}, \subseteq)$. For every $n \in \omega$, we fix a maximal antichain $\mathcal{A}_n \subseteq D_n$. Let M be a countable elementary submodel of a large enough structure containing $\mathbb{P}$, C, and every $\mathcal{A}_n$. Since $(\mathcal{C}, \subseteq)$ is proper, there is a $(M, (\mathcal{C}, \subseteq))$-master condition $C' \subseteq C$. Hence, for every $n \in \omega$, $\mathcal{A}_n \cap M$ is predense below C'. We define $N := \bigcup_{n\in\omega} \bigcup\{B \cap A_n : B \in \mathcal{A}_n \cap M\}$. Then N is $\mathcal{C}$-meager. It is enough to show that $(C' \cap A) \setminus N$ is $\mathcal{C}$-meager. Let $n \in \omega$ and let $B \in \mathcal{C}$ be a region. We make a case-distinction:

Case 1: B and C' are compatible. Without loss of generality, we assume that $B \leq C'$. Since $\mathcal{A}_n \cap M$ is predense below C', there is some element in $B' \in \mathcal{A}_n \cap M$ which is compatible with B. Let $B'' \subseteq B \cap B'$ be a witness. Then $B'' \cap A_n \subseteq B' \cap A_n \subseteq N$. Hence, $B'' \cap ((C' \cap A_n) \setminus N) = \emptyset$.

Case 2: B and C' are incompatible. Then $B \cap C'$ is $\mathcal{C}$-singular. Thus, there is a subregion $B' \subseteq B$ such that $B' \cap C' = \emptyset$ and so $B' \cap ((C' \cap A_n) \setminus N) = \emptyset$.

Therefore, $(C' \cap A_n) \setminus N$ is $\mathcal{C}$-singular for every $n \in \omega$ and so $(C \cap A) \setminus N$ is $\mathcal{C}$-meager. □

Corollary 2.2.20. *Let $(X, \mathcal{C})$ be a proper weak category base. Then $\mathfrak{B}(\mathcal{C})$ is a σ-algebra.*

Proof. Follos directly from Lemma 2.2.13 and Proposition 2.2.19. □

2.2.3 Ikegami's Theorem for weak category bases

In this section, we prove a version of Ikegami's Theorem for weak category bases. The rough idea is to show that for a weak category base $(X, \mathcal{C})$ a subset of X is $\mathcal{C}$-Baire if and only if is $I^*_{\mathcal{C}}$-regular. Then we can use our version of Ikegami's Theorem for uncountable Polish spaces (Theorem 2.1.12) to obtain Judah-Shelah- and Solovay-style characterizations for the $\mathcal{C}$-Baire sets. However, this is only possible if there is already a topology on X that is compatible with $\mathcal{C}$.

Definition 2.2.21. Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a weak category base. We say that $(X, \mathcal{C})$ is *Borel compatible with* X if every region is Borel in X and every Borel set in X is $\mathcal{C}$-Baire.

If X is an uncountable Polish space and $(X, \mathcal{C})$ is a proper weak category base $(X, \mathcal{C})$ which is Borel compatible with X, then $I^*_{\mathcal{C}}$ is a proper σ-ideal on X and so $I^*_{\mathcal{C}}$-regularity is well-defined. Moreover, by Proposition 2.2.10, $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ is a weak category base and a subset of X is $I^*_{\mathcal{C}}$-regular if and only if it is $\mathbb{P}_{I^*_{\mathcal{C}}}$-Baire. Hence, we only have to check that $\mathfrak{B}(\mathcal{C}) = \mathfrak{B}(\mathbb{P}_{I^*_{\mathcal{C}}})$. In fact, we show something even stronger.

Definition 2.2.22. We say that two weak category bases $(X, \mathcal{C})$ and $(X, \mathcal{D})$ are *equivalent* if $I^*_{\mathcal{C}} = I^*_{\mathcal{D}}$ and $\mathfrak{B}(\mathcal{C}) = \mathfrak{B}(\mathcal{D})$.

Note that Morgan defined two category bases $(X, \mathcal{C})$ and $(X, \mathcal{D})$ to be equivalent if they produce the same meager and Baire sets, i.e., $I_{\mathcal{C}} = I_{\mathcal{D}}$ and $\mathfrak{B}(\mathcal{C})$ and $\mathfrak{B}(\mathcal{D})$. Since for category bases, $I_{\mathcal{C}} = I^*_{\mathcal{C}}$, this is equivalent to Definition 2.2.22 for category bases. In the following, we show that $(X, \mathcal{C})$ and $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ are equivalent as weak category bases. We start with a lemma about idealized forcing notions.

Lemma 2.2.23. *Let X be an uncountable Polish space, let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper. Then a subset of X is I-null if and only if it is $I^*_{\mathbb{P}_I}$-small.*

Proof. By definition, a subset of X is I-null if and only if it is $\mathbb{P}_I$-singular. Since $\mathbb{P}_I$ is proper, $\mathcal{N}_I$ is a σ-ideal. Hence, a subset of X is I-null if and only if it is $\mathbb{P}_I$-meager. So in particular, every $\mathbb{P}_I$-meager set is $\mathbb{P}_I$-singular. By Proposition 2.2.17, a subset of X is $\mathbb{P}_I$-meager if and only if it is $I^*_{\mathbb{P}_I}$-small. Therefore, a subset of X is I-null if and only if it is $I^*_{\mathbb{P}_I}$-small. □

Thus, by Proposition 2.2.10 and Lemma 2.2.23, to show that $(X, \mathcal{C})$ and $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ are equivalent, we only need to show that $I^*_{\mathcal{C}} = \mathcal{N}_{I^*_{\mathcal{C}}}$ and that a subset of X is $\mathcal{C}$-Baire if and only if it is $I^*_{\mathcal{C}}$-regular (or $\mathbb{P}_{I^*_{\mathcal{C}}}$-Baire). This is clear if $\mathcal{C}$ is a dense subset of $\mathbb{P}_{I^*_{\mathcal{C}}}$. Since $(X, \mathcal{C})$ is Borel compatible with X, $\mathcal{C}$ is always a subset of $\mathbb{P}_{I^*_{\mathcal{C}}}$. However, there is no reason to think that $\mathcal{C}$ is always a dense subset. In fact, we can characterize the weak category bases for which $\mathcal{C}$ is a dense subset.

Lemma 2.2.24. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X. Then $\mathcal{C}$ is a dense subset of $\mathbb{P}_{I^*_{\mathcal{C}}}$ if and only if every $\mathcal{C}$-meager set is $\mathcal{C}$-singular.*

Proof. We start with the forward direction. Since $\mathcal{C}$ is a dense subset of $\mathbb{P}_{I^*_{\mathcal{C}}}$, a subset of X is $\mathcal{C}$-singular if and only if it is $I^*_{\mathcal{C}}$-null. By Proposition 2.1.6, the $\mathcal{C}$-singular sets form a σ-ideal. Hence, ever $\mathcal{C}$-meager set is $\mathcal{C}$-singular.

We prove the backward direction. Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. Then there is a region $\mathcal{C}$ such that for every subregion $C' \subseteq C$, $C' \cap B$ is not $\mathcal{C}$-meager. Since B is $\mathcal{C}$-Baire, there is a subregion $C' \subseteq C$ such that $C' \setminus B$ is $\mathcal{C}$-meager. By assumption, $C' \cap B$ is $\mathcal{C}$-singular. Hence, there is a subregion $C'' \subseteq C'$ such that $C'' \cap (C' \cap B) = \emptyset$. Then C'' is a subset of B and so $\mathcal{C}$ is dense in $\mathbb{P}_{I^*_{\mathcal{C}}}$. ☐

Next, we give an example of a weak category base $(X, \mathcal{C})$ such that $\mathcal{C}$ is not dense in $\mathbb{P}_{I^*_{\mathcal{C}}}$. Let $X = \omega^\omega$ and $\mathcal{C}$ be the set of non-empty open sets of reals. It is well-known that there are meager sets which are not nowhere dense (take any countable dense set of reals). By Lemma 2.2.24, there is a non-meager Borel set of reals which does not contain a non-empty open set. However, since every Borel set has the Baire property, for every non-meager Borel set B, there is a non-empty open set O such that $O \setminus B$ is meager. We have already seen in the proof of Lemma 2.2.24 that something similar also is true for general weak category bases.

Lemma 2.2.25. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X. Then for every $I^*_{\mathcal{C}}$-positive Borel set B in X, there is a region $C \in \mathcal{C}$ such that $C \setminus B$ is $\mathcal{C}$-meager.*

Proof. Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. Then there is a region $\mathcal{C}$ such that for every subregion $C' \subseteq C$, $C' \cap B$ is not $\mathcal{C}$-meager. Since B is $\mathcal{C}$-Baire, there is a subregion $C' \subseteq C$ such that $C' \setminus B$ is $\mathcal{C}$-meager. ☐

By Lemma 2.2.25, every $I^*_{\mathcal{C}}$-positive Borel set in X contains a region modulo a $\mathcal{C}$-meager set. Therefore, $\mathcal{C}$ can be densely embedded into the quotient algebra $(\mathcal{B}(X)/I^*_{\mathcal{C}})^+$.

Proposition 2.2.26. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X. Then*

$$\begin{aligned} i : (\mathcal{C}, \subseteq) &\longrightarrow (\mathcal{B}(X)/I^*_{\mathcal{C}})^+, \\ C &\longmapsto [C] \end{aligned}$$

is a dense embedding, where $[C]$ is the equivalence class of C in $\mathcal{B}(X)/I^*_{\mathcal{C}}$. In particular, $\mathbb{P}_{I^*_{\mathcal{C}}}$ is proper and forcing equivalent to $(\mathcal{C}, \subseteq)$.

Proof. Let $C, C' \in \mathcal{C}$ be regions. It is clear that if $C \subseteq C'$, then $i(C) \leq i(C')$. If C and C' are compatible, then there is some region $C'' \in \mathcal{C}$ such that $C'' \subseteq C \cap C'$. Then $i(C'') \leq i(C), i(C')$.

Conversely, if $i(C)$ and $i(C')$ are compatible, then there is some $I^*_{\mathcal{C}}$-positive Borel set B in X such that $B \setminus C$ and $B \setminus C'$ are $I^*_{\mathcal{C}}$-small. By Lemma 2.2.25, there is a region $C'' \in \mathcal{C}$ such that $C'' \setminus B$ is $\mathcal{C}$-meager. Then $C \cap C''$ is either $\mathcal{C}$-singular or contains a region. We suppose for a contradiction that $C \cap C''$ is $\mathcal{C}$-singular. Then

$$C'' \subseteq (C'' \setminus B) \cup (B \setminus C) \cup (C \cap C'') \in I^*_{\mathcal{C}}.$$

But this is impossible since $(X, \mathcal{C})$ is proper. Hence, we can assume without loss of generality that $C'' \subseteq C$. Analogously, we can assume that $C'' \subseteq C'$. Therefore, C and C' are compatible.

Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. Again by Lemma 2.2.25, there is some region $C \in \mathcal{C}$ such that $C \setminus B$ is $\mathcal{C}$-meager. Therefore, $\mathrm{ran}(i)$ is dense in $(\mathcal{B}(X)/I^*_{\mathcal{C}})^+$. Altogether, i is a dense embedding. The second part follows directly from Proposition 1.2.44. □

Now we are finally ready to prove that $(X, \mathcal{C})$ and $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ are equivalent.

Theorem 2.2.27. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X. Then $(X, \mathcal{C})$ and $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ are equivalent.*

Proof. Let $A \subseteq X$. We have to show that A is $I^*_{\mathcal{C}}$-small if and only if A is $I^*_{\mathcal{C}}$-null and that A is $\mathcal{C}$-Baire if and only if A is $I^*_{\mathcal{C}}$-regular. We start with the former. First assume that A is $I^*_{\mathcal{C}}$-null. Let $C \in \mathcal{C}$ be a region. Then there is an $I^*_{\mathcal{C}}$-positive Borel set $B \subseteq C$ such that $B \cap A = \emptyset$. By Lemma 2.2.25, there is a region $C' \in \mathcal{C}$ such that $C' \setminus B$ is $\mathcal{C}$-meager. Then $C \cap C'$ is not $\mathcal{C}$-singular and so there is a region $C'' \subseteq C \cap C'$. Then C'' is a subregion of C and $C'' \cap A = (C' \setminus B) \cup (B \cap A)$ is $\mathcal{C}$-meager. Therefore, A is $I^*_{\mathcal{C}}$-small.

Next we assume that A is $\mathcal{C}$-singular. Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. By Lemma 2.2.25, there is a region $C \in \mathcal{C}$ such that $C \setminus B$ is $\mathcal{C}$-meager. Since A is $\mathcal{C}$-singular, there is a subregion $C' \subseteq C$ such that $C' \cap A = \emptyset$. Then $C' \setminus B$ is $I^*_{\mathcal{C}}$-small and so $C' \cap B$ is an $I^*_{\mathcal{C}}$-positive Borel set in X which is disjoint from A. Therefore, A is $I^*_{\mathcal{C}}$-null. By Proposition 2.1.6, $\mathcal{N}_{I^*_{\mathcal{C}}}$ is a σ-ideal. Hence, every $\mathcal{C}$-meager set is also $I^*_{\mathcal{C}}$-null. Now assume that A is $I^*_{\mathcal{C}}$-small. Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. Again by Lemma 2.2.25, there is a region $C \in \mathcal{C}$ such that $C \setminus B$ is $I^*_{\mathcal{C}}$-small. Since A is $I^*_{\mathcal{C}}$-small, there is a subregion $C' \subseteq C$ such that $C' \cap A$ is $\mathcal{C}$-meager. Then $C' \cap B$ is an $I^*_{\mathcal{C}}$-positive Borel set in X and $(C' \cap B) \cap A$ is $\mathcal{C}$-meager. In particular, it is $I^*_{\mathcal{C}}$-null. Thus, there is an $I^*_{\mathcal{C}}$ positive Borel set $B' \subseteq C' \cap B$ such that $B' \cap ((C' \cap B) \cap A) = \emptyset$ and so $B' \cap A = \emptyset$. Therefore, A is $I^*_{\mathcal{C}}$-null.

Now assume that A is $\mathcal{C}$-Baire. Let B be an $I^*_{\mathcal{C}}$-positive Borel set in X. By Lemma 2.2.25, there is a region $C \in \mathcal{C}$ such that $C \setminus B$ is $\mathcal{C}$-meager. Since A is $\mathcal{C}$-Baire, there is a subregion $C' \subseteq C$ such that either $C' \setminus A$ or $C' \cap A$ is $\mathcal{C}$-meager. Without loss of generality, we assume the former. Then $M := (C \setminus B) \cup (C' \setminus A)$ is $\mathcal{C}$-meager and so $I^*_{\mathcal{C}}$-null. Hence, there is an $I^*_{\mathcal{C}}$-positive Borel set $B' \subseteq C'$ such that $B' \cap M = \emptyset$. Then $B' \leq B$ and $B' \subseteq A$. In the case $C' \cap A$ is $\mathcal{C}$-meager, we deduce that there is some $B' \leq B$ such that $B' \cap A = \emptyset$. Therefore, A is $I^*_{\mathcal{C}}$-regular.

Finally, we assume that A is $I^*_{\mathcal{C}}$-regular. Let $C \in \mathcal{C}$ be a region. Since A is $I^*_{\mathcal{C}}$-regular, there is an $I^*_{\mathcal{C}}$-positive Borel set $B \subseteq C$ such that either $B \subseteq A$ or $B \cap A = \emptyset$. By Lemma 2.2.25, there is a region $C' \in \mathcal{C}$ such that $C' \setminus B$ is $\mathcal{C}$-meager. Then either $C' \setminus A$ or $C' \cap A$ is $I^*_{\mathcal{C}}$-small. If $C \cap C'$ contains a region, we are done. Thus, we suppose for a contradiction that it does not. Then $C \cap C'$ is $\mathcal{C}$-singular and so

$$C' \subseteq (C' \setminus B) \cup (C' \cap B) \subseteq (C' \setminus B) \cup (C' \cap C) \in I^*_{\mathcal{C}}.$$

But this is impossible since $(X, \mathcal{C})$ is proper. Therefore, A is $\mathcal{C}$-Baire. □

Since $(X, \mathcal{C})$ and $(X, \mathbb{P}_{I^*_{\mathcal{C}}})$ are equivalent as category bases, we can now apply our results from Section 2.1.2 to $(X, \mathcal{C})$. Hence, $\mathcal{C}$-Baireness behaves similarly to the most other regularity properties for the first two levels of the projective hierarchy.

Corollary 2.2.28. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X.*

(a) The $\mathcal{C}$-Baire sets form a σ-algebra on X containing all analytic and co-analytic sets in X.

(b) In L, there is a $\mathbf{\Delta}^1_2(X)$ set which is not $\mathcal{C}$-Baire.

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\mathbf{\Sigma}^1_2(X)$ sets are $\mathcal{C}$-Baire.

Proof. Follows directly from Proposition 2.1.6 and Theorem 2.2.27. $\square$

Moreover, we can use our version of Ikegami's Theorem for uncountable Polish space (Theorem 2.1.12) to obtain a version of Ikegami's Theorem for weak category bases.

Corollary 2.2.29 (Ikegami's Theorem for weak category bases)**.** *Let X be an uncountable Polish subspace of the Baire space and let $(X, \mathcal{C})$ be a proper weak category base which is Borel compatible with X such that $\{c \in \mathsf{BC} : B_c \in I^*_{\mathcal{C}}\}$ is $\mathbf{\Sigma}^1_2(2^\omega)$.*

(a) Every $\mathbf{\Delta}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ such that X is coded in $\mathrm{L}[r]$ and every region $C \in \mathcal{C}$, there is an $I^*_{\mathcal{C}}$-quasi-generic element over $\mathrm{L}[r]$ in C.

(b) Every $\mathbf{\Sigma}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ such that X is coded in $\mathrm{L}[r]$, the set $\{x \in X : x$ is not $I^*_{\mathcal{C}}$-quasi-generic over $\mathrm{L}[r]\}$ is $I^*_{\mathcal{C}}$-small.

Proof. We start with proving (a). By Theorems 2.1.12 and 2.2.27, every $\mathbf{\Delta}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ such that X is coded in $\mathrm{L}[r]$ and every $I^*_{\mathcal{C}}$-positive Borel set B in X, there is an $I^*_{\mathcal{C}}$-quasi-generic element over $\mathrm{L}[r]$ in B. Hence, it is enough to show that for every $r \in \omega^\omega$ such that X is coded in $\mathrm{L}[r]$ and every $I^*_{\mathcal{C}}$-positive Borel set B in X, there is an $I^*_{\mathcal{C}}$-quasi-generic element over $\mathrm{L}[r]$ in B if and only if for every $r \in \omega^\omega$ such that X is coded in $\mathrm{L}[r]$ and every region $C \in \mathcal{C}$, there is an $I^*_{\mathcal{C}}$-quasi-generic element over $\mathrm{L}[r]$ in C. The forward direction is clear. Let $r \in \omega^\omega$ be a real such that X is coded in $\mathrm{L}[r]$, let B be an $I^*_{\mathcal{C}}$-positive Borel set in X, and let $c \in \mathsf{BC}$ be a Borel code for B. By Lemma 2.2.25, there is a region $C \in \mathcal{C}$ such that $C \setminus B$ is $\mathcal{C}$-meager. Let $c' \in \mathsf{BC}$ be a Borel code for C. Then there is some $x \in C$ which is $I^*_{\mathcal{C}}$-quasi-generic over $\mathrm{L}[r, c, c']$. Since $C \setminus B$ is an $I^*_{\mathcal{C}}$-small Borel set coded in $\mathrm{L}[c, c']$, $x \notin C \setminus B$. Hence, $x \in B$ and x is $I^*_{\mathcal{C}}$-quasi-generic over $\mathrm{L}[r]$.

Item (b) follows directly from Theorems 2.1.12 and 2.2.27. $\square$

2.2.4 Weak category bases satisfying the c.c.c.

In this section, we improve two of the results of Section 2.2.3 for weak category bases which satisfy the c.c.c. First, we show that if the weak category base in the statement of Theorem 2.2.27 also satisfies the c.c.c., then $(X, \mathbb{P}_{I_{\mathcal{C}}})$ is a category base. Second, we prove a version of Ikegami's Theorem for weak category bases satisfying the c.c.c., which links $\mathcal{C}$-Baireness to $(\mathcal{C}, \subseteq)$-generic objects. We start with the former.

Proposition 2.2.30. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base which satisfies the c.c.c. and is Borel compatible with X. Then $(X, \mathbb{P}_{I_\mathcal{C}})$ is a category base. In particular, $(X, \mathcal{C})$ is equivalent to a proper category base which satisfies the c.c.c. and is Borel compatible with X.*

Proof. Since $(X, \mathcal{C})$ satisfies the c.c.c., $I^*_\mathcal{C} = I_\mathcal{C}$. Hence, $(X, \mathbb{P}_{I_\mathcal{C}})$ is a weak category base. By Theorem 2.2.27, it is enough to show that $(X, \mathbb{P}_{I_\mathcal{C}})$ is a category base satisfying the c.c.c. We first show that $(X, \mathbb{P}_{I_\mathcal{C}})$ satisfies the c.c.c. Since $(X, \mathcal{C})$ satisfies the c.c.c. as a weak category base, $(\mathcal{C}, \subseteq)$ satisfies the c.c.c. as a forcing notion. By Proposition 2.2.26, $(\mathcal{C}, \subseteq)$ and $\mathbb{P}_{I_\mathcal{C}}$ are forcing equivalent. Hence, $\mathbb{P}_{I_\mathcal{C}}$ satisfies the c.c.c. as a forcing notion and so $(X, \mathbb{P}_{I_\mathcal{C}})$ satisfies the c.c.c. as a weak category base. It remains to check that $(X, \mathbb{P}_{I_\mathcal{C}})$ is a category base. Let B be an $I_\mathcal{C}$-positive Borel set in X and let $\mathcal{A} \subseteq \mathbb{P}_{I_\mathcal{C}}$ be a disjoint family. Since $(X, \mathbb{P}_{I_\mathcal{C}})$ satisfies the c.c.c., $\mathcal{A}$ is countable. We make a case-distinction:

Case 1: $B \cap \bigcup \mathcal{A}$ contains a region from $\mathbb{P}_{I_\mathcal{C}}$. Then $B \cap \bigcup \mathcal{A}$ is $I_\mathcal{C}$-positive. Since $\mathcal{A}$ is countable, there is some $C \in \mathcal{A}$ such that $B \cap C$ is $I_\mathcal{C}$-positive. Therefore, $B \cap C$ contains a region from $\mathbb{P}_{I_\mathcal{C}}$.

Case 2: $B \cap \bigcup \mathcal{A}$ does not contain a region from $\mathbb{P}_{I_\mathcal{C}}$. Since $\mathcal{A}$ is countable, $B \cap \bigcup \mathcal{A}$ is a Borel set in X. Hence, $B \cap \bigcup \mathcal{A}$ is $I_\mathcal{C}$-small. Then $B \setminus (B \cap \bigcup \mathcal{A})$ is a subregion of B which is disjoint from $\bigcup \mathcal{A}$.

Therefore, $(X, \mathbb{P}_{I_\mathcal{C}})$ is a category base. □

Next, we work on our version of Ikegami's Theorem for weak category bases satisfying the c.c.c. Our goal is to replace $I_\mathcal{C}$-quasi-genericity with some kind of $\mathcal{C}$-genericity in the statement of Ikegami's Theorem for weak category bases (Corollary 2.2.29). We do this in two steps. First, we use our version of Ikegami's Theorem for idealized forcing notions satisfying the c.c.c. (Corollary 2.1.15) to replace $I_\mathcal{C}$-quasi-genericity with $\mathbb{P}_{I_\mathcal{C}}$-genericity. However, to use Corollary 2.1.15, $I_\mathcal{C}$ must to be Borel generated.

Lemma 2.2.31. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base satisfying the c.c.c. such that $(X, \mathcal{C})$ is Borel compatible with X. Then $I_\mathcal{C}$ is Borel generated.*

Proof. It is enough to show every $\mathcal{C}$-singular set is contained in some $\mathcal{C}$-singular which is Borel. Let $A \subseteq X$ be $\mathcal{C}$-singular. By Lemma 2.2.16, there is an essentially disjoint family $\mathcal{A} \subseteq \mathcal{C}$ such that $A \subseteq X \setminus \bigcup_{C \in \mathcal{A}} C$ and $X \setminus \bigcup_{C \in \mathcal{A}} C$ is $\mathcal{C}$-singular. Since $(X, \mathcal{C})$ satisfies the c.c.c., $\mathcal{A}$ is countable and so $X \setminus \bigcup_{C \in \mathcal{A}} C$ is Borel in X. □

Corollary 2.2.32. *Let X be an uncountable Polish subspace of the Baire space, let $a \in \omega$ such that X is $\Pi^0_2(a)$, and let $(X, \mathcal{C})$ be a proper weak category base which satisfies the c.c.c. and is Borel compatible with X such that $\{c \in \mathsf{BC} : B_c \in I_\mathcal{C}\}$ is $\Sigma^1_2(a)$ and $\mathbb{P}_{I_\mathcal{C}}$ satisfies the c.c.c. in every inner model of ZFC containing a.*

(a) *Every $\boldsymbol{\Delta}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, there is a $\mathbb{P}_{I_\mathcal{C}}$-generic element over $\mathrm{L}[r]$.*

(b) *Every $\boldsymbol{\Sigma}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, the set $\{x \in X : x$ is not $\mathbb{P}_{I_\mathcal{C}}$-generic over $\mathrm{L}[r]\}$ is $I_\mathcal{C}$-small.*

Proof. Follows directly from Corollary 2.1.15, Theorem 2.2.27, and Lemma 2.2.31. □

Now the next step is to replace $\mathbb{P}_{I_\mathcal{C}}$-genericity with some kind of $\mathcal{C}$-genericity to get characterizations which can be fully expressed in terms of weak category bases. To do this, we must define $\mathcal{C}$-genericity first.

Definition 2.2.33. Let X be an uncountable Polish space and let $(X,\mathcal{C})$ be a proper weak category base such that $(X,\mathcal{C})$ is Borel compatible with X. We say that an element $x \in X$ is *$\mathcal{C}$-generic over* V if there is a $(\mathcal{C},\subseteq)$-generic filter G_x over V such that for every $C \in \mathcal{C}$, $C \in G_x$ if and only if $x \in C$.

Recall that $(\mathcal{C},\subseteq)$ and $\mathbb{P}_{I_\mathcal{C}}$ are forcing equivalent by Proposition 2.2.26. We can even show that they produce the same generic elements.

Proposition 2.2.34. *Let X be an uncountable Polish space and let $(X,\mathcal{C})$ be a proper weak category base such that $(X,\mathcal{C})$ is Borel compatible with X and the ideal of $\mathcal{C}$-singular sets is Borel generated. Then an element in X is $\mathcal{C}$-generic over* V *if and only if it is $\mathbb{P}_{I^*_\mathcal{C}}$-generic over* V*.*

Proof. We first show that every $\mathcal{C}$-generic omits all $I^*_\mathcal{C}$-small Borel sets in X. We start with $\mathcal{C}$-singular Borel sets. Let $x \in X$ be $\mathcal{C}$-generic over V, let $G_x := \{C \in \mathcal{C} : x \in C\}$, and let B be a $\mathcal{C}$-singular Borel set in X. Then the set $D := \{C \in \mathcal{C} : C \cap B = \emptyset\}$ is dense in $(\mathcal{C},\subseteq)$. Hence, G_x meets D. Let $C \in G_x \cap D$ be a witness. Then $x \in C$ and $C \cap B = \emptyset$. Hence, $x \notin B$. Now let B be a $\mathcal{C}$-meager Borel set in X. Then there are $\mathcal{C}$-singular sets N_n such that $B = \bigcup_{n\in\omega} N_n$. By assumption, there are $\mathcal{C}$-singular Borel sets B_n such that $N_n \subseteq B_n$. Then $B \subseteq \bigcup_{n\in\omega} B_n$ and for every $n \in \omega$, $x \notin B_n$. Thus, $x \notin B$. Finally, let B be an $I^*_\mathcal{C}$-small Borel set in X and let $D := \{C \in \mathcal{C} : C \cap B \in I^*_\mathcal{C}\}$. Then D is dense in $(\mathcal{C},\subseteq)$ and so there is a region $C \in G_x \cap D$. Since $C \cap B$ is $\mathcal{C}$-meager, $x \notin C \cap B$. Hence, $x \notin B$. Therefore, x omits all $I^*_\mathcal{C}$-small Borel sets in X.

Now we show that an element in X is $\mathcal{C}$-generic over V if and only if it is $\mathbb{P}_{I^*_\mathcal{C}}$-generic over V. Let $j : \mathbb{P}_{I^*_\mathcal{C}} \to (\mathcal{B}(X)/I^*_\mathcal{C})^+$ and $i : (\mathcal{C},\subseteq) \to (\mathcal{B}(X)/I^*_\mathcal{C})^+$ be the dense embeddings from Propositions 1.2.44 and 2.2.26, respectively. We start with the forward direction. Let x be a $\mathcal{C}$-generic element over V. Then $G_x := \{C \in \mathcal{C} : x \in C\}$ is a $(\mathcal{C},\subseteq)$-generic filter over V. Let $i^*(G_x)$ be the upwards closure of $i[G]$. Since i and j are dense embeddings, $H := j^{-1}(i^*(G_x))$ is a $\mathbb{P}_{I^*_\mathcal{C}}$-generic filter over V. Hence, it is enough to show that $H = \{B \in \mathbb{P}_{I^*_\mathcal{C}} : x \in B\}$. Let B be an $I^*_\mathcal{C}$-positive Borel set in X. First assume that $x \in B$. Since B is $\mathcal{C}$-Baire, the set $D := \{C \in \mathcal{C} : C \cap B \text{ or } C \setminus B \text{ is } \mathcal{C}\text{-meager}\}$ is dense. Let $C \in G_x \cap D$. Then $x \in C$ and either $C \cap B$ or $C \setminus B$ is $\mathcal{C}$-meager. Since $x \in C \cap B$, $C \cap B$ is not $\mathcal{C}$-meager. Hence, $C \setminus B$ is $\mathcal{C}$-meager. Then $i(C) \leq j(B)$ and so $B \in H$. Now assume that $x \notin B$. We suppose for a contradiction that there is a region $C \in G_x$ such that $i(C) \leq j(B)$. Then $x \in C \setminus B$ and $C \setminus B$ is $I^*_\mathcal{C}$-small. But this is not possible. Therefore, there is no $C \in G_x$ with $i(C) \leq j(B)$ and so $B \notin H$.

We prove the backward direction. Let x be a $\mathbb{P}_{I^*_\mathcal{C}}$-generic element over V and let $H_x := \{B \in \mathbb{P}_{I^*_\mathcal{C}} : x \in B\}$ be the $\mathbb{P}_{I^*_\mathcal{C}}$-generic filter witnessing it. By a similar argument, as in the forward direction, it is enough to show that $G := i^{-1}(j^*(H_x)) = \{C \in \mathcal{C} : x \in C\}$, where $j^*(H_x)$ is the upwards closure of $j[H_x]$. Let $C \in \mathbb{C}$ be a region. If $x \in C$, then clearly $C \in G$. So we can assume that $x \notin C$. We suppose for a contradiction that there is an $I^*_\mathcal{C}$-positive Borel set $B \in H_x$ such that $j(B) \leq i(C)$. Then $x \in B \setminus C$ and $B \setminus C$ is $I^*_\mathcal{C}$-small. But this is not possible since x is $\mathbb{P}_{I^*_\mathcal{C}}$-generic. Therefore, there is no $B \in H_x$ such that $j(B) \leq i(C)$ and so $C \notin G$. $\square$

Note that the proof of Proposition 2.2.34 even shows that $(\mathcal{C},\subseteq)$-generic filters are uniquely determined by $\mathcal{C}$-generic elements: let G be a $(\mathcal{C},\subseteq)$-generic filter over V. We have shown that $H := j^{-1}(i^*(G))$ is a $\mathbb{P}_{I^*_\mathcal{C}}$-generic filter over V. By Theorem 1.2.45, there is a unique $\mathbb{P}_{I^*_\mathcal{C}}$-generic element $x \in X$ such that $H = \{B \in \mathbb{P}_{I^*_\mathcal{C}} : x \in B\}$. Since $G \subseteq H$, every $C \in G$ contains x. Let $C \in \mathcal{C}$ be a region containing x. Then $C \in H$ and so there is a region $C' \in G$ such that $i(C') \leq j(C)$. Then $C' \setminus C$ is $I^*_\mathcal{C}$-small. Let D be the set of all subregions of $C' \cap C$. For every subregion $C'' \subseteq C'$, $C'' \setminus C$ is $I^*_\mathcal{C}$-small and so $C'' \cap C$ is $I^*_\mathcal{C}$-positive. Since the intersection of two regions is either $\mathcal{C}$-singular or contains a region, D is dense below C'. Hence, there is some $C'' \in G$ which is a subset of C and so $C \in G$. Therefore, $G = \{C \in \mathcal{C} : x \in C\}$ and by Proposition 2.2.34, x is $\mathcal{C}$-generic. Next, we show that x is unique. Suppose there is another $y \in X$ which is $\mathcal{C}$-generic and $G = \{C \in \mathcal{C} : x \in C\}$. Then $H = \{B \in \mathbb{P}_{I^*_\mathcal{C}} : y \in B\}$. Since H is uniquely determined by x, $y = x$. Therefore, G is uniquely determined by a $\mathcal{C}$-generic element over V.

Now we are almost ready to prove a version of Ikegami's Theorem for weak category bases which links $\mathcal{C}$-Baireness to $\mathcal{C}$-genericity. The only thing missing is to make sure that $\mathcal{C}$-genericity over inner models is well-defined. To ensure this, we consider a class of definable weak category bases.

Definition 2.2.35. Let X be an uncountable Polish subspace of the Baire space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X and let $a \in \omega^\omega$. We say that $(X, \mathcal{C})$ is *provable* $\Sigma^1_2(a)$ if there are formulas φ_X and $\varphi_\mathcal{C}$ with parameter a such that

(a) φ_X is Π^0_2 and $\varphi_\mathcal{C}$ is Σ^1_2,

(b) $X = \{x : \varphi_X(x, a)\}$ and $\mathcal{C} = \{B_c : \varphi_\mathcal{C}(c, a)\}$,

(c) every inner model M of ZFC containing a proves that $(X^M, \mathcal{C}^M)$ is a proper weak category base which is Borel compatible with X^M, where $X^M := \{x : M \models \varphi_X(x, a)\}$ and $\mathcal{C}^M := \{B^M_c : M \models \varphi_\mathcal{C}(c, a)\}$, and

(d) the statement "c is a Borel code and B_c is $I_\mathcal{C}$-small" is $\Sigma^1_2(a)$.

In the following, we often omit the M if it is clear from the context, i.e., if we say that "$(X, \mathcal{C})$ is a category base in M", we mean "$(X^M, \mathcal{C}^M)$ is a category base in M", where X^M and $\mathcal{C}^M$ have the same definition as in Definition 2.2.35. Similarly, we say that an element $x \in X$ is *$\mathcal{C}$-generic over M* if it is $\mathcal{C}^M$-generic over M, i.e, if $\{C \in \mathcal{C}^M : x \in C\}$ is a filter meeting all dense subsets of $(\mathcal{C}^M, \subseteq)$ which are in M.

Corollary 2.2.36 (Ikegami's Theorem for weak category bases satisfying the c.c.c.)**.** *Let X be an uncountable Polish subspace of the Baire space, let $a \in \omega^\omega$, and let $(X, \mathcal{C})$ be a provable $\Sigma^1_2(a)$, proper weak category base which is Borel compatible with X such that $(X, \mathcal{C})$ satisfies the c.c.c. in every inner model of* ZFC *containing a.*

(a) Every $\boldsymbol{\Delta}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, there is a $\mathcal{C}$-generic element over $\mathrm{L}[r]$.

(b) Every $\boldsymbol{\Sigma}^1_2(X)$ set is $\mathcal{C}$-Baire if and only if for every $r \in \omega^\omega$ with $a \in \mathrm{L}[r]$, the set $\{x \in X : x$ is not $\mathcal{C}$-generic over $\mathrm{L}[r]\}$ *is $\mathcal{C}$-meager.*

Proof. By Corollary 2.2.32, we only have to check that any element from X is $\mathcal{C}$-generic over M if and only if it is $\mathbb{P}_{I_\mathcal{C}}$-generic over M. Let M be an inner model of ZFC containing a. Since $(X, \mathcal{C})$ is provable $\Sigma^1_2(a)$, in M, $(X, \mathcal{C})$ is a proper weak category base which is Borel compatible with X. Let $\mathbb{P}^M_{I_\mathcal{C}}$ be $\mathbb{P}_{I_\mathcal{C}}$ defined in M. By Proposition 2.2.34, an element from X is $\mathcal{C}$-generic over M if and only if it is $\mathbb{P}^M_{I_\mathcal{C}}$-generic over M. Moreover, $I_\mathcal{C}$, defined in M, coincides with $I_\mathcal{C}$ on Borel sets. Thus, a Borel set coded in M is in $\mathbb{P}^M_{I_\mathcal{C}}$ if and only if it is in $\mathbb{P}_{I_\mathcal{C}}$ and so an element from X is $\mathbb{P}^M_{I_\mathcal{C}}$-generic over M if and only if it is $\mathbb{P}_{I_\mathcal{C}}$-generic over M. $\square$

2.2.5 Dichotomies

Besides I-regularity, Khomskii also introduced a second class of regularity properties for σ-ideals in [Kho12]. While I-regularity is a generalization of the Baire property, this second class is a generalization of the perfect set property. In this section, we compare both types of regularity properties for the $I^*_{\mathcal{C}}$-ideal in weak category bases. Our main result is that they coincide for projective pointclasses. We start with the definition.

Definition 2.2.37. Let X be an uncountable Polish space and let I be a proper σ-ideal on X. We say that a set $A \subseteq X$ satisfies the *I-dichotomy* if A is either I-small or there is an I-positive Borel set in X which is a subset of A.

While it is clear from the definition that the set of sets satisfying the I-dichotomy is closed under countable unions, it is not clear whether it is closed under complements. For this reason, these regularity properties are sometimes called asymmetric. Note that for I-regularity it is exactly the other way around. It is clear from the definition that the set of I-regular sets is closed under complements, but not whether it is closed under countable unions. By Proposition 2.1.6, if $\mathbb{P}_I$ is proper, then the I-regular sets form a σ-algebra. However, this is not necessarily true for the set of sets satisfying the I-dichotomy: if an uncountable set $A \subseteq \omega^\omega$ satisfies the **ctbl**-dichotomy, then there is an uncountable Borel set $B \subseteq A$. Since every Borel set has the perfect set property, there is a perfect set $F \subseteq B \subseteq A$. Hence, a set of reals has the perfect set property if and only if it satisfies the **ctbl**-dichotomy. By Theorem 1.2.57, it is not provable in ZFC that the set of sets having the set property is closed under complements. Hence, it is not provable in ZFC that the sets satisfying the **ctbl**-dichotomy form an algebra. Moreover, it is not provable in ZFC that every co-analytic set of reals has the perfect set property. Therefore, it is not necessarily true that for every projective pointclass Γ, every Γ set is I-regular if and only if every Γ set satisfies the I-dichotomy. However, the backward direction is true if I is Borel generated.

Proposition 2.2.38 (Folklore)**.** *Let X be an uncountable Polish space, let I be a proper σ-ideal on X which is Borel generated, and let Γ be a projective pointclass. If every $\Gamma(X)$ set satisfies the I-dichotomy, then every $\Gamma(X)$ set is I-regular.*

Proof. Let $A \subseteq X$ be a $\Gamma(X)$ set and let B be an I-positive Borel set in X. Then $B \cap A$ is also in $\Gamma(X)$. Hence, either $B \cap A$ is I-small or there is an I-positive Borel $B' \subseteq B \cap A$. In the latter case, we are done. So we can assume that $B \cap A$ is I-small. Since I is Borel generated, there is an I-small Borel set $C \subseteq X$ containing $B \cap A$. Then $B \setminus C$ is an I-positive Borel set which is contained in A. □

We have already seen that the converse of Proposition 2.2.38 is not true in general. However, it is true for the $I^*_{\mathcal{C}}$-ideal in weak category bases $(X, \mathcal{C})$. We can even show something stronger.

Proposition 2.2.39. *Let X be an uncountable Polish space and let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X and $I^*_{\mathcal{C}}$ is Borel generated. Then every $\mathcal{C}$-Baire set satisfies the $I^*_{\mathcal{C}}$-dichotomy.*

Proof. Let $A \subseteq X$ be $\mathcal{C}$-Baire which is $I^*_{\mathcal{C}}$-positive. Then there is a region $C \in \mathcal{C}$ such that $C \setminus A$ is $\mathcal{C}$-meager. Since $I^*_{\mathcal{C}}$ is Borel generated, there is an $I^*_{\mathcal{C}}$-small Borel set $M \subseteq X$ containing $C \setminus A$. Then $C \setminus M$ is an $I^*_{\mathcal{C}}$-positive Borel set which is contained in A. □

Corollary 2.2.40. *Let X be an uncountable Polish space, let $(X, \mathcal{C})$ be a proper weak category base such that $(X, \mathcal{C})$ is Borel compatible with X and $I^*_{\mathcal{C}}$ is Borel generated, and let Γ be a projective pointclass. Every $\Gamma(X)$ set is $\mathcal{C}$-Baire if and only if every $\Gamma(X)$ set satisfies the $I^*_{\mathcal{C}}$-dichotomy. In particular, every analytic set in X satisfies the $I^*_{\mathcal{C}}$-dichotomy.*

Proof. Follows directly from Corollary 2.2.28 and Propositions 2.2.38 and 2.2.39. $\square$

2.2.6 Direct implications between regularity statements

In this section, we study implications between regularity statements for weak category bases. There are many such implication results in the literature, especially for the second level of the projective hierarchy (cf. Figure 1.1). Many of the proofs of these implications go via Ikegami's Theorem. Hence, one could argue that these proofs are "indirect", since they actually prove statements about quasi-generic reals. However, there are also a few proofs that work directly with the regularity properties. These "direct" proofs have the advantage that they often work not only for the second level of the projective hierarchy, but for all projective pointclasses. For example, Brendle and Löwe showed, in [BL99], that for every projective pointclass Γ, $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{C})$. The goal of this section is to generalize Brendle and Löwe's result to obtain a sufficient criterion for when the analogue is true for weak category bases. To prepare this, we first give a sketch of their proof. Roughly speaking, they turned the standard proof that Hechler forcing adds Cohen reals into a topological version. More precisely, let $h : \omega^\omega \to 2^\omega$ be defined as

$$h(x)(n) := x(n) \pmod 2.$$

It is well-known that if x is a Hechler real, then $h(x)$ is a Cohen real. Moreover, h has many nice topological properties, e.g., h is continuous, h maps basic open sets to basic open sets, etc. Here it does not matter whether we equip ω^ω with the standard or the dominating topology, however for obvious reasons we are more interested in the dominating topology. Brendle and Löwe used these topological properties to show that for every set of reals $A \subseteq 2^\omega$, if $h^{-1}(A)$ has the Baire property in the dominating topology, then A has the Baire property in the standard topology. Since h is continuous, this directly implies their theorem. For more details, see [BL99, Theorem 3.1].

So the core of Brendle and Löwe's proof was the function h. The following definition generalizes the key property of h to general weak category bases.

Definition 2.2.41. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases. We say that a function $h : X \to Y$ is *weakly category preserving* if for every $A \subseteq Y$ for which $h^{-1}(A)$ is $\mathcal{C}$-Baire, A is $\mathcal{D}$-Baire. Let $\alpha > 0$ be an ordinal. A sequence $\langle h_\beta : \beta < \alpha \rangle$ of functions from $(X, \mathcal{C})$ to $(Y, \mathcal{D})$ is called *weakly category preserving* if for every $A \subseteq Y$ for which $h_\beta^{-1}(A)$ is $\mathcal{C}$-Baire for every $\beta < \alpha$, A is $\mathcal{D}$-Baire.

Note that if $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are proper weak category bases and $h : X \to Y$ is a bijection, then h is weakly category preserving if and only if for every $\mathcal{C}$-Baire set $A \subseteq X$, $f[A]$ is $\mathcal{D}$-Baire. Hence, for bijective functions, the definition of a weakly category preserving function is equivalent to the definition of what one would usually call a preserving function for weak category bases. In the words of Definition 2.2.41, Brendle and Löwe used a continuous weakly category preserving function to show that for every projective pointclass Γ, $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{C})$. Using the same argument, we can show that the existence of a weakly category preserving sequence of Borel functions is enough to prove the analogous implications for weak category bases.

Lemma 2.2.42. *Let X and Y be uncountable Polish spaces, let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases such that $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are Borel compatible with X and Y, respectively. If there is a weakly category preserving sequence of Borel functions from X to Y, then for every projective pointclass Γ, if every $\Gamma(X)$ set is $\mathcal{C}$-Baire, then every $\Gamma(Y)$ set is $\mathcal{D}$-Baire.*

Proof. Let $\alpha > 0$, $\langle h_\beta : \beta < \alpha \rangle$ be a weakly category preserving sequence of Borel functions from X to Y, let Γ be a projective pointclass such that every $\Gamma(X)$ set is $\mathcal{C}$-Baire, and let $A \subseteq Y$ be $\Gamma(Y)$. We have to show that A is $\mathcal{D}$-Baire. Let $\beta < \alpha$. Since Γ is closed under preimages by Borel functions, $h_\beta^{-1}(A)$ is in $\Gamma(X)$ for every $\beta < \alpha$. Hence, $h_\beta^{-1}(A)$ is $\mathcal{C}$-Baire for every $\beta < \alpha$. Since $\langle h_\beta : \beta < \alpha \rangle$ is weakly category preserving, A is $\mathcal{D}$-Baire. $\square$

Next, we investigate the question of when weakly category preserving functions exists. To show that the modulo 2 function h is weakly category preserving, Brendle and Löwe used the fact that h maps basic open sets in the dominating topology to basic open sets in the Cantor space and so induces a function $h' : \mathbb{D} \to \mathbb{C}$. This function is order preserving and for every $p \in \mathbb{D}$ and every $s \leq h'(p)$, there is a $p' \leq p$ such that $h'(p') \leq s$. We call such a function a projection.

Definition 2.2.43. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. A *projection from $\mathbb{P}$ to $\mathbb{Q}$* is an order preserving function $f : \mathbb{P} \to \mathbb{Q}$ such that for every $p \in \mathbb{P}$ and every $q \leq f(p)$, there is a $p' \leq p$ such that $h(p') \leq q$.

Note that if there is a projection from $\mathbb{P}$ to $\mathbb{Q}$, then forcing with $\mathbb{P}$ adds a $\mathbb{Q}$-generic filter: let $f : \mathbb{P} \to \mathbb{Q}$ be a projection, let G be a $\mathbb{P}$-generic filter, and let H be the upwards closure of $f[G]$. We shall show that H is a $\mathbb{Q}$-generic filter over V. It is clear that H is a filter. Let $D \subseteq \mathbb{Q}$ be open dense. It is enough to show that G meets $f^{-1}(D)$. Let $p \in \mathbb{P}$. Then there is a $q \in D$ such that $q \leq f(p)$. Since f is a projection, there is a $p' \leq p$ such that $f(p') \leq q$. Then $f(p') \in D$ and so $f^{-1}(D)$ is dense. Therefore, G meets $f^{-1}(D)$.

In the following, we consider projections from and to weak category bases. Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be weak category bases. We say that a function from $\mathcal{C}$ to $\mathcal{D}$ is a *projection* if it is a projection from $(\mathcal{C}, \subseteq)$ to $(\mathcal{D}, \subseteq)$. The rough idea is to show that if $\langle h_\beta : \beta < \alpha \rangle$ is a sequence of functions from X to Y and $\langle \overline{h}_\beta : \beta < \alpha \rangle$ is a sequence of projections from $\mathcal{C}$ to $\mathcal{D}$ such that for every $\beta < \alpha$, h_β interacts nicely with $\overline{h}_\beta$, then $\langle h_\beta : \beta < \alpha \rangle$ is weakly category preserving (cf. Theorem 2.2.46). To prepare this, we first prove two lemmas about small sets.

Lemma 2.2.44. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases, let $h : X \to Y$ be a function and let $\overline{h} : \mathcal{C} \to \mathcal{D}$ be a projection such that for every $C \in \mathcal{C}$, there is some region $C' \subseteq C$ such that $h[C'] \subseteq \overline{h}(C)$, and let $A \subseteq Y$. Then if A is $I^*_{\mathcal{D}}$-small, then $h^{-1}(A)$ is $I^*_{\mathcal{C}}$-small.*

Proof. First, we assume that A is $\mathcal{D}$-singular. Let $C \in \mathcal{C}$ be a region. Since A is $\mathcal{D}$-singular, there is some region $D \subseteq \overline{h}(C)$ such that $D \cap A = \emptyset$. By assumption, there are subregions $C'' \subseteq C' \subseteq C$ such that $h[C''] \subseteq \overline{h}(C') \subseteq D$. Then $C'' \cap h^{-1}(A) \subseteq h^{-1}(D) \cap h^{-1}(A) = h^{-1}(D \cap A) = \emptyset$ and so $h^{-1}(A)$ is $\mathcal{C}$-singular. Next, we assume that A is $\mathcal{D}$-meager. Then there are $\mathcal{D}$-singular sets $N_n \subseteq Y$ such that $A = \bigcup_{n \in \omega} N_n$. Let $M_n := h^{-1}(N_n)$. Then all M_n are $\mathcal{C}$-singular and $h^{-1}(A) = \bigcup_{n \in \omega} M_n$. Therefore, $h^{-1}(A)$ is $\mathcal{C}$-meager. Finally, we assume that A is $I^*_{\mathcal{D}}$-small. Let $C \in \mathcal{C}$ be a region. Since A is $I^*_{\mathcal{D}}$-small, there is some region $D \subseteq \overline{h}(C)$ such that $D \cap A$ is $\mathcal{C}$-meager. By assumption, there are subregions $C'' \subseteq C' \subseteq C$ such that $h[C''] \subseteq \overline{h}(C') \subseteq D$. Then $C'' \cap h^{-1}(A) \subseteq h^{-1}(D) \cap h^{-1}(A) = h^{-1}(D \cap A)$ and $h^{-1}(D \cap A)$ is $\mathcal{C}$-meager. Therefore, $h^{-1}(A)$ is $I^*_{\mathcal{C}}$-small. $\square$

The question of whether the converse of Lemma 2.2.44 holds is more complicated. We shall answer it later (cf. Corollary 2.2.47). For now, we investigate the special case in which A is $\mathcal{D}$-Baire. This can be proved if we additionally assume that the image of $\overline{h}$ is dense in $(\mathcal{D}, \subseteq)$.

Lemma 2.2.45. *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases, let $\alpha > 0$ be an ordinal, let $\langle h_\beta : \beta < \alpha \rangle$ be a sequence of functions from X to Y, let $\langle \overline{h}_\beta : \beta < \alpha \rangle$ be a sequence of projections from $\mathcal{C}$ to $\mathcal{D}$ such that*

(a) for every $\beta < \alpha$ and every $C \in \mathcal{C}$, there is some region $C' \subseteq C$ such that $h_\beta[C'] \subseteq \overline{h}_\beta(C)$ and

(b) for every $D \in \mathcal{D}$, there are $\beta < \alpha$ and $C \in \mathcal{C}$ such that $\overline{h}_\beta(C) \subseteq D$,

and let $A \subseteq Y$. Then if A is $I^*_{\mathcal{D}}$-positive and $\mathcal{D}$-Baire, then there is some $\beta < \alpha$ such that $h_\beta^{-1}(A)$ is $I^*_{\mathcal{C}}$-positive.

Proof. Since A is $\mathcal{D}$-Baire but not $I^*_{\mathcal{D}}$-small, there is a region $D \in \mathcal{D}$ such that $D \setminus A$ is $\mathcal{D}$-meager. Let $\beta < \alpha$ and $C \in \mathcal{C}$ such that $\overline{h}_\beta(C) \subseteq D$ and let $C' \subseteq C$ be a subregion such that $h_\beta[C'] \subseteq \overline{h}_\beta(C)$. By Lemma 2.2.44, $C' \setminus h_\beta^{-1}(A) \subseteq h_\beta^{-1}(D \setminus A)$ is $\mathcal{C}$-meager. Since C' is $I^*_{\mathcal{C}}$-positive, $h_\beta^{-1}(A)$ has to be $I^*_{\mathcal{C}}$-positive as well. □

Now we are done with the preparations for the main theorem of this section.

Theorem 2.2.46. *Let X and Y be uncountable Polish spaces, let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases such that $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are Borel compatible with X and Y, respectively, and $I^*_{\mathcal{D}}$ is Borel generated, let $\alpha > 0$ be an ordinal, let $\langle h_\beta : \beta < \alpha \rangle$ be a sequence of Borel functions from X to Y, and let $\langle \overline{h}_\beta : \beta < \alpha \rangle$ be a sequence of projections from $\mathcal{C}$ to $\mathcal{D}$ such that*

(a) for every $\beta < \alpha$ and every $C \in \mathcal{C}$, there is some region $C' \subseteq C$ such that $h_\beta[C'] \subseteq \overline{h}_\beta(C)$ and

(b) for every $D \in \mathcal{D}$, there are $\beta < \alpha$ and $C \in \mathcal{C}$ such that $\overline{h}_\beta(C) \subseteq D$.

Then $\langle h_\beta : \beta < \alpha \rangle$ is weakly category preserving. Moreover, for every projective pointclass Γ, if every $\Gamma(X)$ set is $\mathcal{C}$-Baire, then every $\Gamma(Y)$ set is $\mathcal{D}$-Baire.

Proof. The second part follows directly from the first part and Lemma 2.2.42. Hence, it is enough to prove the first part. By Theorem 2.2.27, it is enough to show that if for every $\beta < \alpha$, $h_\beta^{-1}(A)$ is $I^*_{\mathcal{C}}$-regular, then A is $I^*_{\mathcal{D}}$-regular. Let B be an $I^*_{\mathcal{D}}$-positive Borel set in Y. By Lemma 2.2.45, there is a $\beta < \alpha$ such that $C := h_\beta^{-1}(B)$ is $I^*_{\mathcal{C}}$-positive. Since h_β is Borel, C is an $I^*_{\mathcal{C}}$-positive Borel set in X. Hence, there is an $I^*_{\mathcal{C}}$-positive Borel set $C' \subseteq C$ such that either $C' \subseteq h_\beta^{-1}(A)$ or $C' \cap h_\beta^{-1}(A) = \emptyset$. Then $h_\beta[C'] \subseteq B$ and $h_\beta[C']$ is analytic in Y. By Lemma 2.2.44 and Corollary 2.2.40, $h_\beta[C']$ is $I^*_{\mathcal{D}}$-positive and satisfies the $I^*_{\mathcal{C}}$-dichotomy. Hence, there is an $I^*_{\mathcal{D}}$-positive Borel $B' \subseteq h_\beta[C']$. Then $B' \subseteq h_\beta[C'] \subseteq B$ and it remains to check that either $B' \subseteq A$ or $B' \cap A = \emptyset$. We make a case-distinction:

Case 1: $C' \cap h_\beta^{-1}(A) = \emptyset$. We assume for a contradiction that $h_\beta[C'] \cap A \neq \emptyset$. Let $y \in h_\beta[C'] \cap A$. Then there is some $x \in C'$ such that $h_\beta(x) = y$. Since $h_\beta(x) \in A$, $x \in h_\beta^{-1}(A)$. But this is impossible since $C' \cap h_\beta^{-1}(A) = \emptyset$. Therefore, $B' \cap A \subseteq h_\beta[C'] \cap A = \emptyset$.

Case 2: $C' \subseteq h_\beta^{-1}(A)$. Then $B' \subseteq h_\beta[C'] \subseteq h_\beta[h_\beta^{-1}(A)] \subseteq A$. □

Note that Theorem 2.2.46 implies Brendle and Löwe's result: let $\mathcal{C}$ and $\mathcal{D}$ be the set of non-empty basic open sets in the dominating topology and the Cantor space, respectively. Then $(\omega^\omega, \mathcal{C})$ and $(2^\omega, \mathcal{D})$ are category bases satisfying the c.c.c. and a set is $\mathcal{C}$-Baire ($\mathcal{D}$-Baire) if and only if it has the Baire property in the dominating topology (in the Cantor space). Let $h : \omega^\omega \to 2^\omega$ be the modulo 2 function from above and let $\overline{h} : \mathcal{C} \to \mathcal{D}$ be the function induced by h. It is clear that h and $\overline{h}$ satisfy the requirements of Theorem 2.2.46. Hence, for every projective pointclass Γ, $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{C})$.

We conclude this section with two corollaries of Theorem 2.2.46. The first corollary answers the question whether the converse of Lemma 2.2.44 is true.

Corollary 2.2.47. *Let X and Y be uncountable Polish spaces, let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be proper weak category bases such that $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are Borel compatible with X and Y, respectively, and $I^*_{\mathcal{D}}$ is Borel generated, let $\alpha > 0$ be an ordinal, let $\langle h_\beta : \beta < \alpha \rangle$ be a sequence of Borel functions from X to Y, let $\langle \overline{h}_\beta : \beta < \alpha \rangle$ be a sequence of projections from $\mathcal{C}$ to $\mathcal{D}$ such that*

(a) for every $\beta < \alpha$ and every $C \in \mathcal{C}$, there is some region $C' \subseteq C$ such that $h_\beta[C'] \subseteq \overline{h}_\beta(C)$, and

(b) for every $D \in \mathcal{D}$, there are $\beta < \alpha$ and $C \in \mathcal{C}$ such that $\overline{h}_\beta(C) \subseteq D$,

and let $A \subseteq Y$. Then A is $I^*_{\mathcal{D}}$-small if and only if for every $\beta < \alpha$, $h_\beta^{-1}(A)$ is $I^*_{\mathcal{C}}$-small.

Proof. The forward direction follows directly from Lemma 2.2.44. We prove the backward direction. Assume that for every $\beta < \alpha$, $h_\beta^{-1}(A)$ is $I^*_{\mathcal{C}}$-small. Then for every $\beta < \alpha$, $h_\beta^{-1}(A)$ is $\mathcal{C}$-Baire. By Theorem 2.2.46, A is $\mathcal{D}$-Baire and so by Lemma 2.2.45, A is $I^*_{\mathcal{D}}$-small. $\square$

In their proof that for every projective pointclass Γ, $\Gamma(\mathbb{D})$ implies $\Gamma(\mathbb{C})$, Brendle and Löwe left it to the reader to verify that if a set $A \subseteq 2^\omega$ is non-meager in the Cantor space, then $h^{-1}(A)$ is non-meager in the dominating topology, where h is the modulo 2 function from above (cf. [BL99, Theorem 3.1]). Note that this follows directly from Proposition 2.2.47: let $(\omega^\omega, \mathcal{C})$, $(2^\omega, \mathcal{D})$, h, and $\overline{h}$ be defined as before. Since $(\omega^\omega, \mathcal{C})$ satisfies the c.c.c., $I^*_{\mathcal{C}} = I_{\mathcal{C}}$. Hence, a set is $\mathcal{C}$-meager if and only if it is meager in the dominating topology. Analogously, a set is $\mathcal{D}$-meager if and only if it is meager in the Cantor space. Clearly, h and $\overline{h}$ satisfy the requirements of Proposition 2.2.47. Hence, a set $A \subseteq 2^\omega$ is meager in the Cantor space if and only if $h^{-1}(A)$ is meager in the dominating topology.

The second corollary is about the relationships between the cardinal characteristics of $I^*_{\mathcal{C}}$ and $I^*_{\mathcal{D}}$.

Corollary 2.2.48. *Let $(X, \mathcal{C})$, $(Y, \mathcal{D})$, $\langle h_\beta : \beta < \alpha \rangle$, and $\langle \overline{h}_\beta : \beta < \alpha \rangle$ be as in Proposition 2.2.47. Then*

(a) $\mathrm{add}(I^*_{\mathcal{C}}) \leq \mathrm{add}(I^*_{\mathcal{D}})$,

(b) $\mathrm{cov}(I^*_{\mathcal{C}}) \leq \mathrm{cov}(I^*_{\mathcal{D}})$, *and*

(c) $\mathrm{non}(I^*_{\mathcal{C}}) \geq \mathrm{non}(I^*_{\mathcal{D}})$.

Proof. We start with proving (a). Let $\mathscr{F}$ be a family of $I^*_{\mathcal{D}}$-small sets such that $\bigcup \mathscr{F}$ is $I^*_{\mathcal{D}}$-positive and let $\beta < \alpha$. By Proposition 2.2.47, $\mathscr{F}' := \{h_\beta^{-1}(A) : A \in \mathscr{F}\}$ is a family of $I^*_{\mathcal{C}}$-small sets and $\bigcup \mathscr{F}' = h_\beta^{-1}(\bigcup \mathscr{F})$ is $I^*_{\mathcal{C}}$-positive. Therefore, $\mathrm{add}(I^*_{\mathcal{C}}) \leq \mathrm{add}(I^*_{\mathcal{D}})$.

Next, we prove (b). Let $\mathscr{F}$ be a family of $I^*_{\mathcal{D}}$-small sets such that $\bigcup \mathscr{F} = Y$ and let $\beta < \alpha$. By Proposition 2.2.47, $\mathscr{F}' := \{h_\beta^{-1}(A) : A \in \mathscr{F}\}$ is a family of $I^*_{\mathcal{C}}$-small sets and $\bigcup \mathscr{F}' = X$. Therefore, $\mathrm{cov}(I^*_{\mathcal{C}}) \leq \mathrm{cov}(I^*_{\mathcal{D}})$.

Finally, we prove (c). Let $A \subseteq X$ be $I^*_{\mathcal{C}}$-positive and let $\beta < \alpha$. By Proposition 2.2.47, $A' := h_\beta[A]$ is $I^*_{\mathcal{D}}$-positive. Moreover, $|A'| \leq |A|$. Therefore, $\mathrm{non}(I^*_{\mathcal{C}}) \geq \mathrm{non}(I^*_{\mathcal{D}})$. □

Question 2.2.49. *Under the assumptions of Corollary 2.2.48, is* $\mathrm{cof}(I^*_{\mathcal{C}}) \geq \mathrm{cof}(I^*_{\mathcal{D}})$*?*

2.3 Amoeba forcing

2.3.1 Definitions and basics

In Section 2.3, we study Judah and Repický's amoeba regularity. The main goal is to show that if every $\mathbf{\Sigma}^1_2$ set is amoeba regular, then for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. We shall prove this in Section 2.3.2. In this section, we introduce amoeba regularity and prove Solovay- and Judah-Shelah-style characterizations for it. Amoeba forcing was first introduced by Martin and Solovay in their paper [MS70], where they also introduced Martins's axiom. They used amoeba forcing to show that Martin's axiom implies that the additivity number of the Lebesgue null ideal is $2^{\aleph_0}$. At that time, however, it was not yet called amoeba forcing. The name "amoeba" was first mentioned by Truss in [Tru77], a few years later. The idea behind this name is the following: originally, amoeba forcing was introduced as the set of open sets of Lebesgue measure less than some fixed positive number ε, ordered by reversed inclusion. Now for every open set $O \subseteq \mathbb{R}$ with $\mu(O) < \varepsilon$ and every Lebesgue null set $N \subseteq \mathbb{R}$, one can find a bigger open set $O' \supseteq O$ with $\mu(O') < \varepsilon$ containing N as a subset. So amoeba conditions can "absorb" Lebesgue null sets.

Nowadays, many different versions of amoeba forcing can be found in the literature. Truss had shown in [Tru88] that the most common variants are all forcing equivalent. In this work, we shall mostly work with the following variant:

Definition 2.3.1. *Amoeba forcing* is the partial order of all pruned trees $T \subseteq 2^{<\omega}$ with $\mu([T]) > \frac{1}{2}$, ordered by inclusion. We denote it by $\mathbb{A}$.

The size of the bound in Definition 2.3.1 does not really matter. We can be replace $\frac{1}{2}$ with any other positive number $\varepsilon < 1$ and obtain a variant of amoeba forcing $\mathbb{A}_\varepsilon$ which is forcing equivalent to $\mathbb{A}$ (cf. [Tru88, Theorem 3.3]). Later in Section 2.3.2, we shall introduce another variant of amoeba forcing. Here, we study some basic properties of $\mathbb{A}$ and amoeba regularity.

Proposition 2.3.2 (Martin-Solovay)**.** *Amoeba forcing satisfies the c.c.c.*

Proof. Cf., e.g., [Kun11, Proof of III.3.28]. □

Perhaps the most important property of amoeba forcing is that forcing with amoeba forcing causes the union of all ground model Lebesgue null to be Lebesgue null: let M be a transitive model of ZFC, let $\mathcal{N}^M$ be the ideal of Lebesgue null sets in M, and let G be an $\mathbb{A}$-generic filter over M. Then for every Lebesgue null set $N \in \mathcal{N}^M$, the set $D_N := \{T \in \mathbb{A} : [T] \cap N = \emptyset\}$ is dense in $\mathbb{A}$. Hence, there is a $T \in G$ such that $[T] \cap N = \emptyset$. Let $T_G := \bigcap G$. Then T_G is a pruned tree, $\mu([T_G]) = \frac{1}{2}$, and $[T_G] \cap N = \emptyset$. Hence, $[T_G] \cap \bigcup \mathcal{N}^M = \emptyset$ and so, in $M[G]$, $\mu(\bigcup \mathcal{N}^M) \leq \frac{1}{2}$. Since $\mathbb{A}$ and $\mathbb{A}_\varepsilon$ are forcing equivalent for every $0 < \varepsilon < 1$, there is an $\mathbb{A}_\varepsilon$-generic filter G_ε in $M[G]$ for every $0 < \varepsilon < 1$. We can repeat the above argument for these filters. Then for every $0 < \varepsilon < 1$, $\mu(\bigcup \mathcal{N}^M) \leq \varepsilon$ in $M[G]$. Therefore, $\bigcup \mathcal{N}^M$ is Lebesgue null in $M[G]$.

We call the tree T_G from above an *amoeba real over* M. Note that $G = \{T \in \mathbb{A} : T_G \subseteq T\}$ and so every $\mathbb{A}$-generic filter over M is uniquely determined by an amoeba real over M. Strictly speaking amoeba reals are trees, not reals. However, trees can be coded as reals. To do this, we fix a recursive enumeration $\langle s_k : k \in \omega \rangle$ of $2^{<\omega}$. Let $T \subseteq 2^{<\omega}$ be a tree. Then there is a unique real $c \in 2^\omega$ such that $c(k) = 1$ if and only if $s_k \in T$. We say that this c codes T. From now on, we shall often identify trees with their code and so treat them as reals. Let $\mathbf{R}$ be the set of all pruned trees $T \subseteq 2^{<\omega}$ with $\mu([T]) = \frac{1}{2}$. Then we can think of $\mathbf{R}$ as a subset of the reals and every amoeba real lives in $\mathbf{R}$. Moreover, $\mathbf{R}$ with the induced topology is even a Polish space.

Proposition 2.3.3. *The set* $\mathbf{R}$ *is a* Π^0_2 *subset of the Baire space. In particular,* $\mathbf{R}$ *equipped with the induced topology is a Polish space.*

Proof. A real $c \in 2^\omega$ is in $\mathbf{R}$ if it is a code for a pruned tree T_c on 2 with $\mu([T_c]) = \frac{1}{2}$. Clearly, the statement "T_c is a pruned tree" is $\Pi^0_2(c)$. Moreover, $\mu([T_c]) \leq \frac{1}{2}$ if and only if for every $n > 0$, there are pairwise different $s_0, \dots s_m \in 2^{<\omega} \setminus T_c$ with length k such that $\frac{m+1}{2^k} \geq \frac{1}{2} - \frac{1}{n}$ and $\mu([T_c]) \geq \frac{1}{2}$ if and only if for every $s_0, \dots, s_m \in 2^{<\omega} \setminus T_c$ with length k, $\frac{m+1}{2^k} \leq \frac{1}{2}$. Hence, the statement "$\mu([T_c]) = \frac{1}{2}$" is $\Pi^0_2(c)$ as well and so $\mathbf{R}$ is Π^0_2. By Theorem 1.2.2, $\mathbf{R}$ equipped with the induced topology is a Polish space. $\square$

Additionally, we can use $\mathbb{A}$ to define a second structure on $\mathbf{R}$. First, we need a function from $\mathbb{A}$ to $\mathcal{P}(\mathbf{R})$. We define $\langle T \rangle := \{S \in \mathbf{R} : S \subseteq T\}$ for every $T \in \mathbb{A}$. Note that for every $T, T' \in \mathbb{A}$, $T' \leq T$ if and only if $\langle T' \rangle \subseteq \langle T \rangle$. Hence, for all incompatible conditions $T, T' \in \mathbb{A}$, there is no $T'' \in \mathbb{A}$ such $\langle T'' \rangle \subseteq \langle T \rangle \cap \langle T' \rangle$. Therefore, $\mathcal{C}_\mathbb{A} := \{\langle T \rangle : T \in \mathbb{A}\}$ is not a base for a topology on $\mathbf{R}$. However, Judah and Repický showed that $(\mathbf{R}, \mathcal{C}_\mathbb{A})$ is a category base.

Theorem 2.3.4 (Judah-Repický)**.** *The pair* $(\mathbf{R}, \mathcal{C}_\mathbb{A})$ *is a category base.*

Proof. Cf. [JR95, Lemma 1.1]. $\square$

Since $(\mathbf{R}, \mathcal{C}_\mathbb{A})$ is a category base, $\mathcal{C}_\mathbb{A}$ introduces a regularity property on $\mathbf{R}$. Judah and Repický used this regularity property to define amoeba regularity. We say that a set $A \subseteq \mathbf{R}$ is *amoeba regular* if it is $\mathcal{C}_\mathbb{A}$-Baire. Thus, a set $A \subseteq \mathbf{R}$ is amoeba regular if and only if for every $T \in \mathbb{A}$ there is a $T' \leq T$ such that $\langle T' \rangle \setminus A$ or $\langle T' \rangle \cap A$ is $\mathcal{C}_\mathbb{A}$-meager. Recall that a set $A \subseteq \mathbf{R}$ is $\mathcal{C}_\mathbb{A}$-singular if for every $T \in \mathbb{A}$ there is a $T' \leq T$ such that $\langle T' \rangle \cap A = \emptyset$ and that a set is $\mathcal{C}_\mathbb{A}$-meager if it is a countable union of $\mathcal{C}_\mathbb{A}$-singular sets. We have already seen in Section 2.2, that for every category base, the Baire sets form a σ-algebra containing all regions. So, in particular, the amoeba regular sets form a σ-algebra and every region is amoeba regular. Next, we compare $(\mathbf{R}, \mathcal{C}_\mathbb{A})$ with the topology on $\mathbf{R}$.

Lemma 2.3.5. *For every* $T \in \mathbb{A}$, $\langle T \rangle$ *is closed in* $\mathbf{R}$ *and* $\mathcal{C}_\mathbb{A}$*-abundant.*

Proof. We start with the former. Let $T \in \mathbb{A}$ and let $S \notin \langle T \rangle$. Then there is a $k \in \omega$ such that $s_k \in S \setminus T$. Let $c \in 2^\omega$ be the code for S. Then $[c{\restriction}k+1] \cap \mathbf{R}$ is an open set in $\mathbf{R}$ which contains S and is disjoint from $\langle T \rangle$. Hence, $\langle T \rangle$ is closed in $\mathbf{R}$.

It remains to show that for every $T \in \mathbb{A}$, $\langle T \rangle$ is $\mathcal{C}_\mathbb{A}$-abundant. We suppose for a contradiction that there is a $T \in \mathbb{A}$ such that $\langle T \rangle$ is $\mathcal{C}_\mathbb{A}$-meager. Then there are $\mathcal{C}_\mathbb{A}$-singular sets N_n such that $\langle T \rangle = \bigcup_{n \in \omega} N_n$. We shall define a decreasing sequence of conditions $\langle T_n : n \in \omega \rangle$ by recursion: let

$T_0 := T$. If T_n is already defined, then let $T_{n+1} \leq T_n$ such that $\langle T_{n+1}\rangle \cap N_n = \emptyset$. Such a T_{n+1} exists since N_n is $\mathcal{C}_{\mathbb{A}}$-singular.

Then for every $n \in \omega$, $\langle T_{n+1}\rangle \cap N_n = \emptyset$. Let $S := \bigcap_{n\in\omega} T_n$ then S is a pruned tree on 2 and $\mu([S]) \geq \frac{1}{2}$. Without loss of generality, $\mu([S]) = \frac{1}{2}$. Then for every $n \in \omega$, $S \in \langle T_n\rangle$. In particular, $S \in \langle T\rangle$. Hence, there is some $n \in \omega$ such that $S \in N_n$. But this is not possible since $S \in \langle T_{n+1}\rangle$. Therefore, every region is $\mathcal{C}_{\mathbb{A}}$-abundant. □

By Lemma 2.3.5, every region is closed. However, the converse is not true. For example, for any $T, T' \in \mathbb{A}$ with $\mu([T] \cap [T']) = \frac{1}{2}$, $\langle T\rangle \cap \langle T'\rangle$ is closed, but there is no $T'' \in \mathbb{A}$ such that $\langle T''\rangle = \langle T\rangle \cap \langle T'\rangle$. Nevertheless, we can show that every Borel set in $\mathbf{R}$ is amoeba regular.

Proposition 2.3.6. *Every Borel set in $\mathbf{R}$ is amoeba regular. Moreover, $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is Borel compatible with $\mathbf{R}$.*

Proof. The second part follows directly from the first part and Lemma 2.3.5. So we only have to show that every Borel set in $\mathbf{R}$ is amoeba regular. Since the $\mathcal{C}_{\mathbb{A}}$-Baire sets form a σ-algebra, it is enough to show that every open set in $\mathbf{R}$ is $\mathcal{C}_{\mathbb{A}}$-Baire. Let $s \in 2^{<\omega}$, let $T \in \mathbb{A}$, and let $c \in 2^\omega$ be the code for T. We make a case distinction:

Case 1: $s \subseteq c$ and $\mu([T] \cap [s]) > 0$. Let $T' \leq T$ be such that $\mu([T'] \setminus [s]) < \frac{1}{2}$. Then $\langle T\rangle \subseteq ([s] \cap \mathbf{R})$ and so $\langle T\rangle \setminus ([s] \cap \mathbf{R}) = \emptyset$.

Case 2: $s \subseteq c$ and $\mu([T] \cap [s]) = 0$. Let T' be the unique pruned tree such that $[T'] = [T] \setminus [s]$. Then $T' \leq T$ and $\langle T'\rangle \cap ([s] \cap \mathbf{R}) = \emptyset$.

Case 3: there is some $n < \mathrm{lh}(s)$ such that $s(n) = 0 \neq c(n)$. Let $T' \leq T$ be such that $\mu([T'] \setminus [c{\restriction}n + 1]) < \frac{1}{2}$. Then for every $S \in \langle T'\rangle$, $s_n \in S$ and so $\langle T'\rangle \cap ([s] \cap \mathbf{R}) = \emptyset$.

Case 4: there is some $n < \mathrm{lh}(s)$ such that $s(n) = 1 \neq c(n)$. Then $\langle T\rangle \cap ([s] \cap \mathbf{R}) = \emptyset$. □

Proposition 2.3.7. *The category base $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is proper. So in particular, $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is a proper category base which satisfies the c.c.c. and is Borel compatible with $\mathbf{R}$.*

Proof. The second part follows directly from the first part and Propositions 2.3.2 and 2.3.6. Hence, we only have to show that $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is proper, i.e., we have to show that $(\mathcal{C}_{\mathbb{A}}, \subseteq)$ is a proper forcing notion, $I_{\mathcal{C}_{\mathbb{A}}}$ is a proper σ-ideal, and that every region is $\mathcal{C}_{\mathbb{A}}$-abundant. Since $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ satisfies the c.c.c., $(\mathcal{C}_{\mathbb{A}}, \subseteq)$ satisfies the c.c.c. and is therefore proper. By Lemma 2.3.5, every region is $\mathcal{C}_{\mathbb{A}}$-abundant. Hence, we only have to show that every singleton is $\mathcal{C}_{\mathbb{A}}$-meager. But this is clear. □

Corollary 2.3.8.

(a) The amoeba regular sets form a σ-algebra on $\mathbf{R}$ containing all analytic and co-analytic sets in $\mathbf{R}$.

(b) In L, there is a $\mathbf{\Delta}^1_2(\mathbf{R})$ set which is not amoeba regular.

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\mathbf{\Sigma}^1_2(\mathbf{R})$ sets are amoeba regular.

Proof. Follows directly from Corollary 2.2.28 and Proposition 2.3.7. □

By Corollary 2.3.8, we are in the same situation as for most other regularity properties, i.e., every analytic set is amoeba regular and the question whether every $\mathbf{\Delta}^1_2(\mathbf{R})$ set is amoeba regular cannot be answered in ZFC. We write $\Gamma(\mathbb{A})$ for the statement "every $\Gamma(\mathbf{R})$ set is amoeba regular", where Γ is a projective pointclass. We conclude this section with Solovay- and Judah-Shelah-style characterizations for amoeba regularity.

Theorem 2.3.9.

(a) Every $\mathbf{\Delta}^1_2(\mathbf{R})$ set is amoeba regular if and only if for every real $r \in \omega^\omega$, there is an amoeba real over $\mathrm{L}[r]$.

(b) Every $\mathbf{\Sigma}^1_2(\mathbf{R})$ set is amoeba regular if and only if for every real $r \in \omega^\omega$, the set $\{S \in \mathbf{R} : S$ is not an amoeba real over $\mathrm{L}[r]\}$ is $\mathcal{C}_{\mathbb{A}}$-meager.

Proof. It is enough to check that $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ satisfies the requirements of Ikegami's Theorem for weak category bases satisfying the c.c.c. (Corollary 2.2.36). By Proposition 2.3.7, $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is a proper category base which is Borel compatible with $\mathbf{R}$. Hence, it remains to show that $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is provable Σ^1_2 and that $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ satisfies the c.c.c. in every inner model of ZFC. Let $\varphi_{\mathbf{R}}$ be the Π^0_2 formula defining $\mathbf{R}$ from the proof of Proposition 2.3.3 and let $\varphi_{\mathcal{C}_{\mathbb{A}}}(c)$ be the statement "c is a Borel code and there is a $T \in \mathbb{A}$ such that $B_c = \langle T \rangle$". Then $\varphi_{\mathcal{C}_{\mathbb{A}}}$ is Σ^1_2 and $\mathcal{C}_{\mathbb{A}} = \{B_c : \varphi_{\mathcal{C}_{\mathbb{A}}}(c)\}$. Let M be an inner model of ZFC. By Proposition 2.3.7, in M, $(\mathbf{R}, \mathcal{C}_{\mathbb{A}})$ is a proper category base which satisfies the c.c.c. and is Borel compatible with $\mathbf{R}$.

It remains to show that the statement "c is a Borel code and B_c is $\mathcal{C}_{\mathbb{A}}$-meager" is Σ^1_2. By Lemma 2.2.16, a set $A \subseteq \mathbf{R}$ is $\mathcal{C}_{\mathbb{A}}$-meager if and only if there are maximal antichains $\mathcal{A}_n \subseteq \mathbb{A}$ such that $A \subseteq \bigcup_{n\in\omega} \mathbf{R} \setminus \bigcup_{T \in \mathcal{A}_n} \langle T \rangle$. Since $\mathbb{A}$ satisfies the c.c.c., every antichain can be coded as a single real. Moreover, two conditions $T, T' \in \mathbb{A}$ are incompatible if and only if $\mu([T] \cap [T']) \leq \frac{1}{2}$. Hence, the statement "$\mathcal{A}$ is a maximal antichain" is Π^1_1. Therefore, the statement "c is a Borel code and B_c is $\mathcal{C}_{\mathbb{A}}$-meager" is Σ^1_2. □

2.3.2 Amoeba forcing and inaccessibles

In this section, we determine the consistency strength of the statements $\mathbf{\Delta}^1_2(\mathbb{A})$ and $\mathbf{\Sigma}^1_2(\mathbb{A})$. We start with the former. By Theorem 2.3.9, every $\mathbf{\Delta}^1_2(\mathbf{R})$ set is amoeba regular if and only if for every $r \in \omega^\omega$, there is an amoeba real over $\mathrm{L}[r]$. The consistency strength of the latter is well-known.

Theorem 2.3.10 (Folklore)**.** *For every real $r \in \omega^\omega$, the set $\{x \in 2^\omega : x$ is not a random real over $\mathrm{L}[r]\}$ is Lebesgue null if and only if for every real $r \in \omega^\omega$, there is an amoeba real over* $\mathrm{L}[r]$.

Corollary 2.3.11. *The following are equivalent:*

(a) Every $\mathbf{\Sigma}^1_2(2^\omega)$ set is Lebesgue measurable,

(b) every $\mathbf{\Delta}^1_2(\mathbf{R})$ set is amoeba regular, and

(c) for every real $r \in \omega^\omega$, there is an amoeba real over $\mathrm{L}[r]$.

Proof. Follows directly from Theorems 1.2.54, 2.3.9, and 2.3.10. □

Theorem 2.3.10 is an old folklore result. However, it has never been properly documented in the literature. A proof of the forward direction can be found implicitly in [BJ92] and a full proof can be found in Beese's Master's thesis [Bee17]. We include a proof for the sake of completeness. Before we can start with the proof, we need some additional definitions and lemmas.

Lemma 2.3.12 (Truss)**.** *Let M be a transitive model of ZFC. If there is an open set $O \subseteq 2^\omega$ with $\mu(O) < \frac{1}{2}$ such that for every open set $U \subseteq 2^\omega$ coded in M with $\mu(U) < \frac{1}{2}$, there is a finite set $F \subseteq 2^{<\omega}$ such that $O \cup U = O \cup \bigcup_{s \in F} [s]$, then for every Cohen real x over $M[O]$, $M[O][x]$ contains an amoeba real over M.*

Proof. Cf. [BJ92, Lemma 4.1.6]. □

Let $\mathcal{S}_0$ be the set of function from ω to $\mathcal{P}(2^{<\omega})$ such that for every $n \in \omega$, $f(n) \subseteq 2^n$ and $\sum_{n\in\omega} |f(n)| \cdot 2^{-n} < \frac{1}{2}$. For $f, g \in \mathcal{S}_0$, we say that f *eventually covers* g, denoted by $\leq^*$, if

$$f \leq^* g \iff \exists m \in \omega \forall n > m(f(n) \subseteq g(n)).$$

Then $\leq^*$ is an ordering on $\mathcal{S}_0$. Let M be a transitive model of ZFC. A function $f \in \mathcal{S}_0$ is called a *covering real over* M if it eventually covers all functions in $\mathcal{S}_0 \cap M$, i.e., if for every $g \in \mathcal{S}_0 \cap M$, $g \leq^* f$. We can reformulate Lemma 2.3.12 using covering reals.

Lemma 2.3.13. *Let M be a transitive model of ZFC, let $f \in \mathcal{S}_0$ be a covering real over M, and let x be a Cohen real over $M[f]$. Then $M[f][x]$ contains an amoeba real over M.*

Proof. Let $O := \bigcup_{n\in\omega} \bigcup_{s\in f(n)} [s]$. It is enough to show that O satisfies the requirements of Lemma 2.3.12. Let $U \subseteq 2^\omega$ be an open set coded in M with $\mu(U) < \frac{1}{2}$. Then U is a disjoint union of basic open sets. Let $U = \bigcup_{n\in\omega}[s_n]$ be such a union. We define $g \in \mathcal{S}_0$ by

$$s \in g(n) \iff \mathrm{lh}(s) = n \text{ and } \exists m \in \omega(s = s_m).$$

Since f is a covering real over M, $g \leq^* f$. Hence, there is an $m \in \omega$ such that for every $n > m$, $g(n) \subseteq f(n)$. We define $F := \bigcup_{n\leq m} g(n)$. Then $O \cup U = O \cup \bigcup_{s\in F}[s]$. Therefore, O satisfies the requirements of Lemma 2.3.12 and so there is an amoeba real over M. □

Lemma 2.3.13 provides an alternative approach to prove Theorem 2.3.9: the crucial point is to show that if every $\mathbf{\Delta}^1_2(\mathbf{R})$ set is amoeba regular, then for every $r \in \omega^\omega$, there is an amoeba real over $\mathrm{L}[r]$. By Lemma 2.3.13, it is enough to show that if $\mathbf{\Delta}^1_2(\mathbb{A})$ holds, then for every $r \in \omega^\omega$, there is a Cohen real and a covering real over $\mathrm{L}[r]$. We shall show later in this that $\mathbf{\Delta}^1_2(\mathbb{A})$ implies $\mathbf{\Delta}^1_2(\mathbb{D})$ and so $\mathbf{\Delta}^1_2(\mathbb{C})$. Hence, if $\mathbf{\Delta}^1_2(\mathbb{A})$ holds, then the necessary Cohen reals exists. Moreover, we shall show in the proof of Theorem 2.3.10 that if $\mathbf{\Sigma}^1_2(\mathbb{B})$ holds, then the necessary covering reals exist. By Corollary 2.3.11, $\mathbf{\Delta}^1_2(\mathbb{A})$ implies $\mathbf{\Sigma}^1_2(\mathbb{B})$. However, this uses Theorem 2.3.9, which we want to prove.

Question 2.3.14. *Can we show, without using Theorem 2.3.9, that if $\mathbf{\Delta}^1_2(\mathbb{A})$ holds, then for every $r \in \omega^\omega$, there is a covering real over* $\mathrm{L}[r]$.

Brendle and Löwe have done something similar for Hechler forcing. In [BL99], they showed that $\mathbf{\Delta}^1_2(\mathbb{D})$ implies that for every $r \in \omega^\omega$, there is a Hechler real over $\mathrm{L}[r]$. Instead of Lemma 2.3.13 they used a different result from Truss. Truss also showed for every transitive model M of ZFC, if $f \in \omega^\omega$ dominates all reals in M and $x \in \omega^\omega$ is a Cohen real over $M[f]$, then g defined by $g(n) := f(n) + x(n)$ is a Hechler real over M (cf. [Tru77, Lemma 6.2]). In order to show that $\mathbf{\Delta}^1_2(\mathbb{D})$ implies that the necessary dominating reals exists, Brendle and Löwe showed that $\mathbf{\Delta}^1_2(\mathbb{D})$ implies $\mathbf{\Delta}^1_2(\mathbb{L})$. By the Judah-Shelah-style characterization for Laver forcing, this provides us the necessary dominating reals. To try something similar for $\mathbf{\Delta}^1_2(\mathbb{A})$ and covering reals, we would need a forcing notion whose quasi-generics are covering reals. Unfortunately, it is not known if such a forcing notion exists.

The idea for proving the backward direction of Theorem 2.3.10 is to show that if $\mathbf{\Sigma}^1_2(\mathbb{B})$ holds, then the requirements of Lemma 2.3.13 are satisfied. In order to show that the necessarily covering reals exist, we need one last lemma.

Lemma 2.3.15 (Bartoszyński)**.** *There are functions $\alpha : \mathcal{S}_0 \to \mathcal{N}$ and $\alpha^* : \mathcal{N} \to \mathcal{S}_0$ such that for every $f \in \mathcal{S}_0$ and every $N \in \mathcal{N}$, if $\alpha(f) \subseteq N$, then $f \leq^* \alpha^*(N)$. Moreover, if $f \in \mathrm{L}[r]$, then $\alpha(f)$ is a Borel set coded in $\mathrm{L}[r]$.*

Proof. Cf. [BJ95, Lemma 2.3.3] □

Now we are ready to prove Theorem 2.3.10.

Proof of Theorem 2.3.10. We start with the backward direction. Let $r \in \omega^\omega$ and let N be the set of non-random reals over $\mathrm{L}[r]$. Then N is the union of all Lebesgue null Borel sets coded in $\mathrm{L}[r]$. We shall show that for every $0 < \varepsilon < 1$, $\mu(N) \leq \varepsilon$. Let $0 < \varepsilon < 1$ and let B be a Lebesgue null Borel set coded in $\mathrm{L}[r]$. Then the set $D_B := \{T \in \mathbb{A}_\varepsilon : [T] \cap B = \emptyset\}$ is dense and in $\mathrm{L}[r]$. Since there is an amoeba real over $\mathrm{L}[r]$, there is an $\mathbb{A}_\varepsilon$-generic filter G_ε over $\mathrm{L}[r]$. Let $S := \bigcap G_\varepsilon$. Then there is some $T \in D_B$ such that $S \in \langle T \rangle$. Hence, $[S] \cap B = \emptyset$ and so $[S] \cap N = \emptyset$. Therefore, $\mu(N) \leq \varepsilon$. Since ε was arbitrarily chosen, N is Lebesgue null.

We prove the forward direction. Since every $\mathbf{\Sigma}^1_2(\omega^\omega)$ set is Lebesgue measurable, for every $r \in \omega^\omega$, there is a Cohen real over $\mathrm{L}[r]$. By Lemma 2.3.13, it is enough to show that for every $r \in \omega^\omega$, there is a covering real over $\mathrm{L}[r]$. Let $r \in \omega^\omega$. By Theorem 1.2.53, the set of non-random reals over $\mathrm{L}[r]$ is Lebesgue null. Hence, there is a Lebesgue null set N which contains every Lebesgue null Borel set coded in $\mathrm{L}[r]$. Let $\alpha : \mathcal{S}_0 \to \mathcal{N}$ and $\alpha^* : \mathcal{N} \to \mathcal{S}_0$ be the functions from Lemma 2.3.15. We define $f := \alpha^*(N)$. We shall show that f is a covering real over $\mathrm{L}[r]$. Let $g \in \mathcal{S}_0 \cap \mathrm{L}[r]$. Then $\alpha(f)$ is a Lebesgue null Borel set coded in $\mathrm{L}[r]$ and so $\alpha(f) \subseteq N$. Therefore, $g \leq^* \alpha^*(N) = f$. □

Next, we investigate the consistency strength of the statement $\mathbf{\Sigma}^1_2(\mathbb{A})$. We show that $\mathbf{\Sigma}^1_2(\mathbb{A})$ holds if and only if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. We have already proved the backward direction in Corollary 2.3.8. So we only have to show the forward direction. Brendle and Löwe have shown that the analogue is true for $\mathbf{\Sigma}^1_2(\mathbb{D})$ and $\mathbf{\Sigma}^1_2(\mathbb{E})$.

Theorem 2.3.16 (Brendle-Löwe)**.** *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(\omega^\omega)$ set has the Baire property in the dominating topology,

(b) every $\mathbf{\Sigma}^1_2(\omega^\omega)$ set has the Baire property in the eventually different topology, and

(c) for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Proof. Cf. [BL99, Theorems 11 & 12] and [BL11, Theorem 7]. □

The proofs that the statements $\mathbf{\Sigma}^1_2(\mathbb{D})$ and $\mathbf{\Sigma}^1_2(\mathbb{E})$ both imply that for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$ are similar. Both of them use something we call a *Brendle-Łabędzki Lemma*.

Lemma 2.3.17 (Łabędzki-Repický)**.** *For every $A \in [\omega]^\omega$, the set $X_A := \{x \in \omega^\omega : \mathrm{ran}(x) \cap A = \emptyset\}$ is meager in the dominating topology. Moreover, for every maximal almost disjoint family $\mathscr{A}$ and every $\mathbb{D}$-meager set $M \subseteq \omega^\omega$, there are only countably many $A \in \mathscr{A}$ such that $X_A \subseteq M$.*

Proof. Cf. [ŁR95, Theorem 6.2]. □

Lemma 2.3.18 (Brendle)**.** *For every $g \in \omega^\omega$, the set $X_g := \{x \in \omega^\omega : \exists^\infty k \in \omega(x(k) = g(k))\}$ is meager in the eventually different topology. Moreover, for every pairwise eventually different family $\mathscr{E}$ and every $\mathbb{E}$-meager set $A \subseteq \omega^\omega$, there are only countably many $g \in \mathscr{E}$ such that $X_g \subseteq A$.*

Proof. Cf. [Łab96, Theorem 4.7]. □

So roughly speaking, a Brendle-Łabędzki Lemma consists of a function f mapping a real x to a small Borel set which is coded in $\mathrm{L}[x]$, a notion of canonical families, and the statement "if A is small and $\mathscr{F}$ is a canonical family, then there are only countably many $x \in \mathscr{F}$ such that $f(x) \subseteq A$". The idea is the following: if $r \in \omega^\omega$ and $\{x_\alpha : \alpha < \aleph_1^{\mathrm{L}[r]}\}$ is a canonical family in $\mathrm{L}[r]$, then for every $\alpha < \aleph_1^{\mathrm{L}[r]}$, $f(x_\alpha)$ is a small Borel set coded in $\mathrm{L}[r]$. Let $A := \bigcup\{f(x_\alpha) : \alpha < \aleph_1^{\mathrm{L}[r]}\}$. If A is small, then there are only countably many $\alpha \in \aleph_1^{\mathrm{L}[r]}$ such that $f(x_\alpha) \subseteq A$ and so $\aleph_1^{\mathrm{L}[r]}$ has to be countable. In the case of Hechler forcing, A does not contain any Hechler reals over $\mathrm{L}[r]$. So by the Solovay-style characterization, A is meager in the dominating topology if $\mathbf{\Sigma}^1_2(\mathbb{D})$ holds. Therefore, $\mathbf{\Sigma}^1_2(\mathbb{D})$ implies that for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$ and analogously for $\mathbf{\Sigma}^1_2(\mathbb{E})$.

We shall prove a Brendle-Łabędzki Lemma for localization forcing in 2.5.2. However, for amoeba forcing we take a different approach using Theorem 2.2.46. We shall show that for every projective pointclass Γ, $\Gamma(\mathbb{A})$ implies $\Gamma(\mathbb{D})$.

Theorem 2.3.19. *For every projective pointclass Γ, if every $\Gamma(\mathbf{R})$ set is amoeba regular, then every $\Gamma(\omega^\omega)$ has the Baire property in the dominating topology.*

Corollary 2.3.20. *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(\mathbf{R})$ set is amoeba regular,

(b) for every $r \in \omega^\omega$, the set $\{P \in \mathbf{R} : P$ is not an amoeba real over $\mathrm{L}[r]\}$ is $\mathcal{C}_{\mathbb{A}}$-meager, and

(c) for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Proof. Follows directly from Corollary 2.3.8 and Theorems 2.3.9, 2.3.16, and 2.3.19. □

Note that $\mathbf{\Sigma}^1_2(\mathbb{A})$ and $\mathbf{\Sigma}^1_2(\mathbb{D})$ are equivalent. However, the converse of Theorem 2.3.19 is not true in general: by Corollary 2.3.11, $\mathbf{\Delta}^1_2(\mathbb{A})$ is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{B})$. Moreover, it is well-known that $\mathbf{\Delta}^1_2(\mathbb{D})$ does not imply $\mathbf{\Sigma}^1_2(\mathbb{B})$ (cf. Figure 2.1). Therefore, $\mathbf{\Delta}^1_2(\mathbb{D})$ does not imply $\mathbf{\Delta}^1_2(\mathbb{A})$.

We spend the rest of this section proving Theorem 2.3.19. The idea is to define a regularity property for a different variant of amoeba forcing $\mathbb{A}_\infty$ and then use Theorem 2.2.46 twice. First to show that $\Gamma(\mathbb{A})$ implies $\Gamma(\mathbb{A}_\infty)$ and then again to show that $\Gamma(\mathbb{A}_\infty)$ implies $\Gamma(\mathbb{D})$, where Γ is a projective pointclass.

Definition 2.3.21. Let $\mathbb{A}_\infty$ be the set $\{(O, \varepsilon) : O \subseteq \mathbb{R}$ is open, $\varepsilon \in \mathbb{R} \cup \{\infty\}$, and $\mu(O) < \varepsilon\}$ ordered by

$$(O', \varepsilon') \leq (O, \varepsilon) :\iff O \subseteq O' \text{ and } \varepsilon' \leq \varepsilon.$$

Like $\mathbb{A}$, $\mathbb{A}_\infty$ also satisfies the c.c.c. and adds an open set which contains every Lebesgue null set from the ground model. The only difference is that the open set for $\mathbb{A}_\infty$ lives on the real line and not on the Cantor space. However, this makes no difference; Truss showed in in [Tru88] that $\mathbb{A}$ and $\mathbb{A}_\infty$ are forcing equivalent. Nevertheless, it is sometimes more convenient to work with $\mathbb{A}_\infty$ instead of $\mathbb{A}$. For example, we shall use in the proof of Proposition 2.3.24 that the real line can be partitioned in infinitely many parts of the same positive Lebesgue measure. Obviously, this cannot be done for the Cantor space.

Next, we define a regularity property for $\mathbb{A}_\infty$. We start with the topological space. Let $\mathbf{R}_\infty := \{U \subseteq \mathbb{R} : U \text{ is open}\}$. Then for every transitive model M of ZFC and every $\mathbb{A}_\infty$-generic filter G over M, $U_G := \bigcup\{O : \exists\varepsilon((O,\varepsilon) \in G)\} \in \mathbf{R}_\infty$ and $G = \{(O,\varepsilon) : O \subseteq U \text{ and } \mu(U) < \varepsilon\}$. Hence, generic filters for $\mathbb{A}_\infty$ can be uniquely determined by elements from $\mathbf{R}_\infty$. As before, we can code the elements by reals: For the rest of this section, we fix a recursive enumeration $\langle (a_k, b_k) : k \in \omega\rangle$ of the basic open intervals with rational endpoints. By this we mean a recursive enumeration of pairs of rational numbers. We can find such an enumeration since we can recursively code rational numbers as naturals. For every $U \in \mathbf{R}_\infty$, there is a unique real $c \in 2^\omega$ such that for every $k \in \omega$, $c(k) = 1$ if and only if $(a_k, b_k) \subseteq U$. We say that c codes U. As usual, we shall often identify the elements of $\mathbf{R}_\infty$ with their codes.

Lemma 2.3.22. *The set $\mathbf{R}_\infty$ with the induced topology is a Polish space.*

Proof. A real $c \in 2^\omega$ codes a set in $\mathbf{R}_\infty$ if and only if for every $k, \ell \in \omega$, if $c(k) = 1$ and $(a_\ell, b_\ell) \subseteq (a_k, b_k)$, then $c(\ell) = 1$. Hence, $\mathbf{R}_\infty$ is $\mathbf{\Pi}^0_2(\omega^\omega)$ and so a Polish subspace of the Baire space. $\square$

In order to define a regularity property for $\mathbb{A}_\infty$, we shall again define a category base and then use its regularity property. We define $\mathcal{C}_{\mathbb{A}_\infty} := \{[O,\varepsilon] : (O,\varepsilon) \in \mathbb{A}_\infty\}$, where $[O,\varepsilon] := \{U \in \mathbf{R} : O \subseteq U$ and $\mu(U) \leq \varepsilon\}$.

Lemma 2.3.23. *The pair $(\mathbf{R}_\infty, \mathcal{C}_{\mathbb{A}_\infty})$ is a proper category base which satisfies the c.c.c. and is Borel compatible with $\mathbf{R}_\infty$. Moreover, the ideal of $\mathcal{C}_{\mathbb{A}_\infty}$-meager sets is Borel generated.*

Proof. The second part follows directly from Lemma 2.2.31. Hence, we only have to show the first part. We start with proving that $(\mathbf{R}_\infty, \mathcal{C}_{\mathbb{A}_\infty})$ is a category base. It is clear that $\mathbf{R}_\infty = \bigcup \mathcal{C}_{\mathbb{A}_\infty}$. Let $C \in \mathcal{C}_{\mathbb{A}_\infty}$ and let $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{A}_\infty}$ be a disjoint family such that $|\mathcal{C}| < |\mathcal{C}_{\mathbb{A}_\infty}|$. Since $\mathbb{A}_\infty$ satisfies the c.c.c., $\mathcal{C}$ is countable. We first assume that $C \cap \bigcup \mathcal{C}_{\mathbb{A}_\infty}$ contains some $[O,\varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$. If there is no $[O',\varepsilon'] \in \mathcal{C}$ such that $[O,\varepsilon] \cap [O',\varepsilon']$ contains some element from $\mathcal{C}_{\mathbb{A}_\infty}$, then for every $[O',\varepsilon'] \in \mathcal{C}$, $\mu(O \cup O') \geq \min\{\varepsilon, \varepsilon'\}$. Hence, for every $[O',\varepsilon'] \in \mathcal{C}$, either $[O,\varepsilon] \cap [O',\varepsilon']$ is empty or for every $U \in [O,\varepsilon] \cap [O',\varepsilon']$, $\mu(U) = \min\{\varepsilon,\varepsilon'\}$. Let $U \in [O,\varepsilon]$ such that $\mu(U) < \varepsilon$ and $\mu(U) \neq \varepsilon'$ for every $[O',\varepsilon'] \in \mathcal{C}$. Such a U exists since $\mathcal{C}$ is countable. Then $U \notin [O,\varepsilon] \setminus \bigcup \mathcal{C}$. But this is a contradiction.

We now assume that $C \cap \bigcup \mathcal{C}_{\mathbb{A}_\infty}$ does not contain any element from $\mathcal{C}_{\mathbb{A}_\infty}$. Hence, $[O,\varepsilon] \cap [O',\varepsilon']$ also contains no element from $\mathcal{C}_{\mathbb{A}_\infty}$ for every $[O',\varepsilon'] \in \mathcal{C}$. Then for every $[O',\varepsilon'] \in \mathcal{C}$, $\mu(O \cup O') \geq \min\{\varepsilon,\varepsilon'\}$. Since $\mathcal{C}$ is countable, we can find some $U \in [O,\varepsilon]$ such that $\mu(U) < \varepsilon$ and for every $[O',\varepsilon'] \in \mathcal{C}$, $\mu(U \cup O') > \min\{\varepsilon,\varepsilon'\}$. Then $[U,\varepsilon] \subseteq [O,\varepsilon]$ and for every $[O',\varepsilon'] \in \mathcal{C}$, $[U,\varepsilon] \cap [O'\varepsilon'] = \emptyset$.

Next, we show that $(\mathbf{R}_\infty, \mathcal{C}_{\mathbb{A}_\infty})$ is proper. Let $[O,\varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$ be a region. It is clear that $(\mathcal{C}_{\mathbb{A}_\infty}, \subseteq)$ is proper since $\mathbb{A}_\infty$ satisfies the c.c.c. We show that every singleton is $\mathcal{C}_{\mathbb{A}_\infty}$-meager. Let $U \in \mathbf{R}_\infty$ and let $[O,\varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$ be a region. Without loss of generality, we can assume that $U \in [O,\varepsilon]$. Then we can either increase O or decrease ε to find a $(O',\varepsilon') \leq (O,\varepsilon)$ such that $U \notin [O',\varepsilon']$. Therefore, $\{U\}$ is $\mathcal{C}_{\mathbb{A}_\infty}$-singular. So it remains to show that every region is $\mathcal{C}_{\mathbb{A}_\infty}$-abundant. We suppose for a contradiction that there is some $(O,\varepsilon) \in \mathbb{A}_\infty$ such that $[O,\varepsilon]$ is $\mathcal{C}_{\mathbb{A}_\infty}$-meager. Then there are $\mathcal{C}_{\mathbb{A}_\infty}$-singular sets $N_n \subseteq \mathbf{R}_\infty$ such that $[O,\varepsilon] = \bigcup_{n\in\omega} N_n$. Since the N_n are $\mathcal{C}_{\mathbb{A}_\infty}$-singular, there is a decreasing sequence $\langle (O_n,\varepsilon_n) : n \in \omega\rangle$ such that $(O_0,\varepsilon_0) = (O,\varepsilon)$ and for every $n \in \omega$, $(O_{n+1},\varepsilon_{n+1}) \leq (O_n,\varepsilon_n)$ and $[O_{n+1},\varepsilon_{n+1}] \cap N_n = \emptyset$. Let $U := \bigcup_{n\in\omega} O_n$. Then for every $n \in \omega$, $O_n \subseteq U$ and $\mu(U) \leq \varepsilon_n$. Hence, $U \in [O_n,\varepsilon_n]$ for every $n \in \omega$. But this is impossible since $[O,\varepsilon] \cap \bigcup_{n>0}[O_n,\varepsilon_n] = \emptyset$. Therefore, $[O,\varepsilon]$ is non-$\mathcal{C}_{\mathbb{A}_\infty}$-meager.

Finally, we show that $\mathcal{C}_{\mathbb{A}_\infty}$ is Borel compatible with $\mathbf{R}_\infty$. First, we show that every region is closed in $\mathbf{R}_\infty$. Let $[O,\varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$ be a region, let $U \notin [O,\varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$, and let $c \in 2^\omega$ be a code for U. Then $O \nsubseteq U$ or $\mu(U) > \varepsilon$. We make a case-distinction:

Case 1: $O \nsubseteq U$. Then there is some $n \in \omega$ such that $(a_n, b_n) \subseteq O$ and $(a_n, b_n) \nsubseteq U$. Hence, $[c\restriction(n+1)] \cap \mathbf{R}_\infty$ is open in $\mathbf{R}_\infty$, contains U, and is disjoint from $[O, \varepsilon]$.

Case 2: $\mu(U) > \varepsilon$. Then there is some $n \in \omega$ such that $\mu(\bigcup\{(a_k, b_k) : k < n \text{ and } c(k) = 1\}) > \varepsilon$. Hence, $[c\restriction n] \cap \mathbf{R}_\infty$ is open in $\mathbf{R}_\infty$, contains U, and is disjoint from $[O, \varepsilon]$.

In both cases there is an open set containing U which is disjoint from $[O, \varepsilon]$. Therefore, $[O, \varepsilon]$ is closed in $\mathbf{R}_\infty$.

It remains to show that every Borel set in $\mathbf{R}_\infty$ is $\mathcal{C}_{\mathbb{A}_\infty}$-Baire. By Theorem 2.2.8, the $\mathcal{C}_{\mathbb{A}_\infty}$-Baire sets form a σ-algebra. Hence, we only have to check that every basic open set is $\mathcal{C}_{\mathbb{A}_\infty}$-Baire. Let $s \in 2^{<\omega}$, let $[O, \varepsilon] \in \mathcal{C}_{\mathbb{A}_\infty}$ be a region, and let $c \in 2^\omega$ be a code for O. We make a case-distinction:

Case 1: $s \subseteq c$. Let $\varepsilon' \in \mathbb{R}$ be such that for every $n < \mathrm{lh}(s)$ with $s(n) = 0$, $\mu(O) < \varepsilon' < \mu(O \cup (a_n, b_n))$. Then $(O, \varepsilon') \leq (O, \varepsilon)$ and $[O, \varepsilon'] \subseteq [s] \cap \mathbf{R}_\infty$.

Case 2: There is some $n < \mathrm{lh}(s)$ such that $s(n) = 1 \neq c(n)$. Let $\varepsilon' \in \mathbb{R}$ be such that $\mu(O) < \varepsilon' < \mu(O \cup (a_n, b_n))$. Then $(O, \varepsilon') \leq (O, \varepsilon)$ and $[O, \varepsilon'] \cap ([s] \cap \mathbf{R}_\infty)$ is empty.

Case 3: There is some $n < \mathrm{lh}(s)$ such that $s(n) = 0 \neq c(n)$. Then $[O, \varepsilon] \cap ([s] \cap \mathbf{R}_\infty)$ is empty. $\square$

We associate the regularity property defined from $(\mathbf{R}_\infty, \mathcal{C}_{\mathbb{A}_\infty})$ with $\mathbb{A}_\infty$. As usual, for every projective pointclass Γ, we write $\Gamma(\mathbb{A}_\infty)$ for the statement "every $\Gamma(\mathbf{R}_\infty)$ set is $\mathcal{C}_{\mathbb{A}_\infty}$-Baire". Now we are ready to show that for every projective pointclass Γ, $\Gamma(\mathbb{A}_\infty)$ implies $\Gamma(\mathbb{D})$.

Proposition 2.3.24. *For every projective pointclass Γ, if every $\Gamma(\mathbf{R}_\infty)$ set is $\mathcal{C}_{\mathbb{A}_\infty}$-Baire, then every $\Gamma(\omega^\omega)$ has the Baire property in the dominating topology.*

Proof. Let $\mathcal{C}_{\mathbb{D}} := \{[n, f] : (n, f) \in \mathbb{D}\}$. Then $(\omega^\omega, \mathcal{C}_{\mathbb{D}})$ is a proper category base which satisfies the c.c.c. and is Borel compatible with ω^ω. Moreover, the meager ideal in the dominating topology is Borel generated. Note that for every dense subset $D \subseteq \mathbb{A}_\infty$, $(\mathbf{R}_\infty, \{[O, \varepsilon] : (O, \varepsilon) \in D\})$ is a proper category base which is Borel compatible with $\mathbf{R}_\infty$ and equivalent to $(\mathbf{R}_\infty, \mathcal{C}_{\mathbb{A}_\infty})$. Hence, by Theorem 2.2.46, it is enough to find a dense subset $D \subseteq \mathbb{A}_\infty$, a Borel function $h : \mathbf{R}_\infty \to \omega^\omega$ and a projection $\overline{h} : D \to \mathbb{D}$ such that

(a) for every $(O, \varepsilon) \in D$, $h[O, \varepsilon] \subseteq [\overline{h}(O, \varepsilon)]$ and

(b) $\overline{h}[D]$ is dense in $\mathbb{D}$.

We start with the definition of h. Let $\{I^n_k \subseteq \mathbb{R} : k, n \in \omega\}$ be a recursive family of pairwise disjoint open intervals with rational endpoints such that for every $k, n \in \omega$ $\mu(I^n_k) = 2^{-2n}$. We define $h : \mathbf{R}_\infty \to \omega^\omega$ by

$$h(U)(n) := \begin{cases} 0 & \text{if } \mu(U) = \infty, \\ \min\{k \in \omega : \forall \ell \geq k(I^n_\ell \nsubseteq U)\} & \text{otherwise.} \end{cases}$$

We show that h is Borel. Let $(m, f) \in \mathbb{D}$. We have to show that $h^{-1}([m, f])$ is Borel in $\mathbf{R}_\infty$. Let $U \in \mathbf{R}_\infty$. Then $U \in h^{-1}([m, f])$ if and only if $\mu(U) = \infty$ and $\langle 0 : n \in \omega \rangle \in [m, f]$ or $\mu(U) < \infty$ and for every $n < m$, $f(n) = \min\{k \in \omega : \forall l \geq k(I^n_l \nsubseteq U)\}$ and for every $n \in \omega$, $f(n) \leq \min\{k \in \omega : \forall l \geq k(I^n_l \nsubseteq U)\}$. Hence, $h^{-1}([m, f])$ is Borel in ω^ω and so also Borel in $\mathbf{R}_\infty$.

Next, we define the dense subset of $\mathbb{A}_\infty$ that will be the domain of $\overline{h}$. Let

$$D := \{(O, \varepsilon) \in \mathbb{A}_\infty : \exists n \in \omega(\sum_{m \geq n} 2^{-2m} < \varepsilon - \mu(O) < 2^{-2(n-1)})\}.$$

The domain of $\overline{h}$ will be a subset of D. Before we define it, we first show that D is dense in $\mathbb{A}_\infty$. Let $(O,\varepsilon) \in \mathbb{A}_\infty \setminus D$. Without loss of generality, we can assume that $\varepsilon - \mu(O) \leq 1$. Let $n \in \omega$ be minimal such that $\varepsilon - \mu(O) \geq 2^{-2(n-1)}$. Hence,

$$\varepsilon - \mu(O) \geq 2^{-2(n-1)} > \frac{2^{-2(n-1)}}{3} = \sum_{m \geq n} 2^{-2m}.$$

Since $2^{-2(n-1)} > \sum_{m \geq n} 2^{-2m}$, we can find an $\varepsilon' > 0$ such that

$$\sum_{m \geq n} 2^{-2m} < \varepsilon' - \mu(O) < 2^{-2(n-1)}.$$

Then $(O,\varepsilon') \leq (O,\varepsilon)$ and $(O,\varepsilon) \in D$. Therefore, D is dense in $\mathbb{A}_\infty$. For every $(O,\varepsilon) \in D$, let $n_{(O,\varepsilon)} \in \omega$ such that

$$\sum_{m \geq n_{(O,\varepsilon)}} 2^{-2m} < \varepsilon - \mu(O) < 2^{-2(n_{(O,\varepsilon)}-1)}.$$

Now we define the domain of $\overline{h}$. Let

$$D' := \{(O,\varepsilon) \in D : \forall U \in [O,\varepsilon](h(O)\restriction n_{(O,\varepsilon)} = h(U)\restriction n_{(O,\varepsilon)})\}.$$

We show that D' is dense in $\mathbb{A}_\infty$. Let $(O,\varepsilon) \in D$. Without loss of generality, we can assume that $n_{(O,\varepsilon)} > 0$. Let $g \in \omega^\omega$ such that for every $n \in \omega$, $g(n) := \max\{k \in \omega : \mu(I^n_k \setminus O) < 2^{1-2n}\}$. We define

$$O' := O \cup \bigcup\{I^n_{g(n)} : n \geq n_{(O,\varepsilon)}\}.$$

Let $\varepsilon', \varepsilon'' > 0$ such that $\varepsilon'' < \varepsilon' < \varepsilon$, there is some $n \in \omega$ such that

$$\sum_{m \geq n} 2^{-2m} < \varepsilon'' - \mu(O') < \varepsilon' - \mu(O') < 2^{1-2n},$$

and for every $n < n_{(O,\varepsilon)}$ and every $k \geq h(O')(n)$, $\mu(O' \cup I^k_n) \geq \varepsilon'$. Then $(O'\varepsilon'') \leq (O',\varepsilon) \leq (O,\varepsilon)$ and $(O'\varepsilon''), (O',\varepsilon) \in D$. We show that $(O',\varepsilon'') \in D'$. Let $U \in [O',\varepsilon'']$ and let $n < n_{(O',\varepsilon'')}$. If $n < n_{(O,\varepsilon)}$, then by choice of ε', for every $k \geq h(O')(n)$, $\mu(U \cup I^k_n) \geq \varepsilon' > \varepsilon''$. Hence, $h(O')(n) = h(U)(n)$. If $n \geq n_{(O,\varepsilon)}$, then $h(O')(n) = g(n)$. Since $n < n_{(O',\varepsilon'')}$, $\varepsilon'' - \mu(O') < 2^{1-2n}$. Hence, for every $k \geq g(n)$, $\mu(O' \cup I^n_k) > \varepsilon''$ and so $h(O')(n) = g(n) = h(U)(n)$. Therefore, $(O',\varepsilon'') \in D'$ and so D' is dense in $\mathbb{A}_\infty$.

Now we are ready to define $\overline{h}$. Let $\overline{h} : D' \to \mathbb{D}$ be defined by $\overline{h}(O,\varepsilon) := (n_{(O,\varepsilon)}, h(O))$. It remains to show that $\overline{h}$ is a projection and that (a) and (b) from above are satisfied. We first show that $\overline{h}$ is a projection. Let $(O,\varepsilon), (O',\varepsilon') \in D'$ be such that $(O,\varepsilon) \leq (O',\varepsilon')$. Then for every $n \in \omega$, $h(O)(n) \leq h(O')(n)$. Since $\varepsilon' - \mu(O') \leq \varepsilon - \mu(O)$, $n_{(O,\varepsilon)} \leq n_{(O',\varepsilon')}$. It remains to check that $h(O)\restriction n_{(O,\varepsilon)} = h(O')\restriction n_{(O,\varepsilon)}$. But this follows directly, since $(O,\varepsilon) \in D'$. Therefore, $\overline{h}(O,\varepsilon) \leq \overline{h}(O',\varepsilon')$ and so $\overline{h}$ is order-preserving.

Let $(O,\varepsilon) \in D'$ and $(n,f) \in \mathbb{D}$ such that $(n,f) \leq \overline{h}(O,\varepsilon)$. We define $O' := O \cup \bigcup\{I^{n'}_k : n' \geq n_{(O,\varepsilon)} \wedge f(n') = k+1\}$. Then $(O',\varepsilon) \leq (O,\varepsilon)$ and $h(O') = f$. By decreasing ε, we can find some

$\varepsilon' \leq \varepsilon$ such that $(O', \varepsilon') \leq (O, \varepsilon)$ and for every $n' < n$ and every $k \geq h(O')(n)$, $\mu(O' \cup I_k^{n'}) > \varepsilon'$. Since D' is dense in $\mathbb{A}_\infty$, there is some $(O'', \varepsilon'') \in D'$ such that $(O'', \varepsilon'') \leq (O', \varepsilon')$. Then for every $n' \in \omega$, $f(n') = h(O')(n') \leq h(O'')(n')$. Moreover, $n < n_{(O'', \varepsilon'')}$ and $f \restriction n = h(O'') \restriction n$. Hence, $\overline{h}(O'', \varepsilon'') \leq (m, f)$ and so $\overline{h}$ is a projection.

Next, we show (a). Let $(O, \varepsilon) \in D'$ and let $x \in h[O, \varepsilon]$. Then there is some $U \in [O, \varepsilon]$ such that $h(U) = x$. Since $O \subseteq U$, $h(O)(n) \leq h(U)(n)$ for every $n \in \omega$. Since $(O, \varepsilon) \in D'$, $h(O) \restriction n_{(O,\varepsilon)} = h(U) \restriction n_{(O,\varepsilon)}$. Hence, $x \in [\overline{h}(O, \varepsilon)]$.

Finally, we show (b). Let $(m, f) \in \mathbb{D}$. Without loss of generality, $m > 0$. We define $O := \bigcup\{I_k^n : f(n) = k+1\}$ and $\varepsilon := \mu(O) + 2^{-2(m-1)}$. Then $(O, \varepsilon) \in D'$ and $\overline{h}(O, \varepsilon) = (m, f)$. □

It remains to show that for every projective pointclass Γ, $\Gamma(\mathbb{A})$ implies $\Gamma(\mathbb{A}_\infty)$. To do so, we fix a recursive family $\{A_k^n : n, k \in \omega\}$ of open independent subsets of 2^ω with $\mu(J_k^n) = 2^{-(n+1)}$ (cf. Section 1.2.6) for the rest of this section. Moreover, let $\mathcal{I}^{\ell,n}$ be the set of finite unions of intervals with rational endpoints with measure $\leq 4^{\ell - n}$ and let $\{U_k^{\ell,n} : k, \ell, n \in \omega\}$ be a recursive family of open sets such that for every $\ell, n \in \omega$, $\{U_k^{\ell,n} : k \in \omega\}$ enumerates $\mathcal{I}^{\ell,n}$ and each element of $\mathcal{I}^{\ell,n}$ occurs infinitely often in this enumeration. For every $\ell \in \omega$, we define functions $h_\ell : \mathbf{R} \cup \mathbb{A} \to \mathbf{R}_\infty$ and $\overline{h}_\ell : \mathbb{A} \to \mathbb{A}_\infty$ by

$$\begin{aligned} h_\ell(S) :=& \bigcup\{U_k^{\ell,n} : \mu(A_k^n \cap [S]) = 0\} \text{ and} \\ \overline{h}_\ell(T) :=& (h_\ell(T), \sup\{\mu(h_\ell(S)) : S \leq T\}). \end{aligned}$$

Truss used these functions to show that if there is a $\mathbb{A}$-generic filter, then there is also an $\mathbb{A}_\infty$-generic filter over the same model. We shall use them as the functions which are required for Theorem 2.2.46. Before doing so, let us study some properties of h_ℓ and $\overline{h}_\ell$. We start with a lemma that summarizes the properties from Truss' proof which are relevant to us.

Lemma 2.3.25 (Truss)**.**

(a) For every $\ell \in \omega$, $\overline{h}_\ell$ is a projection.

(b) For every $\ell \in \omega$, if $T \in \mathbb{A}$ and $\mu(h_\ell(T)) < \varepsilon$, then there is some $S \leq T$ such that $\overline{h}_\ell(S) \leq (h_\ell(T), \varepsilon)$.

(c) For every $S \in \mathbf{R} \cup \mathbb{A}$ and every $n \in \omega$, the set $\{k : \mu(A_k^n \cap [S]) = 0\}$ has size $\leq 2^{n+1}$.

We define $K_m(S) := \{(n, k) \in m \times \omega : \mu(A_k^n \cap [S]) = 0\}$ for every pruned tree S on 2 and every $m \in \omega$.

(d) For every $T \in \mathbb{A}$ and every $n < m \in \omega$, there is some $k \in \omega$ such that for every $j \geq k$, $K_m(T') = K_m(T) \cup \{(n, j)\}$, where $S \subseteq T$ is the unique pruned tree such that $[S] = [T] \setminus A_j^n$.

Proof. Cf. [Tru88, Proof of Theorem 4.3 and Lemmas 4.4, 4.5, and 4.8]. □

In [Tru88], (a) and (b) of Lemma 2.3.25 are only proved for $\ell = 0$. However, the proofs for $\ell > 0$ are exactly the same. We leave it to the reader to verify this. Items (a) and (b) of Lemma 2.3.25 make sure that requirements (a) and (c) of Theorem 2.2.46 are satisfied. To check (b), we need another lemma.

Lemma 2.3.26. *Let $T \in \mathbb{A}$, let $\ell \in \omega$, and let $\overline{h}_\ell(T) = (O, \varepsilon)$. Then there is some $T' \leq T$ such that for every $S \in \langle T' \rangle$, $\mu(h_\ell(S)) \leq \varepsilon$.*

Proof. Let $m \in \omega$ such that

$$\mu(h_\ell(T)) + \sum_{n \geq m} \frac{2^{n+1}}{4^{l-n}} \leq \varepsilon.$$

By Lemma 1.2.27, we have for every $T' \in \mathbb{A}$ and every $n < m$, that

$$\lim_{k\to\infty} \mu(A_k^n \cap [T']) = \frac{1}{2^{n+1}} \mu([T']) > \frac{1}{2^{n+1}} \frac{1}{2} \geq \frac{1}{2^{m+1}}.$$

Now we can use (d) of Lemma 2.3.25 repeatedly to find a $T' \leq T$ such that $K_m(T') = K_m(T)$ and $\mu([T]) - \frac{1}{2} < \frac{1}{2^{m+1}}$. Then there are only finitely many pairs $(n, k) \in m \times \omega$ such that $0 < \mu(A_k^n \cap [T']) \leq \mu([T']) - \frac{1}{2}$. We can again use (d) of Lemma 2.3.25 repeatedly to find a $T'' \leq T'$ such that $K_m(T'') = K_m(T)$ and for every $k \in \omega$ and every $n < m$, if $\mu(A_k^n \cap [T']) > 0$, then $\mu([T''] \cap A_k^n) > \mu([T'']) - \frac{1}{2}$. Then for every $S \in \langle T'' \rangle$, $K_m(S) = K_m(T'') = K_m(T)$. Hence, by (c) of Lemma 2.3.25, we have

$$\mu(h_\ell(S)) \leq \mu(h_\ell(T)) + \sum_{n\geq m} \frac{|\{k : \mu(A_k^n \cap [S]) = 0\}|}{4^{l-n}} \leq \mu(h_\ell(T)) + \sum_{n\geq m} \frac{2^{n+1}}{4^{l-n}} \leq \varepsilon.$$

Therefore, $T'' \leq T$ and for every $S \in \langle T'' \rangle$, $\mu(h_\ell(S)) \leq \varepsilon$. □

Now we are ready to show that for every projective pointclass Γ, $\Gamma(\mathbb{A})$ implies $\Gamma(\mathbb{A}_\infty)$. This completes the proof of Theorem 2.3.19.

Proposition 2.3.27. *For every projective pointclass Γ, if every $\Gamma(\mathbf{R})$ set is amoeba regular, then every $\Gamma(\mathbf{R}_\infty)$ set is $\mathcal{C}_{\mathbb{A}_\infty}$-Baire.*

Proof. By Theorem 2.2.46, it is enough to show that

(a) $\langle h_\ell : \ell \in \omega \rangle$ is a sequence of Borel functions from $\mathbf{R}$ to $\mathbf{R}_\infty$,

(b) $\langle \overline{h}_\ell : \ell \in \omega \rangle$ is a sequence of projections from $\mathbb{A}$ to $\mathbb{A}_\infty$,

(c) for every $\ell \in \omega$ and every $T \in \mathbb{A}$, there is a $T' \leq T$ such that $h_\ell[\langle T' \rangle] \subseteq [\overline{h}_\ell(T)]$, and

(d) $\bigcup_{\ell\in\omega} \overline{h}_\ell[\mathbb{A}]$ is dense in $\mathbb{A}_\infty$.

We start with proving (a). Let $\ell \in \omega$. We have to show that $h_\ell \restriction \mathbf{R}$ is Borel. Let $\psi(S, i)$ be a formula for the statement

$$\exists k, n(\mu(A_k^n \cap [S]) = \emptyset \wedge (a_i, b_i) \subseteq U_k^{\ell,n}),$$

and let $\psi'(S, x) := \forall i \in \omega(\psi(P, i) \to x(i) = 1)$. Then ψ' is arithmetical. Let $c \in 2^\omega$ be a code for an element from $\mathbf{R}_\infty$. Then

$$\begin{aligned}(S, c) \in h_\ell \restriction \mathbf{R} &\iff \psi'(S, c) \wedge \forall x \in \omega^\omega(\psi(S, y) \to \forall i \in \omega(c(i) = 1 \to x(i) = 1)) \\ &\iff \psi'(S, c) \wedge \forall i(c(i) = 1 \to \exists x \in \omega^\omega(\forall n(\psi(S, z(n))) \wedge (a_i, b_i) \subseteq \bigcup_{n\in\omega}(a_{z(n)}, b_{z(n)}))).\end{aligned}$$

Hence, $h_\ell \restriction \mathbf{R}$ is a $\mathbf{\Delta}_1^1((\omega^\omega)^2)$ set and so it is a Borel function.

Item (b) follows from Lemma 2.3.25. Now we show (c). Let $\ell \in \omega$, let $T \in \mathbb{A}$, and let $\overline{h}_\ell(T) = (O, \varepsilon)$. By Lemma 2.3.12, there is a $T' \leq T$ such that for every $S \in \langle T' \rangle$, $\mu(h_\ell(S)) \leq \varepsilon$. Then $h_\ell[\langle T' \rangle] \subseteq [\overline{h}_\ell(T)]$.

It remains to show (d). Let $(O, \varepsilon) \in \mathbb{A}_\infty$. Then there is some $\ell \in \omega$ and some $x \in \omega^\omega$ such that $O = \bigcup_{n>1} U^{\ell,n}_{x(n)}$. Let $T \subseteq 2^{<\omega}$ be the unique pruned tree such that $[T] = 2^\omega \setminus \bigcup_{n>1} A^n_{x(n)}$. Then $T \in \mathbb{A}$ and by Lemma 1.2.26, $h_\ell(T) = O$. By (b) of Lemma 2.3.25, there is some $T' \leq T$ such that $\overline{h}_\ell(T') \leq (h_\ell(T'), \varepsilon) = (O, \varepsilon)$. Hence, $\overline{h}_\ell(T') \leq (O, \varepsilon)$. □

Note that by the Brendle-Łabędzki Lemma for Hechler forcing (Lemma 2.3.17), $\mathrm{add}(\mathcal{M}_\mathbb{D}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}_\mathbb{D}) = 2^{\aleph_0}$, where $\mathcal{M}_\mathbb{D}$ is the meager ideal in the dominating topology. In general, the existence of a Brendle-Łabędzki Lemma implies that the additivity number of the corresponding ideal is $\aleph_1$ and that the cofinality number is $2^{\aleph_0}$. However, we have no Brendle-Łabędzki Lemma for amoeba forcing. Nevertheless, we can show that the additivity number of the amoeba ideal is $\aleph_1$.

Corollary 2.3.28. *Let $I_\mathbb{A}$ be the σ-ideal of all $\mathcal{C}_\mathbb{A}$-meager sets. Then* $\mathrm{add}(I_\mathbb{A}) = \aleph_1$.

Proof. Let $I_{\mathbb{A}_\infty}$ be the σ-ideal of all $\mathcal{C}_{\mathbb{A}_\infty}$-meager sets. By Corollary 2.2.48, $\mathrm{add}(I_\mathbb{A}) \leq \mathrm{add}(I_{\mathbb{A}_\infty})$ and $\mathrm{add}(I_{\mathbb{A}_\infty}) \leq \mathrm{add}(\mathcal{M}_\mathbb{D})$. Since $\mathrm{add}(\mathcal{M}_\mathbb{D}) = \aleph_1$, $\mathrm{add}(I_\mathbb{A}) = \aleph_1$. □

2.4 Amoeba forcing for category

2.4.1 Definitions and basics

In addition to showing that Martin's axiom implies that the additivity number of the Lebesgue null ideal is $2^{\aleph_0}$, Martin and Solovay also showed in [MS70] that Martin's axiom implies that the additivity number of the meager ideal is $2^{\aleph_0}$. Nowadays, it is well-known that the additivity number of the Lebesgue ideal is smaller than the additivity number of the meager ideal (cf. [BJ95, Theorem 2.3.7]). So it would be enough to show it for the Lebesgue null ideal. However, this was not known at that time and was only proven more than ten years later by Bartoszyński in [Bar84]. Still, Martin and Solovay's proofs for the Lebesgue null ideal and the meager ideal are very similar. The only difference is the forcing notion which is used. For the Lebesgue null ideal they used amoeba forcing and for the meager ideal they used a forcing notion which is called *amoeba forcing for category* or sometimes *universally meager forcing.* The goal of this section is to define a regularity property for amoeba forcing for category and to prove Judah-Shelah- and Solovay-style characterizations for it. We start with the definition of amoeba forcing for category.

Definition 2.4.1. *Amoeba forcing for category* is the partial order of all pairs (σ, E), where σ is a finite sequence of elements from $2^{<\omega}$ and E is an open dense subset of $2^{<\omega}$, ordered by

$$(\sigma', E') \leq (\sigma, E) \iff \sigma' \supseteq \sigma,\ E' \subseteq E, \text{ and } \forall n \in \mathrm{dom}(\sigma' \setminus \sigma)(\sigma'(n) \in E).$$

We denote it by $\mathbb{UM}$.

Note that the intersection of two open dense subsets of $2^{<\omega}$ is open dense as well. Hence, for every $(\sigma, E), (\sigma, E') \in \mathbb{UM}$, $(\sigma, E \cap E') \leq (\sigma, E), (\sigma, E')$ and so (σ, E) and (σ, E') are compatible. Since $2^{<\omega}$ is countable, $\mathbb{UM}$ satisfies the c.c.c. In fact, $\mathbb{UM}$ is even σ-linked, i.e, there are countably many sets $A_n \subseteq \mathbb{UM}$ such that $\mathbb{UM} = \bigcup_{n\in\omega} A_n$ and for every $n \in \omega$, any two conditions in A_n are compatible.

We have already seen in Section 2.3.1 that forcing with amoeba forcing causes the union of all ground model Lebesgue null sets to be Lebesgue null. The same is true for amoeba forcing for category and the meager ideal: let M be a transitive model of ZFC, let $\mathcal{M}^M$ be the meager ideal in M, let G be a $\mathbb{UM}$-generic filter over M, and $x_G := \bigcup\{\sigma : \exists E((\sigma, E) \in G)\}$. Then $x_G \in (2^{<\omega})^\omega$. For every $n \in \omega$, we define $O_G^n := \bigcup\{[s] : \exists m \geq n(s = x_G(m))\}$. Let $n \in \omega$. Then for every $s \in 2^{<\omega}$, the set $\{(\sigma, E) \in \mathbb{UM} : \exists m < \mathrm{lh}(\sigma)(m \geq n \wedge s \subseteq \sigma(m))\}$ is dense in $\mathbb{UM}$. Hence, O_G^n is open dense. Let $O_G := \bigcap_{n\in\omega} O_G^n$. By the Baire Category Theorem, O_G is dense. Let $A \in \mathcal{M}^M$ be nowhere dense in the ground model. Then the set $\{(\sigma, E) \in \mathbb{UM} : \forall s \in E(A \cap [s] = \emptyset)\}$ is dense in $\mathbb{UM}$. Hence, there is some $n \in \omega$ such that for every $m \geq n$, $A \cap [x_G(m)] = \emptyset$ and so O_G^n is disjoint from A. Then O_G is disjoint from A as well. Since A was arbitrarily chosen, $\mathcal{M}^M$ is disjoint from O_G. Therefore, $\mathcal{M}^M$ is meager.

We call x_G from above a $\mathbb{UM}$-generic real over M. Note that $G = \{(\sigma, E) \in \mathbb{UM} : \sigma \subseteq x_G$ and $\forall n \geq \mathrm{lh}(\sigma)(x_G(n) \in E)\}$ and so every $\mathbb{UM}$-generic filter over M is uniquely determined by a $\mathbb{UM}$-generic real over M. As for amoeba reals, x_G is not a real, but we can identify every element of $(2^{<\omega})^\omega$ with a real using the canonical enumeration of $2^{<\omega}$. So we can think of the element of $(2^{<\omega})^\omega$ as reals. Let $\mathbf{U}$ be the set of all $x \in (2^{<\omega})^\omega$ such that for every $s \in 2^{<\omega}$ and every $n \in \omega$, there is some $m \in \omega$ such that $s \subseteq x(m)$, i.e., $\{s : \exists n \in \omega(s = x(n))\}$ is dense in $(2^{<\omega}, \subseteq)$. Hence, $\mathbf{U}$ is a Π_2^0 set of reals and so a Polish subspace of the Baire space. We call this topology the *standard topology on* $\mathbf{U}$. Note that the sets of the form $[\sigma] := \{x \in \mathbf{U} : \sigma \subseteq x\}$ for $\sigma \in (2^{<\omega})^{<\omega}$ form a clopen basis for the standard topology on $\mathbf{U}$. Unless otherwise stated, $\mathbf{U}$ will be always equipped with the standard topology.

We can use amoeba forcing for category to define a second topology on $\mathbf{U}$. For every $(\sigma, E) \in \mathbb{UM}$, we define $[\sigma, E] := \{x \in \mathbf{U} : \sigma \subseteq x$ and $\forall n \geq \mathrm{lh}(\sigma)(x(n) \in E)\}$ and let $\mathcal{C}_\mathbf{U} := \{[\sigma, E] : (\sigma, E) \in \mathbb{UM}\}$. Unlike $\mathcal{C}_\mathbb{A}$, $\mathcal{C}_\mathbf{U}$ generates a topology.

Proposition 2.4.2. *The set $\mathcal{C}_\mathbf{U}$ is a basis of a topology on $\mathbf{U}$. Moreover, every element of $\mathcal{C}_\mathbf{U}$ is clopen in this topology.*

Proof. For the first part, we have to show that the intersection of two elements of $\mathcal{C}_\mathbf{U}$ is either empty or in $\mathcal{C}_\mathbf{U}$. Let $(\sigma, E), (\sigma', E') \in \mathbb{UM}$ such that $[\sigma, E] \cap [\sigma', E'] \neq \emptyset$. Then $\sigma \subseteq \sigma'$ or $\sigma' \subseteq \sigma$. Without loss of generality, we assume the former. Let $x \in [\sigma, E] \cap [\sigma', E']$. Then $\sigma' \subseteq x$ and for every $n \geq \mathrm{lh}(\sigma)$, $x(n) \in E$. Hence, for every $n \in \mathrm{dom}(\sigma' \setminus \sigma)$, $\sigma'(n) \in E$ and so $(\sigma', E \cap E') \leq (\sigma, E), (\sigma', E')$. Therefore, $[\sigma', E \cap E'] = [\sigma, E] \cap [\sigma', E']$.

For the second part, we have to show that every element of $\mathcal{C}_\mathbf{U}$ is closed in the topology generated by $\mathcal{C}_\mathbf{U}$. Let $(\sigma, E) \in \mathbb{UM}$. Then we have

$$\mathbf{U} \setminus [\sigma, E] = \bigcup\{[\sigma', 2^{<\omega}] : (\sigma \nsubseteq \sigma' \wedge \sigma' \nsubseteq \sigma) \vee \exists n \in \mathrm{dom}(\sigma' \setminus \sigma)(\sigma'(n) \notin E)\}.$$

Therefore, $[\sigma, E]$ is clopen in this topology. $\square$

We call the topology generated by $\mathcal{C}_\mathbf{U}$ the $\mathbb{UM}$-*topology on* $\mathbf{R}$. Note that for every $\sigma \in (2^{<\omega})^{<\omega}$, $[\sigma, 2^{<\omega}] = [\sigma]$. Hence, every open set in the standard topology is also open in the $\mathbb{UM}$-topology and every basic open set in the $\mathbb{UM}$-topology is closed in the standard topology. However, $\mathbf{U}$ equipped with the $\mathbb{UM}$-topology is not a Polish space.

Proposition 2.4.3. *The cardinality of a dense set in the $\mathbb{UM}$-topology is at least $\mathfrak{d}$. In particular, the set $\mathbf{U}$ equipped with the $\mathbb{UM}$-topology is not separable.*

Proof. Let $D \subseteq \mathbf{U}$ be dense and let $h : \mathbf{U} \to \omega^\omega$ be defined by

$$h(x)(n) := \min\{\mathrm{lh}(s) : s \in \mathrm{ran}(x) \wedge \forall m < n(s(m) = 0) \wedge s(n) = 1\}.$$

We define $\mathscr{F} := \{h(x) : x \in D\}$. Let $y \in \omega^\omega$ and let

$$E_y := \{s \in 2^{<\omega} : \forall n \in \omega((\forall m < n(s(m) = 0) \wedge s(n) = 1) \to \mathrm{lh}(s) > y(n))\}.$$

Then E_y is open dense in $2^{<\omega}$ and so $(\emptyset, E_y) \in \mathbb{UM}$. Since D is dense, $D \cap [\emptyset, E_y] \neq \emptyset$. Let $x \in D \cap E_y$. Then $h(x)$ eventually dominates y. Hence, $\mathscr{F}$ is a dominating family and so $\mathfrak{d} \leq |\mathscr{F}| \leq |D|$. Therefore, the cardinality of any dense set in the $\mathbb{UM}$-topology is at least $\mathfrak{d}$. □

In the rest of this section, we prove some basic properties for Baire property in the $\mathbb{UM}$-topology. To do this, we first show that $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is proper a category base which is Borel compatible with $\mathbf{U}$.

Lemma 2.4.4. *The pair $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is a proper category base which is Borel compatible with $\mathbf{U}$.*

Proof. Since $\mathcal{C}_\mathbf{U}$ is a basis for a topology on $\mathbf{U}$, $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is a category base. By Proposition 2.2.10, a subset of $\mathbf{U}$ has the Baire property in the $\mathbb{UM}$-topology if and only if it is $\mathcal{C}_\mathbf{U}$-Baire. Moreover, a subset of $\mathbf{U}$ is nowhere dense or meager in the $\mathbb{UM}$-topology if and only if it is $\mathcal{C}_\mathbf{U}$-singular or $\mathcal{C}_\mathbf{U}$-meager, respectively. We have to show that $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is proper and Borel compatible with $\mathbf{U}$. We start with the latter. Recall that every region in $\mathcal{C}_\mathbf{U}$ is closed in $\mathbf{U}$. Hence, we only have to check that Borel set in $\mathbf{U}$ has the Baire property in the $\mathbb{UM}$-topology. Let $B \subseteq \mathbf{U}$ be Borel in $\mathbf{U}$. Since every open set in $\mathbf{U}$ is open in the $\mathbb{UM}$-topology, B is Borel in the $\mathbb{UM}$-topology. Hence, B has the Baire property in the $\mathbb{UM}$-topology. Therefore, $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is Borel compatible with $\mathbf{U}$.

It remains to show that $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is proper. Since $\mathbb{UM}$ satisfies the c.c.c., $(\mathcal{C}_\mathbf{U}, \subseteq)$ is proper as a forcing notion. Clearly, every singleton in $\mathbf{U}$ is $\mathcal{C}_\mathbf{U}$-singular. So we only have to check that every region is $\mathcal{C}_\mathbf{U}$-abundant. We suppose for a contradiction that there is some (σ, E) such that (σ, E) is $\mathcal{C}_\mathbf{U}$-meager. Then there are $\mathcal{C}_\mathbf{U}$-singular sets $N_n \subseteq \mathbf{U}$ such that $\bigcup_{n\in\omega} N_n$. Since the N_n are $\mathcal{C}_\mathbf{U}$-singular, we can recursively define a decreasing sequence $\langle(\sigma_n, E_n) : n \in \omega\rangle$ such that $(\sigma_0, E_0) = (\sigma, E)$ and $N_n \cap [\sigma_{n+1}, E_{n+1}] = \emptyset$. Let $x := \bigcup_{n\in\omega} \sigma_n$. Then for every $n \in \omega$, $x \in [\sigma_n, E_n]$. Thus, $x \in [\sigma, E]$ and so there is some $n \in \omega$ such that $x \in N_n$. But this is not possible since $N_n \cap [\sigma_{n+1}, E_{n+1}] = \emptyset$. Therefore, every region is $\mathcal{C}_\mathbf{U}$-abundant. □

Corollary 2.4.5.

(a) The sets which have the Baire property in the $\mathbb{UM}$-topology form a σ-algebra on $\mathbf{U}$ containing all analytic and co-analytic sets in $\mathbf{U}$.

(b) In L, there is a $\mathbf{\Delta}^1_2(\mathbf{U})$ set which does not have the Baire property in the $\mathbb{UM}$-topology.

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\mathbf{\Sigma}^1_2(\mathbf{U})$ sets have the Baire property in the $\mathbb{UM}$-topology.

Proof. Follows directly from Corollary 2.2.28 and Lemma 2.4.4. □

By Corollary 2.4.5, the Baire property in the $\mathbb{UM}$-topology behaves similarly as most other regularity properties, i.e., every analytic set has the Baire property in the $\mathbb{UM}$-topology and the question if every $\mathbf{\Delta}^1_2(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology cannot be answered in ZFC. As usual, for every projective pointclass Γ, we write $\Gamma(\mathbb{UM})$ for the statement "every $\Gamma(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology". We conclude this section with Solovay- and Judah-Shelah-style characterizations for the Baire property in the $\mathbb{UM}$-topology.

Theorem 2.4.6.

(a) Every $\mathbf{\Delta}^1_2(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology if and only if for every real $r \in \omega^\omega$, there is a $\mathbb{UM}$-generic real over $\mathrm{L}[r]$.

(b) Every $\mathbf{\Sigma}^1_2(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology if and only if for every real $r \in \omega^\omega$, the set $\{x \in \mathbf{U} : x$ is not a $\mathbb{UM}$-generic real over $\mathrm{L}[r]\}$ is meager in the $\mathbb{UM}$-topology.

Proof. By Ikegami's Theorem for weak category bases satisfying the c.c.c. (Corollary 2.2.36), it is enough to show that $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is provable Σ^1_2 and that $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ satisfies the c.c.c. in every inner model of ZFC. Let $\varphi_\mathbf{U}$ be the Π^0_2 formula defining $\mathbf{U}$ and let $\varphi_{\mathcal{C}_\mathbf{U}}(c)$ be the statement "c is a Borel code and there is some $(\sigma, E) \in \mathbb{UM}$ such that $B_c = [\sigma, E]$". Then $\varphi_{\mathcal{C}_\mathbf{U}}$ is Σ^1_2 and $\mathcal{C}_\mathbf{U} = \{B_c : \varphi_{\mathcal{C}_\mathbf{U}}(c)\}$. Let M be an inner model of ZFC. By Lemma 2.4.4, $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ is a proper category base which is Borel compatible with $\mathbf{U}$ in M. Since $\mathbb{UM}$ satisfies the c.c.c. in M, $(\mathbf{U}, \mathcal{C}_\mathbf{U})$ satisfies the c.c.c. in M as well.

It remains to show that the statement "c is a Borel code and B_c is $\mathcal{C}_\mathbf{U}$-meager" is Σ^1_2. By Lemma 2.2.16, a set $A \subseteq \mathbf{U}$ is $\mathcal{C}_\mathbf{U}$-meager if and only if there are maximal antichains $\mathcal{A}_n \subseteq \mathbb{UM}$ such that $A \subseteq \bigcup_{n\in\omega} \mathbf{U} \setminus \bigcup\{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\}$. Since $\mathbb{UM}$ satisfies the c.c.c., every antichain can be coded as a single real. Moreover, two conditions $(\sigma, E), (\sigma', E')$ are compatible if and only if $\sigma \subseteq \sigma'$ and for every $n \in \mathrm{dom}(\sigma' \setminus \sigma)$, $\sigma'(n) \in E$ or vice versa. Hence, the statement "$\mathcal{A}$ is a maximal antichain" is Π^1_1 and so the statement "c is a Borel code and B_c is $\mathcal{C}_\mathbf{U}$-meager" is Σ^1_2. □

2.4.2 Amoeba forcing for category and inaccessibles

In this section, we investigate the consistency strength of $\mathbf{\Delta}^1_2(\mathbb{UM})$ and $\mathbf{\Sigma}^1_2(\mathbb{UM})$. The goal is to show that $\mathbf{\Delta}^1_2(\mathbb{UM})$ is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{C})$ and that $\mathbf{\Sigma}^1_2(\mathbb{UM})$ holds if and only if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. We start with the former. Brendle and Löwe proved the following characterization of $\mathbf{\Sigma}^1_2(\mathbb{C})$.

Theorem 2.4.7 (Brendle-Löwe)**.** *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(2^\omega)$ set has the Baire property in Cantor space,

(b) every $\mathbf{\Delta}^1_2(\omega^\omega)$ set has the Baire property in the dominating topology, and

(c) for every $r \in \omega^\omega$, there is a Hechler real over $\mathrm{L}[r]$.

Proof. Cf. [BL99, Theorem 5.8]. □

Therefore, it is enough to show that $\mathbf{\Delta}^1_2(\mathbb{UM})$ holds if and only if for every $r \in \omega^\omega$, there is a Hechler real over $\mathrm{L}[r]$. By Theorem 2.4.6, $\mathbf{\Delta}^1_2(\mathbb{UM})$ holds if and only if for every real $r \in \omega^\omega$, there is a $\mathbb{UM}$-generic real over $\mathrm{L}[r]$. Truss showed that amoeba forcing for category adds Hechler reals.

Theorem 2.4.8 (Truss)**.** *Let M be a transitive model of ZFC. If there is a $\mathbb{UM}$-generic real over M, then there is a Hechler real over M. Moreover, if x is a Cohen real over M and f is a Hechler real over $M[x]$, then $M[x][f]$ contains a $\mathbb{UM}$-generic real over M.*

Proof. Cf. [Tru77, Theorem 6.5]. □

The converse of the first part of Theorem 2.4.8 is not true in general. We have already seen that after forcing with amoeba forcing for category the union of all ground model meager sets is meager and this is not true for Hechler forcing (cf. [BJ95, Theorem 3.5.4]). Therefore, amoeba forcing for category and Hechler forcing are not forcing equivalent. However, by the second part of Theorem 2.4.8, the two step iteration of Hechler forcing adds $\mathbb{UM}$-generic reals. This is enough to show that $\mathbf{\Delta}^1_2(\mathbb{UM})$ holds if and only if for every $r \in \omega^\omega$, there is a Hechler real over $\mathrm{L}[r]$.

Corollary 2.4.9. *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(2^\omega)$ set has the Baire property in Cantor space,

(b) every $\mathbf{\Delta}^1_2(\omega^\omega)$ set has the Baire property in the dominating topology, and

(c) every $\mathbf{\Delta}^1_2(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology.

Proof. By Theorems 2.4.6 and 2.4.7, we only have to show that for every $r \in \omega^\omega$, there is a Hechler real over $\mathrm{L}[r]$ if and only if for every $r \in \omega^\omega$, there is a $\mathbb{UM}$-generic real over $\mathrm{L}[r]$. The backward direction follows directly from Theorem 2.4.8. So we only have to prove the forward direction. Let $r \in \omega^\omega$. By assumption there are $f, f' \in \omega^\omega$ such that f is a Hechler real over $\mathrm{L}[r]$ and f' is a Hechler real over $\mathrm{L}[r, f]$. Then $\mathrm{L}[r, f]$ contains a Cohen real x over $\mathrm{L}[r]$. Note that f' is also a Hechler real over $\mathrm{L}[r, x]$. By Theorem 2.4.8, there is a $\mathbb{UM}$-generic real over $\mathrm{L}[r]$. □

It remains to show that $\mathbf{\Sigma}^1_2(\mathbb{UM})$ holds if and only if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. To do this, we take a similar approach as for amoeba forcing. The idea is again to use Theorem 2.2.46 to show that for every projective pointclass Γ, $\Gamma(\mathbb{UM})$ implies $\Gamma(\mathbb{D})$.

Theorem 2.4.10. *For every projective pointclass Γ, if every $\Gamma(\mathbb{UM})$ set has the Baire property in the $\mathbb{UM}$-topology, then every $\Gamma(\omega^\omega)$ set has the Baire property in the dominating topology.*

Proof. For every $n \in \omega$, let t_n be the binary sequence of n consecutive 0's followed by a a single 1. We define $h : \mathbf{U} \to \omega^\omega$ and $\overline{h} : \mathbb{UM} \to \mathbb{D}$ by

$$h(x)(n) := \min\{\mathrm{lh}(s) : s \in \mathrm{ran}(x) \wedge t_n \subseteq s\} \text{ and}$$
$$\overline{h}(\sigma, E) := (n_{(\sigma,E)}, f_{(\sigma,E)}),$$

where $n_{(\sigma,E)}$ is maximal such that for every $x, x' \in [\sigma, E]$, $h(x)\restriction n = h(x')\restriction n$ and $f_{(\sigma,E)}(n) := \min\{h(x)(n) : x \in [\sigma, E]\}$. By Theorem 2.2.46, it is enough to check that

(a) $\overline{h}$ is a projection,

(b) for every (σ, E), $h[\sigma, E] \subseteq [\overline{h}(\sigma, E)]$,

(c) $h[\mathbb{UM}]$ is dense in $\mathbb{D}$, and

(d) h is Borel.

We start with proving (a). It is clear that $\overline{h}$ is order-preserving. Let $(\sigma, E) \in \mathbb{UM}$ and let $(n, f) \leq \overline{h}(\sigma, E)$. Then for every $n_{(\sigma,E)} \leq m < n$, there is an $s_m \in E$ such that $t_m \subseteq s_m$, and $\mathrm{lh}(s) = f(m)$. We define $\sigma' := \sigma^\frown\langle s_m : n_{(\sigma,E)} \leq m < n\rangle$ and $E' := \{s \in E : \exists m \in \omega(t_m \subseteq s \wedge \mathrm{lh}(s) \geq f(n))\}$. Then E' is still dense in $2^{<\omega}$ and so $(\sigma', E') \leq (\sigma, E)$. By definition, $n_{(\sigma',E')} \geq n$, $f_{(\sigma',E')}\restriction n = f\restriction n$, and for every $m \geq n$, $f_{(\sigma',E')}(m) \geq f(m)$. Hence, $\overline{h}(\sigma', E') \leq (n, f)$.

Next, we show (b). Let $(\sigma, E) \in \mathbb{UM}$ and let $x \in [\sigma, E]$. Then $h(x) \restriction n_{(\sigma,E)} = f_{(\sigma,E)} \restriction n_{(\sigma,E)}$ and for every $n \in \omega$, $h(x)(n) \geq f_{(\sigma,E)}$. Hence, $x \in [\overline{h}(\sigma, E)]$.

Now we prove (c). Let $(n, f) \in \mathbb{D}$ and let for every $m < n$, $s_m \in 2^{<\omega}$ such that $t_m \subseteq s_m$ and $\mathrm{lh}(s_m) = f(m)$. We define $\sigma := \langle s_m : m < n \rangle$ and $E := \{s \in 2^{<\omega} : \exists m \in \omega(t_m \subseteq s \wedge \mathrm{lh}(s) \geq f(m))\}$. Then $n_{(\sigma,E)} = n$ and for every $m \in \omega$, $f_{(\sigma,E)}(m) = f(m)$. Hence, $\overline{h}(\sigma', E') = (n, f)$.

Finally, we show (d). Let $s \in \omega^{<\omega}$. Then

$$h^{-1}([s]) = \bigcap \{[\sigma] : s \subseteq f_{(\sigma,E)} \restriction n_{(n,E)}\}$$

and so $h^{-1}([s])$ is closed in $\mathbf{U}$. Hence, h is Borel. □

Corollary 2.4.11. *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(\mathbf{U})$ set has the Baire property in the $\mathbb{UM}$-topology,

(b) for every $r \in \omega^\omega$, the set $\{x \in \mathbb{UM} : x$ is not a $\mathbb{UM}$-generic real over $\mathrm{L}[r]\}$ is meager in the $\mathbb{UM}$-topology, and

(c) for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Proof. Follows directly from Theorems 2.3.16, 2.4.6, and 2.4.10 and Corollary 2.4.5. □

We conclude this section with a corollary about the additivity number of the meager ideal in the $\mathbb{UM}$-topology. As for amoeba forcing, we do not have a Brendle-Łabędzki Lemma for amoeba forcing for category. However, we can again use Corollary 2.2.48.

Corollary 2.4.12. *Let $\mathcal{M}_{\mathbb{UM}}$ be the meager ideal in the $\mathbb{UM}$-topology. Then* $\mathrm{add}(\mathcal{M}_{\mathbb{UM}}) = \aleph_1$.

Proof. By Corollary 2.2.48, $\mathrm{add}(\mathcal{M}_{\mathbb{UM}}) \leq \mathrm{add}(\mathcal{M}_{\mathbb{D}})$. Since $\mathrm{add}(\mathcal{M}_{\mathbb{D}}) = \aleph_1$, $\mathrm{add}(\mathcal{M}_{\mathbb{UM}}) = \aleph_1$. □

2.5 Localization forcing

2.5.1 Definitions and basics

In this section, we introduce a regularity property for localization forcing and prove some basic properties for it. In particular, we prove Judah-Shelah- and Solovay-style characterizations. We start with some definitions.

A *slalom* is a function f from ω to $[\omega]^{<\omega}$ such that for every $n \in \omega$, $|f(n)| \leq n + 1$. We write **Loc** for the set of all slaloms. We say that a slalom $f \in \mathbf{Loc}$ *localizes a real* $x \in \omega^\omega$ if for all but finitely many $n \in \omega$, $x(n) \in f$. Let M be a transitive model of ZFC. A slalom is called a *localizing real over M* if it localizes all reals in M. This concept was used by Bartoszyński in [Bar84] to show that the additivity number of the Lebesgue null ideal is smaller than the additivity number of the meager ideal. The canonical forcing notion to add a localizing real is *localization forcing*.

Definition 2.5.1. *Localization forcing* is the partial order of all pairs $(\sigma, E) \in ([\omega]^{<\omega})^{<\omega} \times [\omega^\omega]^{<\omega}$ such that for every $n \in \mathrm{dom}(\sigma)$, $|\sigma(n)| = n + 1$ and $|E| \leq \mathrm{lh}(\sigma) + 1$ ordered by

$$(\sigma', E') \leq (\sigma, E) :\iff \sigma \subseteq \sigma' \wedge E \subseteq E' \wedge \forall x \in E \forall n \in \mathrm{dom}(\sigma' \setminus \sigma)(x(i) \in \sigma'(i)).$$

We denote localization forcing by $\mathbb{LOC}$ and write $\mathrm{dom}(\mathbb{LOC})$ for the set $\{\sigma : \exists E \in [\omega^\omega]^{<\omega}((\sigma, E) \in \mathbb{LOC})\}$.

Let M be a transitive model of ZFC and let G be a $\mathbb{LOC}$-generic filter over M. We say that $f_G := \bigcup\{\sigma : \exists E((\sigma, E) \in G)\}$ is a $\mathbb{LOC}$-*generic real over* M. Since $G = \{(\sigma, E) \in \mathbb{LOC} : \sigma \subseteq f_G$ and $\forall x \in E \forall n \geq \mathrm{lh}(\sigma)(x(n) \in f_G(n))\}$, $M[G] = M[f_G]$ and so every $\mathbb{LOC}$-generic filter over M is uniquely determined by a $\mathbb{LOC}$-generic real over M. Moreover, every $\mathbb{LOC}$-generic real over M is a localizing real: let $x \in \omega^\omega$ be a real in M. Then $D_x := \{(\sigma, E) \in \mathbb{LOC} : x \in E\}$ is dense in $\mathbb{LOC}$ and so there is some $(\sigma, E) \in G$ such that $x \in E$. Hence, for every $n \geq \mathrm{lh}(\sigma)$, $x(n) \in f_G(n)$. Therefore, f_G is a localizing real over M.

The converse is not true in general. Amoeba forcing adds localizing reals, but no $\mathbb{LOC}$-generic reals (cf. [Tru88, Theorem 4.1] and [BJ95, p. 106]). Therefore, not every localizing real is a $\mathbb{LOC}$-generic real. Conversely, localization forcing adds amoeba reals (cf. [Tru88, Theorem 4.2]). Moreover, localization forcing satisfies the c.c.c.: for every $\sigma \in \mathrm{dom}(\mathbb{LOC})$, let $A_\sigma := \{(\sigma', E) : (\sigma', E) \leq (\sigma, E)$ and $2|E| \leq \mathrm{lh}(\sigma) + 1\}$. Then $\mathbb{LOC} = \bigcup\{A_\sigma : \sigma \in \mathrm{dom}(\mathbb{LOC})\}$ and so it is enough to show that for every $\sigma \in \mathrm{dom}(\mathbb{LOC})$, the elements in A_σ are pairwise compatible. Let $\sigma \in \mathrm{dom}(\mathbb{LOC})$. We only have to show that for every $E, E' \in [\omega^\omega]^{<\omega}$ with $(\sigma, E), (\sigma, E') \in A_\sigma$, (σ, E) and (σ, E') are compatible. Since $|E| + |E'| \leq \mathrm{lh}(\sigma) + 1$, $(\sigma, E \cup E') \in \mathbb{LOC}$ and $(\sigma, E \cup E') \leq (\sigma, E), (\sigma, E')$. Therefore, localization forcing satisfies the c.c.c. In fact, this even shows that localization forcing is σ-linked.

From now on, we shall often identify slaloms with reals using the canonical bijections between ω and $[\omega]^{\leq n+1}$. This identification induces a topology on $\mathbf{Loc}$, which we call the *standard topology*. Thus, by definition, $\mathbf{Loc}$ equipped with the standard topology is homeomorphic to the Baire space and so the set $\{[\sigma] : \sigma \in \mathrm{dom}(\mathbb{LOC})\}$ is a clopen basis, where $[\sigma] := \{f \in \mathbf{Loc} : \sigma \subseteq x\}$. Unless stated otherwise, we always assume that $\mathbf{Loc}$ is equipped with the standard topology. Next, we use $\mathbb{LOC}$ to define a second topology on $\mathbf{Loc}$. For every $(\sigma, E) \in \mathbb{LOC}$, let

$$[\sigma, E] := \{f \in \mathbf{Loc} : f\restriction\mathrm{lh}(\sigma) = \sigma \text{ and } \forall x \in E \forall n \geq \mathrm{lh}(\sigma)(x(n) \in f(n))\},$$

and let $\mathcal{C}_{\mathbf{Loc}} := \{[\sigma, E] : (\sigma, E) \in \mathbb{LOC}\}$.

Proposition 2.5.2. *The set $\mathcal{C}_{\mathbf{Loc}}$ is a basis of a topology on $\mathbf{Loc}$. Moreover, every element of $\mathcal{C}_{\mathbf{Loc}}$ is clopen in this topology.*

Proof. For the first part, it is enough to show that the intersection of two elements from $\mathcal{C}_{\mathbf{Loc}}$ is either empty or a union of elements of $\mathcal{C}_{\mathbf{Loc}}$. Let $(\sigma, E), (\sigma', E') \in \mathbb{LOC}$ such that $[\sigma, E] \cap [\sigma', E'] \neq \emptyset$ and let $m := |E| + |E'|$. We show that $[\sigma, E] \cap [\sigma', E'] = A := \bigcup\{[f\restriction m, E \cup E'] : f \in [\sigma, E] \cap [\sigma', E']\}$. It is clear that $[\sigma, E] \cap [\sigma', E'] \subseteq A$. Let $f \in A$ and let $f' \in [\sigma, E] \cap [\sigma', E']$ such that $[f'\restriction m, E \cup E']$. Then for every $x \in E \cup E'$, for every $n \geq \mathrm{lh}(\sigma)$, $x(n) \in f'(n)$ and for every $n \geq \mathrm{lh}(\sigma')$, $x(n) \in f'(n)$. Moreover, $f\restriction m = f'\restriction m$ and for every $x \in E \cup E'$ and every $n \geq m$, $x(n) \in f(n)$. Hence, $f \in [\sigma, E] \cap [\sigma', E']$.

We show the second part. Let $(\sigma, E) \in \mathbb{LOC}$. Then

$$\mathbf{Loc} \setminus [\sigma, E] = \bigcup\{[\sigma', \emptyset] : (\sigma \nsubseteq \sigma' \vee \sigma' \nsubseteq \sigma) \vee \exists x \in E \exists n \in \mathrm{dom}(\sigma' \setminus \sigma)(x(n) \notin \sigma'(n))\}.$$

Therefore, $[\sigma, E]$ is closed and so clopen. ☐

Note that for every $\sigma \in \mathrm{dom}(\mathbb{LOC})$, $[\sigma, \emptyset] = [\sigma]$. Hence, every basic open set in the standard topology is clopen in the localizing topology and so every Borel set in the standard topology is Borel in the localizing topology. Conversely, by the proof of Proposition 2.5.2, for every $(\sigma, E) \in \mathbb{LOC}$, $[\sigma, E]$ is closed in the standard topology. However, there are open sets in the localizing topology which are not even Borel in the standard topology: let $A \subseteq 2^\omega$ be not $\mathbf{\Sigma}^1_1(\omega^\omega)$ and let $O := \bigcup_{x\in A}[\emptyset, \{x\}]$. Then O is open in the localizing topology. We suppose for a contradiction that O is Borel in the standard topology. Then

$$A = \{x \in 2^\omega : \exists f \in O \forall n(x(n) \in f(n) \wedge 2 \nsubseteq f(n))\}.$$

and so A is $\mathbf{\Sigma}^1_1(\omega^\omega)$. But this is a contradiction. Therefore, O is not Borel in the standard topology. Moreover, the localizing topology is not even Polish.

Proposition 2.5.3. *The cardinality of a dense set in the localizing topology is at least $\mathfrak{d}$. In particular, the set* **Loc** *equipped with the localizing topology is not separable.*

Proof. Let $D \subseteq \mathbf{Loc}$ be dense. We define $\mathscr{F} := \{x \in \omega^\omega : \exists f \in D \forall n(x(n) = \max(f(n)) + 1)\}$. Let $x \in \omega^\omega$. Since D is dense, there is some $f \in D \cap [\emptyset, \{x\}]$. Then for every $n \in \omega$, $x(n) \in f(n)$. Hence, there is some $y \in \mathscr{F}$ such that y eventually dominates x and so $\mathscr{F}$ is a dominating family. Therefore, $\mathfrak{d} \leq |\mathscr{F}| \leq |D|$ and so every dense set is uncountable. □

In the rest of this section, we prove some basic properties for Baire property in the localizing topology. To do this, we first show that $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is proper a category base which is Borel compatible with **Loc**.

Lemma 2.5.4. *The pair $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is a proper category base which is Borel compatible with* **Loc**.

Proof. Since $\mathcal{C}_{\mathbf{Loc}}$ is a basis for a topology on **Loc**, $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is a category base. By Proposition 2.2.10, a subset of **Loc** has the Baire property in the localizing topology if and only if it is $\mathcal{C}_{\mathbf{Loc}}$-Baire. Moreover, a subset of **Loc** is nowhere dense or meager in the $\mathbb{LOC}$-topology if and only if it is $\mathcal{C}_{\mathbf{Loc}}$-singular or $\mathcal{C}_{\mathbf{Loc}}$-meager, respectively. We have to show that $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is proper and Borel compatible with **Loc**. We start with the latter. We have already seen that every region is closed in **Loc**. Moreover, every Borel set in the standard topology is Borel in the localizing topology. Hence, every Borel set in the standard topology has the Baire property in the localizing topology. Therefore, $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is Borel compatible with **Loc**.

It remains to show that $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is proper. Since $\mathbb{LOC}$ satisfies the c.c.c., $(\mathcal{C}_{\mathbf{Loc}}, \subseteq)$ is proper as a forcing notion. Clearly, every singleton in **Loc** is $\mathcal{C}_{\mathbf{Loc}}$-singular. So we only have to check that every region is $\mathcal{C}_{\mathbf{Loc}}$-abundant. We suppose for a contradiction that there is some (σ, E) such that (σ, E) is $\mathcal{C}_{\mathbf{Loc}}$-meager. Then there are $\mathcal{C}_{\mathbf{Loc}}$-singular sets $N_n \subseteq \mathbf{Loc}$ such that $\bigcup_{n\in\omega} N_n$. Since the N_n are $\mathcal{C}_{\mathbf{Loc}}$-singular, we can recursively define a decreasing sequence $\langle(\sigma_n, E_n) : n \in \omega\rangle$ such that $(\sigma_0, E_0) = (\sigma, E)$ and $N_n \cap [\sigma_{n+1}, E_{n+1}] = \emptyset$. Let $x := \bigcup_{n\in\omega} \sigma_n$. Then for every $n \in \omega$, $x \in [\sigma_n, E_n]$. Thus, $x \in [\sigma, E]$ and so there is some $n \in \omega$ such that $x \in N_n$. But this is not possible since $N_n \cap [\sigma_{n+1}, E_{n+1}] = \emptyset$. Therefore, every region is $\mathcal{C}_{\mathbf{Loc}}$-abundant. □

Corollary 2.5.5.

(a) The set which have the Baire property in the localizing form a σ-algebra on **Loc** *containing all analytic and co-analytic sets in* **Loc**.

(b) In L, *there is a $\mathbf{\Delta}^1_2(\mathbf{Loc})$ set which does not have the Baire property in the localizing topology.*

(c) If for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$, then all $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ sets have the Baire property in the localizing topology.

Proof. Follows directly from Corollary 2.2.28 and Lemma 2.5.4. $\square$

Thus, the Baire property in the localizing topology behaves similarly as most other regularity properties, i.e., every analytic set has the Baire property in the localizing topology and the question if every $\mathbf{\Delta}^1_2(\mathbf{Loc})$ set has the Baire property in the localizing topology cannot be answered in ZFC. As before, we write $\Gamma(\mathbb{LOC})$ for the statement "every $\Gamma(\mathbf{Loc})$ set has the Baire property in the localizing topology", where Γ is a projective pointclass. We conclude this section with Solovay- and Judah-Shelah-style characterizations for the Baire property in the localizing topology.

Theorem 2.5.6.

(a) Every $\mathbf{\Delta}^1_2(\mathbf{Loc})$ set has the Baire property in the localizing topology if and only if for every real $r \in \omega^\omega$, there is an $\mathbb{LOC}$-generic real over $\mathrm{L}[r]$.

(b) Every $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ set has the Baire property in the localizing topology if and only if for every real $r \in \omega^\omega$, the set $\{f \in \mathbf{Loc} : f$ is not an $\mathbb{LOC}$-generic real over $\mathrm{L}[r]\}$ is meager in the localizing topology.

Proof. By Ikegami's Theorem for weak category bases satisfying the c.c.c. (Corollary 2.2.36), it is enough to show that $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is provable Σ^1_2 and satisfies the c.c.c. in every inner model of ZFC. Let $\varphi_{\mathbf{Loc}}$ be any Π^0_2 formula which is always true and let $\varphi_{\mathcal{C}_{\mathbf{Loc}}}(c)$ be the statement "c is a Borel code and there is some $(\sigma, E) \in \mathbb{LOC}$ such that $B_c = [\sigma, E]$". Then $\varphi_{\mathcal{C}_{\mathbf{Loc}}}$ is Σ^1_2 and $\mathcal{C}_{\mathbf{Loc}} = \{B_c : \varphi_{\mathcal{C}_{\mathbf{Loc}}}(c)\}$. Let M be an inner model of ZFC. By Lemma 2.5.4 in M, $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ is a proper category base which is Borel compatible with $\mathbf{Loc}$ in M. Since $\mathbb{LOC}$ satisfies the c.c.c. in M, $(\mathbf{Loc}, \mathcal{C}_{\mathbf{Loc}})$ satisfies the c.c.c. in M as well.

It remains to show that the statement "c is a Borel code and B_c is $\mathcal{C}_{\mathbf{Loc}}$-meager" is Σ^1_2. By Lemma 2.2.16, a set $A \subseteq \mathbf{Loc}$ is $\mathcal{C}_{\mathbf{Loc}}$-meager if and only if there are antichains $\mathcal{A}_n \subseteq \mathbb{LOC}$ such that $A \subseteq \bigcup_{n\in\omega} \mathbf{Loc} \setminus \bigcup\{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\}$. Since $\mathbb{LOC}$ satisfies the c.c.c., every antichain can be coded as a single real. Moreover, two conditions $(\sigma, E), (\sigma', E') \in \mathbb{LOC}$ are compatible if and only if $\sigma \subseteq \sigma'$ and for every $x \in E$ and every $n \in \mathrm{dom}(\sigma' \setminus \sigma)$, $x(n) \in \sigma'(n)$ or vice versa. Hence, the statement "$\mathcal{A}$ is a maximal antichain in $\mathbb{LOC}$" is Π^1_1 and so the "c is a Borel code and B_c is $\mathcal{C}_{\mathbf{Loc}}$-meager" is Σ^1_2. $\square$

2.5.2 Localization forcing and inaccessibles

In this section, we investigate the consistency strength of $\mathbf{\Delta}^1_2(\mathbb{LOC})$ and $\mathbf{\Sigma}^1_2(\mathbb{LOC})$. The goal is to show that $\mathbf{\Delta}^1_2(\mathbb{LOC})$ is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{B})$ and that $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ holds if and only if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. We start with $\mathbf{\Delta}^1_2(\mathbb{LOC})$. By Theorem 2.5.6, $\mathbf{\Delta}^1_2(\mathbb{LOC})$ holds if and only if for every $r \in \omega^\omega$, there is a $\mathbb{LOC}$-generic real over $\mathrm{L}[r]$. Brendle and Löwe have shown that the latter is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{B})$.

Theorem 2.5.7 (Brendle-Löwe)**.** *Every $\mathbf{\Sigma}^1_2(2^\omega)$ set is Lebesgue measurable if and only if for every $r \in \omega^\omega$, there is a localizing real over $\mathrm{L}[r]$.*

Proof. Cf. [BL11, Lemma 10]. $\square$

Corollary 2.5.8. *The following are equivalent:*

(a) Every $\mathbf{\Sigma}^1_2(2^\omega)$ set is Lebesgue measurable,

(b) *every* $\mathbf{\Delta}^1_2(\mathbf{R})$ *set is amoeba regular, and*

(c) *every* $\mathbf{\Delta}^1_2(\mathbf{Loc})$ *set has the Baire property in the localizing topology.*

Proof. This follows directly from Corollary 2.3.11 and Theorems 2.5.6 and 2.5.7. $\square$

In the rest of this section, we show that $\mathbf{\Sigma}^1_2(\mathbb{LOC})$ holds if and only if for every $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. To do this, we take a different approach than for amoeba forcing and amoeba forcing for category. The idea is to generalize the proof of the Brendle-Łabędzki Lemma for eventually different forcing (Lemma 2.3.18) to obtain a Brendle-Łabędzki Lemma for localization forcing. First, we need a family of small sets. For every $x \in \omega^\omega$, we define $X_x := \{f \in \mathbf{Loc} : \exists^\infty n \in \omega (x(n) \notin f(n))\}$. Let $x \in \omega^\omega$. Then X_x is Borel in $\mathbf{Loc}$ with Borel code in $\mathrm{L}[x]$. Moreover, since $D_x := \{(\sigma, E) : x \in E\}$ is dense in $\mathbb{LOC}$, X_x is nowhere dense in the localizing topology.

Lemma 2.5.9. *Let $\mathscr{E}$ be a pairwise eventually different family and let $A \subseteq \mathbf{Loc}$ be meager in the localizing topology. Then there are only countably many $g \in \mathscr{E}$ such that $X_g \subseteq A$.*

Proof. By Lemma 2.2.16, there are maximal antichains $\mathcal{A}_n$ such that

$$A \cap \bigcap_{n\in\omega} \bigcup \{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\} = \emptyset.$$

Since $\mathbb{LOC}$ satisfies the c.c.c, for every $n \in \omega$, $\mathcal{A}_n$ is of the form $\mathcal{A}_n = \{(\sigma^n_m, E^n_m) : m \in \omega\}$. We define for every finite $M \subseteq \omega^2$, $E_M := \bigcup\{E^n_m : (n, m) \in M\}$. We say that M covers $g \in \omega^\omega$ if for all but finitely many $k \in \omega$, there is a $x \in E_M$ such that $x(k) = g(k)$. Since $\mathscr{E}$ is a pairwise eventually different family, each $M \subseteq \omega^2$ only covers finitely many $g \in \mathscr{E}$. Hence, there are only countably many $g \in \mathscr{E}$ such that g is covered by some $M \subseteq \omega^2$.

Let $g \in \mathscr{E}$ such that g is not covered by any $M \subseteq \omega^2$. We shall construct a sequence $\langle \tau_n : n \in \omega\rangle$ such that for every $n \in \omega$,

(a) $\tau_n \in \mathrm{dom}(\mathbb{LOC})$,

(b) $\tau_n \subsetneq \tau_{n+1}$,

(c) there is some $k \in \mathrm{dom}(\tau_{n+1} \setminus \tau_n)$ such that $g(k) \notin \tau_{n+1}(k)$,

(d) for every $k < n$, there is an $m_k \in \omega$ such that $\sigma^k_{m_k} \subseteq \tau_n$ and for every $x \in E^k_{m_k}$ and every $\ell \in \mathrm{dom}(\tau_n \setminus \sigma^k_{m_k})$, $x(\ell) \in \tau_k(\ell)$, and

(e) $(\tau_n, E_{\{(k,m_k):k<n\}}) \in \mathbb{LOC}$.

If $\langle \tau_n : n \in \omega\rangle$ is such a sequence, then $\bigcup_{n\in\omega} \tau_n \in X_g \cap \bigcap_{n\in\omega} \bigcup\{[\sigma, E] : (\sigma, E) \in \mathcal{A}_n\}$ and so $X_g \nsubseteq A$. Thus, it is enough to show that such a sequence exists. We define the sequence recursively. Let $\tau_0 := \emptyset$. Assume that τ_n is already defined. Let $M := \{(k, m_k) : k \le n\}$. Since M does not cover g, there are infinitely many $k \in \omega$ such that for every $x \in E_M$, $x(k) \neq g(k)$. Let $\ell \ge \mathrm{lh}(\tau_n)$ be minimal with that property. Then there is some $\tau'_n \in \mathrm{dom}(\mathbb{LOC})$ such that $(\tau'_n, E_M) \le (\tau_n, E_M)$ and $g(\ell) \notin \tau'_n(\ell)$. Since $\mathcal{A}_n$ is a maximal antichain, there is some $m \in \omega$ such that (τ'_n, E_M) and (σ^n_m, E^n_m). Let $(\sigma, E) \le (\tau'_n, E_M), (\sigma^n_m, E^n_m)$ be a witness. We set $m_n := m$ and $\tau_{n+1} := \sigma$. $\square$

Lemma 2.5.9 is our Brendle-Łabędzki Lemma for localization forcing. We can use it to show that $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ implies that for every real $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Theorem 2.5.10. *The following are equivalent:*

(a) every $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ set has the Baire property in the localizing topology,

(b) for every real $r \in \omega^\omega$, the set $\{f \in \mathbf{Loc} : f$ is not a $\mathbb{LOC}$-generic real over $\mathrm{L}[r]\}$ is meager in the localizing topology,

(c) for every real $r \in \omega^\omega$, the set $\{f \in \mathbf{Loc} : f$ is not a localizing real over $\mathrm{L}[r]\}$ is meager in the localizing topology, and

(d) for every real $r \in \omega^\omega$, $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Proof. By Corollary 2.5.5 and Theorem 2.5.6, it remains to show that (b) implies (c) and that (c) implies (d). The former is trivial. Let $r \in \omega^\omega$, let N be the set of non-localizing reals over $\mathrm{L}[r]$, and let $\{g_\alpha : \alpha < \aleph_1^{\mathrm{L}[r]}\}$ be a pairwise eventually different family in $\mathrm{L}[r]$. Then for every $\alpha < \aleph_1^{\mathrm{L}[r]}$, X_{g_α} does not contain any localizing reals over $\mathrm{L}[r]$. Hence, for every $\alpha < \aleph_1^{\mathrm{L}[r]}$, $X_{g_\alpha} \subseteq N$. By Lemma 2.5.9, $\aleph_1^{\mathrm{L}[r]}$ has to be countable and so $\aleph_1^{\mathrm{L}[r]} < \aleph_1$. □

Moreover, we can use Lemma 2.5.9 to determine the additivity and cofinality numbers of the meager ideal in the localizing topology.

Proposition 2.5.11. *Let $\mathcal{M}_{\mathbb{LOC}}$ be the meager ideal in the localizing topology. Then* $\mathrm{add}(\mathcal{M}_{\mathbb{LOC}}) = \aleph_1$ *and* $\mathrm{cof}(\mathcal{M}_{\mathbb{LOC}}) = 2^{\aleph_0}$.

Proof. Follows directly from Lemma 2.5.9. □

2.6 Open questions

We conclude this chapter with a few open questions concerning (weak) category bases, amoeba regularity, and the regularity properties for amoeba forcing for category and localization forcing. For the sake of having all questions in one place, we start by restating Questions 2.2.14, 2.2.49, and 2.3.14.

Question 2.2.14. *Let $(X, \mathcal{C})$ be a weak category base. Is $I^*_{\mathcal{C}}$ is always a σ-ideal?*

Question 2.2.49. *Under the assumptions of Corollary 2.2.48, is* $\mathrm{cof}(I^*_{\mathcal{C}}) \geq \mathrm{cof}(I^*_{\mathcal{D}})$*?*

Question 2.3.14. *Can we show, without using Theorem 2.3.9, that if $\mathbf{\Delta}^1_2(\mathbb{A})$ holds, then for every $r \in \omega^\omega$, there is a covering real over $\mathrm{L}[r]$.*

Question 2.6.1.

(a) Let $\mathbb{P}$ be an arboreal forcing notion on 2 (or ω) and let $\mathcal{C}_{\mathbb{P}} := \{[T] : T \in \mathbb{P}\}$. Is $(2^\omega, \mathcal{C}_{\mathbb{P}})$ (or $(\omega^\omega, \mathcal{C}_{\mathbb{P}})$) always a category base?

(b) Let X be an uncountable Polish space and let I be a proper σ-ideal. Is $(X, \mathbb{P}_I)$ always a category base?

Question 2.6.2. *Does a proper weak category base satisfy the c.c.c. if and only if every pairwise disjoint family of regions is countable? (Cf. Proposition 2.2.5.)*

Question 2.6.3. *Is every proper weak category base equivalent to a category base? (Cf. Proposition 2.2.30.)*

Question 2.6.4.

(a) Is there a Brendle-Łabędzki Lemma for amoeba regularity?

(b) Is there a Brendle-Łabędzki Lemma for the Baire property in the $\mathbb{UM}$*-topology?*

Question 2.6.5. *Let* $I_{\mathbb{A}}$ *be the* σ*-ideal of all* $\mathcal{C}_{\mathbb{A}}$*-meager sets and let* $\mathcal{M}_{\mathbb{UM}}$ *be the meager ideal in the* $\mathbb{UM}$*-topology. Are* $\mathrm{cof}(I_{\mathbb{A}}) = 2^{\aleph_0}$ *and* $\mathrm{cof}(\mathcal{M}_{\mathbb{UM}}) = 2^{\aleph_0}$*? (Cf. Question 2.2.49.)*

Question 2.6.6.

(a) Does $\Gamma(\mathbb{A})$ *imply* $\Gamma(\mathbb{B})$ *for every projective pointclass* Γ*?*

(b) Are $\Gamma(\mathbb{A})$*,* $\Gamma(\mathbb{A}_\infty)$*, and* $\Gamma(\mathbb{LOC})$ *equivalent for every projective pointclass* Γ*?*

(c) Are $\Gamma(\mathbb{D})$ *and* $\Gamma(\mathbb{UM})$ *equivalent for every projective pointclass* Γ*?*

Chapter 3

Descriptive choice principles

Remarks on co-authorship. *The results of this chapter were obtained in collaboration between Ned Wontner and the author. Therefore, all results in this chapter are, unless otherwise stated, joint work with Ned Wontner. Both authors contributed equally.*

The strength of the axiom of choice comes from its global nature. However, this universality is usually not needed when working with concrete mathematical objects such as the reals. In fact, most of basic descriptive set theory does not require the full axiom of choice, but a weaker fragment such as $\mathsf{AC}(\omega^\omega)$, $\mathsf{DC}(\omega^\omega)$, or $\mathsf{AC}_\omega(\omega^\omega)$. As the axiom of choice, $\mathsf{AC}(\omega^\omega)$, $\mathsf{DC}(\omega^\omega)$, and $\mathsf{AC}_\omega(\omega^\omega)$ can be stratified into smaller fragments. Descriptive choice principles are such fragments defined using descriptive pointclasses. The first descriptive choice principles were introduced by Kanovei. In [Kan79], he studied descriptive choice principles for pointclasses in the projective hierarchy and proved a first separation theorem.

In this chapter, we shall compare descriptive choice principles with each other. More precisely, in Section 3.1, we shall give a brief overview on what happens to basic descriptive set theory on the reals if we drop the axiom of choice. In Section 3.2, we shall introduce three families of descriptive choice principles. Moreover, we shall prove a separation theorem which improves Kanovei's result. For this, we shall use special forcing notions which we call slicing. In Section 3.3, we shall introduce a framework to construct forcing notions which have a special absoluteness property for products. This framework generalizes a result from Kanovei and Lyubetsky [KL20]. In Section 3.4, we shall define generalized versions of Jensen forcing and show that these generalizations have similar properties as ordinary Jensen forcing. Finally in Section 3.4.3, we shall use the generalized versions of Jensen forcing and the framework from Section 3.3 to prove that slicing forcing notions exist in L.

3.1 Descriptive set theory without the axiom of choice

Most of basic descriptive set theory on the reals does not require the full axiom of choice and can be carried out in $\mathsf{ZF}+\mathsf{DC}(\omega^\omega)$ or even in $\mathsf{ZF}+\mathsf{AC}_\omega(\omega^\omega)$. However, if we drop $\mathsf{AC}_\omega(\omega^\omega)$, strange things can happen: Feferman and Lévy used a symmetric submodel construction to produce a model of ZF in which the reals are a countable union of countable sets (cf. [FL63]). In this model, every set of reals is a countable union of $\boldsymbol{\Sigma}^0_2$ sets. Hence, every set of reals is Borel. The length of the Borel hierarchy is the least ordinal ξ such that every Borel set is $\boldsymbol{\Sigma}^0_\xi$. It is well-known that in

$\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ the length of the Borel hierarchy is always ω_1. However, this is not necessarily true in ZF; in the Feferman-Lévy model, the length of the Borel hierarchy is finite. One might expect it to be 2, since every set is a countable union of $\mathbf{\Sigma}^0_2$ sets. However, Miller showed in [Mil08] that even in ZF the lowest possible length is 4. This means that $\mathbf{\Sigma}^0_2$ is not closed under countable unions in the Feferman-Lévy model. The step of the standard $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ proof that falls apart is where we have to choose representations of infinitely many $\mathbf{\Sigma}^0_2$ sets as countable unions of closed sets at once. This step may fail in ZF, since it is not guaranteed that there is a choice function that will do the job for us. However, this does not post a problem when we are working with finitely many sets. Hence, some closure properties of the Borel hierarchy can be preserved in ZF.

Proposition 3.1.1 (ZF). *Let $\xi \geq 1$ be an ordinal. Then the classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$ are closed under finite unions and intersections and continuous preimages. Moreover, $\mathbf{\Delta}^0_\xi$ is closed under complements.*

Proof. Cf., e.g., [Kec95, Proposition 22.1] and check that the proof does not use any non-trivial choice. □

Miller also showed that the length of the Borel hierarchy is not bounded by ω_1 in ZF. He produced a model of ZF in which the length is ω_2. This is possible because the standard $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ actually shows that the length of the Borel hierarchy is bounded by the least uncountable regular cardinal which happens to be ω_1 in $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$. In Miller's model ω_1 is singular and ω_2 is regular and so the length is not bounded by ω_1. Assuming large cardinals, Miller was even able to produce models of ZF with arbitrarily long Borel hierarchies. For this purpose, he used a model of Gitik [Git80] in which every cardinal is singular.

A key property of the Borel sets in $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ is that every Borel set can be coded as a real (cf. Section 1.2.9). However, this not necessarily true in ZF. Recall that in the Feferman-Lévy model every set is Borel. So if every Borel set were codable, then the decoding function would be a surjection from the reals into their power set. But such a function cannot exist. Hence, it is possible in ZF that the codable Borel sets are a proper subset of the Borel sets. The best we can show is that every $\mathbf{\Sigma}^0_2$ set is codable Borel. Note that this means that the codable Borel sets are not closed under countable unions in the Feferman-Lévy model. Nevertheless, it is provable in ZF that the codable Borel sets form an algebra.

Proposition 3.1.2 (ZF). *The codable Borel sets form an algebra containing every $\mathbf{\Sigma}^0_2$ set.*

Proof. Cf., e.g., [Fre08, Proposition 562D]. □

The projective hierarchy was introduced to close up the Borel sets under continuous images. This is not necessary in the Feferman-Lévy model since the Borel sets are already everything. Note, however, that we did not define the analytic sets as continuous images of Borel sets, but as continuous images of the Baire space. This makes no difference in $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$ (cf. Proposition 1.2.12). In ZF, however, it does. Since every continuous function $f : \omega^\omega \to \omega^\omega$ is uniquely determined by its behavior on a countable dense set, every such function can be coded as a real. Hence, there is a surjection from the reals to the analytic sets of reals. Therefore, there are sets of reals which are not analytic. This means it is not provable in ZF that every Borel set is analytic.

A reason that so many classical theorem about Borel sets fail in ZF is that many of these theorems are actually about codable Borel sets. This suggests that one should focus on codable Borel sets when trying to obtain versions of these classical theorems in ZF. Indeed, it is provable in ZF that a set of reals is codable Borel if and only if it is analytic and co-analytic

Proposition 3.1.3 (ZF)**.** *A set is codable Borel if and only if it is analytic and co-analytic; i.e.,* $\mathcal{B}^* = \mathbf{\Delta}^1_1$.

Proof. Cf., e.g., [Fre08, Theorem 562F]. $\square$

Moreover, Proposition 1.2.12 holds in ZF if we replace Borel with codable Borel.

Proposition 3.1.4 (ZF)**.** *The following are equivalent:*

(a) A is analytic,

(b) there is a codable Borel set $B \subseteq \omega^\omega$ and a continuous function $f : \omega^\omega \to \omega^\omega$ such that $f[B] = A$,

(c) there is a closed set $F \subseteq (\omega^\omega)^2$ such that $\mathrm{proj}_{\omega^\omega}(F) = A$, and

(d) there is a codable Borel set B in $(\omega^\omega)^2$ such that $\mathrm{proj}_{\omega^\omega}(B) = A$.

Proof. Cf., e.g., [Jec03, Lemma 11.6] and replace Borel with codable Borel. $\square$

Note that the above does not justify our definition of analytic sets being the right one in the absence of $\mathsf{AC}_\omega(\omega^\omega)$. However, the author is not aware of a model of ZF in which both version of analytic sets do not coincide and the Borel sets are not closed under continuous images.

Question 3.1.5. *Is there a model of* ZF *in which* $\mathcal{B}^* \subsetneq \mathcal{B} \subsetneq \mathcal{P}(\omega^\omega)$*?*

We have already seen that there is a surjection from the reals to the analytic sets in ZF. By induction, this surjection can be extended to any $\mathbf{\Sigma}^1_n$. Hence, for every $n \in \omega$, $\mathbf{\Sigma}^1_n \neq \mathcal{P}(\omega^\omega)$. Moreover, it is provable in ZF that the projective hierarchy does not collapse at a finite stage even in ZF, i.e., for every $n \geq 1$, we have $\mathbf{\Sigma}^1_n \neq \mathbf{\Pi}^1_n$.

Proposition 3.1.6 (ZF)**.** *For every $n \geq 1$, we have $\mathbf{\Sigma}^1_n \neq \mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n \subsetneq \mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subsetneq \mathbf{\Delta}^1_{n+1}$.*

Proof. Cf. [Kec95, Theorem 37.7] and check that the proof does not use any non-trivial choice. $\square$

The reason why the standard construction of universal sets works for the projective hierarchy but not for the Borel hierarchy is that $\mathbf{\Sigma}^1_{n+1}$ sets are defined from a single $\mathbf{\Pi}^1_n$ set. So the construction still works in ZF since we are always dealing with one object at a time. The $\mathbf{\Sigma}^0_\xi$ sets, on the other hand, are defined from countably many sets of the previous stages. Therefore, the standard construction of a universal $\mathbf{\Sigma}^0_\xi$ set requires some amount of $\mathsf{AC}_\omega(\omega^\omega)$ to check that the constructed set is universal. We run into the same problems when we are working with infinitely many projective sets at the same time. For example, it is not provable in ZF that the analytic sets are closed under countable unions since otherwise every Borel set would always be analytic. This also holds for any other projective pointclass. But just as for the Borel hierarchy closure properties for finite operations can be preserved.

Proposition 3.1.7 (ZF)**.** *For every $n \geq 1$, the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ are closed under finite unions and intersections and preimages by continuous functions. Moreover, $\mathbf{\Sigma}^1_n$ is closed under images by continuous functions and $\mathbf{\Delta}^1_n$ under complements.*

Proof. Cf. [Kec95, Proposition 37.1] and check that the proof does not use any non-trivial choice. $\square$

An important result of basic descriptive set theory is that the lightface and boldface hierarchies define the same sets (cf. Theorem 1.2.19). It is clear that even in ZF for every $n \in \omega$ and every $r \in \omega$, $\Sigma^0_n(r) \subseteq \mathbf{\Sigma}^0_n$. In fact, one can even show that for every $n \in \omega$ and every $r \in \omega$, $\Sigma^0_n(r) \subseteq \mathbf{\Sigma}^0_n \cap \mathcal{B}^*$, where $\mathcal{B}^*$ is the set of all codable Borel sets. Unfortunately, the other direction is not provable in ZF. By a coding argument, one can show that for every $n \in \omega$, there is a surjection from the reals to $\bigcup_{r\in\omega^\omega} \Sigma^0_n(r)$ and so $\bigcup_{r\in\omega^\omega} \Sigma^0_n(r) \neq \mathcal{P}(\omega^\omega)$. Therefore, it is not provable in ZF that for every $n \in \omega$, $\mathbf{\Sigma}^0_n \subseteq \bigcup_{r\in\omega^\omega} \Sigma^0_n(r)$. However, it is provable that $\mathbf{\Sigma}^0_1 = \bigcup_{r\in\omega^\omega} \Sigma^0_1(r)$ and $\mathbf{\Pi}^0_1 = \bigcup_{r\in\omega^\omega} \Pi^0_1(r)$. Then by Proposition 3.1.4, $\mathbf{\Sigma}^1_1 = \bigcup_{r\in\omega^\omega} \Sigma^1_1(r)$ and so for every $n \in \omega$, $\mathbf{\Sigma}^1_n = \bigcup_{r\in\omega^\omega} \Sigma^1_n(r)$ by induction. Therefore, Theorem 1.2.19 can be at least preserved for the projective sets.

We end this section with a list of things that do not necessarily hold in ZF, but are provable in ZFC. Note that most of these things fail in the Feferman-Lévy model.

Fact 3.1.8.

(a) In ZF, a countable union of countable sets is not necessarily countable. In particular, it is possible that the reals are a countable union of countable sets.

(b) In ZF, successor cardinals are not necessarily regular.

(c) In ZF, a Borel set is not necessarily a codable Borel set or a projective set. In particular, it is possible that every set is Borel, i.e., $\mathcal{P}(\omega^\omega) = \mathcal{B}$.

(d) In ZF, the length of the Borel hierarchy is not necessarily ω_1.

(e) Let $\xi \geq 2$. In ZF, $\mathbf{\Sigma}^0_\xi$ is not necessarily closed under countable unions.

(f) In ZF, the codable Borel sets are not necessarily closed under countable unions.

(g) Let $n \geq 1$. In ZF, the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ are not necessarily closed under countable unions and intersections.

3.2 Choice principles for descriptive pointclasses

3.2.1 Descriptive fragments of $\mathsf{AC}_\omega(\omega^\omega)$ and $\mathsf{DC}(\omega^\omega)$

Even $\mathsf{AC}_\omega(\omega^\omega)$ can be stratified into smaller fragments. An example of such a fragment is the axiom of countable choice for countable sets of reals which is sometimes called $\mathsf{CAC}_\omega(\omega^\omega)$ in the literature. It states that every countable family of non-empty countable sets of reals has a choice function. It is well-known that the axiom of countable choice for countable sets of reals is not provable in ZF (cf. [HR98, Form 5]). If we replace the word "countable" by any other collection of sets of reals, we get another fragment of $\mathsf{AC}_\omega(\omega^\omega)$.

Definition 3.2.1. Let $\Xi \subseteq \mathcal{P}(\omega^\omega)$ be a collection of subset of the reals. Then we write $\mathsf{AC}_\omega(\omega^\omega; \Xi)$ for the statement "every countable family of non-empty sets of reals in Ξ has a choice function".

Note that the axiom of countable choice for countable sets of reals is the principle $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$, where $\mathbf{ctbl}$ is, as defined in Section 1.2.12, the set of countable sets of reals. It is clear that if $\Xi' \subseteq \Xi \subseteq \mathcal{P}(\omega^\omega)$, then $\mathsf{AC}_\omega(\omega^\omega; \Xi)$ implies $\mathsf{AC}_\omega(\omega^\omega; \Xi')$. By Propositions 3.1.2 and 3.1.3, it is provable in ZF that $\mathbf{ctbl} \subseteq \mathbf{\Sigma}^0_2 \subseteq \mathbf{\Delta}^1_1$ and so

$$\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_1) \Longrightarrow \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_2) \Longrightarrow \mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl}).$$

Since $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$ is not provable in ZF, $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_2)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_1)$ are also not probable in ZF. Moreover, the Borel and projective hierarchies induce a hierarchy on the family of choice principles $\mathsf{AC}_\omega(\omega^\omega; \Gamma)$, where Γ is a Borel or projective pointclass. Figure 3.1 displays this hierarchy as an implication diagram.

$$\begin{tikzcd}
 & \mathsf{AC}_\omega(\omega^\omega) \arrow[d, Rightarrow] & & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{Proj}) \arrow[d, Rightarrow] & & \\
 & \vdots \arrow[d, Rightarrow] & & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_{n+1}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^1_n) \arrow[dr, Rightarrow] & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_n) \arrow[dl, Rightarrow] & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_n) \arrow[d, Rightarrow] & & \\
 & \vdots \arrow[d, Rightarrow] & & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_2) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^1_1) \arrow[dr, Rightarrow] & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_1) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_1) \arrow[ddddddd, Rightarrow, bend right] \arrow[dddddddrr, Rightarrow, bend right] & \mathsf{AC}_\omega(\omega^\omega; \mathcal{B}) \arrow[l, Rightarrow] \arrow[d, Rightarrow] & \\
 & & \vdots \arrow[d, Rightarrow] & \\
 & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^0_{\xi+1}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_\xi) \arrow[dr, Rightarrow] & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^0_\xi) \arrow[dl, Rightarrow] \\
 & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^0_\xi) \arrow[d, Rightarrow] & \\
 & & \vdots \arrow[d, Rightarrow] & \\
 & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^0_3) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_2) \arrow[d, Rightarrow] & & \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^0_2) \\
 & \mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl}) & &
\end{tikzcd}$$

Figure 3.1: Implication diagram for descriptive fragments of $\mathsf{AC}_\omega(\omega^\omega)$

In Figure 3.1, **Proj** is the sets of all projective sets of reals. The implication from $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Pi}^1_1)$ to $\mathsf{AC}_\omega(\omega^\omega;\mathcal{B})$ was shown by Ikegami and Schlicht in [IS22].[1] Any other implication follows the subset relation. Note that $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Sigma}^0_1)$, $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Pi}^0_1)$, and $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Delta}^0_2)$ are missing in Figure 3.1. The reason for this is that they are provable in ZF. To show this we use Hausdorff's Difference Lemma.

Theorem 3.2.2 (ZF, Hausdorff's Difference Lemma)**.** *Every $\mathbf{\Delta}^0_2$ set $A \subseteq \omega^\omega$ can be written as an α-difference of closed sets for some $\alpha < \omega_1$, i.e., there is a decreasing sequence $\langle C_\beta : \beta < \alpha\rangle$ of closed sets such that*

$$A = \mathrm{Diff}_{\beta<\alpha} C_\beta := \{x \in C_0 : \textit{the least } \beta < \alpha \textit{ such that } x \notin C_\beta \textit{ is odd}\}.$$

Proof. Cf., e.g., [And01, Theorem 7.16]. □

The sequence $\langle C_\beta : \beta < \alpha\rangle$ in Hausdorff's Difference Lemma is not unique. There are two main reasons for this: first, that $\langle C_\beta : \beta < \alpha\rangle$ can always be extended as a sequence without changing $\mathrm{Diff}_{\beta<\alpha} C_\beta$, and second, that for any closed set $F \subseteq \omega^\omega$, $\mathrm{Diff}_{\beta<\alpha} C_\beta = \mathrm{Diff}_{\beta<\alpha}(C_\beta \cup F)$. The first problem can be solved by considering only sequences of the least possible length. To fix the second problem, we require that for every $\beta < \alpha$, C_β is the closure of $\mathrm{Diff}_{\beta\leq\gamma<\alpha} C_\gamma$. These two additional requirements are enough to make the sequence in Hausdorff's Difference Lemma unique. Indeed, one can show that for every $\mathbf{\Delta}^0_2$ set $A \subseteq \omega^\omega$, there is a unique decreasing sequence $\langle C_\beta : \beta < \alpha\rangle$ of minimal length such $A = \mathrm{Diff}_{\beta<\alpha} C_\beta$ and for every $\beta < \alpha$, C_β is the closure of $\mathrm{Diff}_{\beta\leq\gamma<\alpha} C_\gamma$ (cf. [And01, 3.E.1]). We call this sequence the *canonical sequence for A*.

Proposition 3.2.3 (ZF)**.** *The descriptive choice principle $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Delta}^0_2)$ is provable in ZF. Moreover, every (not necessary countable) family of non-empty $\mathbf{\Delta}^0_2$ sets of reals has a choice function in ZF.*

Proof. It is enough to describe how we can pick a canonical element from every $\mathbf{\Delta}^0_2$ set. We fix an enumeration $\{s_n : n \in \omega\}$ of $\omega^{<\omega}$. Let $A \subseteq \omega^\omega$ be a non-empty $\mathbf{\Delta}^0_2$ set and let $\langle C_\beta : \alpha < \beta\rangle$ be the canonical sequence for A. Let $\beta < \alpha$ be minimal such that β is even and $C_{\beta+1} \subsetneq C_\beta$ and let $n \in \omega$ be minimal such that $[s_n] \cap C_\beta \neq \emptyset$ and $[s_n] \cap C_{\beta+1} = \emptyset$. Since $[s_n] \cap C_\beta$ is closed, there is a unique pruned tree $T \subseteq \omega^{<\omega}$ such that $[T] = [s_n] \cap C_\beta$. Then the left-most branch through T yields a real in A. □

Next, we study $\mathsf{DC}(\omega^\omega)$. Just as for $\mathsf{AC}_\omega(\omega^\omega)$, we can use collections Ξ of sets of reals to stratify $\mathsf{DC}(\omega^\omega)$ into smaller fragments. Here, we only consider total relations R which are in Ξ. By this we mean that the image of R under the canonical bijection between $\omega^\omega \times \omega^\omega$ and ω^ω is in Ξ. Note that if Ξ is a descriptive pointclass, then R is in Ξ if and only if R is in Ξ as a subset of $\omega^\omega \times \omega^\omega$.

Definition 3.2.4. Let $\Xi \subseteq \mathcal{P}(\omega^\omega)$ be a set of subsets of the reals. Then we write $\mathsf{DC}(\omega^\omega;\Xi)$ for the statement "for every total relation $R \subseteq \omega^\omega \times \omega^\omega$ in Ξ, there is a sequence $\langle x_k : k \in \omega\rangle \in X^\omega$ such that for every $k \in \omega$, $x_k\ R\ x_{k+1}$".

These choice principles were first introduced by Kanovei. In [Kan79], he studied them for pointclasses in the projective hierarchy. The following theorem lists a few of his results.

[1] Recall that in ZF, $\mathcal{B}$ is not necessarily a subset of $\mathbf{\Pi}^1_1$ (cf. Fact 3.1.8).

Theorem 3.2.5 (ZF, Kanovei)**.**

(a) The choice principle $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_1)$ *is provable in* ZF.

(b) For every $n \geq 1$, $\mathsf{DC}(\omega^\omega;\Pi^1_n)$ *and* $\mathsf{DC}(\omega^\omega;\Sigma^1_{n+1})$ *are equivalent.*

(c) For every $n \geq 1$, $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$ *and* $\mathsf{DC}(\omega^\omega;\mathbf{\Sigma}^1_{n+1})$ *are equivalent.*

(d) For every $n \geq 1$, $\mathsf{DC}(\omega^\omega;\Pi^1_{n+1})$ *implies* $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$.

Proof. Cf. [HR98, Note 61] and [Kan79, Theorems 2.4 & 2.6]. □

Note that by Theorem 3.2.5, $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_1)$ does not imply $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Pi}^1_1)$. The reason why the proof of "$\mathsf{DC}(\omega^\omega)$ implies $\mathsf{AC}_\omega(\omega^\omega)$" cannot be generalized is that we do not in general know the complexity of the relation. Recall that to show that $\mathsf{DC}(\omega^\omega)$ implies $\mathsf{AC}_\omega(\omega^\omega)$, we define for a countable family $\{A_k : k \in \omega\}$ of non-empty sets of reals the relation

$$R := (\omega^\omega \setminus \bigcup_{k\in\omega} A_k) \times A_0 \cup \bigcup_{k\in\omega} (A_k \times A_{k+1}) \subseteq \omega^\omega \times \omega^\omega$$

and then use $\mathsf{DC}(\omega^\omega)$ to obtain the choice function. If we now try to use this proof to show that $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_1)$ implies $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Pi}^1_1)$, then we have to show that R is $\mathbf{\Pi}^1_1$. Actually, by Theorem 3.2.5, it is enough to show that R is $\mathbf{\Sigma}^1_2$ which is clear in ZFC. In ZF, however, $\mathbf{\Pi}^1_1$ is not necessarily closed under countable unions (cf. Fact 3.1.8). Hence, we do not know the complexity of R. The same is true for $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Pi}^1_n)$ for every $n \in \omega$. In fact, we shall show later that even $\mathsf{DC}(\omega^\omega;\mathbf{Proj})$ does not imply $\mathsf{AC}_\omega(\omega^\omega;\mathbf{ctbl})$.

In [Kan79], Kanovei also introduced a family of fragments of $\mathsf{AC}_\omega(\omega^\omega)$. The difference between Kanovei's choice principles and the ones from Definition 3.2.1 is that he only considers families that are themselves in the pointclass.

Definition 3.2.6. Let Γ be a descriptive pointclass. A countable family $\{A_k : k \in \omega\}$ of non-empty sets of reals is *uniformly in* Γ if A_n is in Γ for every $k \in \omega$ and $\bigcup_{k\in\omega}\{k\} \times A_k$ is in Γ as well. We write $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Gamma)$ for the statement "every countable family of non-empty sets of reals which is uniformly in Γ has a choice function".

We call the choice principles from Definition 3.2.6 *uniform* and the ones from Definition 3.2.1 *non-uniform*. It is clear that for every descriptive pointclass Γ, the non-uniform choice principle $\mathsf{AC}_\omega(\omega^\omega;\Gamma)$ implies the uniform one $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Gamma)$ and the converse is true if Γ is closed under countable unions. The advantage of working with families which are uniformly in a descriptive pointclass Γ is that we do not have to worry whether Γ is closed under countable unions or not. This is enough to show that $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$ implies $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\mathbf{\Pi}^1_n)$.

Theorem 3.2.7 (ZF, Kanovei)**.** *For every* $n \geq 1$, $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$ *implies* $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\mathbf{\Pi}^1_n)$ *and similarly for* Π^1_n.

Proof. Let $\{A_k : k \in \omega\}$ be a countable family of non-empty sets which is uniform in Γ, let $X_k := \{k\} \times A_k$, and let $X := \bigcup_{k\in\omega} X_k$. We define

$$R := (\omega^\omega \setminus X) \times X_0 \cup \bigcup_{k\in\omega} X_k \times X_{k+1}.$$

Since X is in $\mathbf{\Pi}^1_n$, R is in $\mathbf{\Sigma}^1_{n+1}$. By Theorem 3.2.5, $\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n)$ and $\mathsf{DC}(\omega^\omega; \mathbf{\Sigma}^1_{n+1})$ are equivalent. Hence, there is a sequence $\langle x_k : k \in \omega\rangle$ of reals such that for every $k \in \omega$, $x_k \; R \; x_{k+1}$. Without loss of generality, $x_0 \in X_0$. For every $k \in \omega$, let $a_k \in \omega^\omega$ be such that for every $\ell \in \omega$, $a_k(\ell) := x_k(\ell+1)$. Then $f := \{(A_k, a_k) : k \in \omega\}$ is the desired choice function. □

Then by Theorems 3.2.5 and 3.2.7, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_1)$ is provable in ZF. Hence, $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_1)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_1)$ are not equivalent. We shall see later that the same is true for $\mathbf{\Pi}^1_n$ with $n \geq 1$. Next, we consider another result of Kanovei. He showed that (b) and (c) of Theorem 3.2.5 are also true for uniform choice principles.

Theorem 3.2.8 (ZF, Kanovei)**.**

(a) For every $n \geq 1$, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Sigma^1_{n+1})$ are equivalent.

(b) For every $n \geq 1$, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Sigma}^1_{n+1})$ are equivalent.

Proof. Cf. [Kan79, Theorem 2.4]. □

In the rest of this section, we investigate the relationships between uniform, non-uniform, and dependent descriptive choice principles. Figure 3.2 displays which relationships we have seen before as an implication diagram. It is not known whether the diagram is complete in the sense that if there is no arrow from a statement to another, then the implication does not hold in ZF. For example, it is not known whether $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_n)$ implies $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$. On the other hand, it is known that $\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n)$ does not imply $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$. Kanovei proved a separation theorem that allowed him to separate dependent choice principles from uniform choice principles of the next level.

Theorem 3.2.9 (Kanovei)**.** *Let $n \geq 1$. There is a model of* $\mathsf{ZF}+\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n)+\neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$.

Proof. Cf. [Kan79, Theorem 1.1]. □

Moreover, Friedman, Gitman, and Kanovei showed in [FGK19] that $\mathsf{AC}_\omega(\omega^\omega)$ does not imply $\mathsf{DC}(\omega^\omega; \Pi^1_2)$. Hence, in particular, for every projective pointclass Γ, $\mathsf{AC}_\omega(\omega^\omega; \Gamma)$ does not imply $\mathsf{DC}(\omega^\omega; \Gamma)$. One of our goals of Chapter 3 is to show that the converse is also not provable in ZF which means that dependent and non-uniform descriptive choice principles are independent from each other. To do this, we use our main theorem of Chapter 3 which generalizes Kanovei's separation theorem.

Theorem 3.2.10. *For every $n \geq 1$, there is a model of* $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.

We shall prove Theorem 3.3.15 in Section 3.2.3. Here, we use it to prove that $\mathsf{DC}(\omega^\omega; \mathbf{Proj})$ does not imply $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$. Note that this does not tell us anything about the relationship between $\mathsf{DC}(\omega^\omega; \mathbf{Proj})$ and $\mathsf{AC}_\omega(\omega^\omega; \Gamma)$ if Γ is a lightface descriptive pointclass.

Corollary 3.2.11. *There is a model of* $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.

Proof. Let Φ be the set of all ZF axioms together with the sentences $\{\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) : n \geq 1\}$ and the sentence $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$. By compactness, it is enough to show that Φ is finitely satisfiable. Let $\Phi_0 \subseteq \Phi$ be a finite subset. Then there are only finitely many $n \in \omega$ such that $\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) \in \Phi_0$. Let n be maximal with that property. By Theorem 3.2.10, there is a model M which satisfies Φ_0. Therefore, Φ_0 is satisfiable and so Φ is finitely satisfiable. □

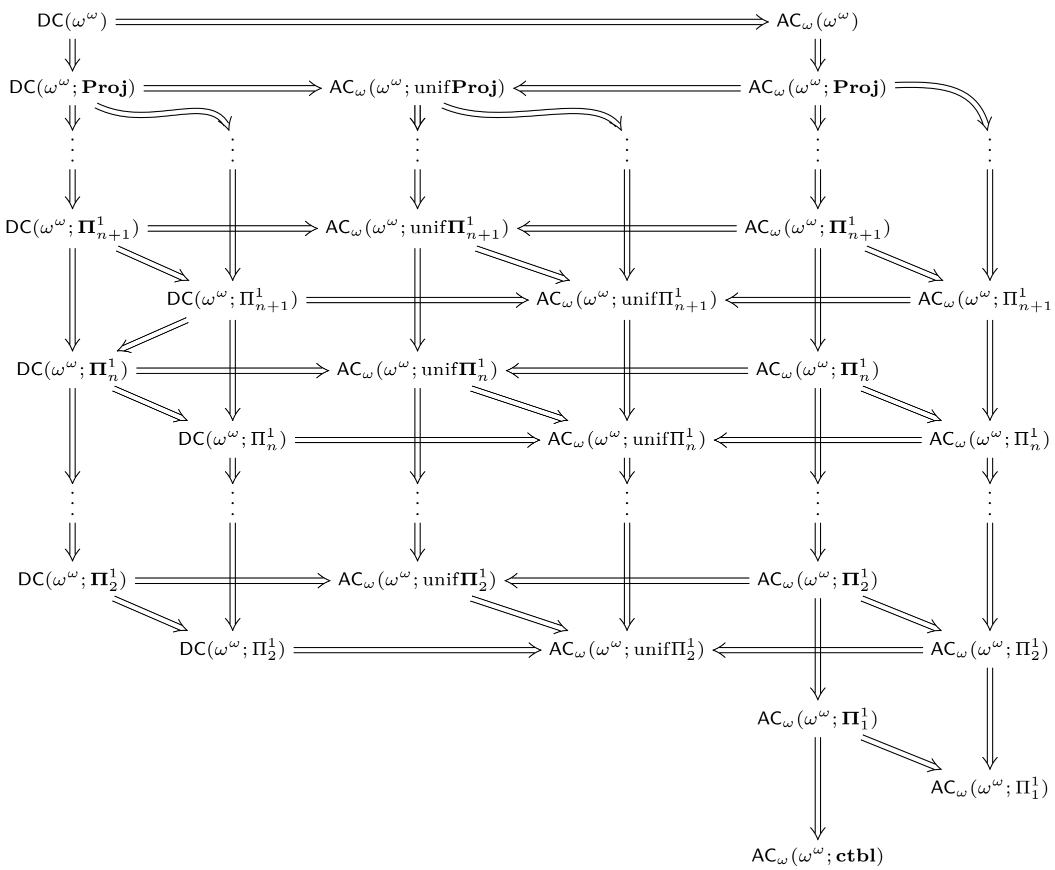

Figure 3.2: Implication diagram of the descriptive fragments of $\mathsf{DC}(\omega^\omega)$ and $\mathsf{AC}_\omega(\omega^\omega)$

3.2.2 Slicing forcing notions

In [Kan79], Kanovei used an inner model construction to prove his separation theorem (Theorem 3.2.5) However, we take a different approach to prove our separation theorem (Theorem 3.2.10). The general idea is to use a symmetric submodel construction. In this section, we introduce a special type of product forcing notions that we shall use for this construction.

Let $\xi \leq \omega_1$ be an ordinal and let $\langle \mathbb{P}_\nu : \nu < \xi \rangle$ be a sequence of forcing notions. The *ω-slice-product of $\langle \mathbb{P}_\nu : \nu < \xi \rangle$ with finite support* is the partial order $\mathbb{P}$ of all partial functions p with finite $\mathrm{dom}(p) \subseteq \xi \times \omega$ such that for every $(\nu, k) \in \xi \times \omega$, $p(\nu, k) \in \mathbb{P}_\nu$ ordered by

$$p \leq q :\iff \forall (\nu, k) \in \mathrm{dom}(q)(p(\nu, k) \leq q(\nu, k)).$$

The *length* of an ω-slice-product is the length of the sequence defining it. Let $\mathbb{Q}$ be a forcing notion. We define the *ω-slice-product of $\mathbb{Q}$ with finite support of length ξ* as the ω-slice-product of $\langle \mathbb{Q} : \nu < \xi \rangle$ with finite support. So an ω-slice is a normal product forcing notion but we add every forcing notion ω-many times.

The rough idea to make $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$ fail is to permute inside of each $\{\nu\} \times \omega$. More precisely, we write $\mathrm{Aut}(\omega_1 \times \omega, \omega)$ for the group of all bijections π of $\omega_1 \times \omega$ such that for every $\nu \in \omega_1$, $\pi[\{\nu\} \times \omega] = \{\nu\} \times \omega$. Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions. Then every $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ induces an automorphism π^* on $\mathbb{P}$: for every $p \in \mathbb{P}$, we set $\pi^*(p) := p'$ with $\mathrm{dom}(p') = \pi[\mathrm{dom}(p)]$ and for $(\nu, k) \in \mathrm{dom}(p')$, $p'(\nu, k) = p(\pi^{-1}(\nu, k))$. Let $\mathcal{G} := \{\pi^* : \pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)\}$ be the group of all such automorphisms on $\mathbb{P}$. A set $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ is a *set of slices* if for every $s \in \mathcal{S}$, s is of the form $X \times \omega$, where $X \subseteq \omega_1$. Each set of slices defines a normal filter. Let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices. For every $s \in \mathcal{S}$, let $H_s := \{\pi^* : \forall (\nu, k) \in s(\pi(\nu, k) = (\nu, k))\} \subseteq \mathcal{G}$ be the subgroup of all π^* such that π point-wise fixes s. Let $\mathcal{F}_\mathcal{S}$ be the filter on the subgroups of $\mathcal{G}$ generated by $\{H_s : s \in \mathcal{S}\}$.

Lemma 3.2.12. *Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions, and let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices. Then $\mathcal{F}_\mathcal{S}$ is normal.*

Proof. Let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ and let $K \in \mathcal{F}_\mathcal{S}$. Then there is an $s \in S$ such that $H_s \subseteq K$. It is enough to show that $H_s \subseteq \pi^* K (\pi^*)^{-1}$. Let $\tau \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ such that $\tau^* \in H_s$. Then for every $(\nu, k) \in s$, $\pi^{-1}(\tau(\pi(\nu, k))) = (\nu, k)$ and so $(\pi^*)^{-1} \circ \tau^* \circ \pi^* \in H_s \subseteq K$. Therefore,

$$\pi^* \circ ((\pi^*)^{-1} \circ \tau^* \circ \pi^*) \circ (\pi^*)^{-1} = \tau^* \in \pi^* K (\pi^*)^{-1}. \qquad \square$$

Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions. For every set of slices $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ and every $\mathbb{P}$-generic filter G over V, we denote the symmetric extension we obtain from G and $\mathcal{F}_\mathcal{S}$ by $\mathrm{V}(G, \mathcal{S})$. The following lemma will be helpful later on.

Lemma 3.2.13. *Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions, let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices, and let G be a $\mathbb{P}$-generic filter over* V*. Then*

(a) for every $s \in \mathcal{S}$, $\mathrm{V}[G{\restriction}s] \subseteq \mathrm{V}(G, \mathcal{S})$, and

(b) for every set of ordinals A, $A \in \mathrm{V}(G, \mathcal{S})$ if and only if there is an $s \in \mathcal{S}$ such that $A \in \mathrm{V}[G{\restriction}s]$.

Proof. We start with proving (a). Let $A \in \mathrm{V}[G{\restriction}s]$. Then there is a $\mathbb{P}{\restriction}s$-name $\dot{A}$ such that $\dot{A}_{G{\restriction}s} = A$. Note that $\dot{A}$ is also a $\mathbb{P}$-name and $\dot{A}_G = A$. Hence, we only have to check that every $\mathbb{P}{\restriction}s$-name is hereditarily symmetric. Let σ be a $\mathbb{P}{\restriction}s$-name, let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ such that $\pi^* \in H_s$, and let p be a condition somewhere occurring in σ. Since π point-wise fixes s, $\pi^*(p) = p$ and so σ is hereditarily symmetric.

Next, we prove (b). The backward direction follows directly from (a). So we only have to show the forward direction. Let A be a set of ordinals in $\mathrm{V}[G]$. Then there is a symmetric $\mathbb{P}$-name $\dot{A}$ such that $\dot{A}_G = A$. Let $s \in \mathcal{S}$ such that $H_s \subseteq \mathrm{sym}(\sigma)$ and let $p \in G$ such that $p \Vdash$ "X is a set of ordinals". We define $\sigma := \{(\check{\xi}, q{\restriction}s) : q \leq p \text{ and } q \Vdash \check{\xi} \in \dot{A}\}$. Then σ is a $\mathbb{P}{\restriction}s$-name. Let $A' := \sigma_{G{\restriction}s}$. We show that $A = A'$. Let $\xi \in A$. Then there is a $q \in G$ such that $q \Vdash \check{\xi} \in \dot{A}$. Without loss of generality, $q \leq p$. Hence, $(\check{\xi}, q{\restriction}s) \in \sigma$ and so $\xi \in A'$.

Now let $\xi \in A'$. Then there is a $q \leq p$ such that $q{\restriction}s \in G{\restriction}s$ and $q \Vdash \check{\xi} \in \dot{A}$. We suppose for a contradiction that $q{\restriction}s \nVdash \check{\xi} \in \dot{A}$. Then there is an $r \leq q{\restriction}s$ such that $r \Vdash \check{\xi} \notin \dot{A}$. Let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ such that $\pi^* \in H_s$ and $\mathrm{dom}(q) \cap \pi[\mathrm{dom}(r)] \subseteq s$. Since $q{\restriction}\mathrm{dom}(q{\restriction}s) = r{\restriction}\mathrm{dom}(q{\restriction}s)$, q and $\pi^*(r)$ are compatible. Moreover, $\pi^*(r) \Vdash \pi^*(\check{\xi}) \notin \pi^*(\dot{A}) = \check{\xi} \notin \dot{A}$. But this is not possible since q and $\pi^*(r)$ are compatible. Therefore, $q{\restriction}s \Vdash \check{\xi} \in \dot{A}$ and so $\xi \in A$. $\square$

By Lemma 3.2.13, every real $x \in \omega^\omega$ is in $\mathrm{V}(G, \mathcal{S})$ if and only if there is an $s \in \mathcal{S}$ such that $x \in \mathrm{V}[G{\upharpoonright}s]$. Note that choice functions for countable families of reals are essentially countable sequences of reals. Hence, choice functions for those families can be coded as reals. Therefore, a countable family $\mathscr{F}$ of non-empty sets of reals has a choice function in $\mathrm{V}(G, \mathcal{S})$ if and only if there is some $s \in \mathcal{S}$ such that $\mathrm{V}[G{\upharpoonright}s]$ contains a choice function for $\mathscr{F}$. We shall exploit this fact later in the proof of Lemma 3.2.19 to make some descriptive choice principles fail in $\mathrm{V}(G, \mathcal{S})$. In contrast, the next concept will be used to preserve other descriptive choice principles.

Definition 3.2.14. Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions and let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices. We say that $\mathbb{P}$ is *n-absolute for $\mathcal{S}$-slices* if for every $\mathbb{P}$-generic filter G over V and every $s \in \mathcal{S}$, every Σ^1_n formula with parameters in $\mathrm{V}[G{\upharpoonright}s]$ is absolute between $\mathrm{V}[G{\upharpoonright}s]$ and $\mathrm{V}[G]$. A set of slices $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ is *unbounded* if for every $s \in \mathcal{S}$, $\{\nu : \{\nu\} \times \omega \subseteq s\}$ is unbounded in ω_1. We say that $\mathbb{P}$ is *n-absolute for slices* if for every unbounded set of slices $\mathcal{S}$, $\mathbb{P}$ is n-absolute for $\mathcal{S}$-slices.

Note that any ω-slice-product with finite support of length ω_1 is 2-absolute for slices, by Shoenfield absoluteness. But we can easily produce counterexamples for $n = 3$: in L, let $\mathbb{P}_0 := \mathbb{B}$, for every $0 < \nu < \omega_1$, let $\mathbb{P}_\nu := \mathbb{C}$, let $\mathbb{P}$ be the ω-slice-product of $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ with finite support, and let G be $\mathbb{P}$-generic over L. Then $\mathrm{L}[G]$ contains a random real over L and $\mathrm{L}[G{\upharpoonright}(\omega_1 \setminus 1) \times \omega]$ does not. Moreover, the statement "there is a random real over L" is Σ^1_3. Therefore, $\mathbb{P}$ is not 3-absolute for slices.

Lemma 3.2.15. *Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions, let G be a $\mathbb{P}$-generic filter, let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be an unbounded set of slices, and let $n \geq 1$. If $\mathbb{P}$ is n-absolute for slices, then for every $s \in \mathcal{S}$, every Σ^1_n-formula with parameters in $\mathrm{V}[G{\upharpoonright}s]$ is absolute between $\mathrm{V}[G{\upharpoonright}s]$, $\mathrm{V}[G]$, and $\mathrm{V}(G, \mathcal{S})$. So in particular, every Σ^1_n-formula with parameters in $\mathrm{V}(G, \mathcal{S})$ is absolute between $\mathrm{V}[G]$ and $\mathrm{V}(G, \mathcal{S})$.*

Proof. The second part follows from the first and Lemma 3.2.13. So we only have to show the first part. Let $s \in \mathcal{S}$ and let φ be a Σ^1_1 formula with parameter in $\mathrm{V}[G{\upharpoonright}s]$. Then there is an arithmetical formula ψ such that $\varphi = \exists x \psi(x)$. Clearly, for every real x in $\mathrm{V}[G{\upharpoonright}s]$, $\psi(x)$ is absolute between $\mathrm{V}[G{\upharpoonright}s]$, $\mathrm{V}[G]$, and $\mathrm{V}(G, \mathcal{S})$. So by upwards-absoluteness, if $\mathrm{V}(G, \mathcal{S}) \models \varphi$, then $\mathrm{V}[G] \models \varphi$ and by downwards-absoluteness, $\mathrm{V}(G, \mathcal{S}) \models \neg\varphi$, then $\mathrm{V}[G{\upharpoonright}s] \models \neg\varphi$. Since $\mathbb{P}$ is n-absolute for slices, φ is absolute between $\mathrm{V}[G{\upharpoonright}s]$ and $\mathrm{V}[G]$. Therefore, φ is absolute between $\mathrm{V}[G{\upharpoonright}s]$, $\mathrm{V}[G]$, and $\mathrm{V}(G, \mathcal{S})$. To prove the lemma, we can now repeat the argument inductively for all Σ^1_{k+1} formulas with $0 < k < n$. $\square$

The rough idea to preserve uniform descriptive choice principles is the following: let $n \geq 2$, let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ of forcing notions such that $\mathbb{P}$ is $(n+1)$-absolute for slices, let G be a $\mathbb{P}$-generic filter, and let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be an unbounded set of slices. Then $\mathrm{V}[G]$ is a model of ZFC and so in particular $\mathrm{V}[G] \models \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\boldsymbol{\Pi}^1_n)$. Note that $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\boldsymbol{\Pi}^1_n)$ is essentially a Π^1_{n+2} statement. So by Lemma 3.2.15 and downwards-absoluteness, $\mathrm{V}(G, \mathcal{S}) \models \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\boldsymbol{\Pi}^1_n)$. We shall give more details in the proof of Lemma 3.2.20. There, we shall show that even $\mathsf{DC}(\omega^\omega; \mathrm{unif}\boldsymbol{\Pi}^1_n)$ is true in $\mathrm{V}(G, \mathcal{S})$. One last ingredient is missing to ensure that $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$ fails in $\mathrm{V}(G, \mathcal{S})$.

Definition 3.2.16. A forcing notion $\mathbb{P}$ is *n-slicing* if there is a sequence of non-atomic forcing notions $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ such that

(a) $\mathbb{P}$ is the ω-slice-product of $\langle \mathbb{P}_\nu : \nu < \omega_1 \rangle$ with finite support,

(b) for every $\nu < \omega_1$, every $\mathbb{P}_\nu$-generic filter G is uniquely determined by a real in $\mathrm{V}[G] \setminus \mathrm{V}$,

(c) $\mathbb{P}$ is n-absolute for slices, and

(d) for every $\mathbb{P}$-generic filter G, the set $\{(\ell, x_G^{(\ell,k)}) : (\ell, k) \in \omega^2\}$ is Π^1_n in $\mathrm{V}[G]$, where $x_G^{(\ell,k)}$ is the generic real defined by $G{\restriction}\{(\ell, k)\}$.

In Section 3.2.3, we shall use n-slicing forcing notions to prove our separation theorem (Theorem 3.2.10). However, it is not clear whether n-slicing forcing notions exist. The following theorem, which we call the Slicing Theorem, guarantees that they at least exist in L.

Theorem 3.2.17 (Slicing Theorem)**.** *Let $n \geq 2$. In* L*, there is an n-slicing forcing notion.*

Before we prove the Slicing Theorem, we use it to show our separation theorem. Afterwards, we spend the rest of Chapter 3 proving the Slicing Theorem.

3.2.3 Separating non-uniform from uniform and dependent choice

In this section, we prove our separation theorem (Theorem 3.2.10) under the assumption that we have already established the Slicing Theorem (Theorem 3.2.17). In particular, we use n-slicing forcing notions to construct models of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$. Throughout this section, we work in L.

We fix a natural number $n \geq 1$. By the Slicing Theorem, there is some $(n+1)$-slicing forcing notion $\mathbb{P}$. Let G be a $\mathbb{P}$-generic filter over L, let $\mathcal{S} := \{(F \cup (\omega_1 \backslash \omega)) \times \omega : F \subseteq \omega \text{ is finite}\}$, and let $N := \mathrm{L}(G, Z)$. We show that N has the desired properties. For every $\ell \in \omega$, let $A_\ell := \{x_G^{(\ell,k)} : k \in \omega\}$, where $x_G^{(\ell,K)}$ is the generic real defined by $G{\restriction}\{(\ell, k)\}$. Then in $\mathrm{L}[G]$, $\mathscr{F} := \{A_\ell : \ell \in \omega\}$ is a countable family of non-empty countable sets which is uniformly in Π^1_{n+1}. We show that the same is true in N, that $\mathscr{F}$ has no choice function in N, and that $\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n)$ is true in N. To do this, we prove the following three lemmas.

Lemma 3.2.18. *In N, $\mathscr{F}$ is uniformly in Π^1_{n+1}.*

Proof. Let $A := \bigcup_{\ell \in \omega} \{\ell\} \times A_\ell$. Since $\mathbb{P}$ is $(n+1)$-slicing, A is Π^1_{n+1} in $\mathrm{L}[G]$. Let φ be a Π^1_{n+1} formula defining A in $\mathrm{L}[G]$. By Lemma 3.2.15, $\varphi(x)$ is absolute between $\mathrm{L}[G]$ and N for every $x \in N$ and by Lemma 3.2.13, $x_G^{(\ell,k)} \in N$ for every $k, \ell \in \omega$. Hence, φ defines A in N as well and so $\mathscr{F}$ is in N. Since A is Π^1_{n+1} in N, $\mathscr{F}$ is uniformly in Π^1_{n+1} in N. □

Lemma 3.2.19. *In N, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$ and $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$ do not hold.*

Proof. Clearly, $\mathscr{F}$ is a countable family of non-empty countable sets in N and by Lemma 3.2.18, $\mathscr{F}$ is uniformly in Π^1_{n+1} in N. Hence, it is enough to show that $\mathscr{F}$ has no choice function in N. We suppose for a contradiction that $\mathscr{F}$ has a choice function f in N. Without loss of generality, we can assume that $\mathrm{dom}(f) = \omega$. Since f is a countable sequence of reals, we can think of f as a real. By Lemma 3.2.13, there is some $s \in \mathcal{S}$ such that $f \in \mathrm{L}[G{\restriction}s]$. Let $\dot{f}$ be a $\mathbb{P}{\restriction}s$-name for f, let $\ell \in \omega$ such $s \cap \ell \times \omega = \emptyset$, and let $k \in \omega$ such that $f(\ell) = x_G^{(\ell,k)}$. Then there is a $p \in G$ such that $p \Vdash$ "f is a choice function for A and $\dot{f}(\check{\ell}) = \dot{x}_G^{(\ell,k)}$", where $\dot{x}_G^{(\ell,k)}$ is the canonical $\mathbb{P}$-name for $x_G^{(\ell,k)}$. Let $k' \in \omega$ such that $k \neq k'$ and $(\ell, k') \notin \mathrm{dom}(p)$, let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ which only swaps (ℓ, k)

and (ℓ, k'), and let $\dot{x}_G^{(\ell,k')}$ be the canonical $\mathbb{P}$-name for $x_G^{(\ell,k')}$. Then $\pi^* \in H_s \subseteq \mathrm{sym}(\dot{f})$ and p and $\pi^*(p)$ are compatible. Moreover, $\pi^*(\dot{x}_G^{(\ell,k)}) = \dot{x}_G^{(\ell,k')}$ and so $\pi^*(p) \Vdash \dot{f}(\check{\ell}) = \dot{x}_G^{(\ell,k')}$. But this is a contradiction. $\square$

Lemma 3.2.20. *In N, $\mathsf{DC}(\mathbb{R}; \mathbf{\Pi}^1_n)$ holds.*

Proof. Let $R \subseteq \omega^\omega \times \omega^\omega$ be a total relation which is $\mathbf{\Pi}^1_n$ in N. Then there is a Π^1_n formula with parameter $r \in N$ which defines R in N. By Lemma 3.2.13, there is some $s \in \mathcal{S}$ such that $r \in \mathrm{L}[G\restriction s]$. Let R' be the set defined by $\varphi(r)$ in $\mathrm{L}[G\restriction s]$ and let $\psi(r)$ be the formula $\forall x \exists y \varphi((x, y), r)$. Then ψ is a Π^1_{n+2} formula which is true in N. Since $\mathbb{P}$ is $(n + 1)$-slicing, Σ^1_{n+1} formulas with parameter in $\mathrm{L}[G\restriction s]$ are absolute between $\mathrm{L}[G\restriction s]$ and N. Hence, by downwards-absoluteness, $\psi(r)$ is true in $\mathrm{L}[G\restriction s]$ and so R' is a total relation in $\mathrm{L}[G\restriction s]$. Since $\mathrm{L}[G\restriction s] \models \mathsf{ZFC}$, there is a sequence $\langle x_k : k \in \omega\rangle$ of reals in $\mathrm{L}[G\restriction s]$ such that $x_k\ R'\ x_{k+1}$ for every $k \in \omega$. By Lemma 3.2.13, $\langle x_k : k \in \omega\rangle$ is in N. Moreover, $\varphi(r)$ is absolute between $\mathrm{L}[G\restriction s]$ and N and so $x_k\ R\ x_{k+1}$ for every $k \in \omega$. Therefore, $\langle x_k : k \in \omega\rangle$ is the sequence we are looking for. $\square$

This completes the proof of our separation theorem (Theorem 3.2.10). Note that the construction of N is similar to the standard construction of a symmetric submodel in which $\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$ fails. The main difference is that we have used an $(\omega_1 \times \omega)$-product instead of an ω^2-product. This extra space will be used in the proof of the Slicing Theorem (Theorem 3.2.17) to ensure that the constructed forcing notion is n-absolute for slices. However, the construction of N would also work for an ω-slice product of length ω if it satisfies an appropriate absoluteness property.

3.3 The Kanovei-Lyubetsky framework

3.3.1 Approximating the forcing relation

In [KL20], Kanovei and Lyubetsky constructed an ω_1-product of variants of almost disjoint forcing with finite support such that for every generic filter G and every unbounded set $e \subseteq \omega_1$, every Σ^1_n formula with parameters in $\mathrm{L}[G\restriction e]$ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G\restriction e]$. The goal of Section 3.3 is to generalize their construction to provide a framework for constructing forcing notions which are n-absolute for slices. The main result is the Kanovei-Lyubetsky Lemma (Lemma 3.3.18), which we shall use in Section 3.4 to prove the Slicing Theorem (Theorem 3.2.17). It will be used to ensure that the forcing notion we construct is n-absolute for slices. In this section, we introduce the framework and prove some basic properties of it. Afterwards, in Section 3.3.2, we prove the Kanovei-Lyubetsky Lemma. Throughout the process we work in L. It should be noted that most proofs are modifications of the results by Kanovei and Lyubetsky.

Definition 3.3.1. A partial order $(\mathcal{M}, \preccurlyeq)$ is a *storage order* if the following conditions are met:

(a) Every element $m \in \mathcal{M}$ is of the form $m = (M, P)$, where $M = \mathrm{L}_\gamma$ for some countable ordinal $\gamma > \omega$ such that $\mathrm{L}_\gamma \models \mathsf{ZFC}^-$ and P is an ω-slice-product with finite support of length $< \omega_1$ which is in M.

(b) The partial order $(\mathcal{M}, \preccurlyeq)$ is Δ^{HC}_1.

(c) If $(M, P) \preccurlyeq (N, Q)$, then $M \subseteq N$, $P \subseteq Q$, incompatible conditions in P remain incompatible in Q, and every predense set $D \subseteq P$ in M remains predense in Q.

A pair $(M, P) \in \mathcal{M}$ is *strictly $\preccurlyeq$-less* than another pair $(N, Q) \in \mathcal{M}$ if $M \subsetneq N$, the length of P is strictly less than the length of Q, and for every $(\nu, k) \in \operatorname{dom}(P)$, $P\restriction\{(\nu, k)\} \subsetneq Q\restriction\{(\nu, k)\}$. We say that a $\preccurlyeq$-increasing sequence $\langle(M_\xi, P_\xi) : \xi < \zeta\rangle$ with $\zeta \leq \omega_1$ is *continuous at limits* if for every limit ordinal $\xi < \zeta$, $P_\xi = \bigcup_{\xi' < \xi} P_{\xi'}$.

(d) For every $(M, P) \in \mathcal{M}$, there is some $(N, Q) \in \mathcal{M}$ such that (M, P) is strictly $\preccurlyeq$-less (N, Q).

(e) Let $\zeta \leq \omega_1$ be a limit ordinal, let $\langle(M_\xi, P_\xi) : \xi < \zeta\rangle$ be a strictly $\preccurlyeq$-increasing sequence from $\mathcal{M}$ which is continuous at limits, and let $P := \bigcup_{\xi<\zeta} P_\xi$. Then:

   (i) If $\zeta < \omega_1$, then there is an M such that $(M, P) \in \mathcal{M}$ and $(M_\xi, P_\xi) \preccurlyeq (M, P)$ for every $\xi < \zeta$.

   (ii) If $\zeta = \omega_1$, then P satisfies the c.c.c. and for every $\xi < \omega_1$, incompatible conditions in P_ξ remain incompatible in P and every predense set $D \subseteq P_\xi$ in M_ξ remains predense in P.

The goal of Section 3.3 is to use storage orders to construct ω-slice-products that are n-absolute for slices. The plan is to build ω-slice-products from below using a strictly $\preccurlyeq$-increasing sequence. More precisely, let $(\mathcal{M}, \preccurlyeq)$ be a storage order. We call a strictly $\preccurlyeq$-increasing sequence $\langle(M_\xi, P_\xi) : \xi < \omega_1\rangle$ which is continuous at limits a *storage sequence*. Then for every storage sequence $\langle(M_\xi, P_\xi) : \xi < \omega_1\rangle$, $\mathbb{P} := \bigcup_{\xi<\omega_1} P_\xi$ is an ω-slice-product of length ω_1. The hard part is to show that $\mathbb{P}$ is n-absolute for slices. This is where the Kanovei-Lyubetsky Lemma comes in. Roughly speaking, it will provide us with sufficient requirements, which we shall specify later, for $\mathbb{P}$ to be n-absolute for slices. The key to proving the Kanovei-Lyubetsky Lemma is to use the way $\mathbb{P}$ was constructed to approximate the ordinary forcing relation for $\mathbb{P}$ with a forcing-like relation for the language of second-order arithmetic, which has more control over the countable fragments of the product. We start with defining a forcing language for second-order arithmetic. The rough idea is to only add constant symbols for names for reals. Let $\mathbb{P}$ be a forcing notion. We say that a $\mathbb{P}$-name σ is a $\mathbb{P}$-*name for a real* if $1_{\mathbb{P}} \Vdash$ "σ is a real".

Definition 3.3.2. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order. For $(M, P) \in \mathcal{M}$, let $\mathcal{FA}^2(M, P)$ be the language of second-order arithmetic augmented with a new constant symbol c_n for every $n \in \omega$, a new constant symbol c_σ for every P-name σ for a real in M, and new second-order quantifiers $\forall^B$ and $\exists^B$ for every countable set $B \subseteq \omega_1$ in M. Let G be a P-generic filter over M and let φ be a formula in $\mathcal{FA}^2(M, P)$. The *valuation of φ by G*, denoted φ_G, is the formula produced by replacing every c_n in φ by n, every c_σ in φ by σ_G, and every Q^B in φ by $Qx \in \omega^\omega \cap M[G\restriction B \times \omega]$, where Q is a second-order quantifier.

The definition of $\mathcal{FA}^2(M, P)$ differs from the standard definition of a forcing language as we have also added new quantifiers on top of the new constant symbols. These extra quantifiers will give us the amount of control over the countable fragments of the product we need. Note that every formula in the language $\mathcal{FA}^2(M, P)$ can be seen as a formula in the standard forcing language for P.

Lemma 3.3.3. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $(M, P), (N, Q) \in \mathcal{M} \in \mathcal{M}$ such that $(M, P) \preccurlyeq (N, Q)$, and let G be a Q-generic filter over N.*

(a) If $\sigma \in M$ is a P-name for a real, then σ is a Q-name for a real and $\sigma_G = \sigma_{G\cap P}$.

(b) If φ is a formula in the language $\mathcal{FA}^2(M, P)$, then φ is a formula in the language $\mathcal{FA}^2(N, Q)$ and $\varphi_G = \varphi_{G\cap P}$.

Proof. Item (b) follows directly from (a). So it is enough to prove (a). Let $\sigma \in M$ be a P-name for a real. Clearly, σ is a Q-name. We have to show that the weakest condition in Q forces that σ is a real. Since the weakest condition in P forces that σ is a real, for every $n \in \omega$, the set $D_n := \{p \in P : \exists m(((\check{n}, \check{m}), p) \in \sigma)\}$ is dense in P and if there are $p, p' \in P$ and $m \neq m' \in \omega$ such that $p \Vdash \sigma(\check{n}) = \check{m}$ and $p' \Vdash \sigma(\check{n}) = \check{m}'$, then p and p' are incompatible. By definition of a storage order, incompatible conditions in P remain incompatible in Q and every predense sets in P remains predense in Q. Hence, σ is also a Q-name for a real. Moreover, $G' := G \cap P$ is a P-generic filter over M and $\sigma_G = \sigma_{G'}$. $\square$

Recall that in the syntactic definition of the forcing relation one usually only considers the cases atomic, $\neg$, $\wedge$, and $\forall$, and treats $\vee$ and $\exists$ as abbreviations. Hence, for a forcing notion $\mathbb{P}$, a condition $p \in \mathbb{P}$, and a sentence in the forcing language φ, $p \Vdash \exists x \varphi(x)$ if $\{p' :$ there is a P-name $\sigma \in M$ such that $p' \Vdash \varphi(\sigma)\}$ is dense below p and analogous for $\vee$. The strong forcing relation $\Vdash_{\mathrm{s}}$ is defined similarly, except that the roles of $\wedge$ and $\vee$ and $\forall$ and $\exists$ are reversed. Then $p \Vdash_{\mathrm{s}} \exists x \varphi(x)$ if there is a $\mathbb{P}$-name σ such that $p \Vdash_{\mathrm{s}} \varphi(\sigma)$ which is a significant difference. In particular, the strong forcing relation is not equivalent to the usual forcing relation. Still, one can show that $p \Vdash_{\mathrm{s}} \varphi$ implies $p \Vdash \varphi$ and that $p \Vdash \varphi$ if and only if $p \Vdash_{\mathrm{s}} \neg\neg\varphi$. Therefore, the usual forcing relation is sometimes called the weak forcing relation.[2] Next, we define forcing-like relations for pairs in the storage order in the style of the strong forcing relation.

Definition 3.3.4. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $(M, P) \in \mathcal{M}$, and let φ be a formula in the language $\mathcal{FA}^2(M, P)$. We say that φ is $\Sigma^1_k(M, P)$ (or $\Pi^1_k(M, P)$) if it is Σ^1_k (or Π^1_k) when we replace every new constant symbol in φ by a free variable and every new quantifier in φ by the corresponding second-order quantifier. Let $p \in P$. For $\Sigma^1_k(M, P)$ and $\Pi^1_k(M, P)$ sentences φ, we define the notion $p \ \mathbf{forc}^M_P \varphi$ recursively on φ:

(a) if φ is $\Sigma^1_1(M, P)$, then $p \ \mathbf{forc}^M_P \varphi$ if and only if $p \Vdash^M_P \varphi$,

(b) if $\varphi = \neg\psi$ and ψ is $\Sigma^1_k(M, P)$, then $p \ \mathbf{forc}^M_P \varphi$ if and only if there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq p$, and $q \ \mathbf{forc}^N_Q \psi$,

(c) if $\varphi = \exists x \psi(x)$ and ψ is $\Pi^1_k(M, P)$, then $p \ \mathbf{forc}^M_P \varphi$ if and only if there is a P-name $\sigma \in M$ for a real such that $p \ \mathbf{forc}^M_P \psi(c_\sigma)$, and

(d) if $\varphi = \exists^B x \psi(x)$ and ψ is $\Pi^1_k(M, P)$, then $p \ \mathbf{forc}^M_P \varphi$ if and only if there is a $(P{\restriction}B \times \omega)$-name $\sigma \in M$ for a real such that $p \ \mathbf{forc}^M_P \psi(c_\sigma)$.

Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $(M, P) \in \mathcal{M}$, let $p \in P$, let $k \geq 1$, and let $\varphi = \forall x \psi(x)$ be a $\Pi^1_k(M, P)$ sentence. Then by definition, $p \ \mathbf{forc}^M_P \varphi$ if and only if there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq p$, and $q \ \mathbf{forc}^N_Q \exists x \neg\psi(x)$ and similarly for $\forall^B x \varphi(x)$. Now we check some basic properties of **forc**.

Lemma 3.3.5. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $(M, P) \in \mathcal{M}$, let $p \in P$, and let $k \geq 1$.*

(a) Let φ be a $\Sigma^1_k(M, P)$ or $\Pi^1_k(M, P)$ sentence, let $(N, Q) \in \mathcal{M}$ such that $(M, P) \preccurlyeq (N, Q)$, and let $q \in Q$. If $q \leq p$ and $p \ \mathbf{forc}^M_P \varphi$, then $q \ \mathbf{forc}^N_Q \varphi$.

[2] The strong forcing relation is often used in computability theory for complexity reasons. We shall talk about that in Chapter 4.

(b) *There is no* $\Sigma^1_k(M,P)$ *sentence such that* $p\ \mathbf{forc}^M_P\varphi$ *and* $p\ \mathbf{forc}^M_P\neg\varphi$.

Proof. Item (b) follows directly from the definition. We prove (a) by induction on φ. First, we assume that φ is a $\Sigma^1_1(M,P)$. Let G be a Q-generic filter over N containing q. Then $G\cap P$ is a P-generic filter over M containing p. Since $p\ \mathbf{forc}^M_P\varphi$, $p \Vdash^M_P \varphi$ and so $M[G\cap P]\models\varphi_{G\cap P}$. By analytic absoluteness, $N[G]\models\varphi_{G\cap P}$ as well. Hence, $N[G]\models\varphi_G$ and so $q\Vdash^N_Q\varphi$.

If $\varphi=\neg\psi$ and ψ is $\Sigma^1_k(M,P)$, then there are no $(N,Q)\in\mathcal{M}$ and $q'\in Q$ such that $(M,P)\preccurlyeq(N,Q)$, $q'\leq p$, and $q'\ \mathbf{forc}^N_Q\psi$. So in particular, there are no $(N',Q')\in\mathcal{M}$ and $q'\in Q'$ such that $(N,Q)\preccurlyeq(N',Q')$, $q'\leq q$ and $q'\ \mathbf{forc}^{N'}_{Q'}\psi$. Therefore, $q\ \mathbf{forc}^N_Q\varphi$.

If $\varphi=\exists x\psi(x)$ and ψ is $\Pi^1_k(M,P)$, then there is a P-name $\sigma\in M$ for a real such that $p\ \mathbf{forc}^M_P\psi(c_\sigma)$. By the induction hypothesis, $q\ \mathbf{forc}^N_Q\psi(c_\sigma)$ and so $q\ \mathbf{forc}^N_Q\varphi$. The case $\varphi=\exists^B x\psi(x)$ is similar. □

The next step is to extend **forc** to forcing notions which are defined storage sequences. We start with the definition of the forcing language.

Definition 3.3.6. Let $(\mathcal{M},\preccurlyeq)$ be a storage order, let $\langle(M_\xi,P_\xi):\xi<\omega_1\rangle$ be a storage sequence, let $\mathbb{P}:=\bigcup_{\xi<\omega_1}P_\xi$. We define $\mathcal{FA}^2(\mathbb{P})$ as the union of the languages $\mathcal{FA}^2(M_\xi,P_\xi)$ with $\xi<\omega_1$. Let $k\geq 1$. We say that a formula in the language $\mathcal{FA}^2(\mathbb{P})$ is $\Sigma^1_k(\mathbb{P})$ (or $\Pi^1_k(\mathbb{P})$) if there is some $\xi<\omega_1$ such that it is $\Sigma^1_k(M,P)$ (or $\Pi^1_k(M,P)$).

Note that for every formula φ in the language $\mathcal{FA}^2(\mathbb{P})$, there is some $\xi<\omega_1$ such that φ is a formula in the language $\mathcal{FA}^2(M_\xi,P_\xi)$. Moreover, if φ is a $\Sigma^1_k(\mathbb{P})$ or $\Pi^1_k(\mathbb{P})$ sentence and there is a condition $p\in\mathbb{P}$ and some $\xi<\omega_1$ such that $p\ \mathbf{forc}^{M_\xi}_{P_\xi}\varphi$, then by Lemma 3.3.5, $p\ \mathbf{forc}^{M_{\xi'}}_{P_{\xi'}}\varphi$ for every $\xi<\xi'<\omega_1$. We use this fact to extend **forc** to $\mathbb{P}$.

Definition 3.3.7. Let $(\mathcal{M},\preccurlyeq)$ be a storage order, let $\langle(M_\xi,P_\xi):\xi<\omega_1\rangle$ be a storage sequence, let $\mathbb{P}:=\bigcup_{\xi<\omega_1}P_\xi$, let $p\in\mathbb{P}$, let $k\geq 1$, and let φ be a $\Sigma^1_k(\mathbb{P})$ or $\Pi^1_k(\mathbb{P})$ sentence. We write $p\ \mathbf{forc}_\xi\varphi$ if $p\in P_\xi$, φ is a $\Sigma^1_k(M_\xi,P_\xi)$ or $\Pi^1_k(M_\xi,P_\xi)$ sentence, and $p\ \mathbf{forc}^{M_\xi}_{P_\xi}\varphi$ and $p\ \mathbf{forc}_\infty\varphi$ if there is a $\xi<\omega_1$ such that $p\ \mathbf{forc}_\xi\varphi$.

Throughout most of the rest of this section, we shall show that $\mathbf{forc}_\infty$ approximates the ordinary forcing relation. The accuracy of this approximation is depends on things we shall specify later. However, before going into details, we show that $\mathbf{forc}_\infty$ satisfies some basic properties of the ordinary forcing relation.

Lemma 3.3.8. *Let* $(\mathcal{M},\preccurlyeq)$ *be a storage order, let* $\langle(M_\xi,P_\xi):\xi<\omega_1\rangle$ *be a storage sequence, let* $\mathbb{P}:=\bigcup_{\xi<\omega_1}P_\xi$, *let* $p\in\mathbb{P}$, *and let* $k\geq 1$.

(a) *Let* φ *be a* $\Sigma^1_k(\mathbb{P})$ *or* $\Pi^1_k(\mathbb{P})$ *sentence and let* $p'\in\mathbb{P}$. *If* $p'\leq p$ *and* $p\ \mathbf{forc}_\infty\varphi$, *then* $p'\ \mathbf{forc}_\infty\varphi$.

(b) *There is no* $\Sigma^1_k(\mathbb{P})$ *sentence such that* $p\ \mathbf{forc}_\infty\varphi$ *and* $p\ \mathbf{forc}_\infty\neg\varphi$.

Proof. Follows directly from Lemma 3.3.5. □

Another important property of the ordinary forcing relation is that for every sentence φ, the set of conditions deciding φ is dense. Before we can prove that something similar is also true for $\mathbf{forc}_\infty$, we need some more definitions.

Definition 3.3.9. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order and let $n > 2$. We say that a storage sequence $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ is *n-complete* if for every $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$ set $D \subseteq \mathcal{M}$, there is a $\xi < \omega_1$ such that either $(M_\xi, P_\xi) \in D$ or there is no $(N, Q) \in D$ such that $(M_\xi, P_\xi) \preccurlyeq (N, Q)$.

Note that if $D \subseteq \mathcal{M}$ is a $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$ set which is additionally dense in $\mathcal{M}$, then the latter case is impossible and so there is some $\xi < \omega_1$ such that $(M_\xi, P_\xi) \in D$. Hence, an n-complete storage sequence is generic for all $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$ sets in L. Since there are ω_1 many $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$ sets, on first sight it may not be obvious that n-complete storage sequences exists. However, storage orders are σ-closed as forcing notions and so we can construct n-complete storage sequences recursively. Moreover, since we are in L, there is a Δ^{HC}_1 well-ordering of HC. Using this well-ordering and with a little bit of extra care, we can even construct n-complete storage sequences of bounded complexity.

Lemma 3.3.10. *For every storage order $(\mathcal{M}, \preccurlyeq)$ and every $n > 2$, there is a $\Delta^{\mathrm{HC}}_{n-1}$, n-complete storage sequence.*

Proof. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $n > 2$, and let $\{(\varphi_\xi, x_\xi) : \xi < \omega_1\}$ be a Δ^{HC}_1 enumeration of the set of all pairs (φ, x), where φ is a Σ_{n-2} formula and $x \in \mathrm{HC}$. Such an enumeration exists since $<_{\mathrm{L}}$ induces a Δ^{HC}_1 well-ordering on HC. We define the required storage sequence recursively. Let (M_0, P_0) be the $<_{\mathrm{L}}$-least pair in $\mathcal{M}$. Now we assume that $\langle (M_{\xi'}, P_{\xi'}) : \xi' < \xi \rangle$ is already defined. If ξ is a limit ordinal, then we set $P_\xi := \bigcup_{\xi' < \xi} P_{\xi'}$ and M as the $<_{\mathrm{L}}$-least countable transitive model of ZFC^- such that $(M_\xi, P_\xi) \in \mathcal{M}$ and $\langle (M_{\xi'}, P_{\xi'}) : \xi' < \xi \rangle$ is in M_ξ. If $\xi = \xi' + 1$, then we set (M_ξ, P_ξ) as the $<_{\mathrm{L}}$-least pair in $\mathcal{M}$ such that $(M_{\xi'}, P_{\xi'}) \preceq (M_\xi, P_\xi)$ and either $(M_\xi, P_\xi) \in D_{\xi'}$ or there is no $(N, Q) \in D_{\xi'}$ such that $(M_\xi, P_\xi) \preccurlyeq (N, Q)$, where $D_{\xi'}$ is the $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$ set defined by $\varphi_{\xi'}$ with parameter $x_{\xi'}$. By construction, $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ is an n-complete storage sequence. It remains to check that it is $\Delta^{\mathrm{HC}}_{n-1}$. To do so, it is enough to show that $(\mathcal{M}, \preccurlyeq)$ and $<_{\mathrm{L}} \cap (\mathrm{HC} \times \mathrm{HC})$ are Δ^{HC}_1. But this follows directly from the assumption that $(\mathcal{M}, \preccurlyeq)$ is Δ^{HC}_1 and Lemma 1.2.39. $\square$

For the rest of this section, the complexity of n-complete storage sequences does not matter. But it shall be important for the construction of an n-slicing forcing notion in Section 3.4. There, we shall use it to bound the complexity of the set of generics.

Now we are almost ready to show that something similar as the decision property is also true for $\mathbf{forc}_\infty$. We shall show that $\mathbf{forc}_\infty$ decides all $\Sigma^1_k(\mathbb{P})$ formulas if k is small enough. The rough idea is to use n-completeness on sets for the form $\{(M, P) \in \mathcal{M} : \exists p \in P (p \ \mathbf{forc}^M_P \neg\varphi)\}$. In order to do this, we have to check first that these sets are $\mathbf{\Sigma}^{\mathrm{HC}}_{n-2}$.

Definition 3.3.11. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order and let $k \geq 1$. We define

$$\mathbf{Forc}(\Sigma^1_k) := \{(M, P, p, \varphi) : (M, P) \in \mathcal{M} \wedge p \in P \wedge \varphi \text{ is a } \Sigma^1_k(M, P) \text{ sentence} \wedge p \ \mathbf{forc}^M_P \varphi\}$$

and $\mathbf{Forc}(\Pi^1_k)$ analogously.

Lemma 3.3.12. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order. Then*

(a) $\mathbf{Forc}(\Sigma^1_1)$, $\mathbf{Forc}(\Pi^1_1)$, and $\mathbf{Forc}(\Sigma^1_2)$ are Δ^{HC}_1 and

(b) for every $k > 1$, $\mathbf{Forc}(\Pi^1_k)$ and $\mathbf{Forc}(\Sigma^1_{k+1})$ are Π^{HC}_{k-1}.

Proof. First, we consider

$$\Sigma^1_k(\mathcal{M}) := \{(M, P, \varphi) : (M, P) \in \mathcal{M} \wedge \varphi \text{ is a } \Sigma^1_k(M, P) \text{ sentence}\}$$

for $k \geq 1$. Since $\mathcal{M}$ is a storage order, $\mathcal{M}$ is Δ_1^{HC}. Let $k \geq 1$ and let $(M, P) \in \mathcal{M}$. Then the question whether φ is a $\Sigma_k^1(M, P)$ sentence can be answered in M. Therefore, $\Sigma_k^1(\mathcal{M})$ is Δ_1^{HC}. By the same argument,

$$\Pi_k^1(\mathcal{M}) := \{(M, P, \varphi) : (M, P) \in \mathcal{M} \wedge \varphi \text{ is a } \Pi_k^1(M, P) \text{ sentence}\}$$

is also Δ_1^{HC} for every $k \geq 1$. Now we are ready to prove (a). We start with $\mathbf{Forc}(\Sigma_1^1)$. Since $\Pi_1^1(\mathcal{M})$ is Δ_1^{HC}, we only check the complexity of "$p\ \mathbf{forc}_P^M \varphi$". By definition, for every $(M, P, \varphi) \in \Sigma_1^1(\mathcal{M})$, $p\ \mathbf{forc}_P^M \varphi$ if and only if $p \Vdash_P^M \varphi$. The latter can be checked again in M. Therefore, $\mathbf{Forc}(\Sigma_1^1)$ is Δ_1^{HC}.

Next, we show that $\mathbf{Forc}(\Pi_1^1)$ is Δ_1^{HC}. Again, it is enough to check the complexity of "$p\ \mathbf{forc}_P^M \varphi$". Let $(M, P, \varphi) \in \Pi_1^1(\mathcal{M})$. Then there is a formula without second-order quantifiers ψ such that either $\varphi = \forall x \psi(x)$ or $\varphi = \forall^B x \psi(x)$. Without loss of generality, we assume the former.

Claim 3.3.13. *For every $p \in P$, $p\ \mathbf{forc}_P^M \varphi$ if and only if there is no $p' \leq p$ such that $p' \Vdash_P^M \exists x \neg\psi(x)$.*

Proof. By definition, $p\ \mathbf{forc}_P^M \varphi$ if and only if there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq p$, and $q\ \mathbf{forc}_Q^N \exists x \neg\psi(x)$. If there is a $p' \leq p$ such that $p' \Vdash_P^M \exists x \neg\psi(x)$, then $p'\ \mathbf{forc}_P^M \exists x \neg\psi(x)$ and so $p\ \mathbf{forc}_P^M \varphi$ does not hold. Hence, we can assume that there is no $p' \leq p$ such that $p' \Vdash_P^M \exists x \neg\psi(x)$. Then the set $D := \{p' \leq p : p' \Vdash_P^M \varphi\}$ is dense below p. Let $(N, Q) \in \mathcal{M}$ such that $(M, P) \preccurlyeq (N, Q)$. We assume for a contradiction that there is some $q \in Q$ such that $q \leq p$ and $q\ \mathbf{forc}_Q^N \exists x \neg\psi(x)$. Then $q \Vdash_Q^N \exists x \neg\psi(x)$. Since $(M, P) \preccurlyeq (N, Q)$, D remains predense in Q. Hence, there is some $r \in D$ which is compatible with q. Let G be a Q-generic filter over N containing r. Then $G' := G \cap P$ is a P-generic filter over M. Since $r \Vdash_P^M \varphi$, $M[G'] \models \varphi_{G'}$. By analytic absoluteness, $N[G] \models \varphi_G$ and so $r \Vdash_Q^N \varphi$. But this is a contradiction since $q \Vdash_Q^N \exists x \neg\psi(x)$. Therefore, there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq p$, and $q\ \mathbf{forc}_Q^N \exists x \neg\psi(x)$ and so $p\ \mathbf{forc}_P^M \varphi$. $\square$

Let $p \in P$. By Claim 3.3.13, $p\ \mathbf{forc}_P^M \varphi$ if and only if there is no $p' \leq p$ such that $p' \Vdash_P^M \exists x \neg\psi(x)$. The latter can be checked in M. Therefore, $\mathbf{Forc}(\Pi_1^1)$ is Δ_1^{HC}.

For (a), it remains to show that $\mathbf{Forc}(\Sigma_2^1)$ is Δ_1^{HC}. Again, we only have to check the complexity of "$p\ \mathbf{forc}_P^M \varphi$". Let $(M, P, \varphi) \in \Pi_1^1(\mathcal{M})$. Then there is a $\Pi_1^1(M, P)$ formula ψ such that either $\varphi = \exists x \psi(x)$ or $\varphi = \exists^B x \psi(x)$. Without loss of generality, we assume the former. By definition, $p\ \mathbf{forc}_P^M \varphi$ if and only if there is a P-name $\sigma \in M$ for a real such that $p\ \mathbf{forc}_P^M \psi(c_\sigma)$. Since $\mathbf{Forc}(\Pi_1^1)$ is Δ_1^{HC}, $\mathbf{Forc}(\Sigma_2^1)$ is Δ_1^{HC} as well.

Next, we prove (b) by induction on k. We start with $k = 2$. As before, we only have to check the complexity of "$p\ \mathbf{forc}_P^M \varphi$". Let $(M, P, \varphi) \in \Pi_2^1(\mathcal{M})$ and let ψ be a $\Sigma_1^1(M, P)$ formula such that $\varphi = \forall x \psi(x)$. By definition, $p\ \mathbf{forc}_P^M \varphi$ if and only if there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$, $(M, P) \preccurlyeq (N, Q)$ such that $q \leq p$, and $q\ \mathbf{forc}_Q^N \exists \neg\psi(x)$. Since $\mathbf{Forc}(\Sigma_2^1)$ is Δ_1^{HC}, $\mathbf{Forc}(\Pi_2^1)$ is Π_1^{HC}. By a similar argument as for $\mathbf{Forc}(\Sigma_2^1)$, $\mathbf{Forc}(\Sigma_3^1)$ is Π_1^{HC} as well. The induction step is analogous to the case $k = 2$. $\square$

Lemma 3.3.14. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $n > 2$, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ be an n-complete storage sequence, let $\mathbb{P} := \bigcup_{\xi < \omega_1} P_\xi$, let $p \in \mathbb{P}$, let $1 < k < n$, and let φ be a $\Sigma_k^1(\mathbb{P})$ sentence. Then*

(a) there is a $q \leq p$ such that either $q\ \mathbf{forc}_\infty \varphi$ or $q\ \mathbf{forc}_\infty \neg\varphi$, and

(b) p $\mathbf{forc}_\infty \neg\varphi$ if and only if there is no $q \leq p$ such that q $\mathbf{forc}_\infty \varphi$.

Proof. We start with proving (a). Let $\xi < \omega_1$ such that $p \in P_\xi$ and φ is $\Sigma^1_k(M_\xi, P_\xi)$ and let

$$D := \{(N, Q) \in \mathcal{M} : (M_\xi, P_\xi) \preccurlyeq (N, Q) \wedge \exists q \in Q(q \leq p \wedge q \ \mathbf{forc}^N_Q \varphi)\}.$$

By Lemma 3.3.12, D is $\mathbf{\Sigma}^{\mathrm{HC}}_{k-1}$. Since $k < n$ and $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ is n-complete, there is some $\xi \leq \xi' < \omega_1$ such that either $(M_{\xi'}, P_{\xi'}) \in D$ or there is no $(N, Q) \in D$ such that $(M_{\xi'}, P_{\xi'}) \preccurlyeq (N, Q)$. We make a case-distinction:

Case 1: $(M_{\xi'}, P_{\xi'}) \in D$. Then there is a condition $p' \in P_{\xi'}$ such that $p' \leq p$ and p' $\mathbf{forc}_{\xi'} \varphi$. Hence, p' $\mathbf{forc}_\infty \varphi$.

Case 2: there is no $(N, Q) \in D$ such that $(M_{\xi'}, P_{\xi'}) \preccurlyeq (N, Q)$. Then there are no $(N, Q) \in \mathcal{M}$ and $q \in Q$ such that $(M_{\xi'}, P_{\xi'}) \preccurlyeq (N, Q)$, $q \leq p$, and q $\mathbf{forc}^N_Q \varphi$. Hence, p $\mathbf{forc}_{\xi'} \neg\varphi$ and so p $\mathbf{forc}_\infty \neg\varphi$.

Next, we show (b). The forward direction follows directly from Lemma 3.3.8. For the backward direction, let $D := \{p' \in \mathbb{P} : p' \leq p \wedge p' \ \mathbf{forc}_\infty \neg\varphi\}$. By (a), D is dense below p. Let $\mathcal{A} \subseteq D$ be a maximal antichain below p. Since $\mathbb{P}$ satisfies the c.c.c., $\mathcal{A}$ is countable and so $\mathcal{A} \in \mathrm{HC}$. Moreover, $\mathrm{HC} = \mathrm{L}_{\omega_1} = \bigcup_{\xi < \omega_1} M_\xi$. Hence, there is some $\xi < \omega_1$ such that $p \in P_\xi$, φ is $\Sigma^1_k(M_\xi, P_\xi)$, $\mathcal{A} \in M_\xi$, $\mathcal{A} \subseteq P_\xi$, and for every $p' \in \mathcal{A}$, p' $\mathbf{forc}_\xi \neg\varphi$. We suppose for a contradiction that p $\mathbf{forc}_\xi \neg\varphi$ does not hold. Then there is some $(N, Q) \in \mathcal{M}$ and a $q \in Q$ such that $(M_\xi, P_\xi) \preccurlyeq (N, Q)$, $q \leq p$, and q $\mathbf{forc}^N_Q \varphi$. Since $\mathcal{A}$ is predense below p in P, $\mathcal{A}$ is also predense below q in Q. Hence, there is some $r \in \mathcal{A}$ which is compatible with q. But this is impossible by Lemma 3.3.8. Therefore, p $\mathbf{forc}_\xi \neg\varphi$ and so p $\mathbf{forc}_\infty \neg\varphi$. ☐

Next, we prove a version of the Truth Lemma for $\mathbf{forc}_\infty$. To do this, we need to define the valuation of formulas in the language $\mathcal{FA}^2(\mathbb{P})$ first. Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ storage sequence, let $\mathbb{P} := \bigcup_{\xi < \omega_1} P_\xi$, let φ be a formula in the language $\mathcal{FA}^2(\mathbb{P})$, and let G be a $\mathbb{P}$-generic filter over L. Then there is some $\xi < \omega_1$ such that φ is a formula in the language $\mathcal{FA}^2(M_\xi, P_\xi)$. By definition of a storage order, $G_\xi := G \cap P_\xi$ is a P_ξ-generic filter over M_ξ. We define the *valuation of φ by G* as $\varphi_G := \varphi_{G_\xi}$. Note that this is well-defined by Lemma 3.3.3.

Theorem 3.3.15. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $n > 2$, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ be an n-complete storage sequence, let $\mathbb{P} := \bigcup_{\xi < \omega_1} P_\xi$, let G be $\mathbb{P}$-generic over* L*, and let φ be a sentence in the language $\mathcal{FA}^2(\mathbb{P})$.*

(a) Let $p \in G$. Then for every $1 \leq k \leq n+1$, if φ is $\Sigma^1_k(\mathbb{P})$ and p $\mathbf{forc}_\infty \varphi$, then $\mathrm{L}[G] \models \varphi_G$.

(b) Let $p \in G$. Then for every $1 \leq k \leq n$, if φ is $\Pi^1_k(\mathbb{P})$ and p $\mathbf{forc}_\infty \varphi$, then $\mathrm{L}[G] \models \varphi_G$.

(c) For every $1 \leq k \leq n$, if φ is $\Sigma^1_k(\mathbb{P})$ and $\mathrm{L}[G] \models \varphi_G$, then there is some $p \in G$ such that p $\mathbf{forc}_\infty \varphi$.

(d) For every $1 \leq k < n$, if φ is $\Pi^1_k(\mathbb{P})$ and $\mathrm{L}[G] \models \varphi_G$, then there is some $p \in G$ such that p $\mathbf{forc}_\infty \varphi$.

Proof. We prove all items simultaneously by induction on φ. First, we assume that φ is $\Sigma^1_1(\mathbb{P})$. If there is a $p \in G$ such that p $\mathbf{forc}_\infty \varphi$, then there is some $\xi < \omega_1$ such that $p \in P_\xi$, φ is $\Sigma^1_1(M_\xi, P_\xi)$, and p $\mathbf{forc}_\xi \varphi$. By definition, p $\mathbf{forc}_\xi \varphi$ if and only if $p \Vdash^{M_\xi}_{P_\xi} \varphi$. Let $G_\xi := G$. Then G_ξ is a P_ξ-generic filter over M_ξ containing p and so $m_\xi[G_\xi] \models \varphi_{G_\xi}$. By analytic absoluteness, $\mathrm{L}[G] \models \varphi_{G_\xi}$ and so $\mathrm{L}[G] \models \varphi_G$.

If conversely, $\mathrm{L}[G] \models \varphi_G$, then let $\xi < \omega_1$ such that φ is $\Sigma^1_1(\mathbb{P})$ and let $G_\xi := G \cap M_\xi$. By analytic absoluteness, $M_\xi[G_\xi] \models \varphi_G$ and so $M_\xi[G_\xi] \models \varphi_{G_\xi}$. Hence, there is some $p \in G_\xi$ such that $p \Vdash^{M_\xi}_{P_\xi} \varphi$. Then by definition, $p\ \mathbf{forc}_\xi \varphi$ and so $p\ \mathbf{forc}_\infty \varphi$.

Next, we assume that φ is $\Pi^1_k(\mathbb{P})$. Then there is a formula φ in the language $\mathcal{FA}^2(\mathbb{P})$ such that either $\varphi = \forall x \psi(x)$ or $\varphi = \forall^B x \psi(x)$. Without loss of generality, we assume the former. If $1 \leq k \leq n$ and there is some $p \in G$ such that $p\ \mathbf{forc}_\infty \varphi$, then by Lemma 3.3.8, there is no $q \leq p$ such that $q\ \mathbf{forc}_\infty \exists x \neg \psi(x)$. By the induction hypothesis, we have $\mathrm{L}[G] \not\models (\exists x \neg \psi(x))_G$ and so $\mathrm{L}[G] \models \varphi_G$.

If conversely, $1 \leq k < n$ and $\mathrm{L}[G] \models \varphi_G$, then $\mathrm{L}[G] \not\models (\exists x \neg \psi(x))_G$. By the induction hypothesis, there is no $p \in G$ such that $p\ \mathbf{forc}_\infty \exists x \neg \psi(x)$. Let $D := \{p \in \mathbb{P} : p\ \mathbf{forc}_\infty \varphi \text{ or } p\ \mathbf{forc}_\infty \exists x \neg \psi(x)\}$. By Lemma 3.3.14, D is dense in $\mathbb{P}$. Hence, there is some $p \in G$ such that $p\ \mathbf{forc}_\infty \varphi$.

Finally, we assume that φ is $\Sigma^1_{k+1}(\mathbb{P})$. Then there is a Π^1_k formula ψ such that either $\varphi = \exists x \psi(x)$ or $\varphi = \exists^B x \psi(x)$. Without loss of generality, we assume the former. If $1 \leq k \leq n$ and there is a $p \in G$ such that $p\ \mathbf{forc}_\infty \varphi$, then there is some $\xi < \omega_1$ such that $p \in P_\xi$, φ is $\Sigma^1_{k+1}(M_\xi, P_\xi)$, and $p\ \mathbf{forc}_\xi \varphi$. Since $p\ \mathbf{forc}_\xi \varphi$, there is a P_ξ-name $\sigma \in M_\xi$ for a real such that $p\ \mathbf{forc}_\xi \psi(c_\sigma)$. By the induction hypothesis, $\mathrm{L}[G] \models (\psi(c_\sigma))_G$ and so $\mathrm{L}[G] \models \varphi_G$.

If conversely, $1 \leq k < n$ and $\mathrm{L}[G] \models \varphi_G$, then there is some real $x \in \mathrm{L}[G]$ such that $\mathrm{L}[G] \models \psi_G(x)$. Since $\mathbb{P}$ satisfies the c.c.c., there is a countable name $\sigma \in \mathrm{L}$ for x. Then $\mathrm{L}[G] \models (\psi(c_\sigma))_G$ and $\sigma \in \mathrm{HC} = \mathrm{L}_{\omega_1} = \bigcup_{\xi<\omega_1} M_\xi$. Hence, there is some $\xi < \omega_1$ such that $\psi(c_\sigma)$ is $\Pi^1_k(M_\xi, P_\xi)$. By the induction hypothesis, there is a $p \in G$ such that $p\ \mathbf{forc}_\infty \psi(c_\sigma)$. Therefore, $p\ \mathbf{forc}_\infty \varphi$. $\square$

As a corollary of Theorem 3.3.15, we get that $\mathbf{forc}_\infty$ approximates the ordinary forcing relation for $\Pi^1_k(\mathbb{P})$ sentences if k is small enough.

Corollary 3.3.16. *Let $(\mathcal{M}, \preccurlyeq)$ be a storage order, let $n > 2$, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ be an n-complete strictly $\preccurlyeq$-increasing sequence which is continuous at limits, let $\mathbb{P} := \bigcup_{\xi<\omega_1} P_\xi$, let $p \in \mathbb{P}$, and let φ be a sentence in the language $\mathcal{FA}^2(\mathbb{P})$.*

(a) For every $1 \leq k \leq n+1$, if φ is $\Sigma^1_k(\mathbb{P})$ and p $\mathbf{forc}_\infty \varphi$, then $p \Vdash_{\mathbb{P}} \varphi$.

(b) For every $1 \leq k \leq n$, if φ is $\Pi^1_k(\mathbb{P})$ and p $\mathbf{forc}_\infty \varphi$, then $p \Vdash_{\mathbb{P}} \varphi$.

(c) For every $1 \leq k \leq n$, if φ is $\Sigma^1_k(\mathbb{P})$ and $p \Vdash_{\mathbb{P}} \varphi$, then there is some $q \leq p$ such that q $\mathbf{forc}_\infty \varphi$.

(d) For every $1 \leq k < n$, if φ is $\Pi^1_k(\mathbb{P})$ and $p \Vdash_{\mathbb{P}} \varphi$, then there is some $q \leq p$ such that q $\mathbf{forc}_\infty \varphi$.

(e) For every $1 \leq k \leq n$, if φ is $\Sigma^1_k(\mathbb{P})$, then $p \Vdash_{\mathbb{P}} \neg\varphi$ if and only if there is no $q \leq p$ such that q $\mathbf{forc}_\infty \varphi$.

(f) For every $1 \leq k < n$, if φ is $\Pi^1_k(\mathbb{P})$, then $p \Vdash_{\mathbb{P}} \varphi$ if and only if p $\mathbf{forc}_\infty \varphi$.

Proof. Items (a) and (b) follow directly from Theorem 3.3.15 and (f) from (a), (e), and Lemma 3.3.14. For (c), let G be a $\mathbb{P}$-generic filter over L containing p. Then $\mathrm{L}[G] \models \varphi_G$. By Theorem 3.3.15, there is some $q \in \mathbb{P}$ such that $q\ \mathbf{forc}_\infty \varphi$. Since G is a filter, there is some $r \in G$ which witnesses that p and q are compatible. Then $r \leq p$ and $r\ \mathbf{forc}_\infty \varphi$. The proof of (d) is similar.

Finally, we prove (e). First, we assume that there is a $q \leq p$ such that $q\ \mathbf{forc}_\infty \varphi$. By (a), $q \Vdash_{\mathbb{P}} \varphi$ and so $q \not\Vdash_{\mathbb{P}} \neg\varphi$. If conversely, $p \not\Vdash_{\mathbb{P}} \neg\varphi$, then there is some $q \leq p$ such that $q \Vdash_{\mathbb{P}} \varphi$. By (c), there is some $r \leq q$ such that $r\ \mathbf{forc}_\infty \varphi$. $\square$

Corollary 3.3.16 completes the first step of constructing ω-slice-products which are n-absolute for slices. In the second and final step, we now use the forcing-like relation to prove the Kanovei-Lyubetsky Lemma. Before we can formally state the Kanovei-Lyubetsky Lemma, we need one last definition.

Definition 3.3.17. We say that a storage order $(\mathcal{M}, \preccurlyeq)$ is *simple* if for every $(M, P) \in \mathcal{M}$, P is an ω-slice-product of a single forcing notion.

Lemma 3.3.18 (Kanovei-Lyubetsky Lemma)**.** *Let $(\mathcal{M}, \preccurlyeq)$ be a simple storage order, let $n > 2$, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ be an n-complete storage sequence, let $\mathbb{P} := \bigcup_{\xi<\omega_1} P_\xi$, let G be $\mathbb{P}$-generic over* L*, and let $e \subseteq \omega_1$ be unbounded. Then every Σ^1_n formula with parameters in $\mathrm{L}[G{\restriction}e \times \omega]$ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G{\restriction}e \times \omega]$. In particular, $\mathbb{P}$ is n-absolute for slices.*

The Kanovei-Lyubetsky Lemma will be an essential tool for the proof of the Slicing Theorem (Theorem 3.2.17). It will be used to construct ω-slice-products which are n-absolute for slices. Kanovei and Lyubetsky's original result can also be used to construct a forcing notion which n-absolute for slices. Recall that they constructed an ω_1-product of variants of almost disjoint forcing with finite support such that for every generic filter G over L and every unbounded set $e \subseteq \omega_1$, every Σ^1_n formula with parameters in $\mathrm{L}[G{\restriction}e]$ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G{\restriction}e]$. Let $\mathbb{Q}$ be this forcing notion and let $f : \omega_1 \to \omega_1 \times \omega$ be the canonical bijection. For every $q \in \mathbb{Q}$, we define $f^*(q) := q'$ with $\mathrm{dom}(q') = f[\mathrm{dom}(q)]$ and for every $(\nu, k) \in \mathrm{dom}(q')$, $q'(\nu, k) = q(f^{-1}(\nu, k))$. Then $\mathbb{Q}^* := \{f^*(q) : q \in \mathbb{Q}\}$ is an $(\omega_1 \times \omega)$-product with finite support and f^* is an isomorphism between $\mathbb{Q}$ and $\mathbb{Q}^*$. Let G^* be a $\mathbb{Q}^*$-generic filter over L and let $G := \{(f^*)^{-1}(q) : q \in G^*\}$. Then G is a $\mathbb{Q}$-generic filter over L and for every $A \subseteq \omega_1 \times \omega$, $\mathrm{L}[G^*{\restriction}s] = \mathrm{L}[G{\restriction}f^{-1}(A)]$. Let $e \subseteq \omega_1$ be unbounded, let $s := e \times \omega$, and let φ be a Σ^1_n formula with parameters in $\mathrm{L}[G{\restriction}s]$. Then $f^{-1}(s)$ is unbounded in ω_1 and so φ is absolute between $\mathrm{L}[G^*] = \mathrm{L}[G]$ and $\mathrm{L}[G^*{\restriction}s] = \mathrm{L}[G{\restriction}f^{-1}(s)]$. Therefore, $\mathbb{Q}^*$ is n-absolute for slices. Unfortunately, $\mathbb{Q}^*$ is not an ω-slice-product. However, with some extra work, the symmetric submodel construction from Section 3.2.3 can be applied to $\mathbb{Q}^*$. By similar arguments as in Section 3.2.3, the resulting symmetric submodel is a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_{n-1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$, but it is not known if $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_n)$ fails in this model.

3.3.2 Absoluteness for slices

In this section, we prove the Kanovei-Lyubetsky Lemma (Lemma 3.3.18). To explain the plan of the proof, we first consider the special case $n = 3$. Let $(\mathcal{M}, \preccurlyeq)$, $\mathbb{P}$, G, and e be as in the Kanovei-Lyubetsky Lemma. We have to show that every Σ^1_3 formula φ with parameters in $\mathrm{L}[G{\restriction}e \times \omega]$ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G{\restriction}e \times \omega]$. For simplicity, we assume that φ has no parameters. Then there is a Π^1_2 formula ψ such that $\varphi = \exists x \psi(x)$. By Shoenfield absoluteness, ψ is absolute between $\mathrm{L}[G{\restriction}e \times \omega]$ and $\mathrm{L}[G]$. Hence, we only have to show that φ is downwards absolute from $\mathrm{L}[G]$ to $\mathrm{L}[G{\restriction}e \times \omega]$. If $\mathrm{L}[G] \models \varphi$, then there is some $y \in \mathrm{L}[G]$ such that $\mathrm{L}[G] \models \psi(y)$. Since $\mathbb{P}$ satisfies the c.c.c., there is a countable $\mathbb{P}$-name for y. Hence, there is countable set $B \subseteq \omega_1$ such that $y \in \mathrm{L}[G{\restriction}B \times \omega]$. By Shoenfield absoluteness, it is enough to show that there is some $y' \in \mathrm{L}[G{\restriction}e \times \omega]$ such that $\mathrm{L}[G] \models \psi(y')$. To do this, we shall prove that for every countably infinite set $C \subseteq \omega_1$ with $B \cap C = \emptyset$, there is such a $y' \in \mathrm{L}[G{\restriction}C \times \omega]$. Since e is uncountable, this completes the proof. By induction, the same argument also works for $n > 3$. Thus, the following lemma is essential for the proof of the Kanovei-Lyubetsky Lemma.

Lemma 3.3.19. *Let $(\mathcal{M}, \preccurlyeq)$ be a simple storage order, let $n > 2$, let $\langle (M_\xi, P_\xi) : \xi < \omega_1 \rangle$ be an n-complete storage sequence, let $\mathbb{P} := \bigcup_{\xi<\omega_1} P_\xi$, let $d \subseteq \omega_1$ be countable, let $b, c \subseteq \omega_1 \setminus d$ be countably infinite, let G be $\mathbb{P}$-generic over* L*, and let ψ be a Π^1_{n-1} formula with parameter $r \in \mathrm{L}[G{\restriction}d \times \omega]$. If there is a real $y \in \mathrm{L}[G{\restriction}(b \cup d) \times \omega]$ such that $\mathrm{L}[G] \models \psi(y, r)$, then there is a real $y' \in \mathrm{L}[G{\restriction}(c \cup d) \times \omega]$ such that $\mathrm{L}[G] \models \psi(y', r)$.*

The rough idea to prove Lemma 3.3.19 is to show that for every $p \in \mathbb{P}$ with $p\ \mathbf{forc}_\infty \exists^B x\psi(x, c_{\dot r})$, there is some $p' \in \mathbb{P}$ such that p and p' are compatible and $p'\ \mathbf{forc}_\infty \exists^C x\psi(x, c_{\dot r})$, where $B := b \cup d$, $C := c \cup d$, and $\dot r$ is a countable $\mathbb{P}$-name for r. To find such a p', we use permutations of some $\zeta < \omega_1$ which map B to C and vice versa. Note that for every ω-slice-product P with finite support of length ζ, we can think of the conditions as $\zeta \times \omega$ matrices. If P is ω-slice-product of a single forcing notion, then we can freely swap around the entries in these matrices without obtaining a matrix that does not represent any condition in P. Recall that we already did something similar to prove Theorem 3.2.10. There, we constructed a symmetric submodel using permutations which swap around the entries inside of rows. Here, we use permutations which swap entire rows with each other. More precisely, let $(\mathcal{M}, \preccurlyeq)$ be a simple storage order, let $(M, P) \in \mathcal{M}$, let ζ be the length of P, and let $f : \zeta \to \zeta$ be a bijection in M. Since $(\mathcal{M}, \preccurlyeq)$ is simple, for every $p \in P$,

$$f^*(p) := \{((f(\nu), k), q) : ((\nu, k), q) \in p\} \in P$$

and so f induces an automorphism on P. Let $\sigma \in M$ be a P-name and let φ be a formula in the language $\mathcal{FA}^2(M, P)$. We write $f^*(\sigma)$ for the P-name in M that we get by replacing every condition p in σ with $f^*(p)$ and $f^*(\varphi)$ for the formula we get by replacing every c_τ in φ with $c_{f^*(\tau)}$ and every Q^B with $Q^{f[B]}$, where Q is a second-order quantifier.

Lemma 3.3.20. *Let $(\mathcal{M}, \preccurlyeq)$ be a simple storage order, let $(M, P) \in \mathcal{M}$, let $p \in P$, let ζ be the length of P, let $f : \zeta \to \zeta$ be a bijection in M, let $k \geq 1$, and let φ be a $\Sigma^1_k(M, P)$ or $\Pi^1_k(M, P)$ sentence. If $p\ \mathbf{forc}^M_P \varphi$, then $f^*(p)\ \mathbf{forc}^M_P f^*(\varphi)$.*

Proof. We prove the lemma by induction on φ. First, we assume that φ is $\Sigma^1_1(M, P)$. Let G be a P-generic filter containing $f^*(p)$. Then $G' := (f^{-1})^*[G]$ is a P-generic filter containing p and $f^*[G'] = G$. Hence, $M[G] = M[G']$ and so $M[G] \models \varphi_{G'}$. Moreover, for every P-name $\sigma \in M$ for a real, $\sigma_G = \sigma_{G'}$. Hence, $M[G] \models (f^*(\varphi))_G$. Therefore, $f^*(p)\ \mathbf{forc}^M_P f^*(\varphi)$.

Next, we assume that φ is $\Pi^1_k(M, P)$. Then there is a formula ψ in the language $\mathcal{FA}^2(M, P)$ such that either $\varphi = \forall x\psi(x)$ or $\varphi = \forall^B x\psi(x)$. Without loss of generality, we assume the former. We suppose for a contradiction that $f^*(p)\ \mathbf{forc}^M_P f^*(\varphi)$ does not hold. Then there is a $(N, Q) \in \mathcal{M}$ and a $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq f^*(p)$, and $q\ \mathbf{forc}^N_Q f^*(\exists x \neg\psi(x))$. By the induction hypothesis, $(f^{-1})^*(q)\ \mathbf{forc}^N_Q (f^{-1})^*(f^*(\exists x \neg\psi(x)))$. Hence, $(f^{-1})^*(q) \leq p$ and $(f^{-1})^*(q)\ \mathbf{forc}^N_Q \exists x \neg\psi(x)$. But this is a contradiction. Therefore, $f^*(p)\ \mathbf{forc}^M_P f^*(\varphi)$.

Finally, we assume that φ is $\Sigma^1_{k+1}(M, P)$. Then there is a $\Pi^1_k(M, P)$ formula ψ such that either $\varphi = \exists x\psi(x)$ or $\varphi = \exists^B x\psi(x)$. Without loss of generality, we assume the latter. Since $p\ \mathbf{forc}^M_P \varphi$, there is a $(P{\restriction}B \times \omega)$-name $\sigma \in M$ such that $p\ \mathbf{forc}^M_P \psi(c_\sigma)$. By the induction hypothesis, $f^*(p)\ \mathbf{forc}^M_P f^*(\psi(c_\sigma))$. Then $f^*(\sigma)$ is a $(P{\restriction}f[B] \times \omega)$-name for a real and $f^*(p)\ \mathbf{forc}^M_P f^*(\psi)(c_{f^*(\sigma)})$. Therefore, $f^*(p)\ \mathbf{forc}^M_P f^*(\varphi)$. □

Proof of Lemma 3.3.19. Let $B := b \cup d$, let $C := c \cup d$, and let $\mathbb{Q} := \mathbb{P}{\restriction}d \times \omega$. Since $\mathbb{P}$ satisfies the c.c.c., $\mathbb{Q}$ satisfies the c.c.c. as well. Hence, there is a countable $\mathbb{Q}$-name $\dot r$ for r. Then $\exists^B x\psi(x, c_{\dot r})$ is a $\Sigma^1_n(\mathbb{P})$ sentence. By assumption, there is some $y \in \mathrm{L}[G{\restriction}B \times \omega]$ such that $\mathrm{L}[G] \models \psi(y, r)$. Hence, $\mathrm{L}[G] \models (\exists^B x\psi(x, c_{\dot{r}}))_G$. By Theorem 3.3.15, there is a $p \in G$ such that p $\mathbf{forc}_\infty \exists^B x\psi(x, c_{\dot{r}})$. Let

$$D := \{p' \leq p : p' \ \mathbf{forc}_\infty \exists^C x\psi(x, c_{\dot{r}})\}.$$

We show that D is dense below p. Let $p' \leq p$. Since b, c, d, B, and C are countable, they are all in HC. Hence, there is some $\xi < \omega_1$ such that $b, c, d, B, C \in M_\xi$, $p' \in P_\xi$, ψ is $\Sigma^1_n(M_\xi, P_\xi)$, and p' $\mathbf{forc}_\xi \exists^B x\psi(x, c_{\dot{r}})$. Let ζ be the length of P_ξ. Without loss of generality, $\zeta \setminus B$ and $\zeta \setminus C$ are countably infinite. Let $f : \zeta \to \zeta$ be a bijection in M_ξ such that $f{\restriction}d$ is the identity, $f[b] = c$, and for every $\nu \in b \cap c$, $f(\nu) = \nu$. By Lemma 3.3.20, $f^*(p')$ $\mathbf{forc}_\xi \exists^C x\psi(x, c_{\dot{r}})$. Since p' and $f^*(p')$ agree on their common domain, they are compatible. Let q be a witness. Then q $\mathbf{forc}_\xi \exists^C x\psi(x, c_{\dot{r}})$ and so q $\mathbf{forc}_\infty \exists^C x\psi(x, c_{\dot{r}})$. Hence, $q \in D$ and so D is dense below p. Thus, there is some $p' \in G$ such that p' $\mathbf{forc}_\infty \exists^C x\psi(x, c_{\dot{r}})$. By Theorem 3.3.15, $\mathrm{L}[G] \models (\exists^C x\psi(x, c_{\dot{r}}))_G$. Therefore, there is a real $y' \in \mathrm{L}[G{\restriction}C \times \omega]$ such that $\mathrm{L}[G] \models \psi(y', r)$. □

The proof of Lemma 3.3.19 completes the preparations for the Kanovei-Lyubetsky Lemma. Now we are finally ready to prove it.

Proof of the Kanovei-Lyubetsky Lemma (Lemma 3.3.18). The second part follows directly from the first. Let $1 \leq k \leq n$ and let φ be a Σ^1_k formula with parameter $r \in \mathrm{L}[G{\restriction}e \times \omega]$. We show that φ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G{\restriction}e \times \omega]$ by induction on k. The case $k = 1$ is clear by analytic absoluteness. If $k > 1$ and $\mathrm{L}[G] \models \varphi$, then there is some Π^1_n formula ψ such that $\varphi = \exists x\psi(x, r)$. Since $\mathbb{P}{\restriction}e \times \omega$ satisfies the c.c.c., there is a countable set $d \subseteq e$ such that $r \in \mathrm{L}[G{\restriction}d \times \omega]$. Let $c \subseteq e \setminus d$ be a countably infinite set. By assumption, there is some $y \in \mathrm{L}[G]$ such that $\mathrm{L}[G] \models \psi(y)$. Again, we can find a countably infinite set $b \subseteq \omega_1$ such that $y \in \mathrm{L}[G{\restriction}(b \cup d) \times \omega]$. By Lemma 3.3.19, there is a $y' \in \mathrm{L}[G{\restriction}(c \cup d) \times \omega]$ such that $\mathrm{L}[G] \models \psi(y')$. Then $y' \in \mathrm{L}[G{\restriction}e \times \omega]$ and so by the induction hypothesis, $\mathrm{L}[G{\restriction}e \times \omega] \models \psi(y')$. Therefore, $\mathrm{L}[G{\restriction}e \times \omega] \models \varphi$. If conversely, $\mathrm{L}[G{\restriction}e \times \omega] \models \varphi$, then by the induction hypothesis and upwards-absoluteness, $\mathrm{L}[G] \models \varphi$. □

3.4 Jensen-like forcing notions

3.4.1 Jensen forcing

The goal of Section 3.4 is to prove the Slicing Theorem (Theorem 3.2.17), i.e., we have to show that for every $n \geq 2$, there are n-slicing forcing notions in L. To do this, we use forcing notions that are constructed in a similar way to Jensen forcing. We introduce these Jensen-like forcing notions in Section 3.4.2. To prepare for this, we give an introduction to ordinary Jensen forcing in this section. Jensen forcing was first introduced by Jensen in [Jen70] to produce a model of ZFC containing a non-constructible Δ^1_3 real. By Shoenfield absoluteness, every Σ^1_2 and Π^1_2 real is constructible. So Jensen's model is optimal in this sense.

Jensen forcing is an arboreal forcing notion. Recall that a forcing notion $\mathbb{P}$ is arboreal if its conditions are perfect trees ordered by inclusion and for every $T \in \mathbb{P}$ and every $t \in T$, there is a $T' \leq T$ such that $t \subseteq \mathrm{stem}(T')$. We start with introducing some notation for arboreal forcing notions. Let $T, T' \in \mathbb{S}$ be perfect trees. If T and T' are compatible, then there is a perfect tree which is contained in $T \cap T'$. Since the union of perfect trees is again a perfect tree, we can even find a maximal such perfect tree. We define the *meet of T and T'* by

$$T \wedge T' := \begin{cases} \text{the maximal perfect tree contained in } T \cap T' & \text{if } T \text{ and } T' \text{ are compatible,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then T and T' are compatible if $T \wedge T' \neq \emptyset$.

Definition 3.4.1. We say that an arboreal forcing notion $\mathbb{P} \subseteq \mathbb{S}$ is *sufficiently closed* if $\mathbb{P}$ is closed under meets, $\mathbb{P}$ is closed under finite unions, and for every $s \in 2^{<\omega}$, $(2^{<\omega})_s \in \mathbb{P}$.

Note that two perfect trees T and T' in a sufficiently closed arboreal forcing notion $\mathbb{P}$ are compatible in $\mathbb{P}$ if and only if $T \wedge T' \neq \emptyset$. Hence, T and T' are compatible in $\mathbb{P}$ if and only if they are compatible in $\mathbb{S}$.

Jensen forcing is constructed as the union of a sequence $\langle P_\xi : \xi < \omega_1 \rangle$ of sufficiently closed arboreal forcing notions. We shall use *fusion* to construct $P_{\xi+1}$ from P_ξ. Recall that a fusion sequence for $\mathbb{S}$ is a sequence $\langle T_k : k \in \omega \rangle$ of conditions in $\mathbb{S}$ such that for every $k \in \omega$, $T_k \leq_n T_{k+1}$, where $S \leq_k T$ if and only if $S \leq T$ and S has the same kth splitting nodes as T. Then the Fusion Lemma says that for every fusion sequence $\langle T_k : k \in \omega \rangle$ for $\mathbb{S}$, $T := \bigcap_{k \in \omega} T_k \in \mathbb{S}$ and for every $k \in \omega$, $T \leq_k T_k$ (cf., e.g., [Jec87, Lemma 3.7]).

Definition 3.4.2. Let $\mathbb{P}$ be a sufficiently closed arboreal forcing notion. The *fusion order of* $\mathbb{P}$ is the set $P \times \omega$ ordered by

$$(S, n) \leq (T, m) :\iff S \leq T \wedge m \leq n \wedge 2^m \cap S = 2^m \cap T.$$

We denote it by $\mathbb{Q}(\mathbb{P})$.

Let $\mathbb{P}$ be a sufficiently closed arboreal forcing notion. Then for every $k \in \omega$, the set $\{(T, n) \in \mathbb{Q}(\mathbb{P}) : \forall (S, n) \leq (T_m)(S \leq_k T)\}$ is dense in $\mathbb{Q}(\mathbb{P})$. Hence, if G is $\mathbb{Q}(\mathbb{P})$-generic filter, then the set $A := \{T \in \mathbb{P} : \exists m (T, m) \in G\}$ contains a fusion sequence for $\mathbb{S}$ and so by the Fusion Lemma, $T_G := \bigcap A$ is a perfect tree. So every $\mathbb{Q}(\mathbb{P})$-generic filter adds a new perfect tree which is defined from a fusion sequence of conditions in $\mathbb{P}$.

Definition 3.4.3. Let M be a countable transitive model of $\mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists", let P be a sufficiently closed arboreal forcing notion in M, let $\mathbb{Q}(P)^{<\omega}$ be the ω-product of $\mathbb{Q}(P)$ with finite support, let H be a $\mathbb{Q}(P)^{<\omega}$-generic filter over M, and let $\langle T_k : k \in \omega \rangle$ be the generic trees added by H. We define $J(P, H)$ as the closure of the set

$$P \cup \{T_k \wedge S : S \in P \wedge k \in \omega \wedge (T_k \wedge S \neq \emptyset)\}$$

under countable unions ordered by inclusion.

Lemma 3.4.4. *Let M be a countable transitive model of ZFC^-+"$\mathcal{P}(\omega)$ exists", let P be a sufficiently closed arboreal forcing notion, let $\mathbb{Q}(P)^{<\omega}$ be the ω-product of $\mathbb{Q}(P)$ with finite support, let H be a $\mathbb{Q}(P)^{<\omega}$-generic filter, let $\langle T_k : k \in \omega \rangle$ be the generic trees added by H, let $P^{<\omega}$ be the ω-product of P with finite support, and let $(J(P, H))^{<\omega}$ be the ω-product of $J(P, H)$ with finite support. Then*

(a) $J(P, H)$ is a sufficiently closed arboreal forcing notion,

(b) $\{T_k : k \in \omega\}$ is a maximal antichain in $J(P, H)$,

(c) *every predense set $D \subseteq P$ in M remains predense in $J(P, H)$, and*

(d) *every predense set $D \subseteq P^{<\omega}$ in M remains predense in $(J(P, H))^{<\omega}$.*

Proof. Cf. [FGK19, Propositions 2.4 & 2.5]. □

We are now ready to give the definition of Jensen forcing. Nowadays, Jensen forcing is usually constructed using a $\diamondsuit$-sequence (cf., e.g., [Jec03, Theorem 28.1]). However, when Jensen forcing was introduced, the $\diamondsuit$-principle had not been defined yet and so Jensen had to do the construction by hand. We follow Jensen's approach and do not use a $\diamondsuit$-sequence because then the construction is easier to generalize. Jensen forcing is defined as the union of a sequence $\langle P_\xi : \xi < \omega_1 \rangle$ of sufficiently closed arboreal forcing notions. We define the sequence recursively in L: let P_0 be the closure of $\mathbb{C}^*$ under finite unions and let γ_0 be the least countable ordinal such that L_{γ_0} is a model of $\mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exits" containing P_0. We assume that for $\xi' < \xi$, we have already defined a pair $(\mathrm{L}_{\gamma_{\xi'}}, P_{\xi'})$ such that $\mathrm{L}_{\gamma_{\xi'}}$ is a model of $\mathsf{ZFC}^- +$"$\mathcal{P}(\omega)$ exits" and $P_{\xi'}$ is a sufficiently closed arboreal forcing notion in $\mathrm{L}_{\gamma_{\xi'}}$. If ξ is a limit ordinal, then we set $P_\xi := \bigcup_{\xi' < \xi} P_{\xi'}$. If otherwise $\xi = \xi' + 1$, we set $P_\xi := J(P_{\xi'}, H)$, where H is the $<_\mathrm{L}$-least $\mathbb{Q}(P_{\xi'})^{<\omega}$-generic filter over $\mathrm{L}_{\gamma_{\xi'}}$. In both cases, let γ_ξ be the least countable ordinal such that $P_\xi \in \mathrm{L}_{\gamma_\xi}$ and $\omega^\omega \cap \mathrm{L}_{\gamma_\xi+1} \not\subseteq \mathrm{L}_{\gamma_\xi}$. Finally, we define Jensen forcing as $\mathbb{J} := \bigcup_{\xi<\omega_1} P_\xi$.

Note that Jensen forcing is an arboreal forcing notion. Hence, every $\mathbb{J}$-generic filter over L is uniquely determined by a real. We call such reals *Jensen reals over* L. Jensen forcing has the special property that after forcing with Jensen forcing over L there is only one Jensen real over L.

Theorem 3.4.5 (Jensen)**.**

(a) *Jensen forcing satisfies the c.c.c.*

(b) *The set of Jensen reals over* L *is* Π^1_2 *in every transitive model of* ZFC *containing* L.

(c) *For every* $\mathbb{J}$*-generic filter* G *over* L*, the set of Jensen reals over* L *is a singleton in* $\mathrm{L}[G]$.

Proof. Cf. [Jen70, Lemmas 6 & 10 and Corollary 9]. □

A model of ZFC containing a non-constructible Δ^1_3 real can now be obtained as follows: let G be a $\mathbb{J}$-generic filter over L and let x_G be the corresponding Jensen generic real over L. By Theorem 3.4.5, x_G is the only Jensen real over L in $\mathrm{L}[G]$ and the set of Jensen reals over L is Π^1_2. Hence, $\{x_G\}$ is Π^1_2 in $\mathrm{L}[G]$ and so x_G is Δ^1_3. Therefore, $\mathrm{L}[G]$ contains a non-constructible Δ^1_3 real.

3.4.2 Jensen-like forcing notions

In this section, we generalize Jensen's construction to obtain forcing notions which share many properties with Jensen forcing, except that the set of generic reals over L has not necessarily complexity Π^1_2, but Π^1_n for some $n \geq 2$. We shall then use these Jensen-like forcing notions in Section 3.4.3 to construct n-slicing forcing notions. In fact, we shall show that a subset of the ω-slice-product of Jensen forcing with finite support of length ω_1 is 2-slicing. In order to increase the complexity of the set of generic reals, we have to understand why the set of Jensen reals over L has complexity Π^1_2. The rough idea is that a real is a Jensen real over L if and only if for every $\xi < \omega_1$, there is some $k \in \omega$ such that $x \in [T^\xi_k]$, where $\langle T^\xi_k : k \in \omega \rangle$ is the sequence of $\mathbb{Q}(P_\xi)$-generic trees used in the construction of $\mathbb{J}$ to define $P_{\xi+1}$. Moreover, the sequence $\langle\langle T^\xi_k : k \in \omega\rangle : \xi < \omega_1\rangle$

was constructed by always picking the $<_{\mathrm{L}}$-least $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter. Thus, this sequence is Δ_1^{HC} and so the set of Jensen reals over L is Π_1^{HC}. So to increase the complexity of the set of generic reals, we have to increase the complexity of the sequence.

Definition 3.4.6. Let $\zeta \leq \omega_1$. We say that $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle$ is a *Jensen-sequence* if for every $\xi < \zeta$,

(a) P_0 is the closure of $\{(2^{<\omega})_s : s \in 2^{<\omega}\}$ under finite unions,

(b) γ_ξ is a countable ordinal such that $\mathrm{L}_{\gamma_\xi} \models \mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists" and $\omega^\omega \cap \mathrm{L}_{\gamma_\xi+1} \nsubseteq \mathrm{L}_{\gamma_\xi}$,

(c) P_ξ is a sufficiently closed arboreal forcing notion in L_{γ_ξ},

(d) if $\xi = \xi' + 1$, then there is a $\mathbb{Q}(P_{\xi'})^{<\omega}$-generic filter $H_{\xi'} \in \mathrm{L}_{\gamma_\xi}$ over $\mathrm{L}_{\gamma_{\xi'}}$ such that $P_\xi = J(P_{\xi'}, H_{\xi'})$, and

(e) if ξ is a limit, then $P_\xi = \bigcup_{\xi' < \xi} P_{\xi'}$.

A forcing notion $\mathbb{P}$ is *Jensen-like* if there is a Jensen-sequence $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$.

Note that Jensen-like forcing notions are arboreal forcing notions. Hence, every generic filter over L for a Jensen-like forcing notion is uniquely determined by a real. In the following, we show that Jensen-like forcing notions have the same properties as Jensen forcing, except that the set of $\mathbb{P}$-generic reals does not necessarily have complexity Π_2^1. It should be noted that most proofs in this Section are not new, but slight modifications of the proofs for ordinary Jensen forcing.

Lemma 3.4.7. *Let $\mathbb{P}$ be a Jensen-like forcing notion, let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$, and let $\xi < \omega$. Then every predense set $D \subseteq P_\xi$ in L_{γ_ξ} is predense in $\mathbb{P}$. Moreover, the same is true for $(P_\xi)^{<\omega}$ and $\mathbb{P}^{<\omega}$, where $(P_\xi)^{<\omega}$ and $\mathbb{P}^{<\omega}$ are the ω-products with finite support of P_ξ and $\mathbb{P}$, respectively.*

Proof. Let $D \subseteq P_\xi$ be a predense set in L_{γ_ξ}. We suppose for a contradiction that there is some $p \in \mathbb{P}$ which is incompatible with every $q \in D$. By definition, there is some $\xi' < \omega$ such that $\xi \leq \xi'$ and $q \in P_{\xi'}$. Then D is not predense in $P_{\xi'}$. Let $\zeta < \omega_1$ be minimal such that $\xi < \zeta$ and D is not predense in P_ζ. Then ζ is a successor ordinal. Let ζ' be the predecessor of ζ. Then D is not predense in $P_{\zeta'}$ and since $\xi \leq \zeta'$, D is in $\mathrm{L}_{\gamma_{\zeta'}}$. By Lemma 3.4.4, D is predense in P_ζ. But this is a contradiction. Therefore, D is predense in $\mathbb{P}$. The case where $D \subseteq P_\xi^{<\omega}$ is similar. ☐

Let $\mathbb{P}$ be a Jensen-like forcing notion and let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$. By Lemma 3.4.7, for every $\mathbb{P}$-generic filter G over L and every $\xi < \omega_1$, $G \cap P_\xi$ is a P_ξ-generic filter over L_{γ_ξ}. Hence, every $\mathbb{P}$-generic real over L is P_ξ-generic over L_{γ_ξ} for every $\xi < \omega_1$. We shall see in Proposition 3.4.10 that the converse is also true. But before we can prove this, we must first show that every Jensen-like forcing notion satisfies the c.c.c.

Proposition 3.4.8. *Let $\mathbb{P}$ be a Jensen-like forcing notion and let $\mathbb{P}^{<\omega}$ be the ω-product of $\mathbb{P}$ with finite support. Then both $\mathbb{P}$ and $\mathbb{P}^{<\omega}$ satisfy the c.c.c.*

Proof. It is enough to show that $\mathbb{P}^{<\omega}$ satisfies the c.c.c. Let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$ and let $\mathcal{A} \subseteq \mathbb{P}^{<\omega}$ be a maximal antichain. Since $\mathbb{P}^{<\omega}$ and $\mathcal{A}$ are subsets of L_{ω_1}, there are elements of L_{ω_2}. Let X be a countable elementary submodel of L_{ω_2}

containing $\langle P_\xi : \xi < \omega_1 \rangle$, $\mathbb{P}$, $\mathbb{P}^{<\omega}$, and $\mathcal{A}$, let M be the transitive collapse of X, and let $\pi : X \to M$ be the collapsing isomorphism. By Theorem 1.2.31, there is a countable ordinal ζ such that $M = \mathrm{L}_\zeta$. Moreover, there is some ordinal $\xi < \zeta$ such that $\mathrm{L}_{\omega_1} \cap X = \mathrm{L}_\xi$. Then $\pi(\omega_1) = \xi$ and for every $x \in X$, if $x \in \mathrm{L}_{\omega_1}$, then $\pi(x) = x$ and if $x \subseteq \mathrm{L}_{\omega_2}$, then $\pi(x) = x \cap \mathrm{L}_\xi$. By elementarity, in X, the domain of $\langle P_\xi : \xi < \omega_1 \rangle$ is ω_1 and $\mathbb{P} = \bigcup_{\xi' < \omega_1} P_{\xi'}$. Hence, for every $\xi' < \xi$, $P_{\xi'} \in X$ and so

$$\pi(\mathbb{P}) = \bigcup_{\xi' < \xi} \pi(P_{\xi'}) = \bigcup_{\xi' < \xi} P_{\xi'} = P_\xi.$$

Let $\mathcal{A}' := \pi(A)$. Then $\mathcal{A}'$ is countable in L and by elementarity, $\mathcal{A}'$ is a maximal antichain in $(P_\xi)^{<\omega}$. Since $\mathrm{L}_\xi = \mathrm{L}_{\omega_1} \cap X$, $\omega^\omega \cap \mathrm{L}_\zeta \subseteq \mathrm{L}_\xi$. By definition, $\gamma_\xi \geq \xi$ and $\omega^\omega \cap \mathrm{L}_{\gamma_\xi + 1} \nsubseteq \mathrm{L}_{\gamma_\xi}$. Thus, $\gamma_\xi \geq \zeta$ and so $\mathcal{A}'$ is in L_{γ_ξ}. By Lemma 3.4.7, $\mathcal{A}'$ remains predense in $\mathbb{P}^{<\omega}$. Therefore, $\mathcal{A} = \mathcal{A}'$ and so $\mathcal{A}$ is countable. □

Corollary 3.4.9. *Let $\mathbb{P}$ be a Jensen-like forcing notion and $\mathbb{Q}$ be the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1. Then $\mathbb{Q}$ satisfies the c.c.c.*

Proof. We suppose for a contradiction that there is an antichain $\mathcal{A} \subseteq \mathbb{Q}$ which is uncountable. By the Δ-system Lemma (cf. [Kun11, Lemma III.2.6]), there is an uncountable set $\mathcal{A}' \subseteq \mathcal{A}$ and a finite set $F \subseteq \omega_1 \times \omega$ such that for every $q \neq q' \in \mathcal{A}$, $\mathrm{dom}(q) \cap \mathrm{dom}(q') = F$. Since $\mathcal{A}'$ is an antichain, for every $q \neq q' \in \mathcal{A}$, $q \restriction F$ and $q' \restriction F$ are incompatible. Hence, $\mathcal{A}'' := \{q \restriction F : q \in \mathcal{A}'\}$ is an uncountable antichain in $\mathbb{Q} \restriction F$. But this contradicts Proposition 3.4.8. □

Let $\mathbb{P}$ be a Jensen-like forcing notion and let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$. By Proposition 3.4.8, $\mathbb{P}$ satisfies the c.c.c. Hence, every antichain in $\mathbb{P}$ is a countable set of trees and so in HC. By definition, the sequence $\langle \gamma_\xi : \xi < \omega_1 \rangle$ is strictly increasing. Therefore, $\bigcup_{\xi < \omega} \mathrm{L}_{\gamma_\xi} = \mathrm{L}_{\omega_1} = \mathrm{HC}$ and so for every maximal antichain $\mathcal{A}$ in $\mathbb{P}$, there is some $\xi < \omega_1$ such that $\mathcal{A}$ is a maximal antichain in P_ξ and $\mathcal{A}$ is in L_{γ_ξ}. We can use this fact to prove the following characterization for $\mathbb{P}$-generic reals.

Proposition 3.4.10. *Let $\mathbb{P}$ be a Jensen-like forcing notion, let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$, and for every $\xi < \omega_1$, let $\langle T^\xi_k : k \in \omega \rangle$ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic sequence which was used to construct $P_{\xi+1}$. Then the following are equivalent:*

(a) a real x is $\mathbb{P}$-generic over L,

(b) for every $\xi < \omega_1$, x is P_ξ-generic over L_{γ_ξ}, and

(c) for every $\xi < \omega_1$, there is some $k \in \omega$ such that $x \in [T^\xi_k]$.

Proof. First, we show that (a) implies (c). Let $G_x := \{S \in \mathbb{P} : x \in [S]\}$ and let $\xi < \omega_1$. Then G_x is $\mathbb{P}$-generic over L. By Lemma 3.4.4, $\{T^\xi_k : k \in \omega\}$ is a maximal antichain in $P_{\xi+1}$ and by Lemma 3.4.7 it remains predense in $\mathbb{P}$. Hence, there is a $k \in \omega$ such that $T^\xi_k \in G_x$ and so $x \in [T^\xi_k]$.

Next, we show that (c) implies (b). Let $\xi < \omega_1$, let $G^\xi_x := \{S \in P_\xi : x \in [S]\}$, and let $D \subseteq P_\xi$ be a dense set in L_{γ_ξ}. We have to show that G^ξ_x meets D. By assumption, there is a $k \in \omega$ such that $x \in [T^\xi_k]$. We define

$$E := \{q \in \mathbb{Q}(P_\xi)^{<\omega} : q(k) = (S, n) \to \forall s \in 2^n \cap S(S_s \in D)\}.$$

Let $q \in \mathbb{Q}(P_\xi)^{<\omega}$ and let $q(k) = (S, n)$. Then for every $s \in 2^n \cap S$, $S_s \in P_\xi$. Since D is dense, for every $s \in 2^n \cap S$, there is a condition in D which is stronger than S_s. Hence, there is a perfect tree S' such that for every $s \in 2^n \cap S$, $S'_s \leq S_s$ and $S'_s \in D$. Since P_ξ is closed under finite unions, $S' \in P_\xi$. We define $q' \in \mathbb{Q}(P_\xi)$ by

$$q'(\ell) := \begin{cases} S'_s & \text{if } \ell = k, \\ q(\ell) & \text{otherwise.} \end{cases}$$

Then $q' \leq q$ and $q' \in E$. Hence, E is dense in $\mathbb{Q}(P_\xi)^{<\omega}$. Let H_ξ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter corresponding to $\langle T_k^\xi : k \in \omega \rangle$. Then there is a $q \in H_\xi \cap E$. Let $(S, n) \in \mathbb{Q}(P_\xi)$ such that $q(k) = (S, n)$ and let $s \in 2^n \cap S$ such that $s \subseteq x$. Then $x \in [S_s]$ and $S_s \in D$. Therefore, $S_s \in G_x^\xi \cap D$ and so x is P_ξ-generic over L_{γ_ξ}.

Finally, we show that (b) implies (a). Let $G_x := \{S \in \mathbb{P} : x \in [S]\}$ and let $\mathcal{A} \subseteq \mathbb{P}$ be a maximal antichain. We have to show that G_x meets $\mathcal{A}$. By Proposition 3.4.8, $\mathbb{P}$ satisfies the c.c.c. and so $\mathcal{A}$ is in HC. Hence, there is some $\xi < \omega_1$ such that $\mathcal{A} \subseteq P_\xi$ and $\mathcal{A} \in \mathrm{L}_{\gamma_\xi}$. Since x is P_ξ-generic over L_{γ_ξ}, $G_x^\xi := \{S \in P_\xi : x \in [S]\}$ is P_ξ-generic over L_{γ_ξ}. Hence, there is some $S \in \mathcal{A}$ such that $x \in [S]$. Therefore, x is $\mathbb{P}$-generic over L. □

By Proposition 3.4.10, we can reduce the complexity of the set of generic reals for Jensen-like forcing notions to the complexity of the Jensen-sequence.

Corollary 3.4.11. *Let $\mathbb{P}$ be a Jensen-like forcing notion and let $\langle(\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1\rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi<\omega_1} P_\xi$. For every $n > 1$, if $\langle(\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1\rangle$ is $\Delta^{\mathrm{HC}}_{n-1}$, then the set of $\mathbb{P}$-generics is Π^1_n in every transitive model of* ZFC *containing* L.

Proof. Let M be a model of ZFC containing L. By Proposition 3.4.10, a real is $\mathbb{P}$-generic over L if and only if x is P_ξ-generic over L_{γ_ξ}. Note that ω_1^{L} and $\mathrm{L}_{\omega_1^{\mathrm{L}}}$ are Σ^{HC}_1 in M. Hence, $\langle(\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1^{\mathrm{L}}\rangle$ is $\Delta^{\mathrm{HC}}_{n-1}$ in M and so the set of $\mathbb{P}$-generic reals over L is $\Delta^{\mathrm{HC}}_{n-1}$ in M. By Theorem 1.2.38, every Π^{HC}_{n-1} set of reals is Π^1_n. Therefore, the set of $\mathbb{P}$-generic reals over L is Π^1_n in M. □

It remains to show that Jensen-like forcing notions add only a single generic real. To prove this, we use the following proposition about the two step product of Jensen-like forcing notions.

Proposition 3.4.12. *Let $\mathbb{P}$ be a Jensen-like forcing notion and let x, y be $\mathbb{P}$-generic reals over* L*. If $x \neq y$, then (x, y) is $(\mathbb{P} \times \mathbb{P})$-generic over* L*.*

Proof. Let $\langle(\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1\rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi<\omega_1} P_\xi$. With a similar argument as in Proposition 3.4.10, it is enough to show that for every $\xi < \omega_1$, (x, y) is $(P_\xi \times P_\xi)$-generic over L_{γ_ξ}. Let $\xi < \omega_1$, let $G^\xi := \{(S, S') \in P_\xi \times P_\xi : x \in [S] \wedge y \in [S']\}$, and let $D \subseteq P_\xi \times P_\xi$ be an open dense set in L_{γ_ξ}. We have to show that G^ξ meets D. Let $m \in \omega$ such that $x{\restriction}m \neq y{\restriction}m$ and let E be the set of all conditions $q \in \mathbb{Q}(P_\xi)^{<\omega}$ such that

(a) for every $\ell \in \mathrm{dom}(q)$, if $q(\ell) = (S, n)$, then $n > m$, and

(b) for every $\ell, \ell' \in \mathrm{dom}(q)$, if $q(\ell) = (S, n)$ and $q(\ell') = (S', n')$, then for every $s \in 2^n \cap S$ and $s' \in 2^{n'} \cap S'$, $(S_s, S'_{s'}) \in D$ or $\ell = \ell'$ and $s = s'$.

Since D is open dense in $P_\xi \times P_\xi$, E is dense in $\mathbb{Q}(P_\xi)^{<\omega}$. Let H_ξ the $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter over L_{γ_ξ} which was used to construct $P_{\xi+1}$ and let $\langle T^\xi_k : k \in \omega\rangle$ be the corresponding $\mathbb{Q}(P_\xi)^{<\omega}$-generic sequence. Then there is some $q \in H_\xi \cap E$. By Proposition 3.4.10, there are $k, k' \in \omega$ such that $x \in [T^\xi_k]$ and $y \in [T^\xi_{k'}]$. Let $(S, n), (S', n') \in \mathbb{Q}(P_\xi)$ such that $q(k) = (S, n)$ and $q(k') = (S', n')$. We define $s := x \restriction n$ and $s' := y \restriction n'$. Then $x \in [S_s]$, $y \in [S'_{s'}]$, and $(S_s, S'_{s'}) \in D$. Therefore, G^ξ meets D and so (x, y) is $(\mathbb{P} \times \mathbb{P})$-generic over L. □

Corollary 3.4.13. *Let $\mathbb{P}$ be a Jensen-like forcing notion. For every $\mathbb{P}$-generic filter G over* L*, the set of $\mathbb{P}$-generic reals over* L *is a singleton in* $\mathrm{L}[G]$.

Proof. Let G be a $\mathbb{P}$-generic filter over L, let x be a $\mathbb{P}$-generic real corresponding to G, and let y be another $\mathbb{P}$-generic real over L such that $x \neq y$. By Proposition 3.4.12, (x, y) is $(\mathbb{P} \times \mathbb{P})$-generic over L. Then y is $\mathbb{P}$-generic over $\mathrm{L}[G]$ and so $y \notin \mathrm{L}[G]$. □

In [KL17], Kanovei and Lyubetsky generalized Jensen's uniqueness property of the Jensen generic reals to products. More precisely, let $\mathbb{J}^{<\omega}$ be the ω-product of $\mathbb{J}$ with finite support. They showed that for every $\mathbb{J}^{<\omega}$-generic filter G over L, a real $y \in \mathrm{L}[G]$ is a Jensen real over L if and only if there is some $k \in \omega$ such that $y = x^k_G$, where x^k_G is the Jensen real which was added by the kth coordinate of G. We show that the same is true for Jensen-like forcing notions. To do this, we use a lemma from Kanovei and Lyubetsky's proof.

Lemma 3.4.14 (Kanovei-Lyubetsky)**.** *Let M be a countable transitive model of* ZFC^- + "$\mathcal{P}(\omega)$ *exists", let $P \in M$ be a sufficiently closed arboreal forcing notion in M, let $P^{<\omega}$ be the ω-product of P with finite support, for every $k \in \omega$, let $\dot{x}^k_G$ be the canonical $P^{<\omega}$-name for the P-generic real which is added by the kth coordinate of a $P^{<\omega}$-generic filter, and let $\dot{y} \in M$ be a $P^{<\omega}$-name for a real such that for every $k \in \omega$, $1_{P^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$. Then for every $\mathbb{Q}(P)^{<\omega}$-filter H over M, in $M[H]$, the set of conditions forcing that $\dot{y} \notin [T_k]$ is dense in $(J(P, H))^{<\omega}$, where $\langle T_k : k \in \omega\rangle$ is the sequence of $\mathbb{Q}(P)$-generic trees corresponding to H.*

Proof. Let $p \in (J(P, H))^{<\omega}$ and let $m \in \omega$. We have to find a condition below p which forces $\dot{y} \notin [T_m]$. By Lemma 3.4.4, $\{T_k : k \in \omega\}$ is dense in $J(P, H)$. Hence, we can assume without loss of generality that for every $k \in \mathrm{dom}(p)$, there are $\ell_k \in \omega$ and $S_k \in P$ such that $p(k) = T_{\ell_k} \wedge S_k$. Moreover, we can assume that there is some $k \in \mathrm{dom}(p)$ such that $\ell_k = m$.

Claim 3.4.15. *There is a condition $q \in H$ such that $\{\ell_k : k \in \mathrm{dom}(p)\} \subseteq \mathrm{dom}(q)$ and for every $k \in \mathrm{dom}(p)$, there is some $s_k \in R_{\ell_k}$ such that*

(a) $\mathrm{lh}(s_k) = n_{\ell_k}$*,*

(b) if $k' \in \mathrm{dom}(p)$ with $k \neq k'$, then $s_k \neq s_{k'}$, and

(c) $(R_{\ell_k})_{s_k} \leq S_k$*,*

where $R_{\ell_k} \in P$ and $n_{\ell_k} \in \omega$ such that $q(\ell_k) = (R_{\ell_k}, n_{\ell_k})$.

Proof. Since $T_{\ell_k} \wedge S_k \neq \emptyset$ for every $k \in \omega$, there is some $q \in H$ such that $\{\ell_k : k \in \mathrm{dom}(p)\} \subseteq \mathrm{dom}(q)$ and for every $k \in \mathrm{dom}(p)$, $q \Vdash \dot{T}_{\ell_k} \wedge \check{S}_k \neq \emptyset$, where $\dot{T}_{\ell_k}$ is the canonical $\mathbb{Q}(P)^{<\omega}$-name for T_{ℓ_k}. It is enough to show that the set of conditions such that there are s_k satisfying (a), (b), and (c) is dense below q. Let $q' \leq q$, let $\ell \in \mathrm{dom}(q')$ and let $A_\ell := \{k \in \mathrm{dom}(p) : \ell_k = \ell\}$. Then there are $R_\ell \in P$ and $n_\ell \in \omega$ such that $q'(\ell) = (R_\ell, n_\ell)$. Since $q' \Vdash \dot{T}_{\ell_k} \wedge \check{S}_k \neq \emptyset$ for every $k \in \mathrm{dom}(p)$, for every

$k \in A_\ell$, there is an $s_k \in R_\ell$ such that $\mathrm{lh}(s_k) \geq n_\ell$ and $(R_\ell)_{s_k} \wedge S_k \neq \emptyset$. Without loss of generality, we can assume that for every $k \neq k' \in A_\ell$, $\mathrm{lh}(s_k) = \mathrm{lh}(s_{k'})$ and $s_k \neq s_{k'}$. Let n'_ℓ be this length and let R'_ℓ be the tree we get by replacing $(R_\ell)_{s_k}$ with $(R_\ell)_{s_k} \wedge S_k$ in R_ℓ for every $k \in A_\ell$. We define $q'' \leq q'$ by

$$q''(\ell) := \begin{cases} (R'_\ell, n'_\ell) & \text{if } A_\ell \neq \emptyset, \\ q'(\ell) & \text{otherwise.} \end{cases}$$

Then q'' satisfies (a), (b), and (c). □

Let $q \in H$ be the condition from Claim 3.4.15. Next, we construct a condition $r_q \in P^{<\omega}$ such that

(a) for every $k \in \mathrm{dom}(p)$, $r_q(k) \leq (R_{\ell_k})_{s_k}$,

(b) there is a finite set $A_q \subseteq \mathrm{dom}(r_q)$ such that for every $s \in R_m \cap 2^{n_m}$, there is some $k \in A_q$ such that $r_q(k) \leq (R_m)_s$, and

(c) for every $k \in \mathrm{dom}(p) \cup A_q$, $r_q \Vdash \dot{y} \notin [r_q(k)]$.

Let $B := (R_m \cap 2^{n_m}) \setminus \{s_k : \ell_k = m\}$, let $A \subseteq \omega \setminus \mathrm{dom}(p)$ such that $|A| = |B|$, and let $\{t_k : k \in A\}$ be an enumeration of B. We define $r \in P^{<\omega}$ by

$$r(k) := \begin{cases} (R_{\ell_k})_{s_k} & \text{if } k \in \mathrm{dom}(p), \\ (R_m)_{t_k} & \text{if } k \in A. \end{cases}$$

Since $1_{P^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$ for every $k \in \omega$, there is a condition $r_q \leq r$ such that for every $k \in \mathrm{dom}(p) \cup A$, $r_q \Vdash \dot{y} \notin [r_q(k)]$. Then r_q satisfies (a), (b), and (c) with $A_q := A \cup \{k \in \mathrm{dom}(p) : \ell_k = m\}$.

Now we can use r_q to define a condition $\overline{q} \leq q$. Let $\ell \in \mathrm{dom}(q)$ such that $A_\ell := \{k \in \mathrm{dom}(p) : \ell_k = \ell\} \neq \emptyset$ and let R'_ℓ be the tree we get by replacing $(R_\ell)_{s_k}$ with $r_q(k)$ in R_ℓ for every $k \in A_\ell$ and if $\ell = m$, then we also replace $(R_\ell)_{t_k}$ with $r_q(k)$ for every $k \in A$. We define $\overline{q} \leq q$ by

$$\overline{q}(\ell) := \begin{cases} (R'_\ell, n_\ell) & \text{if } A_\ell \neq \emptyset, \\ q(\ell) & \text{otherwise.} \end{cases}$$

Similarly, we can construct such a condition for every $q' \leq q$. Let $D \subseteq \mathbb{Q}(P)^{<\omega}$ be the set of all of these conditions. Then D is dense below q. Since H is $\mathbb{Q}(P)^{<\omega}$-generic, $G \cap D \neq \emptyset$. Without loss of generality, we can assume that $\overline{q} \in H$. Then for every $k \in \mathrm{dom}(p)$, $(T_{\ell_k})_{s_k} \leq r_q(k) \wedge S_n$. Hence, p and r_q are compatible.

To complete the proof, it is enough to show that $r_q \Vdash \dot{y} \notin [T_m]$. Let G be a $(J(P,H))^{<\omega}$-generic filter over $M[H]$ containing r_q. By Lemma 3.4.4, every predense subset of $P^{<\omega}$ in M remains predense in $(J(P,H))^{<\omega}$. Hence, $G' := G \cap P^{<\omega}$ is a $P^{<\omega}$-generic filter over M containing r_q. Then, in $M[G']$, for every $k \in \mathrm{dom}(p) \cup A_q$, $\dot{y}_{G'} \notin [r_q(k)]$. By absoluteness, the same is true in $M[H][G]$. Since $\overline{q} \in H$, for every $s \in T_m \cap 2^{n_m}$, there is some $k \in A_q$ such that $(T_m)_s \leq r_q(k)$. Therefore, $\dot{y}_G \notin [T_m]$ and so $r_q \Vdash \dot{y} \notin [T_m]$. □

Proposition 3.4.16. *Let $\mathbb{P}$ be a Jensen-like forcing notion, let $\mathbb{P}^{<\omega}$ be the ω-product of $\mathbb{P}$ with finite support, let G be a $\mathbb{P}^{<\omega}$-generic filter over* L, *and let $\langle x^k_G : k \in \omega \rangle$ be the sequence of $\mathbb{P}$-generic reals corresponding to G. Then for every $\mathbb{P}$-generic real $y \in \mathrm{L}[G]$ over* L, *there is some $k \in \omega$ such that $y = x^k_G$.*

Proof. We suppose for a contradiction that there is a $y \in \mathrm{L}[G]$ such that y is $\mathbb{P}$-generic over L and for every $k \in \omega$, $y \neq x^k_G$. By Proposition 3.4.8, $\mathbb{P}$ satisfies the c.c.c. Hence, there is a countable $\mathbb{P}^{<\omega}$-name $\dot{y}$ for y. Let $\dot{x}^k_G$ be the canonical $\mathbb{P}^{<\omega}$-name for the $\mathbb{P}$-generic real which is added by the kth coordinate of a $\mathbb{P}^{<\omega}$-generic filter. Without loss of generality, we can assume that for every $k \in \omega$, $1_{\mathbb{P}^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$. Let $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \omega_1 \rangle$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi < \omega_1} P_\xi$, let X be a countable elementary submodel of L_{ω_2} containing $\langle P_\xi : \xi < \omega_1 \rangle$, $\mathbb{P}$, $\mathbb{P}^{<\omega}$, $\dot{y}$, and every $\dot{x}^k_G$, let M be the transitive collapse of X, and let $\pi : X \to M$ be the collapsing isomorphism. By Theorem 1.2.31, there is a countable ordinal ζ such that $M = \mathrm{L}_\zeta$. Moreover, there is some ordinal $\xi < \zeta$ such that $\mathrm{L}_{\omega_1} \cap X = \mathrm{L}_\xi$. Then $\pi(\omega_1) = \xi$ and for every $x \in X$, if $x \in \mathrm{L}_{\omega_1}$, then $\pi(x) = x$ and if $x \subseteq \mathrm{L}_{\omega_2}$, then $\pi(x) = x \cap \mathrm{L}_\xi$. Then $\pi(\mathbb{P}) = P_\xi$, $\pi(\dot{y}) = \dot{y}$, and for every $k \in \omega$, $\pi(\dot{x}^k_G) = \dot{x}^k_G \cap (P_\xi)^{<\omega}$. Hence, for every $k \in \omega$, $\pi(\dot{x}^k_G)$ is the canonical $(P_\xi)^{<\omega}$-name for the P_ξ-generic real added by the kth coordinate of G. Let $k \in \omega$. By elementarity, $1_{\mathbb{P}^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$ in X and so $1_{(P_\xi)^{<\omega}} \Vdash \dot{y} \neq \pi(\dot{x}^k_G)$ in L_ζ. With the same argument as in Proposition 3.4.8, $\gamma_\xi \geq \zeta$. Thus, $1_{(P_\xi)^{<\omega}} \Vdash \dot{y} \neq \pi(\dot{x}^k_G)$ in L_{γ_ξ}.

Let $H \in \mathrm{L}_{\gamma_{\xi+1}}$ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter over L_{γ_ξ} such that $(P_\xi)^H = P_{\xi+1}$ and let $\langle T_k : k \in \omega \rangle$ be the sequence of $\mathbb{Q}(P_\xi)$-generic trees corresponding to H. By Proposition 3.4.10, there is some $k \in \omega$ such that $y \in [T_k]$ in $\mathrm{L}[G]$. Then by Lemma 3.4.14, for every $k \in \omega$, the set $D_k := \{p \in P^{<\omega}_{\xi+1} : p \Vdash \dot{y} \notin [T_k]\}$ is dense in $P^{<\omega}_{\xi+1}$. Hence, there is some $p \in G_{\xi+1}$ such that $p \Vdash \dot{y} \notin [T_k]$ and so $\mathrm{L}_{\gamma_{\xi+1}}[G_{\xi+1}] \models y \notin [T_k]$. By analytic absoluteness, $\mathrm{L}[G] \models y \notin [T_k]$. But this is a contradiction. Therefore, there is some $k \in \omega$ such that $y = x^k_G$. ☐

Corollary 3.4.17. *Let $\mathbb{P}$ be a Jensen-like forcing notion, let $\mathbb{Q}$ be the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1, let $\mathcal{I} \subseteq \omega_1 \times \omega$, let G be a $\mathbb{Q}{\restriction}\mathcal{I}$-generic filter over* L, *and for every $i \in \mathcal{I}$, let x^i_G be the $\mathbb{P}$-generic real corresponding to $G{\restriction}\{i\}$. Then for every $\mathbb{P}$-generic real $y \in \mathrm{L}[G]$ over* L, *there is some $i \in \mathcal{I}$ such that $y = x^i_G$.*

Proof. Let $y \in \mathrm{L}[G]$ be $\mathbb{P}$-generic over L. By Corollary 3.4.9, $\mathbb{Q}$ satisfies the c.c.c. and so $\mathbb{Q}{\restriction}\mathcal{I}$ satisfies the c.c.c. as well. Hence, there is a countable $\mathbb{Q}{\restriction}\mathcal{I}$-name $\dot{y}$ for y. Then there is a countably infinite set $\mathcal{J} \subseteq \mathcal{I}$ such that $\dot{y}$ is a $\mathbb{Q}{\restriction}\mathcal{J}$-name. Let $\mathbb{P}^{<\omega}$ be the ω-product of $\mathbb{P}$ with finite support. Since $\mathcal{J}$ is countable, $\mathbb{Q}{\restriction}\mathcal{J}$ is order-isomorphic to $\mathbb{P}^{<\omega}$. Hence, we can apply Proposition 3.4.16 to $\mathbb{Q}{\restriction}\mathcal{J}$. Therefore, there is some $j \in \mathcal{J}$ such that $y = x^j_G$. ☐

3.4.3 Storing Jensen-sequences

In this section, we put everything together to finally prove the Slicing Theorem (Theorem 3.2.17), i.e., we show that for every $n \geq 2$, there are n-slicing forcing notions in L. The general idea is to use Jensen-sequences to define a storage order, and then use the Kanovei-Lyubetsky Lemma (Lemma 3.3.18) to show that there are Jensen-like forcing notions such that the ω-slice-product is n-absolute for slices. We begin by introducing the storage order.

Definition 3.4.18. Let $\mathcal{M}_J$ be the set of all triples (M, P, μ) such that

(a) there is a countable ordinal γ such that $M = \mathrm{L}_\gamma$ and $\mathrm{L}_\gamma \models \mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists",

(b) P is a sufficiently closed arboreal forcing notion in M,

(c) $\mu > 0$ is a countable ordinal in M, and

(d) there is some ordinal $\zeta < \omega_1$ and a Jensen-sequence $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle$ in M such that $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle^\frown (M, P)$ is a Jensen-sequence.

For $(M,P,\mu),(N,Q,\nu) \in \mathcal{M}_J$, we write $(M,P) \preccurlyeq (N,Q)$ if and only if $M \subseteq N$, $\mu \leq \nu$, and either $P = Q$ or there is a Jensen-sequence $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle$ in N with $\zeta < \omega_1$ such that

(a) there is some $\xi < \zeta$ such that $(\mathrm{L}_{\gamma_\xi}, P_\xi) = (M,P)$ and

(b) $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle^\frown (N,Q)$ is a Jensen-sequence.

Recall that a storage order $(\mathcal{M}, \preccurlyeq)$ is simple if for every pair $(M,P) \in \mathcal{M}$, P is an ω-slice-product of a single forcing notion. Hence, for every such P, there is a unique pair (Q_P, μ_P) such that P is the ω-slice-product of Q_P with finite support of length μ_P. From now on, to simplify notation, we identify every pair (M,P) in a simple storage order with the triple (M, Q_P, μ_P). With this identification, $(\mathcal{M}_J, \preccurlyeq_J)$ is a simple storage order.

Lemma 3.4.19. *The pair $(\mathcal{M}_J, \preccurlyeq_J)$ is a simple storage order. Moreover, for every storage sequence $\langle (M_\xi, P_\xi, \mu_\xi) : \xi < \omega_1 \rangle$ in $\mathcal{M}_J$, $\bigcup_{\xi<\omega_1} P_\xi$ is a Jensen-like forcing notion.*

Proof. First, we show that $(\mathcal{M}_J, \preccurlyeq_J)$ is a partial order. It is clear that $\preccurlyeq_J$ is reflexive. Let $(M,P,\mu),(N,Q,\nu) \in \mathcal{M}_J$ such that $(M,P,\mu) \preccurlyeq_J (N,Q,\nu)$ and $(N,Q,\nu) \preccurlyeq_J (M,P,\mu)$, Then $M = N$ and $\mu = \nu$. Moreover, for every Jensen-sequence $\langle (\mathrm{L}_{\gamma_\xi}, P_\xi) : \xi < \zeta \rangle$ and every $\xi' < \xi < \zeta$, $\mathrm{L}_{\gamma_{\xi'}} \subsetneq \mathrm{L}_{\gamma_\xi}$. Hence, $P = Q$ and so $\preccurlyeq_J$ is antisymmetric. Let $(M,P,\mu),(M',P',\mu'),(M'',P'',\mu'') \in \mathcal{M}_J$ such that $(M,P,\mu) \preccurlyeq_J (M',P',\mu') \preccurlyeq_J (M'',P'',\mu'')$. Without loss of generality, we assume $P \neq P' \neq P''$. By definition, $M \subseteq M' \subseteq M''$, $\mu \leq \mu' \leq \mu''$, and there is a Jensen-sequence J in M' such that $\mathrm{lh}(J) < \omega_1$, there is some $\xi < \mathrm{lh}(J)$ such that $J(\xi) = (M,P)$, and $J^\frown(M',P')$ is a Jensen-sequence, and there is a Jensen-sequence J' in M'' such that $\mathrm{lh}(J') < \omega_1$, there is some $\xi' < \mathrm{lh}(J')$ such that $J'(\xi') = (M',P')$, and $J'^\frown(M'',P'')$ is a Jensen-sequence. We define $J'' := J^\frown(J' \restriction \mathrm{lh}(J') \setminus \xi')$. Then J'' is a Jensen-sequence in M'', $J''(\xi) = (M,P)$, and $J''^\frown(M'',P'')$ is a Jensen-sequence. Therefore, $(M,P,\mu) \preccurlyeq_J (M'',P'',\mu'')$ and so $\preccurlyeq_J$ is transitive.

Next, we prove the second part of the lemma. It is enough to show that for every $\zeta \leq \omega_1$ and every strictly $\preccurlyeq_J$-increasing sequence $\langle (M_\xi, P_\xi, \mu_\xi) : \xi < \zeta \rangle$ which is continuous at limits, there is a Jensen sequence which contains $\langle (M_\xi, P_\xi) : \xi < \zeta \rangle$ as an unbounded subsequence. We prove this by induction on ζ. It is clear if $\zeta = 1$. We assume that it is true for all $\zeta' < \zeta$. If $\zeta = \zeta' + 1$, then there is a Jensen sequence J which contains $\langle (M_\xi, P_\xi) : \xi < \zeta' \rangle$ as an unbounded subsequence. We make a case-distinction:

Case 1: $\zeta' = \zeta'' + 1$. Then $\mathrm{lh}(J)$ is a successor ordinal and $J(\mathrm{lh}(J) - 1) = (M_{\zeta''}, P_{\zeta''})$. Since $(M_{\zeta''}, P_{\zeta''})$ is strictly $\preccurlyeq_J$-less than $(M_{\zeta'}, P_{\zeta'})$, there is a Jensen sequence J' such that there is some $\xi < \mathrm{lh}(J')$ such that $J'(\xi) = (M_{\zeta''}, P_{\zeta''})$ and $J'^\frown(M_{\zeta'}, P_{\zeta'})$. Then $J^\frown(J' \restriction \mathrm{lh}(J') \setminus \xi)^\frown(M_{\zeta'}, P_{\zeta'})$ is a Jensen-sequence which contains $\langle (M_\xi, P_\xi) : \xi < \zeta \rangle$ as an unbounded subsequence.

Case 2: ζ' is a limit ordinal. Then $\mathrm{lh}(J)$ is a limit ordinal as well. Since $\langle (M_\xi, P_\xi, \mu_\xi) : \xi < \omega_1 \rangle$ is continuous at limits, $P_{\zeta'} = \bigcup_{\xi<\zeta'} P_\xi$. Hence, $J^\frown(M_{\zeta'}, P_{\zeta'})$ is a Jensen-sequence which contains $\langle (M_\xi, P_\xi) : \xi < \zeta \rangle$ as an unbounded subsequence.

If ζ is a limit, then for every $\zeta' < \zeta$, there is a Jensen sequence $J_{\zeta'}$ which contains $\langle (M_\xi, P_\xi) : \xi < \zeta' \rangle$ as an unbounded subsequence. Let $\alpha < \zeta$ be the largest limit ordinal and for every $\alpha \leq \zeta' < \zeta$, let $\xi_{\zeta'} < \mathrm{lh}(J_{\zeta'+1})$ be such that $J_{\zeta'+1}(\xi_{\zeta'}) = (M_{\zeta'}, P_{\zeta'})$. Then

$$J_\alpha{}^\frown(J_{\alpha+1} \restriction \mathrm{lh}(J_{\alpha+1}) \setminus \xi_\alpha)^\frown \ldots {}^\frown (J_{\alpha+n+1} \restriction \mathrm{lh}(J_{\alpha+n+1}) \setminus \xi_{\alpha+n})^\frown \ldots$$

is a Jensen-sequence which contains $\langle (M_\xi, P_\xi) : \xi < \zeta \rangle$ as an unbounded subsequence.

It remains to show that $(\mathcal{M}_J, \preccurlyeq_J)$ is a storage order. Items (a), (b), and (d) of Definition 3.3.1 are clear. For (c), let $(M,P,\mu) \preccurlyeq_J (N,Q,\nu) \in \mathcal{M}_J$, let P' be the ω-slice product of P with finite support of length μ, and let Q' be the ω-slice product of Q with finite support of length ν. It is enough to show that every predense set $D \subseteq P'$ in M remains predense in Q'. Since $\mu < \omega_1$, P' is order-isomorphic to the ω-product of P with finite support. So we can apply Lemma 3.4.7 here. By the lemma, D remains predense in $Q' \restriction \mu \times \omega$ and so D is also predense in Q'.

For (e), let $\zeta \leq \omega_1$ be a limit ordinal and let $\langle (M_\xi, P_\xi, \mu_\xi) : \xi < \zeta \rangle$ be a strictly $\preccurlyeq_J$-increasing sequence which is continuous at limits. We have already shown that there is a Jensen-sequence J which contains $\langle (M_\xi, P_\xi) : \xi < \zeta \rangle$ as an unbounded subsequence. If $\zeta < \omega_1$, we can find a Jensen sequence J' such that $J \subsetneq J'$. Let $(M_\zeta, P_\zeta) := J'(\mathrm{lh}(J))$. Since ζ is a limit ordinal, $\mathrm{lh}(J)$ is a limit as well. Hence, $P_\zeta = \bigcup_{\xi < \zeta} P_\xi$. Let $\mu_\zeta > \mu_\xi$ for all $\xi < \zeta$. Without loss of generality, we can assume that $\mu_\zeta \in M_\zeta$. Then $(M_\zeta, P_\zeta, \mu_\zeta) \in \mathcal{M}_J$ and for every $\xi < \zeta$, $(M_\xi, P_\xi, \mu_\xi) \preccurlyeq_J (M_\zeta, P_\zeta, \mu_\zeta)$. If $\zeta = \omega_1$, then $\mathbb{P} := \bigcup_{\xi < \omega_1} P_\xi$ is a Jensen-like forcing notion. Hence, we only have to check that predense sets remain predense. Let $\mathbb{Q}$ be the ω-slice product of $\mathbb{P}$ with finite support of length ω_1, let $\xi < \omega_1$, let Q_ξ be the ω-slice product of P_ξ with finite support of length μ_ξ, and let $D \subseteq Q_\xi$ be dense in M_ξ. With a similar argument as above, D is predense in $\mathbb{Q} \restriction \omega \times \mu_\xi$. Hence, D is predense in $\mathbb{Q}$ as well. Therefore, $(\mathcal{M}_J, \preccurlyeq_J)$ is a simple storage order. □

Definition 3.4.20. Let $n \geq 2$. We call a forcing notion $\mathbb{P}$ is *n-Jensen* if

(a) $\mathbb{P}$ is Jensen-like,

(b) the set of $\mathbb{P}$-generic reals over L is Π^1_n in every transitive model of ZFC containing L, and

(c) the ω-slice product of $\mathbb{P}$ with finite support of length ω_1 is n-absolute for slices.

Clearly, Jensen forcing is 2-Jensen. For $n > 2$, we can find n-Jensen forcing notions using Lemma 3.3.10.

Theorem 3.4.21. *For every $n \geq 2$, there is an n-Jensen forcing notion.*

Proof. Since Jensen forcing is 2-Jensen, we can assume that $n > 2$. By Lemma 3.3.10, there is a $\Delta^{\mathrm{HC}}_{n-1}$, n-complete storage sequence $\langle (M_\xi, P_\xi, \mu_\xi) : \xi < \omega_1 \rangle$ in $\mathcal{M}_J$. Then by Lemma 3.4.19, $\mathbb{P} := \bigcup_{\xi < \omega_1} P_\xi$ is a Jensen-like forcing notion and so by Corollary 3.4.11, the set of $\mathbb{P}$-generic reals over L is Π^1_n in every transitive model of ZFC containing L. It remains to show that ω-slice product of $\mathbb{P}$ with finite support of length ω_1 is n-absolute for slices. But this follows directly from the Kanovei-Lyubetsky Lemma (Lemma 3.3.18). Therefore, $\mathbb{P}$ is n-Jensen. □

Let $\mathbb{P}$ be a forcing notions and let $\mathbb{Q}$ be the ω-slice product of $\mathbb{P}$ with finite support of length ω_1. Recall that $\mathbb{P}$ can only be n-slicing if for every $\mathbb{Q}$-generic filter G over L, the set $\{(\ell, x_G^{(\ell,k)}) : k, \ell \in \omega\}$ is Π^1_n in $\mathrm{L}[G]$, where $x_G^{(\ell,k)}$ is the $\mathbb{P}$-generic real which is added by $G \restriction \{(\ell, k)\}$. If $\mathbb{P}$ is an n-Jensen forcing notion, then we only know that the set of $\mathbb{P}$-generic reals over L is Π^1_n in $\mathrm{L}[G]$. However, we can show that every ω-slice-product of an n-Jensen forcing notion with finite support of length ω_1 contains some n-slicing forcing notion as a subset which finally proves the Slicing Theorem.

Proof of the Slicing Theorem (Theorem 3.2.17). Let $n \geq 2$. By Theorem 3.4.21, there is some n-Jensen forcing notion $\mathbb{P}$. For every $\nu < \omega_1$, let

$$\mathbb{P}_\nu := \begin{cases} \{T \in \mathbb{P} : \langle \nu + 1 \rangle \subseteq \mathrm{stem}(T)\} & \text{if } \nu \in \omega, \\ \{T \in \mathbb{P} : \langle 0 \rangle \subseteq \mathrm{stem}(T)\} & \text{otherwise.} \end{cases}$$

We first show that for every $k \in \omega$ and every real $x \in 2^\omega$ with $x(0) = k+1$, x is $\mathbb{P}$-generic over L if and only if x is $\mathbb{P}_k$-generic. Let $k \in \omega$, let $x \in 2^\omega$ such that $x(0) = k+1$, let $G_x := \{T \in \mathbb{P}_k : x \in [T]\}$, and let $H_x := \{T \in \mathbb{P} : x \in [T]\}$. It is enough to show that G_x is $\mathbb{P}_k$-generic over L if and only if H_x is $\mathbb{P}$-generic over L. First, we assume that H_x is $\mathbb{P}$-generic over L. Let $D \subseteq \mathbb{P}_k$ be dense and let $D' := D \cup \{T \in \mathbb{P} : \mathrm{lh}(\mathrm{stem}(T)) \geq 1 \text{ and } T \notin \mathbb{P}_k\}$. Then D' is dense in $\mathbb{P}$ and so there is some $T \in H_x \cap D'$. Since $x \in T$ and $x(0) = k+1$, $T \in \mathbb{P}_k$. Hence, $T \in G_x \cap D'$ and so G_x is $\mathbb{P}_k$-generic over L. Conversely, we assume that G_x is $\mathbb{P}_k$-generic over L. Let $D \subseteq \mathbb{P}$ be dense. Then $D' := \cap \mathbb{P}_k$ is dense in $\mathbb{P}_k$. Hence, there is some $T \in G_x \cap D'$ and so $T \in H_x \cap D$. Therefore, H_x is $\mathbb{P}$-generic over L.

Let $\mathbb{Q}$ be the ω-slice product of $\mathbb{P}$ with finite support of length ω_1 and let $\mathbb{Q}'$ be the ω-slice product of $\langle P_\nu : \nu < \omega_1 \rangle$ with finite support. We show that $\mathbb{Q}'$ is n-slicing. Let G be a $\mathbb{Q}'$-generic filter over L. Note that for every dense set D in $\mathbb{Q}$, $D \cap \mathbb{Q}'$ is dense in $\mathbb{Q}'$. Hence, $H := \{q \in \mathbb{Q} : \exists q' \in G(q' \leq q)\}$ is a $\mathbb{Q}$-generic over L. Since H can be constructed in $\mathrm{L}[G]$ and vice versa, $\mathrm{L}[G] = \mathrm{L}[H]$. With the same argument, we get that for every $\mathcal{I} \subseteq \omega_1 \times \omega$, $\mathrm{L}[G \restriction \mathcal{I}] = \mathrm{L}[H \restriction \mathcal{I}]$. Hence, $\mathbb{Q}'$ is n-absolute for slices. It remains to show that the set $A := \{(\ell, x_G^{(\ell,k)}) : k, \ell \in \omega\}$ is Π^1_n in $\mathrm{L}[G]$, where $x_G^{(\ell,k)}$ is the $\mathbb{P}_\ell$-generic real which is added by $G \restriction \{(\ell,k)\}$. In $\mathrm{L}[G]$, let

$$A' := \{(\ell, x) \in \omega \times 2^\omega : x \text{ is } \mathbb{P}\text{-generic over L} \land x(0) = \ell + 1\}.$$

Then A' is Π^1_n. We show that $A = A'$. Let $(\ell, x) \in A$. Then x is $\mathbb{P}_\ell$-generic over L and $x(0) = \ell+1$. Hence, x is $\mathbb{P}$-generic over L and so $(\ell, x) \in A'$. Now let $(\ell, x) \in A'$. Then x is $\mathbb{P}$-generic over L and $x(0) = \ell + 1$. By Corollary 3.4.17, there is some $k \in \omega$ such that $x = x_H^{(\ell,k)}$, where $x_H^{(\ell,k)}$ is the $\mathbb{P}$-generic real which is added by $H \restriction \{(\ell,k)\}$. Then for every $T \in G \restriction \{(\ell,k)\}$, $x \in [T]$. Hence, $x = x_G^{(\ell,k)}$ and so $(\ell, x) \in A$. Therefore, A is Π^1_n in $\mathrm{L}[G]$. □

3.5 Open questions

We conclude this chapter with a few open questions concerning descriptive choice principles and Jensen-like forcing notions.

Question 3.5.1. *Let $n \geq 1$.*

(a) Are $\mathsf{AC}_\omega(\omega^\omega; \Pi^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \Sigma^1_{n+1})$ equivalent?

(b) Are $\mathsf{AC}_\omega(\omega^\omega; \boldsymbol{\Pi}^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \boldsymbol{\Sigma}^1_{n+1})$ equivalent?

Question 3.5.2. *Let $n \geq 1$.*

(a) Is there a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{Proj})$ which violates $\mathsf{AC}_\omega(\omega^\omega; \Pi^1_1)$?

(b) Is there a model of $\mathsf{ZF} + \mathsf{AC}(\omega^\omega; \boldsymbol{\Pi}^1_n)$ which violates $\mathsf{AC}_\omega(\omega^\omega; \Pi^1_{n+1})$?

(c) Is there a model of $\mathsf{ZF} + \mathsf{AC}(\omega^\omega; \boldsymbol{\Pi}^1_n)$ which violates $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$?

Question 3.5.3. *What are the relationships between the descriptive choice principles for point-classes in the Borel hierarchy?*

Question 3.5.4. *Let $n \geq 2$. Are all n-Jensen forcing notions forcing equivalent?*

Question 3.5.5. *Let $n \geq 2$, let $\mathbb{P}$ be n-Jensen, and let G be a $\mathbb{P}$-generic filter over* L*. Does* $\mathrm{L}[G]$ *contain a non-constructible real of complexity strictly lower than Δ^1_{n+1}?*

Chapter 4

Set-theoretic forcing notions in computability theory

Remarks on co-authorship. *The results of this chapter are, unless otherwise stated, solely due to the author.*

In retrospect, one could say that the first forcing-like construction in computability theory was done by Kleene and Post [KP54], even before the technique of forcing was introduced by Cohen. They used sets which are similar to what we today call Cohen 1-generic reals to show that there is a degree strictly in between the degrees of $\emptyset$ and $\emptyset'$. However, the method of forcing in computability theory was first formalized by Feferman [Fef64], shortly after Cohen's work was published. Since then, forcing and especially Cohen n-generic reals have played an important role in computability theory. However, n-generic reals for other forcing concepts have also been investigated. For example, in [CDHS14], Cholak, Dzhafarov, Hirst, and Slaman studied n-generic reals for a computable version of Mathias forcing. Among other things, they proved that for every $n \geq 3$, every n-generic real for this version of Mathias forcing computes a Cohen n-generic reals. This might be surprising since Miller [Mil81] proved that in set theory Mathias forcing does not add Cohen reals.

In this chapter, we shall investigate computable versions of set-theoretic forcing notions and compare their n-generic reals. More specifically, in Section 4.1, we shall give an introduction to forcing in computability theory, including Section 4.1.2, where we shall take a brief look at arboreal forcing notions in computability theory. In Section 4.2, we shall investigate the relationship between n-generic reals for Cohen forcing and computable versions of Hechler, Laver, Mathias, Miller, Sacks, and Silver forcing and compare the results with the situation in set theory. In particular, we shall show that n-generic for the computable version of Laver forcing computes Cohen n-generic reals, which is also different from set theory. For this purpose, we shall generalize Cholak, Dzhafarov, Hirst, and Slaman's result in Section 4.2.5. Finally, in Section 4.3, we shall briefly summarize our results and state a few open questions.

4.1 Forcing in computability theory

4.1.1 Generic and weakly generic reals

The difference between forcing in set theory and computability theory is that in computability theory forcing is not used to add new elements to some ground model, but to find already existing reals with certain properties. To be more precise, in computability theory, we often have a list of requirements and want to find a real which satisfies all of them. This can usually be done recursively by taking care of the kth requirement at step $k+1$ of the recursion. In many cases, however, these requirements can be reformulated in terms of an initial segment of the real being in a particular set. Hence, every real which is (Cohen) generic for all these sets satisfies the requirements. Moreover, if we only have countably many requirements, then we know that such generic reals exist. Therefore, forcing can be seen as a tool to do many of these constructions at once. In this section, we give an introduction to forcing in computability theory. For more details, we refer the reader to [DM22] and [Sho15].

In computability theory, we usually only consider forcing notions that are countable and whose conditions can be coded as natural numbers. It is common practice to identify these forcing notions with the set of codes of their conditions. Following this practice, we shall usually refer to conditions when we actually mean the codes for these conditions. The typical example of a forcing notion in computability theory is Cohen forcing. Other examples can be obtained from (set theoretic) forcing notions whose conditions can be coded as reals. If $\mathbb{P}$ is such a forcing notion, then we define $\mathbb{P}_\mathrm{c}$ as the set of conditions which are coded by a computable real. To identify $\mathbb{P}_\mathrm{c}$ with a subset of the natural numbers, we code every $p \in \mathbb{P}_\mathrm{c}$ as some $e \in \omega$ such that $x_p = \Phi_e$, where x_p is the real which codes p. Now we define genericity in computability theory. Unlike in set theory, we have to be careful that generic filters exit in V. Therefore, we do not require them to be generic for all sets, but only for those with low complexity.

Definition 4.1.1. Let $\mathbb{P}$ be a forcing notion and let $n \in \omega$. A filter $G \subseteq \mathbb{P}$ is *n-generic for* $\mathbb{P}$ if for every Σ^0_n set $C \subseteq \mathbb{P}$, G meets C or the set of conditions having no extension in C. We say that G is *ω-generic for* $\mathbb{P}$ if it is n-generic for $\mathbb{P}$ for every $n \in \omega$.

In the literature, n-generic filters for Cohen forcing are often just called n-generic. However, this might cause confusion when working with more than one forcing notion at a time. Therefore, in what follows we usually write "Cohen n-generic" when we mean "n-generic for $\mathbb{C}$".

The definition of genericity in computability theory differs from usual the one in set theory. Recall that in set theory a filter is generic if it meets every dense set. So one might have expected a filter to be n-generic if it meets every Σ^0_n dense set. However, such filters are called *weakly n-generic* in computability theory.

Definition 4.1.2. Let $\mathbb{P}$ be a forcing notion and let $n \in \omega$. A filter $G \subseteq \mathbb{P}$ is *weakly n-generic for* $\mathbb{P}$ if G meets every Σ^0_n dense set $D \subseteq \mathbb{P}$. We say that G is *weakly ω-generic for* $\mathbb{P}$ if it is weakly n-generic for $\mathbb{P}$ for every $n \in \omega$.

Proposition 4.1.3. *Let $\mathbb{P}$ be a forcing notion and let $n \in \omega$. Then every n-generic filter for $\mathbb{P}$ is weakly n-generic for $\mathbb{P}$.*

Proof. Let G be an n-generic filter for $\mathbb{P}$ and let $D \subseteq \mathbb{P}$ be dense. Then G meets D or the set of conditions having no extension in D. Since D is dense, there are no conditions which have no extension in D. Hence, G meets D. □

The converse of Proposition 4.1.3 is not true in general. Kurtz showed in [Kur83] that there are weakly Cohen n-generic filters which are not Cohen n-generic. In set theory, however, there is no difference between the "weak" and "strong" definition of genericity.

Proposition 4.1.4. *Let $\mathbb{P}$ be a forcing notion in set theory and let $G \subseteq \mathbb{P}$ be a filter. Then G is $\mathbb{P}$-generic over V if and only if for every set $C \subseteq \mathbb{P}$, G meets C or the set of conditions having no extension in C.*

Proof. The backward direction is analogous to Proposition 4.1.3. Hence, we only have to prove the forward direction. Let $C \subseteq \mathbb{P}$. Then the set $D := C \cup \{p \in \mathbb{P} : \forall q \leq p(q \notin C)\}$ is dense in $\mathbb{P}$ and so G meets D. Hence, G meets C or the set of conditions having no extension in C. □

Note that for Cohen forcing the set D from Proposition 4.1.4 is Σ^0_{n+1}. Hence, every weakly Cohen $(n+1)$-generic filter is Cohen n-generic. The same is true for general forcing notions of low enough complexity.

Proposition 4.1.5. *Let $n \in \omega$ and let $\mathbb{P}$ be a forcing notion such that $\mathbb{P}$ as a set and the ordering are Σ^0_n. Then every weakly $(n+1)$-generic filter for $\mathbb{P}$ is n-generic for $\mathbb{P}$.*

Proof. Let G be a weakly n-generic filter for $\mathbb{P}$ and let $C \subseteq \mathbb{P}$ be a Σ^0_n set. We define

$$D := C \cup \{p \in \mathbb{P} : \forall q \leq p(q \notin C)\}.$$

Then D is a Σ^0_{n+1} dense set. Hence, G meets D and so G meets either C or the set of conditions having no extension in C. Therefore, G is n-generic for $\mathbb{P}$. □

So far, we have only considered n-generic filters. The next step is to define reals for these filters. The idea is the same as in set theory. Recall that in set theory we use functions that map conditions to initial segments of reals to define generic reals. Such functions are called *valuation functions* in computability theory.

Definition 4.1.6. Let $\mathbb{P}$ be a forcing notion. A *valuation function* for $\mathbb{P}$ is a function $v : \mathbb{P} \to 2^{<\omega}$ (or $\omega^{<\omega}$) such that for every $q \leq p \in \mathbb{P}$, $v(p) \subseteq v(q)$ and for every $k \in \omega$, the set $\{p \in \mathbb{P} : \mathrm{lh}(v(p)) > k\}$ is dense.

From now on, we assume that every forcing notion $\mathbb{P}$ is equipped with a valuation function v. For Cohen forcing, we simply use the identity function. As in set theory, we usually define the valuation function by mapping conditions to the longest initial segment of what is already decided by the condition. For example, for forcing notions whose conditions are trees, we use $v(T) = \mathrm{stem}(T)$.

Definition 4.1.7. Let $\mathbb{P}$ be a forcing notion and let $n \in \omega$. A real x is *(weakly) n-generic for* $\mathbb{P}$ if there is a (weakly) n-generic filter G for $\mathbb{P}$ such that $x = \bigcup_{p \in G} v(p)$. We say that x is *(weakly) ω-generic for* $\mathbb{P}$ if it is (weakly) n-generic for $\mathbb{P}$ for every $n \in \omega$.

Next, we show that every n-generic filter defines an n-generic real. However, we cannot prove this for all forcing notions since we need that the complexity of the sets $\{p \in \mathbb{P} : \mathrm{lh}(v(p)) > k\}$ is at most Σ^0_n. To ensure this, we only consider forcing notions of low complexity. Let $n \in \omega$ and let Γ be either Σ^0_n, Π^0_n, or Δ^0_n. We say that a forcing notion $\mathbb{P}$ is Γ if $\mathbb{P}$ as a set and the ordering and valuation function are Γ. Note that what we really mean here is that the set of codes, the ordering on the codes, and the valuation function on the codes are Γ.

Proposition 4.1.8. *Let $n \in \omega$ and let $\mathbb{P}$ be a Δ^0_n forcing notion. If G is a weakly n-generic filter for $\mathbb{P}$, then $\bigcup_{p \in G} v(p)$ is a real.*

Proof. Let G be a weakly n-generic filter for $\mathbb{P}$ and let $x := \bigcup_{p \in G} v(p)$. Since G is a filter, x is a partial function from ω to ω. So we only have to show that $\mathrm{dom}(x) = \omega$. Let $k \in \omega$ and let $D_k := \{p \in \mathbb{P} : \mathrm{lh}(v(p)) > k\}$. Then D_k is a Σ^0_n dense set. Hence, there is some $p \in G$ such that $\mathrm{lh}(v(p)) > k$ and so $k \in \mathrm{dom}(x)$. Therefore, x is a real. $\square$

Note that, similar to in set theory, a real $x \in 2^\omega$ is Cohen n-generic if and only if $G_x := \{s \in \mathbb{C} : s \subseteq x\}$ is a Cohen n-generic filer. However, for a general forcing notion $\mathbb{P}$, it is not necessarily the case that a real x is n-generic for $\mathbb{P}$ if and only if $G_x := \{p \in \mathbb{P} : v(p) \subseteq x\}$ is an n-generic filter for $\mathbb{P}$. For example, if $\mathbb{P}$ consists of trees, then it is possible that there are incompatible trees with the same stem. Hence, G_x is not necessarily a filter. Nevertheless, we can reconstruct the generic filter from the generic real for arboreal forcing notions in set theory. Recall that for an arboreal forcing notion $\mathbb{P}$, a real x is $\mathbb{P}$-generic if and only if $G_x := \{T \in \mathbb{P} : x \in [T]\}$ is a $\mathbb{P}$-generic filter. We want similar characterizations for n-generic reals in computability theory. To obtain these, for a forcing notion $\mathbb{P}$, we say that a real x *satisfies* a condition $p \in \mathbb{P}$ if there is a decreasing sequence of conditions $\langle p_k : k \in \omega\rangle$ such that $p_0 = p$ and $x = \bigcup_{k \in \omega} v(p_k)$. Then for Cohen forcing, a condition $s \in \mathbb{C}$ is satisfied by a real x if and only if $s \subseteq x$, and for an arboreal forcing notion, a condition T is satisfied by x if and only if $x \in [T]$. We say that a forcing notion $\mathbb{P}$ is *separative* if for every $p, q \in \mathbb{P}$ with $p \not\leq q$, there is a $p' \leq p$ such that for every $q' \leq q$ with $\mathrm{lh}(v(p')) < \mathrm{lh}(v(q'))$, $v(p') \not\subseteq v(q')$.

Lemma 4.1.9. *Let $n \in \omega$, let $\mathbb{P}$ be a Δ^0_n separative forcing notion, and let $m > n$. Then a real x is m-generic for $\mathbb{P}$ if and only if $G_x := \{p \in \mathbb{P} : x \text{ satisfies } p\}$ is an m-generic filter for $\mathbb{P}$.*

Proof. The backward direction follows directly from Proposition 4.1.8. We prove the forward direction. Let x be an m-generic real for $\mathbb{P}$ and let G be an m-generic filter for $\mathbb{P}$ such that $x = \bigcup_{p \in G} v(p)$. We show that $G = G_x$. Let $p \in G$. Since $x = \bigcup_{q \in G} v(q)$, there is a decreasing sequence of conditions $\langle p_k : k \in \omega\rangle$ such that $p_0 = p$ and $x = \bigcup_{k \in \omega} v(p_k)$. Hence, x satisfies p and so $p \in G_x$. Conversely, let $p \in G_x$. We define

$$C := \{q \in \mathbb{P} : \forall p' \leq p(\mathrm{lh}(v(q)) < \mathrm{lh}(v(p')) \to v(q) \not\subseteq v(p'))\}.$$

Then C is Π^0_n and so Σ^0_m. Hence, there is a $q \in G$ such that $q \in C$ or q has no extension in C. The former is not possible since p and q are both satisfied by x. Hence, q has no extension in C. Since $\mathbb{P}$ is separative, $q \leq p$. Therefore, $p \in G$ and so $G = G_x$. $\square$

An application of forcing in computability theory is that Cohen 1-generic reals can be used to show that there is a pair of incomparable Turing degrees below $\emptyset'$: it is enough to find $y, z \leq_{\mathrm{T}} \emptyset'$ such that $y \not\leq_{\mathrm{T}} z$ and $z \not\leq_{\mathrm{T}} y$. Let $x \in 2^\omega$ be Cohen 1-generic and let $y, z \in 2^\omega$ such that $y(k) := x(2k)$ and $z(k) := x(2k+1)$ for every $k \in \omega$. We define for every $e \in \omega$, the set $C_e := \{s \in \mathbb{C} : \exists k(\Phi_e^{s_0}(k)\downarrow \neq s_1(k))\}$, where $s_0(k) := s(2k)$ and $s_1(k) := s(2k+1)$. Let $e \in \omega$. Then C_e is a Σ^0_1 set. Hence, there is some $s \subseteq x$ such that either $s \in C_e$ or for every $s' \leq s$, $s' \notin C_e$. In the former case there is some $k \in \omega$ such that $\Phi_e^y(k)\downarrow \neq z(k)$ and in the latter case there is some $k \in \omega$ such that $\Phi_e^y(k)\uparrow$. Therefore, $z \not\leq_{\mathrm{T}} y$ and if we repeat the argument with swapped roles, we obtain $y \not\leq_{\mathrm{T}} z$. It remains to show that $y, z \leq_{\mathrm{T}} \emptyset'$. For this, it would be enough to prove that $\emptyset'$ computes a Cohen 1-generic real. Since there are only countably many Σ^0_1 subsets of $\mathbb{C}$, we can construct Cohen 1-generic filters. With a bit of extra care, we can even construct Cohen 1-generic filters such that the corresponding real is computable in $\emptyset'$.

Proposition 4.1.10. *Let $n \in \omega$ and let $\mathbb{P}$ be a Δ^0_n forcing notion. Then there is some n-generic real for $\mathbb{P}$ which is computable in $\emptyset^{(n)}$.*

Proof. Let $\{C_e : e \in \omega\}$ be a computable enumeration of all Σ^0_n subsets of $\mathbb{P}$. Then

$$D_e := \{p \in \mathbb{P} : p \in C_e \vee \forall q \leq p(q \notin C_e)\}$$

is Δ^0_{n+1} for every $e \in \omega$. We recursively define a decreasing sequence of conditions. Let p_0 be the weakest condition in $\mathbb{P}$. If p_e is already defined, we set p_{e+1} as the condition in D_e which is coded by the least natural number. Then the sequence $\langle p_e : e \in \omega\rangle$ is computable in $\emptyset^{(n)}$. Since $\mathbb{P}$ is Δ^0_n, $x := \bigcup_{e\in\omega} v(p_e)$ is also computable in $\emptyset^{(n)}$. It remains to show that x is n-generic for $\mathbb{P}$. Let $G := \{p \in \mathbb{P} : \exists e(p \geq p_e)\}$. Then G is a n-generic filter for $\mathbb{P}$ and $x = \bigcup_{p\in G} v(p)$. Therefore, x is n-generic for $\mathbb{P}$. ☐

An important tool of forcing in set theory is the forcing relation. It is used to approximate the truth value of statements in the forcing extension from the ground model. The forcing relation in computability theory is used to approximates the truth value of statements about n-generic reals. Before we can define it, we must first introduce the forcing language in computability theory. Unlike in set theory, we use the same forcing language for all forcing notions. Let $\mathcal{FL}_\mathrm{c}$ be the language of second-order arithmetic augmented by a first-order constant symbol $\check{n}$ for every $n \in \omega$ and a second-order constant symbol $\dot{x}$. For a real x and a formula in the language $\mathcal{FL}_\mathrm{c}$, we write $\varphi(x)$ for the formula in the language of second-order arithmetic obtained by replacing $\dot{x}$ by x, and every $\check{n}$ by n. We say that a formula φ in the language $\mathcal{FL}_\mathrm{c}$ is *arithmetical* if $\varphi(x)$ is arithmetical for some x and analogously for Σ^0_n and Π^0_n.

Definition 4.1.11. Let $\mathbb{P}$ be a forcing notion, let $p \in \mathbb{P}$, and let φ be an arithmetical sentence in the language $\mathcal{FL}_\mathrm{c}$. We define the (strong) forcing relation $\Vdash$ recursively on φ:

(a) if φ is bounded, then $p \Vdash \varphi$ if for every real x which satisfies p, $\varphi(x)$ is true,

(b) if φ is not bounded and $\varphi = \psi \vee \chi$, then $p \Vdash \varphi$ if $p \Vdash \psi$ or $p \Vdash \chi$,

(c) if φ is not bounded and $\varphi = \neg\psi$, then $p \Vdash \varphi$ if for every $q \leq p$, $q \nVdash \psi$, and

(d) if $\varphi = \exists n\psi(n)$, then $p \Vdash \varphi$ if there is some $n \in \omega$ such that $p \Vdash \psi(\check{n})$.

Note that in set theory, the roles of $\vee$ and $\wedge$ and $\exists$ and $\forall$ in Definition 4.1.11 are usually reversed. The reason for this is that, in set theory, we usually use the weak forcing because it is equivalent to the semantic forcing relation.

Definition 4.1.12. Let $\mathbb{P}$ be a forcing notion, let $p \in \mathbb{P}$, and let φ be an arithmetical sentence in the language $\mathcal{FL}_\mathrm{c}$. We define the weak forcing relation $\Vdash_\mathrm{w}$ recursively on φ:

(a) if φ is bounded, then $p \Vdash_\mathrm{w} \varphi$ if for every real x which satisfies p, $\varphi(x)$ is true,

(b) if φ is not bounded and $\varphi = \psi \wedge \chi$, then $p \Vdash_\mathrm{w} \varphi$ if $p \Vdash_\mathrm{w} \psi$ and $p \Vdash_\mathrm{w} \chi$,

(c) if φ is not bounded and $\varphi = \neg\psi$, then $p \Vdash_\mathrm{w} \varphi$ if for every $q \leq p$, $q \nVdash_\mathrm{w} \psi$, and

(d) if $\varphi = \forall n\psi(n)$, then $p \Vdash_\mathrm{w} \varphi$ if for every $n \in \omega$, $p \Vdash_\mathrm{w} \psi(\check{n})$.

We shall see later that, as in set theory, both forcing relations in computability theory can be used to approximate the truth (cf. Lemma 4.1.16 and Corollary 4.1.17). However, unlike in set theory, in computability theory it is more common to work with the strong forcing relation because it usually has less complexity. For example, if $s \in \mathbb{C}$ and φ is a Σ^0_n or Π^0_n sentence, then we can compute the complexity of the statements "$s \Vdash \varphi$" and "$s \Vdash_{\mathrm{w}} \varphi$" inductively. If φ is bounded, then $s \Vdash \varphi$ if and only if for every $x \in [s]$, $\varphi(x)$ is true. Since φ is bounded, there is a $k \in \omega$ such that for every $x \in \omega$, $\varphi(x)$ only depends on $x \upharpoonright k$. Hence, whether $s \Vdash \varphi$ can be checked computably if φ is bounded. Hence, the statement "$s \Vdash \varphi$" is Σ^0_1 if φ is Σ^0_1 and so Π^0_1 if φ is Π^0_1. By induction, we get that for every $n \geq 1$, the statement "$s \Vdash \varphi$" is Σ^0_n if φ is Σ^0_n and Π^0_n if φ is Π^0_n. For the weak forcing relation, the "$s \Vdash_{\mathrm{w}} \varphi$" is also Δ^0_1 if φ is bounded and so it is Π^0_1 if φ is Π^0_1. Hence, it is Π^0_2 if φ is Σ^0_1. By induction, we get that for every $n \geq 1$, the statement "$s \Vdash \varphi$" is Π^0_n if φ is Π^0_n and Π^0_{n+1} if φ is Σ^0_n. Therefore, it is often more convenient to work with the strong forcing relation, since n-genericity is defined using Σ^0_n and not Π^0_n sets. Something similar is also true for most of the other forcing notions we consider in the following.

Next, we consider some basic properties of the weak and strong forcing relations in computability theory.

Proposition 4.1.13. *Let $\mathbb{P}$ be a forcing notion, let $p \in \mathbb{P}$, and let φ be an arithmetical sentence in the language $\mathcal{FL}_{\mathrm{c}}$.*

(a) If $q \leq p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.

(b) If $p \Vdash \varphi$, then $p \nVdash \neg\varphi$.

(c) There is some $q \leq p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

(d) If φ is not bounded and $\varphi = \psi \wedge \chi$, then $p \Vdash \varphi$ if and only if $\{q \leq p : q \Vdash \psi\}$ and $\{q \leq p : q \Vdash \chi\}$ are dense below p.

(e) If $\varphi = \forall n \psi(n)$, then $p \Vdash \varphi$ if and only if for every $n \in \omega$, $\{q \leq p : q \Vdash \psi(\check{n})\}$ is dense below p.

(f) We have $p \Vdash_{\mathrm{w}} \varphi$ if and only if $p \Vdash \neg\neg\varphi$.

(g) If $p \Vdash \varphi$, then $p \Vdash_{\mathrm{w}} \varphi$, but not conversely.

(h) If $q \leq p$ and $p \Vdash_{\mathrm{w}} \varphi$, then $q \Vdash_{\mathrm{w}} \varphi$.

(i) If $p \Vdash_{\mathrm{w}} \varphi$, then $p \nVdash_{\mathrm{w}} \neg\varphi$.

(j) There is some $q \leq p$ such that either $q \Vdash_{\mathrm{w}} \varphi$ or $q \Vdash_{\mathrm{w}} \neg\varphi$.

Proof. We start with proving (a) by induction on φ. It is clear if φ is bounded. If φ is not bounded, $\varphi = \neg\psi$, and $p \Vdash \varphi$, then for every $p' \leq p$, $p' \nVdash \psi$. Let $q \leq p$. Then in particular, for every $q' \leq q$, $q' \nVdash \psi$ and so $q \Vdash \varphi$. The cases $\varphi = \psi \vee \chi$ and $\varphi = \exists n \psi(n)$ follow directly from the induction hypothesis.

Items (b) and (c) follow directly from the definition. Next, we prove (d). If φ is not bounded and $\varphi = \psi \wedge \chi$, then we have

$$\begin{aligned}
p \Vdash \psi \wedge \chi &\iff p \Vdash \neg(\neg\psi \vee \neg\chi) \\
&\iff \text{for every } p' \leq p,\ p' \nVdash \neg\psi \vee \neg\chi \\
&\iff \text{for every } p' \leq p,\ p' \nVdash \neg\psi \text{ and } p' \nVdash \neg\chi \\
&\iff \text{for every } p' \leq p, \text{ there are } q, q' \leq p' \text{ such that } q' \Vdash \psi \text{ and } q \Vdash \chi.
\end{aligned}$$

Therefore, $p \Vdash \varphi$ if and only if $\{q \leq p : q \Vdash \psi\}$ and $\{q \leq p : q \Vdash \chi\}$ are dense below p.

Now we prove (e). If $\varphi = \forall n \psi(n)$, then we have

$$\begin{aligned} p \Vdash \forall n \psi(n) &\iff p \Vdash \neg \exists n \neg \psi(n) \\ &\iff \text{for every } p' \leq p,\ p' \nVdash \exists n \neg \psi(n) \\ &\iff \text{for every } p' \leq p \text{ and every } n \in \omega,\ p' \nVdash \neg \psi(\check{n}) \\ &\iff \text{for every } n \in \omega \text{ and every } p' \leq p, \text{ there is a } q \leq p' \text{ such that } q \Vdash \psi(\check{n}). \end{aligned}$$

Therefore, $p \Vdash \varphi$ if and only if for every $n \in \omega$, $\{q \leq p : q \Vdash \psi(\check{n})\}$ is dense below p.

Next, we (f) by induction on φ. It is clear if φ is bounded. If φ is not bounded and $\varphi = \psi \wedge \chi$, then we have

$$\begin{aligned} p \Vdash_{\mathrm{w}} \psi \wedge \chi &\iff p \Vdash_{\mathrm{w}} \psi \text{ and } p \Vdash_{\mathrm{w}} \chi \\ &\iff p \Vdash \neg\neg\psi \text{ and } p \Vdash \neg\neg\chi \\ &\iff \text{for every } p' \leq p, \text{ there are } q, q' \leq p' \text{ such that } q \Vdash \psi \text{ and } q' \Vdash \chi \\ &\iff p \Vdash \psi \wedge \chi. \end{aligned}$$

Moreover, $p \Vdash \neg\neg(\psi \wedge \chi)$ if and only if for every $p' \leq p$, there is a $q \leq p'$ such that $q \Vdash \psi \wedge \chi$. By (a) and (d), the latter is equivalent to $p \Vdash \psi \wedge \chi$. Therefore, $p \Vdash_{\mathrm{w}} \varphi$ if and only if $p \Vdash \neg\neg\varphi$.

If φ is not bounded and $\varphi = \neg\psi$, then we have

$$\begin{aligned} p \Vdash_{\mathrm{w}} \neg\psi &\iff \text{for every } p' \leq p,\ p' \nVdash_{\mathrm{w}} \psi \\ &\iff \text{for every } p' \leq p,\ p' \nVdash \neg\neg\psi \\ &\iff p \Vdash \neg\neg\neg\psi. \end{aligned}$$

Therefore, $p \Vdash_{\mathrm{w}} \varphi$ if and only if $p \Vdash \neg\neg\varphi$.

If $\varphi = \forall n \psi(n)$, then we have

$$\begin{aligned} p \Vdash_{\mathrm{w}} \forall n \psi(n) &\iff \text{for every } n \in \omega,\ p \Vdash_{\mathrm{w}} \psi(\check{n}) \\ &\iff \text{for every } n \in \omega,\ p \Vdash \neg\neg\psi(\check{n}) \\ &\iff \text{for every } n \in \omega \text{ and every } p' \leq p, \text{ there is a } q \leq p' \text{ such that } q \Vdash \psi(\check{n}) \\ &\iff p \Vdash \forall \psi(n). \end{aligned}$$

Moreover, $p \Vdash \neg\neg\forall\psi(n)$ if and only if for every $p' \leq p$, there is a $q \leq p'$ such that $q \Vdash \forall n \psi(n)$. By (a) and (f), the latter is equivalent to $p \Vdash \forall n \psi(n)$. Therefore, $p \Vdash_{\mathrm{w}} \varphi$ if and only if $p \Vdash \neg\neg\varphi$.

Now we prove (g). The first part follows directly from (a) and (f). For the second part, let $s \in \mathbb{C}$ be the empty sequence and let $\varphi := \exists k \dot{x}(k) = 0$. Then $s \nVdash \varphi$, but $s \Vdash_{\mathrm{w}} \varphi$.

Finally, (h), (i), and (j) follow directly from (a), (b), (c), and (f). $\square$

Next, we discuss the Truth Lemma in computability theory. In contrast to set theory, the Truth Lemma in computability theory does not talk about generic filters, but about generic reals. For this purpose, it is common practice to extend the forcing relation to the reals.

Definition 4.1.14. Let $\mathbb{P}$ be a forcing notion, let φ be an arithmetical sentence in the language $\mathcal{FL}_{\mathrm{c}}$, and let x be a real. We write $x \Vdash_{\mathbb{P}} \varphi$ if there is a condition $p \in \mathbb{P}$ such that x satisfies p and $p \Vdash \varphi$.

Roughly speaking, the Truth Lemma in computability theory says that for every n-generic real and every arithmetical sentence φ in the language $\mathcal{FL}_\mathrm{c}$ of low enough complexity, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true. Before we show this, we consider a proposition which shall be helpful for the proof of the Truth Lemma.

Proposition 4.1.15. *Let $\mathbb{P}$ be a forcing notion, let x be a real, and let $n \in \omega$. Then the following are equivalent:*

(a) for every Σ^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ or $x \Vdash_\mathbb{P} \neg\varphi$, and

(b) for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true.

Proof. The direction from (b) to (a) is clear. We prove the other direction by induction on φ. It is clear if φ is bounded. If φ is Π^0_m with $m \leq n$, then there is some Σ^0_m formula ψ such that $\varphi = \neg\psi$. By assumption, $x \Vdash_\mathbb{P} \varphi$ if and only if $x \nVdash_\mathbb{P} \psi$ and by induction hypothesis, $x \nVdash_\mathbb{P} \psi$ if and only if $\psi(x)$ is not true. Therefore, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true. The case φ is Σ^0_m with $0 < m \leq n$ follows directly from the induction hypothesis. □

Now we are ready to prove the Truth Lemma in computability theory.

Lemma 4.1.16 (Truth Lemma)**.** *Let $\mathbb{P}$ be a forcing notion, let $n, m \in \omega$ such that for every Σ^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, the set $\{p \in \mathbb{P} : p \Vdash \varphi\}$ is Σ^0_m, and let x be an m-generic real for $\mathbb{P}$. Then for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true.*

Proof. By Proposition 4.1.15, it is enough to show that every Σ^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ or $x \Vdash_\mathbb{P} \neg\varphi$. Let φ be such a formula and let G be a m-generic filter for $\mathbb{P}$ such that $x = \bigcup_{p \in G} v(p)$. By assumption, the set $\{p \in \mathbb{P} : p \Vdash \varphi\}$ is Σ^0_m. Hence, there is some $p \in G$ such that either $p \Vdash \varphi$ or for every $p' \leq p$, $p' \nVdash \varphi$. In the latter case, $p \Vdash \neg\varphi$. Moreover, since $p \in G$ and $x = \bigcup_{p \in G} v(p)$, there is a decreasing sequence $\langle p_k : k \in \omega\rangle$ of conditions in G such that $p_0 = p$ and $\bigcup_{k \in \omega} v(p_k) = x$. Therefore, x satisfies p and so either $x \Vdash_\mathbb{P} \varphi$ or $x \Vdash_\mathbb{P} \neg\varphi$. □

In set theory, the weak forcing relation is equivalent to the semantic forcing relation. The following corollary shows that something similar is true for the weak forcing in computability theory.

Corollary 4.1.17. *Let $k \in \omega$, let $\mathbb{P}$ be a Δ^0_k separative forcing notion, let $n \in \omega$, and let $m > k$ such that for every Σ^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, the set $\{p \in \mathbb{P} : p \Vdash \varphi\}$ is Σ^0_m, let $p \in \mathbb{P}$, and let φ be a Σ^0_m or Π^0_m sentence in the language $\mathcal{FL}_\mathrm{c}$. Then $p \Vdash_\mathrm{w} \varphi$ if and only if for every m-generic real x for $\mathbb{P}$ which satisfies p, $\varphi(x)$ is true.*

Proof. We start with the forward direction. Let x be an m-generic real for $\mathbb{P}$ which satisfies p. We assume for a contradiction that $\varphi(x)$ is not true. By the Truth Lemma, $x \nVdash_\mathbb{P} \varphi$. We make a case-distinction:

Case 1: φ is Σ^0_n. Then by Proposition 4.1.15, $x \Vdash_\mathbb{P} \neg\varphi$. Let $q \in \mathbb{P}$ such that x satisfies q and $q \Vdash \neg\varphi$. By (g) of Proposition 4.1.13, $q \Vdash_\mathrm{w} \neg\varphi$.

Case 2: φ is Π^0_n. Let ψ be the formula such that $\varphi = \neg\psi$. By Proposition 4.1.15, $x \Vdash_\mathbb{P} \psi$. Let $q \in \mathbb{P}$ such that x satisfies q and $q \Vdash \psi$. By (f) of Proposition 4.1.13, $q \Vdash_\mathrm{w} \neg\neg\psi$.

In both cases, there is a $q \in \mathbb{P}$ such that x satisfies q and $q \Vdash_\mathrm{w} \neg\varphi$. By Lemma 4.1.9, p and q are compatible. But this is a contradiction since $p \Vdash_\mathrm{w} \varphi$. Therefore, $\varphi(x)$ is true.

We prove the backward direction. Let $p' \leq p$ and let x be an m-generic real for $\mathbb{P}$ which satisfies p'. Then x satisfies p and so $\varphi(x)$ is true. By the Truth Lemma, $x \Vdash_{\mathbb{P}} \varphi$. Let $q \in \mathbb{P}$ such that x satisfies q and $q \Vdash \varphi$. By Lemma 4.1.9, p' and q are compatible. Hence, there is a $p'' \leq p'$ such that $p'' \Vdash \varphi$. Therefore, $p \Vdash \neg\neg\varphi$ and so by (f) of Proposition 4.1.13, $p \Vdash_{\mathrm{w}} \varphi$. $\square$

For Cohen forcing, the Truth Lemma says that for every $n \geq 1$, if $x \in 2^\omega$ is Cohen n-generic, then for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ if and only if $\varphi(x)$ is true. It is well-known that the converse is also true. In fact, it is even sometimes used as the definition of n-genericity in the literature.

Proposition 4.1.18 (Jockusch)**.** *Let $n \geq 1$. The following are equivalent:*

(a) A real $x \in 2^\omega$ is Cohen n-generic,

(b) for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ if and only if $\varphi(x)$ is true, and

(c) for every Σ^0_n sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ or $x \Vdash_{\mathbb{P}} \neg\varphi$.

Proof. By Proposition 4.1.15 and the Truth Lemma, it is enough to show that (b) implies (a). Let $x \in 2^\omega$ such that for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ if and only if $\varphi(x)$ is true. Recall that x is Cohen n-generic if and only if $G_x := \{s \in \mathbb{C} : s \subseteq x\}$ is a Cohen n-generic filter. It is clear that G_x is a filter. So we only have to check that for every Σ^0_n set $C \subseteq \mathbb{P}$, G_x either meets C or the set of conditions having no extension in C. Let ψ be the statement "there is some $s \in C$ such that $s \subseteq x$" and let φ be the formula in the language $\mathcal{FL}_{\mathrm{c}}$ obtained by replacing x by $\dot{x}$ in ψ. Then ψ is Σ^0_n and so φ is Σ^0_n as well. We make a case-distinction:

Case 1: $\varphi(x)$ is true. Then there is some $s \in \mathbb{C}$ such that $c \in C$ and $s \subseteq x$. Hence, $s \in G_x \cap C$ and so G_x meets C.

Case 2: $\varphi(x)$ is false. Then $\neg\varphi(x)$ is true and so $x \Vdash_{\mathbb{P}} \neg\varphi$. Let $s \in \mathbb{C}$ such that $s \subseteq x$ and $s \Vdash \neg\varphi$, let $s' \leq s$ and let $y \in 2^\omega$ be a Cohen n-generic real such that $s' \subseteq y$. Then $y \Vdash_{\mathbb{P}} \neg\varphi$ and so by the Truth Lemma, $\varphi(y)$ is false. Hence, $s' \notin C$ and so s has no extension in C. Therefore, G_x meets the set conditions having no extension in C. $\square$

The proof of the direction from (b) to (a) in Proposition 4.1.18 does not work for general forcing notions. However, we can modify it to prove a weaker version of the converse of the Truth Lemma. For this, we need the following lemma.

Lemma 4.1.19. *Let $\mathbb{P}$ be a forcing notion $\mathbb{P}$ and let $p \in \mathbb{P}$. Then there is a real x satisfying p such that for every $n \in \omega$ and every Σ^0_n sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ or $x \Vdash_{\mathbb{P}} \neg\varphi$.*

Proof. Let $\{\varphi_k : k \in \omega\}$ be an enumeration of the set of all sentences in the language $\mathcal{FL}_{\mathrm{c}}$ which are Σ^0_n for some $n \in \omega$. We recursively define a decreasing sequence $\langle p_k : k \in \omega\rangle$ of conditions. Let $p_0 := p$. If p_k is already defined, let $p_{k+1} \leq p_k$ such that $p_{k+1} \Vdash \varphi_k$ or $p_{k+1} \Vdash \neg\varphi_k$ and $\mathrm{lh}(v(p_{k+1})) > k$. Such a condition exists by (c) of Proposition 4.1.13 and the fact that $\{q \in \mathbb{P} : \mathrm{lh}(v(q)) > k\}$ is dense. Then $x := \bigcup_{k \in \omega}$ is a real and for every $k \in \omega$, x satisfies p_k. Hence, x satisfies p and for every $k \in \omega$, $x \Vdash_{\mathbb{P}} \varphi_k$ or $x \Vdash_{\mathbb{P}} \neg\varphi_k$. $\square$

Note that Lemma 4.1.19 can also be proved by taking an ω-generic real and then using Proposition 4.1.15 and the Truth Lemma. However, this only works if the complexity of the forcing relation is bounded, as in the statement of the Truth Lemma. The proof we gave has the advantage that it does not need this additional assumption. Therefore, we can prove the following proposition without any requirements on the complexity of the forcing relation.

Proposition 4.1.20. *Let $n \in \omega$, let $\mathbb{P}$ be a Δ^0_n separative forcing notion such that the set $\{(y,p) : y$ satisfies $p\}$ is Π^0_n, let $m > n$, and let x be a real such that for every Σ^0_m and Π^0_m sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ if and only if $\varphi(x)$ is true. Then x is m-generic for $\mathbb{P}$.*

Proof. By Lemma 4.1.9, it is enough to show that $G_x := \{p \in \mathbb{P} : x$ satisfies $p\}$ is an m-generic filter for $\mathbb{P}$. We first show that for every Σ^0_m set of $C \subseteq \mathbb{P}$, G_x either meets C or the set of conditions that have no extension in C. Let ψ be the statement "there is a $p \in \mathbb{P}$ such that $p \in C$ and x satisfies p". Since the set $\{(y,p) : y$ satisfies $p\}$ is Π^0_n, ψ is Σ^0_{n+1}. Let φ be the formula in the language $\mathcal{FL}_{\mathrm{c}}$ obtained by replacing x by $\dot{x}$ in ψ. Then φ is Σ^0_{n+1}. We make a case-distinction:

Case 1: $\varphi(x)$ is true. Then there is some $p \in \mathbb{P}$ such that $p \in C$ and x satisfies p. Hence, $p \in G_x \cap C$ and so G_x meets C.

Case 2: $\varphi(x)$ is false. Then $\neg\varphi(x)$ is true and so $x \Vdash_{\mathbb{P}} \neg\varphi$. Let $p \in \mathbb{P}$ such that x satisfies p and $p \Vdash \neg\varphi$ and let $q \leq p$. By Proposition 4.1.15 and Lemma 4.1.19, there is a real y satisfying q such that $y \Vdash_{\mathbb{P}} \neg\varphi$ if and only if $\neg\varphi(y)$ is true. Since $q \Vdash \neg\varphi$, $\varphi(y)$ is false. Hence, $q \notin C$ and so p has no extension in C. Therefore, G_x meets the set conditions having no extension in C.

It remains to show that G_x is a filter. For this, we can repeat the argument from the proof of Lemma 4.1.9. □

The Truth Lemma and Proposition 4.1.20 are not sufficient to prove a characterization for n-generic reals as given in Proposition 4.1.18 for Cohen forcing. However, we can use them to obtain such a characterization for ω-generic reals.

Corollary 4.1.21. *Let $n \in \omega$, let $\mathbb{P}$ be a Π^0_n separative forcing notion such that the set $\{(x,p) : x$ satisfies $p\}$ is Π^0_n and for every $m \in \omega$, there is an $m' \in \omega$ such that for every Σ^0_m sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, the set $\{p \in \mathbb{P} : p \Vdash \varphi\}$ is $\Sigma^0_{m'}$, and let x be a real. Then the following are equivalent:*

(a) x is ω-generic for $\mathbb{P}$,

(b) for every $m \in \omega$ and every Σ^0_m sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ or $x \Vdash_{\mathbb{P}} \neg\varphi$, and

(c) for every $m \in \omega$ and every Σ^0_m and Π^0_m sentence φ in the language $\mathcal{FL}_{\mathrm{c}}$, $x \Vdash_{\mathbb{P}} \varphi$ if and only if $\varphi(x)$ is true.

Proof. Follows directly from Propositions 4.1.15 and 4.1.20 and Lemma 4.1.16,. □

4.1.2 Computably arboreal forcing notions

In Section 4.2, we shall study computable versions of set theoretic forcing notions. More than half of these forcing notions will be computable versions of arboreal forcing notions. To prepare this, in this section we investigate arboreal forcing notions in computability theory. The goal is to simplify the Truth Lemma (Lemma 4.1.16) for these forcing notions. Recall that a forcing notion $\mathbb{P}$ is arboreal if its conditions are perfect trees ordered by inclusion and for every $T \in \mathbb{P}$ and every $t \in T$, there is some $S \leq T$ such that $t \subseteq \mathrm{stem}(S)$. To obtain an arboreal forcing notion in computability theory, we need to code trees as natural numbers. For cardinality reasons, it is clear that we cannot code all trees as natural numbers. Therefore, we only consider trees which can be coded as computable reals. We say that a tree on 2 (or ω) is *computable* if its characteristic function is computable. Strictly speaking, the characteristic function of a tree is not a real, but we can use the canonical bijections to identify ω with $2^{<\omega}$ (or $\omega^{<\omega}$). Then every computable tree is coded by a natural number, but this code is not unique. We call a forcing notion $\mathbb{P}$ *computably arboreal* if $\mathbb{P}$ is an arboreal forcing notion, every $T \in \mathbb{P}$ is computable, and for every $T \in \mathbb{P}$, $v(T) = \mathrm{stem}(T)$. As usual, we identify computably arboreal forcing notions with the set of codes of their conditions and so treat computable trees as natural numbers. Next, we prove some basic properties of computably arboreal forcing notions. We start with a lemma about the complexity of the coding.

Lemma 4.1.22.

(a) The set of all perfect computable trees on 2 (or ω) is Π^0_2.

(b) If T, T' are perfect computable trees, then the statement "$T' \subseteq T$" is Π^0_1.

(c) If $T \subseteq 2^{<\omega}$ is a perfect computable tree, then the statement "$t = \mathrm{stem}(T)$" is Δ^0_1.

(d) If $T \subseteq \omega^{<\omega}$ is a perfect computable tree, then the statement "$t = \mathrm{stem}(T)$" is Δ^0_2.

(e) Let $n \geq 2$ and let $\mathbb{P}$ be a computably arboreal forcing notion. If $\mathbb{P}$ as a set is Π^0_n, then $\mathbb{P}$ is a Π^0_n forcing notion.

Proof. We start with proving (a). Let $e \in \omega$. Then Φ_e is the characteristic function of a perfect computable tree on 2 if

(i) Φ_e is a total function with $\mathrm{ran}(\Phi_e) \subseteq 2$,

(ii) for every $t \in 2^{<\omega}$ and every $t' \subseteq t$, if $\Phi_e(t)\downarrow = 1$, then $\Phi_e(t')\downarrow = 1$, and

(iii) for every $t \in 2^{<\omega}$, if $\Phi_e(t)\downarrow = 1$, then there is some $t' \in 2^{<\omega}$ such that $\Phi_e(t'^\frown 0)\downarrow = 1$ and $\Phi_e(t'^\frown 1)\downarrow = 1$.

Therefore, the set of all perfect computable trees on 2 is Π^0_2. The proof for trees on ω is analogous.

Next, we prove (b). Let $e, f \in \omega$ such that Φ_e and Φ_f are characteristic functions for perfect computable trees T_e and T_f, respectively. Then $T_e \leq T_f$ if and only if

$$\forall t(\exists \sigma(\Phi_{e,\sigma}(t)\downarrow = 1) \rightarrow \forall \sigma(\Phi_{f,\sigma}(t) \neq 0)).$$

Hence, the statement "$T_e \leq T_f$" is Π^0_1.

Now we prove (c). Let $e \in \omega$ such that Φ_e is the characteristic function for a perfect computable tree T_e on 2 and let $t \in 2^{<\omega}$. Then $\mathrm{stem}(T_e) = t$ if and only if t is splitting in T_e and for every t' which appears before t in the canonical enumeration of $2^{<\omega}$, $t' \subseteq t$ or $t' \notin T_e$. Since Φ_e is computable, the statement "$\mathrm{stem}(T_e) = t$" is Δ^0_1.

Next, we prove (d). Let $e \in \omega$ such that Φ_e is the characteristic function for a perfect computable tree T_e on ω and let $t \in \omega^{<\omega}$. Then $\mathrm{stem}(T_e) = t$ if and only if t is splitting in T_e and for every $t' \in \omega^{<\omega}$, $t' \subseteq t$ or $t' \notin T_e$. The former is Σ^0_1 and the latter is Π^0_1. Hence, the whole statement is Δ^0_2.

Finally, (e) follows directly from (b) and (d). □

Recall that in set theory, for an arboreal forcing notion, a real x is $\mathbb{P}$-generic if and only if $G_x := \{T \in \mathbb{P} : x \in [T]\}$ is a $\mathbb{P}$-generic filter. We show that the analogue is true for n-generic reals for computably arboreal forcing notions. To do this, we first prove that computably arboreal forcing notions are separative.

Lemma 4.1.23. *Every computably arboreal forcing notion is separative.*

Proof. Let $\mathbb{P}$ be a computably arboreal forcing notion and let $S, T \in \mathbb{P}$ such that $S \not\leq T$. Then there is an $s \in S \setminus T$. Since $\mathbb{P}$ is arboreal, there is some $S' \leq S$ such that $s \subseteq \mathrm{stem}(S')$. Let $T' \leq T$ such that $\mathrm{lh}(\mathrm{stem}(S')) < \mathrm{lh}(\mathrm{stem}(T'))$. Then $s \notin T'$ and so $\mathrm{stem}(S') \not\subseteq \mathrm{stem}(T')$. Therefore, $\mathbb{P}$ is separative. $\square$

Now let $n \in \omega$, let $\mathbb{P}$ be a Δ^0_n computably arboreal forcing notions, and let x be a real. By Lemma 4.1.9, for every $m > n$, x is m-generic for $\mathbb{P}$ if and only if $G_x := \{T \in \mathbb{P} : x \text{ satisfies } T\}$ is an m-generic filter for $\mathbb{P}$. Note that x satisfies T if and only if $x \in [T]$. Hence, for every $m > n$, x is m-generic for $\mathbb{P}$ if and only if $G_x := \{T \in \mathbb{P} : x \in [T]\}$ is an m-generic filter for $\mathbb{P}$. We can even improve this to $m \geq n$.

Lemma 4.1.24. *Let $n \geq 2$, let $\mathbb{P}$ be a Δ^0_n computably arboreal forcing notions, and let x be a real. Then for every $m \geq n$, x is m-generic for $\mathbb{P}$ if and only if $G_x := \{T \in \mathbb{P} : x \in [T]\}$ is an m-generic filter for $\mathbb{P}$.*

Proof. The backward direction follows directly from Proposition 4.1.8. We prove the forward direction. Let x be an m-generic real for $\mathbb{P}$ and let G be an m-generic filter for $\mathbb{P}$ such that $x = \bigcup_{T \in G} \mathrm{stem}(T)$. We show that $G = G_x$. Let $T \in G$. Since $x = \bigcup_{S \in G} \mathrm{stem}(S)$, $x \in [T]$ and so $T \in G_x$. Conversely, let $T \in G_x$. We define

$$C := \{S \in \mathbb{P} : \mathrm{stem}(S) \notin T\}.$$

Then C is Σ^0_n and so there is an $S \in G$ such that $S \in C$ or S has no extension in C. The former is not possible since $x \in [S] \cap [T]$. Hence, S has no extension in C and so $S \leq T$. Therefore, $T \in G$ and so $G = G_x$. $\square$

In the rest of this section, we simplify the Truth Lemma (Lemma 4.1.16) for computably arboreal forcing notions. Since all the computably arboreal forcing notions we shall study in Section 4.2 are Π^0_2, we focus on Π^0_2 computably arboreal forcing notions. First, we compute the complexity of the forcing relation.

Lemma 4.1.25. *Let $\mathbb{P}$ be a Π^0_2 computably arboreal forcing notion on the Cantor space, let $T \in \mathbb{P}$, and let φ be an arithmetical sentence in the language $\mathcal{FL}_\mathrm{c}$.*

(a) If φ is bounded, then the statement "$T \Vdash \varphi$" is Δ^0_1.

(b) If φ is Σ^0_1, then the statement "$T \Vdash \varphi$" is Σ^0_1.

(c) If φ is Π^0_1, then the statement "$T \Vdash \varphi$" is Π^0_1.

(d) If φ is Σ^0_2, then the statement "$T \Vdash \varphi$" is Σ^0_2.

(e) If $n \geq 2$ and φ is Π^0_n, then the statement "$T \Vdash \varphi$" is Π^0_{n+1}.

(f) If $n > 2$ and φ is Σ^0_n, then the statement "$T \Vdash \varphi$" is Σ^0_{n+1}.

Proof. We start with proving (a). Since φ is bounded, there is a $k \in \omega$ such that for every $x \in 2^\omega$, the truth value of $\varphi(x)$ only depends on $x \restriction k$. Hence, $T \Vdash \varphi$ if and only if for every $t \in T$ with $\mathrm{lh}(t) = k$, $\varphi(t^\frown\langle 0 : n \in \omega\rangle)$ is true. Moreover, this k can be found computably and so whether $T \Vdash \varphi$ can be checked computably as well. Therefore, the statement "$T \Vdash \varphi$" is Δ^0_1.

Item (b) follows directly from the first. Next, we prove (c). Let ψ be the bounded sentence ψ in the language $\mathcal{FL}_{\mathrm{c}}$ such that $\varphi = \neg\exists n\psi(n)$. By definition, $T \Vdash \varphi$ if and only if for every $S \leq T$, $S \nVdash \exists n\psi(n)$. If there is an $S \leq T$ such that $S \Vdash \exists n\psi(n)$, then there is some $n \in \omega$ such that $S \Vdash \psi(\check{n})$. Since $\psi(\check{n})$ is bounded, there is a $k \in \omega$ such that for every $x \in 2^\omega$, the truth value of $\psi(x, \check{n})$ only depends on $x{\restriction}k$. Let $s \in S$ such that $\mathrm{lh}(s) = k$ and let $T' := \{t \in T : t \subseteq s \vee s \subseteq t\}$. Then $T' \leq T$ and for every $x \in [T']$, $\psi(x, \check{n})$ is true. Hence, $T' \Vdash \psi(\check{n})$ and so $T' \Vdash \exists n\psi(n)$. Therefore, $T \Vdash \varphi$ if and only if for every $s \in T$, $\{t \in T : t \subseteq s \vee s \subseteq t\} \nVdash \exists n\psi(n)$. By (b), the latter is Π^0_1.

Item (d) follows directly from (c). Finally, we prove (e) and (f) simultaneously by induction on φ. If φ is Π^0_2, then there is a Σ^0_2 sentence ψ in the language $\mathcal{FL}_{\mathrm{c}}$ such that $\varphi = \neg\psi$. By definition, $T \Vdash \varphi$ if and only if for every $S \leq T$, $S \nVdash \psi$. Since $\mathbb{P}$ is Π^0_2 the statement "$S \leq T$" is Π^0_2 and by (d), the statement "$S \nVdash \psi$" is Π^0_2. Hence, the whole statement is Π^0_3. The case φ is Σ^0_n with $n > 2$ follows directly from the induction hypothesis. If φ is Π^0_n with $n > 2$, then there is a Σ^0_n sentence ψ in the language $\mathcal{FL}_{\mathrm{c}}$ such that $\varphi = \neg\psi$. By definition, $T \Vdash \varphi$ if and only if for every $S \leq T$, $S \nVdash \psi$. Hence by the induction hypothesis, the statement "$T \Vdash \varphi$" is Π^0_{n+1}. $\square$

Note that the proof of (a) of Lemma 4.1.25 does not work for computably arboreal forcing notions on the Baire space. This is principally because there are bounded sentences φ such that there is no $k \in \omega$ such that for every $x \in \omega^\omega$, the truth value of $\varphi(x)$ only depends on $x{\restriction}k$. An example of such a sentence is $\varphi := \forall n < \dot{x}(0)(\dot{x}(n) = 0)$ since for every $x \in \omega^\omega$, the truth value of $\varphi(x)$ depends on $x{\restriction}x(0)$. Nevertheless, we can show that (e) and (f) of Lemma 4.1.25 are also true for computably arboreal forcing notions on the Baire space.

Lemma 4.1.26. *Let $\mathbb{P}$ be a Π^0_2 computably arboreal forcing notion on the Baire space, let $T \in \mathbb{P}$, and let φ be an arithmetical sentence in the language $\mathcal{FL}_{\mathrm{c}}$.*

(a) If φ is bounded, then the statement "$T \Vdash \varphi$" is Π^0_1.

(b) If φ is Σ^0_1, then the statement "$T \Vdash \varphi$" is Σ^0_2.

(c) If φ is Π^0_1, then the statement "$T \Vdash \varphi$" is Π^0_2.

(d) If $n \geq 2$ and φ is Σ^0_n, then the statement "$T \Vdash \varphi$" is Σ^0_{n+1}.

(e) If $n \geq 2$ and φ is Π^0_n, then the statement "$T \Vdash \varphi$" is Π^0_{n+1}.

Proof. We start with proving (a). Since φ is bounded, for every $x \in 2^\omega$, there is a $k \in \omega$ such that the truth value of $\varphi(x)$ only depends on $x{\restriction}k$. Hence, $T \Vdash \varphi$ if and only if for every $t \in T$, if t has enough information to answer φ, then $\varphi(t^\frown\langle 0 : n \in \omega\rangle)$ is true. Since it can be checked computably whether some $t \in T$ has enough information to answer φ, the statement "$T \Vdash \varphi$" is Π^0_1.

Item (b) follows directly from (a). Now we prove (c). Let ψ be the bounded sentence ψ in the language $\mathcal{FL}_{\mathrm{c}}$ such that $\varphi = \neg\exists n\psi(n)$. By definition, $T \Vdash \varphi$ if and only if for every $S \leq T$, $S \nVdash \exists n\psi(n)$. If there is an $S \leq T$ such that $S \Vdash \exists n\psi(n)$, then there is some $n \in \omega$ such that $S \Vdash \psi(\check{n})$. Let $x \in [S]$. Then $\psi(x, \check{n})$ is true. Since $\psi(\check{n})$ is bounded, there is a $k \in \omega$ such that the truth value of $\varphi(x)$ only depends on $x{\restriction}k$. Let $T' := \{t \in T : t \subseteq x{\restriction}k \vee x{\restriction}k \subseteq t\}$. Then $T' \leq T$ and for every $y \in [T']$, $\psi(y, \check{n})$ is true. Hence, $T' \Vdash \psi(\check{n})$ and so $T' \Vdash \exists n\psi(n)$. Therefore, $T \Vdash \varphi$ if and only if for every $s \in T$, $\{t \in T : t \subseteq s \vee s \subseteq t\} \nVdash \exists n\psi(n)$. By (b), the latter is Π^0_2.

Finally, we prove (d) and (e) simultaneously by induction on φ. The cases φ is Σ^0_2 and φ is Σ^0_n with $n > 2$ follow directly from (c) and the induction hypothesis, respectively. If φ is Π^0_n with

$n \geq 2$, then there is a Σ^0_n sentence ψ in the language $\mathcal{FL}_\mathrm{c}$ such that $\varphi = \neg\psi$. By definition, $T \Vdash \varphi$ if and only if for every $S \leq T$, $S \nVdash \psi$. Hence by the induction hypothesis, the statement "$T \Vdash \varphi$" is Π^0_{n+1}. $\square$

Now we can use Lemmas 4.1.25 and 4.1.26 to simplify the Truth Lemma (Lemma 4.1.16) for Π^0_2 computably arboreal forcing notions. Moreover, can use Lemma 4.1.24 to improve Proposition 4.1.20 and Corollary 4.1.17.

Corollary 4.1.27. *Let $\mathbb{P}$ be a Π^0_2 computably arboreal forcing notion.*

(a) If $n \geq 2$ and x is an $(n+1)$-generic real for $\mathbb{P}$, then for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true.

(b) If $n \geq 3$ and x is a real such that for every Σ^0_n and Π^0_n sentence φ in the language $\mathcal{FL}_\mathrm{c}$, $x \Vdash_\mathbb{P} \varphi$ if and only if $\varphi(x)$ is true, then x is n-generic for $\mathbb{P}$.

(c) If $n \geq 2$, $T \in \mathbb{P}$, and φ is a Σ^0_n or Π^0_n sentence in the language $\mathcal{FL}_\mathrm{c}$, then $T \Vdash_\mathrm{w} \varphi$ if and only if for every $(n+1)$-generic real x for $\mathbb{P}$, $\varphi(x)$ is true.

Proof. Item (a) follows directly from the Truth Lemma (Lemma 4.1.16) and Lemmas 4.1.25 and 4.1.26. The proofs of (b) and (c) are similar to the proofs of Proposition 4.1.20 and Corollary 4.1.17, respectively, but instead of Lemma 4.1.9 we use Lemma 4.1.24. $\square$

4.2 Separating generic reals for computable versions of set-theoretic forcing notions

4.2.1 The situation in set theory

Throughout Section 4.2, we investigate the relationships between n-generic reals for Cohen forcing and computable versions of Hechler, Laver, Mathias, Miller, Sacks, and Silver forcing, and compare the results to set theory. In preparation, this section outlines the relationships between these forcing notions in set theory. Table 4.1 summarizes the situation in set theory. An × indicates that the forcing notion listed above the column does not add a real listed in the row and a ✓ indicates that the forcing notion always adds such a real. Note that we have already talked about the relationships between these forcing notions in Chapter 2. However, since we talked about their regularity properties there, we compared their quasi-generic reals and not their generic reals. This makes a big difference for Laver, Miller, Sacks, and Silver forcing, because their quasi-generic and generic reals do not coincide. If we compare Table 4.1 with Figure 2.1, there are a several differences. For example, Cohen forcing does not add Miller reals, but it adds quasi-generic reals for Miller forcing (unbounded reals). So we have that $\mathbf{\Delta}^1_2(\mathbb{C})$ implies $\mathbf{\Delta}^1_2(\mathbb{M})$, even though Cohen forcing does not add Miller reals.

In the following, we briefly discuss why a forcing notion from Table 4.1 does or does not add generic reals for the other forcing notions. We start with Sacks forcing. Sacks forcing does not add dominating, unbounded, or splitting reals (cf. [Hal17, Lemmas 23.2 & 23.3]). Since every other forcing notion in Table 4.1 adds unbounded or splitting reals, Sacks forcing does not add generic reals for these forcing notions. Moreover, Sacks showed in [Sac71] that Sacks reals are minimal, i.e., if $x \in 2^\omega$ is a Sacks real, then for every real $y \in V[x]$, $y \in V$ or $x \in V[y]$.

	$\mathbb{C}$	$\mathbb{D}$	$\mathbb{L}$	$\mathbb{M}$	$\mathbb{R}$	$\mathbb{S}$	$\mathbb{V}$
Cohen real	✓	✓	×	×	×	×	×
Hechler real	×	✓	×	×	×	×	×
Laver real	×	×	✓	×	×	×	×
Miller real	×	×	×	✓	×	×	×
Mathias real	×	×	×	×	✓	×	×
Sacks real	×	×	×	×	×	✓	×
Silver real	×	×	×	×	×	×	✓
dominating real	×	✓	✓	×	✓	×	×
unbounded real	✓	✓	✓	✓	✓	×	×
splitting real	✓	✓	✓	×	✓	×	✓

Table 4.1: Relationships between Cohen, Hechler, Laver, Mathias, Miller, Sacks, and Silver forcing in set theory

Next, we consider Silver forcing. Silver forcing adds splitting reals, but no dominating or unbounded reals (cf. [Hal17, Lemmas 22.2 & 22.3]). Hence, Silver forcing does not add Cohen, Hechler, Laver, Mathias, or Miller reals. Moreover, Silver reals are minimal (cf. [Gri71, Corollary 5.5]). Thus, Silver forcing does not add Sacks reals, because otherwise there would be a Sacks real x such that $V[x]$ contains a Silver real, and this is not possible.

Now we consider Miller forcing. Miller forcing adds unbounded reals, but no dominating or splitting reals (cf. [Hal17, Lemmas 25.2 & 25.3]). Hence, Miller forcing does not add Cohen, Hechler, Laver, Mathias, or Silver reals. Moreover, Miller proved in [Mil84] that Miller reals are minimal. Thus, by a similar argument as for Silver forcing, Miller forcing does not add Sacks reals.

Next, we consider Cohen forcing. Cohen forcing adds unbounded and splitting reals, but no dominating reals (cf. [Hal17, Lemmas 22.1, 22.2, & 22.3]). Hence, Cohen forcing does not add Hechler, Laver, or Mathias reals. Moreover, Cohen reals are not minimal (cf. [Hal17, Related results 114]) and if x is a Cohen real, then for every real $y \in V[x]$, $y \in V$ or $V[y]$ contains a Cohen real (cf. [Sam76, Lemma 1.9]). Thus, Cohen forcing does not minimal reals. So in particular, Cohen forcing does not add Miller, Sacks, or Silver reals.

Now we consider Laver forcing. Laver forcing adds dominating, unbounded, and splitting reals (cf. [BJ95, Lemma 7.3.28]). Moreover, Gray showed in [Gra80] that Laver reals are minimal. Hence, Laver forcing does not add Cohen, Miller, Sacks, or Silver reals. Since Hechler forcing adds Cohen reals, Laver forcing does not add Hechler reals. Furthermore, Laver forcing does not add Mathias reals because they are not minimal (cf. [Mat77, Corollary 8.3]).

Next, we consider Mathias forcing. Mathias forcing adds dominating, unbounded, and splitting reals (cf. [Hal17, Lemma 26.1]), but no Cohen reals (cf. [Hal17, Corollary 26.8]). Hence, Mathias forcing does not add Hechler reals. Moreover, Mathias forcing does not add minimal reals (cf. [Hal17, Lemma 26.10 & Related result 152] and [Gro87, Theorem 5]). Thus, Mathias forcing does not add Laver, Miller, Sacks, or Silver reals.

Finally, we consider Hechler forcing. Hechler forcing adds dominating, unbounded, and splitting reals (cf. [Jec03, p. 278]). Moreover, Palumbo showed in [Pal13] that if $x \in \omega^\omega$ is a Hechler real, then for every real $y \in V[x]$, $y \in V$ or $V[y]$ contains a Cohen real. Hence, Hechler forcing adds Cohen reals, but no Laver, Mathias, Miller, Sacks, or Silver reals.

4.2.2 Dominating, unbounded, and splitting reals in computability theory

We have seen in Section 4.2.1 that dominating, unbounded, and splitting reals are quite useful for checking whether a forcing notion adds generic reals for another forcing notion. For example, Cohen forcing adds unbounded and splitting reals and Sacks forcing does not (cf. Table 4.1), and so Sacks forcing does not add Cohen reals. In the following sections, we shall use similar arguments to separate n-generic reals from each other. However, we cannot use reals that are dominating, unbounded, or splitting for all reals for these arguments, because such reals do not exist in V. Instead, we use reals that are dominating, unbounded, or splitting for Σ^0_n reals. To prepare this, in this section we study these kinds of dominating, unbounded, and splitting reals. We show that they behave similarly to dominating, unbounded, and splitting reals, except that each real which is not dominated by any Σ^0_n real computes a real which splits all Σ^0_n reals.

Definition 4.2.1. Let $n \geq 1$.

(a) We say that a real $f \in \omega^\omega$ is *n-dominating* if f dominates every $g \in \omega^\omega$ which is Σ^0_n.

(b) We say that a real $f \in \omega^\omega$ is *n-unbounded* if there is no $g \in \omega^\omega$ which is Σ^0_n and dominates f.

(c) We say that a real $A \in [\omega]^\omega$ is *n-splitting* if A splits every $B \in [\omega]^\omega$ which is Σ^0_n.

Note that 1-splitting reals are also known under a different name in computability theory. An infinite set is called *immune* if it contain no Σ^0_1 set and it is called *bi-immune* if both it and its complement are immune. One can show that a real is 1-splitting if and only if it is bi-immune. This even generalizes to arbitrary $n \geq 1$.

Proposition 4.2.2 (Folklore)**.** *Let $A \in [\omega]^\omega$ and let $n \geq 1$. The following are equivalent:*

(a) A is n-splitting,

(b) A splits all infinite Δ^0_n sets,

(c) A and its complement do not contain an infinite Δ^0_n set, and

(d) A and its complement do not contain an infinite Σ^0_n set.

Proof. The direction from (a) to (b) is clear. We prove the direction from (b) to (c). Let $B \in [\omega]^\omega$ be a Δ^0_n set. Since A splits B, $B \cap A$ and $B \setminus A$ are infinite. Hence, A and its complement do not contain B.

Next, we prove the direction from (c) to (d). It is enough to show that every infinite Σ^0_n set contains an infinite Δ^0_n set. Let $B \in [\omega]^\omega$ be a Σ^0_n set. Then B is computably enumerable in $\emptyset^{(n-1)}$. Hence, B the range of a total function $f \in \omega^\omega$ which is computable in $\emptyset^{(n-1)}$. We define

$$C := \{f(k) : k \in \omega \text{ and } \forall k' < k(f(k') < f(k))\}.$$

Then $C \subseteq \operatorname{ran}(f) = B$ and C is computable in $\emptyset^{(n-1)}$. Hence, C is Δ^0_n. It remains to check that C is infinite. We suppose for a contradiction that C is finite. Let m be the maximal element in C. Since $\operatorname{ran}(f)$ is infinite, there is a least $k \in \omega$ such that $f(k) > m$. Then for every $k' < k$, $f(k') \leq m < f(k)$ and so $f(k) \in C$. But this is a contradiction and so C is infinite.

Finally, we prove the direction from (d) to (a). Let $B \in [\omega]^\omega$ be a Σ^0_n set. We have to show that for every $k \in \omega$, there are $m, m' \geq k$ such that $m \in B \cap A$ and $m' \in B \setminus A$. Let $k \in \omega$ and let $B_k := \{m \in B : m \geq k\}$. Then B_k is Σ^0_n. Hence, neither A nor its complement contain B_k. Therefore, there are $m, m' \geq k$ such that $m \in B_k \cap A \subseteq B \cap A$ and $m' \in B_k \setminus A \subseteq B \setminus A$. $\square$

By Proposition 4.2.2, for every $n \geq 1$, a real is n-splitting if and only if it splits every infinite Δ^0_n set. The analogue is also true for n-dominating and n-unbounded reals.

Proposition 4.2.3 (Folklore)**.** *Let $f \in \omega^\omega$ and let $n \geq 1$. Then*

(a) f is n-dominating if and only if f dominates every $g \in \omega^\omega$ which is Δ^0_n, and

(b) f is n-unbounded if and only if there is no $g \in \omega^\omega$ which is Δ^0_n and dominates f.

Proof. It is enough to show that for every increasing $g \in \omega^\omega$ real which is Σ^0_n, there is some $g' \in \omega^\omega$ such that g' is Δ^0_n and for every $k \in \omega$, $g(k) \leq g'(k)$. Since g is Σ^0_n, $\mathrm{ran}(g)$ is Σ^0_n as well. We have already seen in the proof of Proposition 4.2.2 that there is an infinite Δ^0_n set $A \subseteq \mathrm{ran}(g)$. Let $g' \in \omega^\omega$ be the real enumerating A in ascending order. Then g' is Δ^0_n and for every $k \in \omega$, $g(k) \leq g(k)$. $\square$

Like 1-splitting reals, 1-unbounded reals are also closely connected to a well-known concept in computability theory. An infinite set $A \in [\omega]^\omega$ is called *hyperimmune* if p_A is not dominated by a computable real, where p_A is the function enumerating A in ascending order. By Proposition 4.2.3, A is hyperimmune if and only if p_A is 1-unbounded. Moreover, 1-dominating reals can be used to characterize high sets. A set $A \in [\omega]^\omega$ is called *high* if $A' \geq_{\mathrm{T}} \emptyset''$. Martin proved in [Mar66] the following characterization of high sets.

Theorem 4.2.4 (Martin)**.** *A set is high if and only if it computes a 1-dominating real.*

Proof. Cf., e.g., [Soa16, Theorem 4.5.6]. $\square$

It is clear that every n-dominating real is n-unbounded. Hence, every high set computes a hyperimmune set. Moreover, Jockusch proved in [Joc69] that every hyperimmune set computes a bi-immune set. So in particular, every 1-unbounded real computes a 1-splitting real. This might be surprising since the analogous result is not true for unbounded and splitting reals. Indeed, it is well-known that the existence of unbounded reals does not imply the existence splitting reals. For example, Miller forcing adds unbounded reals, but does not add splitting reals (cf. Table 4.1). Jockusch's result can even be generalized to n-unbounded and n-splitting reals. To prove this, we approximate $\emptyset^{(n-1)}$ by computable sets. Let $A \subseteq \omega$ be a set of natural numbers. We define the approximation of the jump of A with $\sigma \in \omega$ steps as $A'[\sigma] := \{e \leq \sigma : \Phi^A_{e,\sigma}(e)\downarrow\}$. Then $A'[\sigma]$ is a computable set for every $\sigma \in \omega$ and for every $k \in \omega$, there is a $\sigma \in \omega$ such that $A' \cap k = A'[\sigma] \cap k$. We can iterate this to approximate the nth jump of A. For $n > 1$ and $\sigma_0, \dots, \sigma_{n-1} \in \omega$, we recursively define

$$A^{(n)}[\sigma_0, \dots, \sigma_{n-1}] := (A^{(n-1)}[\sigma_0, \dots, \sigma_{n-2}])'[\sigma_{n-1}].$$

Then $A^{(n)}[\sigma_0, \dots, \sigma_{n-1}]$ is a computable set for every $n \geq 1$ and $\sigma_0, \dots, \sigma_{n-1} \in \omega$. Moreover, we can show that if $\sigma_0, \dots, \sigma_{n-1} \in \omega$ are chosen correctly, then $A^{(n)}[\sigma_0, \dots, \sigma_{n-1}]$ approximates the nth jump of A.

Lemma 4.2.5 (Folklore)**.** *Let $A \subseteq \omega$ be a set of natural numbers and let $n \geq 1$. Then for every $k \in \omega$, there are $\sigma_0, \ldots, \sigma_{n-1} \in \omega$ such that for every $m < n$, $\sigma_m \in B_m$ and $A^{(n)} \cap k = A^{(n)}[\sigma_0, \ldots, \sigma_{n-1}] \cap k$.*

Proof. We prove the lemma by induction on n. It is clear for $n = 1$. Let $n \geq 1$ such that the induction hypothesis holds and let $k \in \omega$. Then there is some $\sigma \in \omega$ such that for every $\sigma' \geq \sigma$ and every $e < k$, $\Phi_e^{A^{(n)}}(e)$ converges if and only if $\Phi_{e,\sigma'}^{A^{(n)}}(e)$ converges. Since B_n is infinite, there is some $\sigma_n \in B_n$ such that $\sigma_n \geq \sigma$. By induction hypothesis, there are $\sigma_0, \ldots, \sigma_{n-1} \in \omega$ such that $A^{(n)} \cap \sigma_n = A^{(n)}[\sigma_0, \ldots, \sigma_{n-1}] \cap \sigma_n$ and for every $m < n$, $\sigma_m \in B_m$. Let $s \in 2^{<\omega}$ be such that for every $m < \sigma_n$, $s(m) = 1$ if and only if $m \in A^{(n)}$. Then for every $e < k$,

$$\Phi_e^{A^{(n)}}(e) \iff \Phi_{e,\sigma_n}^{A^{(n)}}(e)\downarrow \iff \Phi_{e,\sigma_n}^{s}(e)\downarrow \iff \Phi_{e,\sigma_n}^{A^{(n)}}(e)\downarrow.$$

Therefore, $A^{(n+1)} \cap k = A^{(n+1)}[\sigma_0, \ldots, \sigma_n] \cap k$. □

Proposition 4.2.6. *Let $n \geq 1$. Every n-unbounded real computes an n-splitting real.*

Proof. Let $f \in \omega^\omega$ be n-unbounded. Without loss of generality, we can assume that f is strictly increasing. By Proposition 4.2.2, it is enough to show that there is an $A \in [\omega]^\omega$ such that $A \leq_{\mathrm{T}} f$ and A and its complement do not contain an infinite Δ_n^0 set. We fix a recursive bijection $g : \omega \to \omega^{n-1}$. For every $k \in \omega$, let $g(k) = \langle \sigma_m^k : m < n-1 \rangle$ and let

$$B_k := \begin{cases} \emptyset & \text{if } n = 1, \\ \emptyset^{(n-1)}[\sigma_0^k, \ldots, \sigma_{n-2}^k] & \text{otherwise.} \end{cases}$$

We recursively define a sequence $\langle (I_e, O_e) : e \in \omega \rangle$. Let $I_0 := \emptyset$ and $O_0 := \emptyset$. If (I_e, O_e) is already defined, then we recursively define a sequence $\langle (I_{e+1}^k, O_{e+1}^k) : k \leq f(e+1)+1 \rangle$. Let $I_{e+1}^0 := I_e$ and $O_{e+1}^0 := O_e$. If (I_{e+1}^k, O_{e+1}^k) is already defined, then we check whether

$$\{m \geq e : \Phi_{e,\sigma_{n-1}^k}^{B_k}(m)\downarrow = 1\} \setminus (I_{e+1}^k \cup O_{e+1}^k)$$

contains at least two elements. If this is the case, then let m be the smallest such element and m' the second smallest, and define $I_{e+1}^{k+1} := I_{e+1}^k \cup \{m\}$ and $O_{e+1}^k := O_{e+1}^k \cup \{m'\}$. Otherwise, we do nothing and set $I_{e+1}^{k+1} := I_{e+1}^k$ and $O_{e+1}^{k+1} := O_{e+1}^k$. Finally, we define $I_{e+1} := I_{e+1}^{f(e+1)+1}$ and $O_{e+1} := O_{e+1}^{f(e+1)+1}$.

Let $A := \bigcup_{e \in \omega} I_e$. Then $A \leq_{\mathrm{T}} f$ and so it remains to show that neither A nor its complement contain a Δ_n^0 set. Let $C \in [\omega]^\omega$ be Δ_n^0. Then C is computable in $\emptyset^{(n-1)}$ and so there is a strictly increasing $x \in \omega^\omega$ such that $x \leq_{\mathrm{T}} \emptyset^{(n-1)}$ and for every $e \in \omega$, $\Phi_{x(e)}^{\emptyset^{(n-1)}}$ is the characteristic function of C. Let $h \in \omega^\omega$ be such that for every $e \in \omega$, $h(e)$ is the least natural number such that there is some $\sigma_{n-1}^{h(e)} < m \leq h(e)$ with

(a) $|\{k : x(e) \leq k < m\} \cap C| \geq 2x(e)^2$,

(b) $C \cap m = \{k < m : \Phi_{x(e),\sigma_{n-1}^{h(e)}}^{\emptyset^{(n-1)}}(k)\downarrow = 1\}$, and

(c) $\emptyset^{(n-1)} \cap m = B_{h(e)} \cap m$.

Note that h exists by Lemma 4.2.5. Moreover, $h \leq_{\mathrm{T}} \emptyset^{(n-1)}$ and so h is Δ^0_n. Let $f' := f \circ x$. Then f' is n-unbounded and so there is some $e > 0$ such that $h(e) \leq f'(e)$. Let $m \in \omega$ such that $\sigma^{h(e)}_{n-1} < m \leq h(e)$ and (a), (b), and (c) hold. Then

$$\{k \geq x(e) : \Phi^{B_{h(e)}}_{e,\sigma^{h(e)}_{n-1}}(m)\downarrow = 1\} \cap m = \{k : x(e) \leq k < m\} \cap C$$

and $|\{k : x(e) \leq k < m\} \cap C| \geq 2x(e)^2$. Hence,

$$\{k \geq x(e) : \Phi^{B_{h(e)}}_{e,\sigma^{h(e)}_{n-1}}(m)\downarrow = 1\} \setminus (I^{h(e)-1}_{x(e)} \cup O^{h(e)-1}_{x(e)})$$

contains at least two elements and so $I_{x(e)+1} \cap C \neq \emptyset$ and $O_{x(e)+1} \cap C \neq \emptyset$. Since $A \cap O_{x(e)+1} = \emptyset$, both A and its complement meet C. $\square$

The converse of Proposition 4.2.6 is not true. We shall see in Section 4.2.4 that there are reals which compute an n-splitting real, but do not compute n-unbounded reals (cf. Propositions 4.2.25 and 4.2.27). Moreover, there are reals which compute n-unbounded reals, but do not compute n-dominating reals. To show this, we consider Cohen n-generic reals. It is well-known in set theory that Cohen forcing adds unbounded and splitting reals, but no dominating reals (cf. Table 4.1). The analogue is true for Cohen n-generic reals. Jockusch proved in [Joc80, Lemma 2.6] that no Cohen 2-generic real computes a real which dominates all computable reals and Kurtz showed in [Kur83, Corollary 2.7] that every Cohen n-generic reals computes an n-unbounded real.

Proposition 4.2.7 (Jockusch)**.** *No Cohen 2-generic real computes a 1-dominating real.*

Proposition 4.2.8 (Kurtz)**.** *Let $n \geq 1$. Every Cohen n-generic real computes an n-unbounded real.*

Corollary 4.2.9. *Let $n \geq 1$. Every Cohen n-generic real computes an n-unbounded real.*

Proof. Follows directly from Propositions 4.2.6 and 4.2.8. $\square$

Therefore, the relationship between n-dominating, n-unbounded, and n-splitting reals is similar to that between dominating, unbounded, and splitting reals, except that n-unbounded reals compute n-splitting reals.

4.2.3 Cohen, computable Mathias and computable Sacks forcing

In this section, we investigate the relationship between n-generic reals for Cohen, computable Mathias, and computable Sacks forcing. Recall that in set theory Cohen, Mathias, and Sacks forcing do not add generic reals for any of the other two (cf. Table 4.1). However, the analogue does not hold in computability theory. Cholak, Dzhafarov, Hirst, and Slaman showed in [CDHS14] that for every $n \geq 3$, every n-generic real for computable Mathias forcing computes a Cohen n-generic real. Otherwise the situation is similar to set theory, except that it is not known whether n-generic reals for computable Mathias forcing can compute n-generic reals for computable Sacks forcing. Table 4.2 summarizes this; an × indicates that no n-generic real listed above the column computes a real listed in the row, a ✓ indicates that every n-generic real listed above the column computes a real listed in the row, and a ? indicates that it is not known. In the following, we give a brief introduction to computable Mathias and computable Sacks forcing and discuss the presented results in Table 4.2. It should be noted that all results in this section are not new and were known before this work. We start with the definition of computable Mathias forcing.

	$\mathbb{C}$ n-gen.	$\mathbb{R}_c$ n-gen.	$\mathbb{S}_c$ n-gen.
$\mathbb{C}$ n-gen.	✓	✓	×
$\mathbb{R}_c$ n-gen.	×	✓	×
$\mathbb{S}_c$ n-gen.	×	?	✓
1-dom.	×	✓	×
2-dom.	×	×	×
n-unb.	✓	✓	×
n-split.	✓	✓	×

Table 4.2: Relationships between Cohen, Mathias, and Sacks n-generic reals for $n \geq 4$

Definition 4.2.10. *Computable Mathias forcing*, denoted by $\mathbb{R}_c$, is the partial order of all pairs $(F, E) \in [\omega]^{<\omega} \times [\omega]^{\omega}$ such that E is computable and $\min(F) < \max(E)$ ordered by

$$(F', E') \leq (F, E) \iff F' \cap (\max(F) + 1) = F,\ E' \subseteq E, \text{ and } F' \setminus F \subseteq E'.$$

and with valuation function $v : \mathbb{R}_c \to 2^{<\omega}$ defined by $v(F, E) := s$, where $s \in 2^{<\omega}$ such that $\mathrm{lh}(s) = \min(E)$ and for every $n < \mathrm{lh}(s)$, $n \in F$ if and only if $s(n) = 1$.

Computable Mathias forcing is a Π^0_2 forcing notion (cf. [CDHS14, Section 2]). As for Cohen forcing, we write "Mathias n-generic" instead of "n-generic for $\mathbb{R}_c$". Note that every Mathias n-generic real $x \in 2^\omega$ is the characteristic function of an infinite set of natural numbers A. Let G be the Mathias n-generic filter corresponding to x. Then $A = \bigcup\{F : \exists E((F, E) \in G)\}$ and for every $(F, E) \in G$, $F \subseteq G \subseteq F \cup E$. In practice, it is often more convenient to work with A instead of x. We say that an infinite set of natural numbers is Mathias n-generic if its characteristic function is Mathias n-generic. It is well-known that if $A \in [\omega]^\omega$ is Mathias 3-generic, then A is high (cf. [BKHLS06, Corollary 6.7]) and so by Theorem 4.2.4, every Mathias 3-generic real computes a 1-dominating real. Recall that by Proposition 4.2.7, no Cohen 2-generic real computes a 1-dominating real. Hence, no Cohen 2-generic real computes a Mathias 3-generic real. Cholak, Dzhafarov, Hirst, and Slaman even showed in [CDHS14, Corollary 5.1] that also no Cohen 1-generic real computes a Mathias 3-generic real.

Corollary 4.2.11 (Cholak-Dzhafarov-Hirst-Slaman)**.** *No Cohen 1-generic real computes a Mathias 3-generic real.*

As mentioned before, Cholak, Dzhafarov, Hirst, and Slaman also showed in [CDHS14, Theorem 5.2] that for every $n \geq 3$, every Mathias n-generic real computes a Cohen n-generic real.

Theorem 4.2.12 (Cholak-Dzhafarov-Hirst-Slaman)**.** *Let $n \geq 3$. Every Mathias n-generic real computes a Cohen n-generic real.*

In Section 4.2.5, we shall generalize Theorem 4.2.12 to show that every n-generic real for a computable version of Laver forcing computes a Cohen n-generic real. Note that, like Mathias forcing, in set theory Laver forcing does not add Cohen reals (cf. Table 4.1). Hence, Mathias forcing is not the only forcing notion whose relationship to Cohen forcing is different in set theory and computability theory.

Recall that in set theory Mathias forcing adds dominating reals (cf. Table 4.1). Indeed, if $A \in [\omega]^\omega$ is $\mathbb{R}$-generic over V, then the function enumerating A in ascending order is dominating over V. The rough idea is that for every $x \in \omega^\omega \cap \mathrm{V}$ and every $(F, E) \in \mathbb{R}$, we can thin out E until the function enumerating $F \cup E$ in ascending order dominates x. So one might expect that every Mathias n-generic real computes an n-dominating real. In fact, one can use the argument from set theory to show that every Mathias 3-generic real computes a 1-dominating reals. However, it does not work for Mathias 3-generic reals and 2-dominating reals, because if $x \in \omega^\omega$ is Δ^0_2 and 1-unbounded, then there is no condition $(F, E) \in \mathbb{R}_\mathrm{c}$ such that the function enumerating $F \cup E$ in ascending order dominates x. One can even show that no Mathias 3-generic computes a 2-dominating real. To show this, we consider another result of Cholak, Dzhafarov, Hirst, and Slaman from [CDHS14]. They proved that no Mathias 3-generic real computes $\emptyset'$. Now it is enough to prove that every 2-dominating real computes $\emptyset'$.

Lemma 4.2.13 (Folklore)**.** *Let $n \geq 1$. Every $(n+1)$-dominating real computes $\emptyset^{(n)}$.*

Proof. Let $f \in \omega^\omega$ be $(n+1)$-dominating and let $g \in \omega^\omega$ be defined by

$$g(e) := \begin{cases} \min\{\sigma : \Phi^{\emptyset^{(n-1)}}_{e,\sigma}(e)\downarrow\} & \text{if } e \in \emptyset^{(n)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is computable in $\emptyset^{(n)}$ and so g is Σ^0_{n+1}. Since f is $(n+1)$-dominating, f dominates g. Hence, there is some $k \in \omega$ such that for every $e \in \omega$, $g(e) < f(e)$. Without loss of generality, we can assume that for every $e \in \omega$, $g(e) < f(e)$. Then $e \in \emptyset^{(n)}$ if and only if $\Phi^{\emptyset^{(n-1)}}_{e,f(e)}(e)$ converges. Therefore, f computes $\emptyset^{(n)}$. □

Corollary 4.2.14. *No Mathias 3-generic real computes a 2-dominating real.*

Proof. Follows directly from Lemma 4.2.13 and the fact that Mathias 3-generic reals cannot compute $\emptyset'$. □

Next, we consider computable Sacks forcing. Note that when Sacks introduced Sacks forcing in [Sac71], he studied Sacks forcing not only in the context of set theory, but also in the context of computability theory. Since then, Sacks forcing is one of the better known forcing notions in computability theory. For more details on Sacks forcing in computability theory, we refer the reader to [Odi83].

Definition 4.2.15. *Computable Sacks forcing*, denoted by $\mathbb{S}_\mathrm{c}$, is the partial order of all computable perfect trees on 2 ordered by inclusion and equipped with $\mathrm{stem}(T)$ as the valuation function.

It is clear that computable Sacks forcing is a computably arboreal forcing notion. By Lemma 4.1.22, it is a Π^0_2 computably arboreal forcing notion. As usual, we write "Sacks n-generic" instead of "n-generic for $\mathbb{S}_\mathrm{c}$". To compare Sacks n-generic reals to Cohen and Mathias n-generic reals, we first investigate whether Sacks n-generic reals can compute n-dominating, n-unbounded, or n-splitting reals. Recall that in set theory, Sacks forcing does not add dominating, unbounded, or splitting reals. The analogue is true for Sacks n-generic reals. In fact, one can translate the proofs from set theory into computability theory.

Proposition 4.2.16. *No Sacks 4-generic real computes a 2-splitting real.*

Proof. Let $x \in 2^\omega$ be 3-Sacks generic and let $A \in [\omega]^\omega$ be computable in x. Then there is an $e \in \omega$ such that Φ_e^x is the characteristic function of A. Let ψ be the statement "Φ_e^x is a total function with $\mathrm{ran}(\Phi_e^x) \subseteq 2$". Then ψ is Π_2^0. Let φ be the formula in the language $\mathcal{FL}_\mathrm{c}$ obtained by replacing x by $\dot{x}$ and e by $\check{e}$ in ψ. Then φ is a Π_2^0 sentence and $\varphi(x)$ is true. By Corollary 4.1.27, $x \Vdash_{\mathbb{S}_\mathrm{c}} \varphi$. Hence, there is some $T \in \mathbb{S}_\mathrm{c}$ such that $x \in [T]$ and $T \Vdash \varphi$. We define

$$\begin{aligned} D_0 :=& \{S \in \mathbb{S}_\mathrm{c} : \forall k \exists n \geq k (S \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 0)\}, \\ D_1 :=& \{S \in \mathbb{S}_\mathrm{c} : \forall k \exists n \geq k (S \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 1)\}, \text{ and} \\ D :=& D_0 \cup D_1. \end{aligned}$$

For every $S \in \mathbb{S}_\mathrm{c}$ and every Π_1^0 sentence in the language $\mathcal{FL}_\mathrm{c}$, the statement "$S \Vdash_\mathrm{w} \psi$" is Π_1^0. Hence, D is Π_3^0. By Lemma 4.1.24, $G_x := \{S \in \mathbb{S}_\mathrm{c} : x \in [S]\}$ is a Sacks 4-generic filter and so there is some $S \in G_x$ such either $S \in D$ or S has no extension in D. Without loss of generality, we can assume that $S \leq T$. We suppose for a contradiction that S has no extension in D. Then in particular, S has no extension in D_0. We recursively define a decreasing sequence $\langle S_k : k \in \omega \rangle$ of conditions. Let $S_0 := S$. If S_k is already defined, then let $\{s_m : m < 2^k\}$ be the set of the $(k+1)$st splitting notes of S_k, let $m < 2^k$, and let $i < 2$. Since S has no extension in D_0, $S_k^{m,i} := \{s \in S_k : s \subseteq s_m$ or $s_m{}^\frown i \subseteq s\}$ is not in D_0. Hence, there is some $n \geq k$ such that $S_k^{m,i} \nVdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 0$. By Corollary 4.1.27, there is some Sacks 3-generic real $y \in [S_k^{m,i}]$ such that $\Phi_e^y(n) = 0$. Thus, there is some $s \in S_k^{m,i}$ such that there is some $n < \mathrm{lh}(s)$ such that $n \geq k$ and $\Phi_{e,\mathrm{lh}(s)}^s(n) = 0$. Let $s_m^i \in S_k^{m,i}$ be minimal in the canonical enumeration with that property. We define

$$S_{k+1} := \{s \in S_k : \exists m < 2^k \exists i < 2 (s \subseteq s_m^i \vee s_m^i \subseteq s)\}.$$

Let $S' := \bigcap_{k \in \omega} S_k$. Then $s \in S'$ if and only if $s \in S_{\mathrm{lh}(s)+1}$. Hence, S' is computable and so $S' \in \mathbb{S}_\mathrm{c}$. Moreover, for every $k \in \omega$, there is some $n \in \omega$ such that for every $y \in [S_{k+1}]$, $\Phi_e^y(n) = 0$. By Corollary 4.1.27, $S_{k+1} \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 1$. Hence, for every $k \in \omega$, there is some $n \in \omega$ such that $S' \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 1$ and so $S' \in D_1$. But this is a contradiction. Therefore, $S \in D$.

Let $A_0 := \{n \in \omega : S \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 0\}$ and $A_1 := \{n \in \omega : S \Vdash_\mathrm{w} \Phi_{\check{e}}^{\dot{x}}(n) \neq 1\}$. Then A_0 and A_1 are Π_1^0. Since $S \in D$, at least A_0 or A_1 is infinite. Without loss of generality, we can assume that A_0 is infinite. Let $n \in A_0$. By Corollary 4.1.27, $\Phi_e^x(n) \neq 0$. Hence, $\Phi_e^x(n)\downarrow = 1$ and so $A_0 \subseteq A$. Therefore, A does not split A_0. ☐

Thus, by Corollary 4.2.9, every Cohen 2-generic real computes a 2-splitting real. Hence, no Sacks 4-generic real can compute a Cohen 2-generic real. By Theorem 4.2.12, no Sacks 4-generic real can compute a Mathias 3-generic real. We can even improve this by looking at n-unbounded reals.

Proposition 4.2.17 (Folklore)**.** *No Sacks 3-generic real computes a 1-unbounded real.*

Proof. Let $x \in 2^\omega$ be Sacks 3-generic and let $f \in \omega^\omega$ be computable in x. Then there is an $e \in \omega$ such that $f = \Phi_e^x$. Let ψ be the statement "Φ_e^x is a total function". Then ψ is Π_2^0. Let φ be the formula in the language $\mathcal{FL}_\mathrm{c}$ obtained by replacing x by $\dot{x}$ and e by $\check{e}$ in ψ. Then φ is a Π_2^0 sentence and $\varphi(x)$ is true. By Corollary 4.1.27, $x \Vdash_{\mathbb{S}_\mathrm{c}} \varphi$. Hence, there is some $T \in \mathbb{S}_\mathrm{c}$ such that $x \in [T]$ and $T \Vdash \varphi$. Let

$$D := \{S \in \mathbb{S}_\mathrm{c} : \forall n \exists \ell \forall s \in S \cap 2^\ell (\Phi_{e,\mathrm{lh}(s)}^s(n)\downarrow)\}.$$

Then D is Π^0_2. We show that D is dense below T. Let $S \leq T$. We recursively define a decreasing sequence $\langle S_n : n \in \omega\rangle$ of conditions. Let $S_0 := S$. If S_n is already defined, then let $\{s_m : m < 2^n\}$ be the set of the $(n+1)$st splitting notes of S_n, let $m < 2^n$, let $i < 2$, and let $S_n^{m,i} := \{s \in S_n : s \subseteq s_m$ or $s_m{}^\frown i \subseteq s\}$. Since $S_n^{m,i} \leq T$, $S_n^{m,i} \Vdash \varphi$. Let $y \in [S_n^{m,i}]$ be Sacks 3-generic. By Corollary 4.1.27, $\varphi(y)$ is true and so $\Phi_e^y(n)$ converges. Hence, there is some $s \in S_n^{m,i}$ such that $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges. Let $s^i_m \in S_n^{m,i}$ be minimal in the canonical enumeration with that property. We define

$$S_{n+1} := \{s \in S_n : \exists m < 2^n \exists i < 2 (s \subseteq s^i_m \vee s^i_m \subseteq s)\}.$$

Then $S' := \bigcap_{n\in\omega} S_n \in \mathbb{S}_\mathrm{c}$ and for every $n \in \omega$, there is some $k \in \omega$ such that for every $s \in S_{k+1} \cap 2^k$, $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges. Hence, $S' \in D$ and so D is dense below T. Since x is Sacks 3-generic, there is some $S \in D$ such that $x \in [S]$. For every $n \in \omega$, let $\ell_n \in \omega$ be minimal such that for all $s \in S \cap 2^{\ell_n}$, $\Phi^s_{e,\mathrm{lh}(s)}(n){\downarrow} < \ell_n$. Let $g \in \omega^\omega$ be defined by $g(n) := \ell_n$. Then g is computable and for every $n \in \omega$, $f(n) < g(n)$. □

Corollary 4.2.18. *No Sacks 3-generic real computes a Cohen 1-generic real or a Mathias 3-generic real.*

Proof. Follows directly from Propositions 4.2.8 and 4.2.17 and Theorem 4.2.12. □

Next, we investigate whether Cohen n-generic reals can compute Sacks n-generic reals. Recall that in set theory, Cohen forcing does not add Sacks reals (cf. Table 4.1). To show this, we have used the fact that if x is Cohen real, then for every real $y \in V[x] \setminus V$, $y \in V$ or $V[y]$ contains a Cohen real. Martin showed that something similar is also true for Cohen n-generic reals.

Theorem 4.2.19 (Martin)**.** *Let $n \geq 2$, let $x \in 2^\omega$ be Cohen n-generic, and let $y \in \omega^\omega$ such that $0 <_\mathrm{T} y \leq_\mathrm{T} x$. Then there is some $z \leq_\mathrm{T} y$ which is Cohen n-generic.*

Proof. Cf. [Joc80, Theorem 4.1]. □

Note that by Theorem 4.2.19, a Cohen 2-generic real $x \in 2^\omega$ can only compute a real $y \in \omega^\omega$ if y already computes a Cohen 2-generic real. We can use this to show that no Cohen 2-generic real computes Sacks 3-generic real.

Corollary 4.2.20. *No Cohen 2-generic real computes a Sacks 3-generic real.*

Proof. Let $x \in 2^\omega$ be Cohen 2-generic. We suppose for a contradiction that there is a Sacks 3-generic real $y \leq_\mathrm{T} x$. By Corollary 4.2.18, $y <_\mathrm{T} x$ and by Theorem 4.2.19, there is a Cohen 2-generic real $z \leq_\mathrm{T} y$. But this contradicts Corollary 4.2.18. Therefore, x does not compute a Sacks 3-generic real. □

Recall that in set theory one can show that the Cohen forcing does not only add no Sacks reals, but also no minimal reals. In computability theory, there is a concept similar to minimal reals. We say that a real $x \geq_\mathrm{T} \emptyset$ has *minimal degree* if for every $y \leq_\mathrm{T} x$, either $x \leq_\mathrm{T} \emptyset$ or $x \leq_\mathrm{T} y$. As in theory, no Cohen 2-generic real computes a real which has minimal degree.

Corollary 4.2.21. *No Cohen 2-generic real computes a real which has minimal degree.*

Proof. Follows directly from Theorem 4.2.19 and the fact that Cohen 2-generic reals have not minimal degree (cf. [Yu06, Proposition 2.2]). □

We conclude this section by showing that Sacks 3-generic reals have minimal degree. Again, the proof is a translation of the set-theoretic proof that Sacks reals are minimal into computability theory.

Proposition 4.2.22 (Folklore)**.** *Every Sacks* 3*-generic real has minimal degree.*

Proof. Let $x \in 2^\omega$ be Sacks 3-generic and let $y \in 2^\omega$ be computable in x. Then there is an $e \in \omega$ such that $y = \Phi_e^x$. Let ψ be the statement "Φ_e^x is a total function with $\mathrm{ran}(\Phi_e^x) \subseteq 2$". Then ψ is Π_2^0. Let φ be the formula in the language $\mathcal{FL}_\mathrm{c}$ obtained by replacing x by $\dot{x}$ and e by $\check{e}$ in ψ. Then φ is a Π_2^0 sentence and $\varphi(x)$ is true. By Corollary 4.1.27, $x \Vdash_{\mathbb{S}_\mathrm{c}} \varphi$. Hence, there is some $T \in \mathbb{S}_\mathrm{c}$ such that $x \in [T]$ and $T \Vdash \varphi$. Let

$$C := \{S \in \mathbb{S}_\mathrm{c} : \forall n \forall s, s' \in S((\Phi^s_{e,\mathrm{lh}(s)}(n)\downarrow \wedge \Phi^{s'}_{e,\mathrm{lh}(s')}(n)\downarrow) \to \Phi^s_{e,\mathrm{lh}(s)}(n) = \Phi^{s'}_{e,\mathrm{lh}(s')}(n))\}.$$

Then C is Π_1^0. Since x is Sacks 3-generic, there is some $T' \in \mathbb{S}_\mathrm{c}$ such that $x \in [T']$ and either $T' \in C$ or T' has no extension in C. Without loss of generality, we can assume that $T' \leq T$. We make a case-distinction:

Case 1: $T' \in C$. We define a real $z \in 2^\omega$ by $z(n) := t(n)$, where t is the least $t \in T'$ in the canonical enumeration such that $\Phi^t_{e,\mathrm{lh}(t)}(n)$ converges. Then z is computable. We show that $y = z$. Let $n \in \omega$ and let $s \subseteq x$ such that $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges. Since $T' \in C$, for every $t \in T'$, if $\Phi^t_{e,\mathrm{lh}(t)}(n)$ converges, then $\Phi^s_{e,\mathrm{lh}(s)}(n) = \Phi^t_{e,\mathrm{lh}(t)}(n)$. Hence, $y(n) = z(n)$ for every $n \in \omega$ and so y is computable.

Case 2: T' has no extension in C. Let D be the set of all $S \leq T$ such that for every $s \in S$, if s is splits in S, then there are $s_0, s_1 \in S$ such that for every $i < 2$, $s^\frown i \subseteq s_i$ and for every $t \subsetneq s_i$, $t \subseteq s$ or t does not split in S and there is some $n < \min\{\mathrm{lh}(s_0), \mathrm{lh}(s_1)\}$ such that $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n)$ and $\Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$ converges and $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n) \neq \Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$. Then D is Π_2^0. We show that D is dense below T'. Let $S \leq T'$. We recursively define a decreasing sequence $\langle S_k : k \in \omega\rangle$ of conditions. Since T' has no extension in C, $S \notin C$. Hence, there are $s, s' \in S$ such that for some $n < \min\{\mathrm{lh}(s), \mathrm{lh}(s')\}$, $\Phi^s_{e,\mathrm{lh}(s)}(n)$ and $\Phi^{s'}_{e,\mathrm{lh}(s')}(n)$ converges and $\Phi^s_{e,\mathrm{lh}(s)}(n) \neq \Phi^{s'}_{e,\mathrm{lh}(s')}(n)$. Let (s, s') be minimal with that property. We set

$$S_0 := \{t \in S : t \subseteq s \vee s \subseteq t\} \cup \{t \in S : t \subseteq s' \vee s' \subseteq t\}.$$

If S_k is already defined, then let $\{s_m : m < 2^k\}$ be the set of the $(k+1)$st splitting notes of S_k, let $m < 2^k$, let $i < 2$, and let $S_k^{m,i} := \{s \in S_k : s \subseteq s_m \text{ or } {s_m}^\frown i \subseteq s\}$. Since S' has no extension in C, $S_k^{m,i} \notin C$. Hence, there are $s, s' \in S_k^{m,i}$ such that for some $n < \min\{\mathrm{lh}(s), \mathrm{lh}(s')\}$, $\Phi^s_{e,\mathrm{lh}(s)}(n)$ and $\Phi^{s'}_{e,\mathrm{lh}(s')}(n)$ converges and $\Phi^s_{e,\mathrm{lh}(s)}(n) \neq \Phi^{s'}_{e,\mathrm{lh}(s')}(n)$. Let (s^i_m, t^i_m) be minimal with that property. We define

$$S_{k+1} := \{s \in S_k : \exists m < 2^k \exists i < 2((s \subseteq s^i_m \vee s^i_m \subseteq s) \vee (s \subseteq t^i_m \vee t^i_m \subseteq s))\}.$$

Then $S' := \bigcap_{k \in \omega} S_k \in D$ and so D is dense below T'. Since x is Sacks 3-generic, there is some $S \in D$ such that $x \in [S]$. We define a real $z \in 2^\omega$ by $z(n) := s(n)$, where s is the least $s \in S$ such that $\mathrm{lh}(s) > n$, s splits in S, and for every $m < \mathrm{lh}(s)$, if $\Phi^s_{e,\mathrm{lh}(s)}(m)$ converges, then $\Phi^s_{e,\mathrm{lh}(s)}(m) = y(m)$. Then z is computable in y. We show that $x = z$. Let $n \in \omega$ and let $s \subseteq x$ such that $\mathrm{lh}(s) > n$ and s splits in S. Since $S \in D$, $s(n) = z(n)$ and so $x(n) = z(n)$. Therefore, x is computable in y. $\square$

4.2.4 Computable Silver forcing

In this section, we introduce a computable version of Silver forcing, which we call computable Silver forcing, and compare its n-generic reals to Cohen, Mathias, and Sacks n-generic reals. Recall that in set theory, Silver forcing does not add Cohen, Mathias, or Sacks reals, and Cohen, Mathias, and Sacks forcing do not add Silver reals (cf. Table 4.1). We show that no Cohen 2-generic or Sacks 4-generic real computes a Silver 3-generic real and that no Silver 3-generic real computes a Cohen 1-generic, Mathias 3-generic, or Sacks 4-generic real. Hence, the situation is analogous to set theory, except that it is not known whether Mathias n-generic reals can compute n-generic reals for computable Silver forcing (cf. Table 4.3). Note that most of the proofs in this section are translations of the set-theoretic proofs into computability theory.

We start with the definition of computable Silver forcing. Recall that a tree $T \subseteq 2^{<\omega}$ is uniform if for every $t, t' \in T$ with $\mathrm{lh}(t) = \mathrm{lh}(t')$ and every $i \in 2$, $t^\frown i \in T$ if and only if $t'^\frown i \in T$. Hence, for every uniform tree $T \subseteq 2^{<\omega}$, there is a unique function $f_T : \omega \to 3$ which determines T. If T is additionally computable, then f_T is computable as well. In the following, we always identify T with f_T and write $T(\ell)$ when we mean $f_T(\ell)$. We say that $\ell \in \omega$ is a *splitting level* for T if $T(\ell) = 2$.

Definition 4.2.23. *Computable Silver forcing*, denoted by $\mathbb{V}_{\mathrm{c}}$, is the partial order of all uniform computable perfect trees on 2 ordered by inclusion and equipped with $\mathrm{stem}(T)$ as the valuation function.

It is clear that computable Silver forcing is a computably arboreal forcing notion. Similar as before, we write "Silver n-generic" instead of "n-generic for $\mathbb{V}_{\mathrm{c}}$". Before we compare Silver n-generic reals to Cohen, Mathias, and Sacks n-generic reals, we first compute the complexity of computable Silver forcing.

Lemma 4.2.24. *Computable Silver forcing is a Π^0_2 forcing notion.*

Proof. By Lemma 4.1.22, it is enough to show that for every computable perfect tree $T \subseteq 2^{<\omega}$, the statement "T is uniform" is Π^0_2. Let $e \in \omega$ such that Φ_e is a characteristic function for a tree $T_e \subseteq 2^{<\omega}$. Then T_e is uniform if and only if

$$\forall t, t' (t, t' \in T_e \to \forall i \in 2 (t^\frown i \in T_e \leftrightarrow t'^\frown i \in T_e)).$$

Since Φ_e is a total computable function, the statement "T_e is uniform" is Π^0_2. $\square$

Now we compare Silver n-generic reals to Cohen, Mathias, and Sacks n-generic reals. To do this, we investigate whether Silver n-generic reals can compute n-dominating, n-unbounded, or n-splitting reals. Recall that in set theory, Silver forcing adds splitting reals, but no dominating or unbounded (cf. Table 4.1). We show that the analogue is true for Silver n-generic reals. We start with n-splitting reals.

Proposition 4.2.25. *Let $n \geq 3$. Every Silver n-generic real computes an n-splitting real.*

Proof. Let $x \in 2^\omega$ be Silver n-generic, let $f : 2^{<\omega} \to [\omega]^{<\omega}$ be defined by

$$m \in f(s) :\Longleftrightarrow m < \mathrm{lh}(s) \text{ and } \sum_{k \leq m} s(k) \equiv 1 \pmod 2,$$

and let $A := \bigcup_{k\in\omega} f(x{\restriction}n)$. Then f is computable and A is computable in x. We show that A is n-splitting. Let $B \in [\omega]^\omega$ be Δ^0_n, let $k \in \omega$, and let

$$D_k := \{T \in \mathbb{V}_\mathrm{c} : |B \cap f(\mathrm{stem}(T))| \geq k \text{ and } |B \setminus f(\mathrm{stem}(T))| \geq k\}.$$

Then D_k is Σ^0_n. Let $T \in \mathbb{V}_\mathrm{c}$ and let $\{\ell_i : i < 2k\} \subseteq \{\ell \in \omega : T(\ell) = 2\}$ and $\{m_i : i < 2k\} \subseteq B$ such that $\ell_0 < m_0 < \ell_1 < \cdots < \ell_{2k-1} < m_{2k-1}$. Then there is some $S \leq T$ such that for every $\ell \in \omega$,

$$S(\ell) := \begin{cases} T(\ell) & \text{if } \ell > m_{2k-1} \text{ or } T(\ell) \in 2, \\ 0 & \text{if } \ell \leq m_{2k+1},\ \ell \notin \{\ell_i : i < 2k\}, \text{ and } T(\ell) = 2, \\ 0 & \text{if there is an } i < k \text{ such that } \ell = \ell_i \text{ and } |\{\ell' < \ell : T(\ell') = 1\}| \text{ is odd}, \\ 0 & \text{if there is an } i \geq k \text{ such that } \ell = \ell_i \text{ and } |\{\ell < \ell_i : T(\ell) = 1\}| \text{ is even}, \\ 1 & \text{otherwise.} \end{cases}$$

By definition, $\{m_i : i < k\} \subseteq B \cap f(\mathrm{stem}(S))$ and $\{m_i : k \leq i < 2k\} \subseteq B \setminus f(\mathrm{stem}(S))$. Hence, $S \in D_k$ and so D_k is dense. Since x is Silver n-generic, there is some $T \in D$ such that $x \in [T]$. Hence, $|B \cap A| \geq k$ and $|B \setminus A| \geq k$ for every $k \in \omega$. Therefore, A splits every Δ^0_n set. By Proposition 4.2.2, A is n-splitting. ☐

Corollary 4.2.26. *No Sacks* 4*-generic real computes a Silver* 3*-generic real.*

Proof. Follows directly from Propositions 4.2.16 and 4.2.25. ☐

Next, we show that no Silver 3-generic real computes a 1-dominating or 1-unbounded real. Since every 1-dominating real is 1-unbounded, it is enough to prove that no Silver 3-generic real computes a 1-unbounded reals.

Proposition 4.2.27. *No Silver* 3*-generic real computes a* 1*-unbounded real.*

Proof. Let $x \in 2^\omega$ be Silver 3-generic and let $f \in \omega^\omega$ be computable in x. Then there is an $e \in \omega$ such that $f = \Phi^x_e$. Let ψ be the statement "Φ^x_e is a total function". Then ψ is Π^0_2. Let φ be the formula in the language $\mathcal{FL}_\mathrm{c}$ obtained by replacing x by $\dot{x}$ and e by $\check{e}$ in ψ. Then φ is a Π^0_2 sentence and $\varphi(x)$ is true. By Corollary 4.1.27, $x \Vdash_{\mathbb{V}_\mathrm{c}} \varphi$. Hence, there is some $T \in \mathbb{V}_\mathrm{c}$ such that $x \in [T]$ and $T \Vdash \varphi$. Let

$$D := \{S \in \mathbb{V}_\mathrm{c} : \forall n \exists \ell \forall s \in S \cap 2^\ell (\Phi^s_{e,\mathrm{lh}(s)}(n){\downarrow})\}.$$

Then D is Π^0_2. We show that D is dense below T. Let $S \leq T$. We recursively define a decreasing sequence $\langle S_n : n \in \omega\rangle$ of conditions. Let $S_0 := S$. If S_n is already defined, then let $\ell \in \omega$ be the $(n+1)$st splitting level and let $\{s_k : k < 2^{n+1}\} = S_{n,k} \cap 2^{\ell+1}$. We recursively define a sequence $\langle (S_{n,k}, \ell_k) : k \leq 2^{n+1}\rangle$. Let $(S_{n,0}, \ell_0) := (S_n, \ell+1)$. If $(S_{n,k}, \ell_k)$ is already defined, then let $y \in [S_{n,k}]$ such that $s_k \subseteq y$ and y is Silver 3-generic. Then $y \Vdash_{\mathbb{V}_\mathrm{c}} \varphi$ and so $\varphi(y)$ is true by Corollary 4.1.27. Hence, $\Phi^y_e(n)$ converges and so there is some $s \subseteq y$ such that $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges. Let $s \in S_{n,k}$ be minimal in the canonical enumeration such that $\mathrm{lh}(s) \geq \ell_k$, $s_k \subseteq s$, and $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges, let $\ell_{k+1} := \mathrm{lh}(s)$, and let $S_{n,k+1}$ be defined by

$$S_{n,k+1}(m) := \begin{cases} s(m) & \text{if } \ell_k \leq m < \mathrm{lh}(s), \\ S_{n,k}(m) & \text{otherwise.} \end{cases}$$

Then $S_{n,2^{n+1}} \leq S_n$ and for every $s \in S_{n,2^{n+1}} \cap 2^{\ell_{2^{n+1}}}$, $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges. We set $S_{n+1} := S_{n,2^{n+1}}$.

Let $S' := \bigcap_{n\in\omega} S_n$. Then $S' \leq S$ and $S' \in D$. Hence, D is dense below T. Since x is Silver 3-generic, there is some $S \in D$ such that $x \in [S]$. For every $n \in \omega$, let $\ell_n \in \omega$ be minimal such that for all $s \in S \cap 2^{\ell_n}$, $\Phi^s_{e,\mathrm{lh}(s)}(n)\downarrow < \ell_n$. Let $g \in \omega^\omega$ be defined by $g(n) := \ell_n$. Then g is computable and for every $n \in \omega$, $f(n) < g(n)$. □

Corollary 4.2.28.

(a) No Silver 3-generic real computes a Cohen 1-generic real.

(b) No Cohen 2-generic real computes a Silver 3-generic real.

(c) No Silver 3-generic real computes a Mathias 3-generic real.

Proof. Item (a) follows directly from Propositions 4.2.8 and 4.2.27, (b) from (a) and Theorem 4.2.19, and (c) from (a) and Theorem 4.2.12 □

Finally, we show that Silver 3-generic reals do not compute Sacks 4-generic reals. To do this, we first prove that Silver 3-generic reals have minimal degree.

Proposition 4.2.29. *Every Silver 3-generic real has minimal degree.*

Proof. Let $x \in 2^\omega$ be Silver 3-generic and let $y \in 2^\omega$ be computable in x. Then there is an $e \in \omega$ such that $y = \Phi^x_e$. Let ψ be the statement "Φ^x_e is a total function with $\mathrm{ran}(\Phi^x_e) \subseteq 2$". Then ψ is Π^0_2. Let φ be the formula in the language $\mathcal{FL}_\mathrm{c}$ obtained by replacing x by $\dot{x}$ and e by $\check{e}$ in ψ. Then φ is a Π^0_2 sentence and $\varphi(x)$ is true. By Corollary 4.1.27, $x \Vdash_{\mathbb{V}_\mathrm{c}} \varphi$. Hence, there is some $T \in \mathbb{V}_\mathrm{c}$ such that $x \in [T]$ and $T \Vdash \varphi$. Let

$$C := \{S \in \mathbb{V}_\mathrm{c} : \forall n \forall \ell \forall s, s' \in S \cap 2^\ell ((|\{m : s(m) \neq s'(m)\}| = 1 \wedge \Phi^s_{e,\mathrm{lh}(s)}(n)\downarrow \wedge \Phi^{s'}_{e,\mathrm{lh}(s')}(n)\downarrow) \\ \rightarrow \Phi^s_{e,\mathrm{lh}(s)}(n) = \Phi^{s'}_{e,\mathrm{lh}(s')}(n))\}.$$

Then C is Π^0_1. Since x is Silver 3-generic, there is some $T' \in \mathbb{S}_\mathrm{c}$ such that $x \in [T']$ and either $T' \in C$ or T' has no extension in C. Without loss of generality, we can assume that $T' \leq T$. We make a case-distinction:

Case 1: $T' \in C$. We define a real $z \in 2^\omega$ by $z(n) := t(n)$, where t is the least element of T' in the canonical enumeration such that $\Phi^t_{e,\mathrm{lh}(t)}(n)$ converges. Then z is computable. We show that $y = z$. Let $n \in \omega$, let $s \subseteq x$ such that $\Phi^s_{e,\mathrm{lh}(s)}(n)$ converges, and let $t \in T'$ such that $\Phi^t_{e,\mathrm{lh}(t)}(n)$ converges. We suppose for a contradiction that $\Phi^s_{e,\mathrm{lh}(s)}(n) \neq \Phi^t_{e,\mathrm{lh}(t)}(n)$. Without loss of generality, we can assume that $\mathrm{lh}(s) = \mathrm{lh}(t)$ and $k := |\{m : s(m) \neq t(m)\}|$ is minimal. Since $T' \in C$, $k > 1$. Let $\ell < \mathrm{lh}(s)$ be minimal such that $s(m) \neq t(m)$ and let $t' \in 2^{\mathrm{lh}(t)}$ be defined by

$$t'(m) := \begin{cases} 1 - t(m) & \text{if } m = \ell, \\ t(m) & \text{otherwise.} \end{cases}$$

Then $t' \in T'$ and $|\{m : s(m) \neq t'(m)\}| = k - 1$. Without loss of generality, we can assume that $\Phi^{t'}_{e,\mathrm{lh}(t')}(n)$ converges. Since $T' \in C$, $\Phi^t_{e,\mathrm{lh}(t)}(n) = \Phi^{t'}_{e,\mathrm{lh}(t')}(n)$. But this contradicts the minimality of k. Hence, $\Phi^s_{e,\mathrm{lh}(s)}(n) = \Phi^t_{e,\mathrm{lh}(t)}(n)$ and so $y(n) = z(n)$. Therefore, y is computable.

Case 2: T' has no extension in C. Let D be the set of all $S \leq T$ such that for every $s \in S$, if s is splits in S, then there are $s_0, s_1 \in S$ such that for every $i < 2$, $s^\frown i \subseteq s_i$ and for every $t \subsetneq s_i$, $t \subseteq s$ or t does not split in S and there is some $n < \min\{\mathrm{lh}(s_0), \mathrm{lh}(s_1)\}$ such that $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n)$ and $\Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$ converges and $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n) \neq \Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$. Then D is Π^0_2. We show that D is dense below T'. Let $S \leq T'$. We recursively define a decreasing sequence $\langle S_k : k \in \omega\rangle$ of conditions. Since T' has no extension in C, $S \notin C$. Hence, there are $s, s' \in S$ such that $\mathrm{lh}(s) = \mathrm{lh}(s')$, $|\{m : s(m) \neq s'(m)\}| = 1$, and there is some $n < \min\{\mathrm{lh}(s), \mathrm{lh}(s')\}$, $\Phi^{s}_{e,\mathrm{lh}(s)}(n)$ and $\Phi^{s'}_{e,\mathrm{lh}(s')}(n)$ converges and $\Phi^{s}_{e,\mathrm{lh}(s)}(n) \neq \Phi^{s'}_{e,\mathrm{lh}(s')}(n)$. Let (s, s') be minimal with that property. We set

$$S_0 := \{t \in S : t \subseteq s \vee s \subseteq t\} \cup \{t \in S : t \subseteq s' \vee s' \subseteq t\}.$$

If S_k is already defined, then let $\ell \in \omega$ be the $(n+1)$st splitting level and let $\{s_k : k < 2^{n+1}\} = S_{n,k} \cap 2^{\ell+1}$. We recursively define a sequence $\langle (S_{n,k}, \ell_k) : k \leq 2^{n+1}\rangle$. Let $(S_{n,0}, \ell_0) := (S_n, \ell + 1)$. If $(S_{n,k}, \ell_k)$ is already defined, then let $S^m_{n,k} := \{s \in S_{n,k} : s \subseteq s_m \text{ or } s_m \subseteq s\}$. Since T' has no extension in C, $S^m_{n,k} \notin C$. Hence, there are $s, s' \in S^m_{n,k}$ such that $\mathrm{lh}(s) = \mathrm{lh}(s') \geq \ell_k$, $|\{m : s(m) \neq s'(m)\}| = 1$, and there is some $n < \min\{\mathrm{lh}(s), \mathrm{lh}(s')\}$ such that $\Phi^{s}_{e,\mathrm{lh}(s)}(n)$ and $\Phi^{s'}_{e,\mathrm{lh}(s')}(n)$ converges and $\Phi^{s}_{e,\mathrm{lh}(s)}(n) \neq \Phi^{s'}_{e,\mathrm{lh}(s')}(n)$. Let (s^i_m, t^i_m) be minimal with that property, let $\ell_{k+1} := \mathrm{lh}(s)$, and let $S_{n,k+1}$ be defined by

$$S_{n,k+1}(m) := \begin{cases} s(m) & \text{if } \ell_k \leq m < \mathrm{lh}(s) \text{ and } s(m) = s'(m), \\ 2 & \text{if } \ell_k \leq m < \mathrm{lh}(s) \text{ and } s(m) \neq s'(m), \\ S_{n,k}(m) & \text{otherwise.} \end{cases}$$

Then $S_{n,2^{n+1}} \leq S_n$ and for every $s \in S_{n,2^{n+1}} \cap 2^\ell$, there are $s_0, s_1 \in S_{n,2^{n+1}}$ such that for every $i < 2$, $s^\frown i \subseteq s_i$ and for every $t \subsetneq s_i$, $t \subseteq s$ or t does not split in $S_{n,2^{n+1}}$ and there is some $n < \min\{\mathrm{lh}(s_0), \mathrm{lh}(s_1)\}$ such that $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n)$ and $\Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$ converges and $\Phi^{s_0}_{e,\mathrm{lh}(s_0)}(n) \neq \Phi^{s_1}_{e,\mathrm{lh}(s_1)}(n)$. We set $S_{n+1} := S_{n,2^{n+1}}$.

Let $S' := \bigcap_{n\in\omega} S_n$. Then $S' \leq S$ and $S' \in D$. Hence, D is dense below T. Since x is Silver 3-generic, there is some $S \in D$ such that $x \in [S]$. We define a real $z \in 2^\omega$ by $z(n) := s(n)$, where s is the least $s \in S$ such that $\mathrm{lh}(s) > n$, s splits in S, and for every $m < \mathrm{lh}(s)$, if $\Phi^{s}_{e,\mathrm{lh}(s)}(m)$ converges, then $\Phi^{s}_{e,\mathrm{lh}(s)}(m) = y(m)$. Then z is computable in y. We show that $x = z$. Let $n \in \omega$ and let $s \subseteq x$ such that $\mathrm{lh}(s) > n$ and s splits in S. Since $S \in D$, $s(n) = z(n)$ and so $x(n) = z(n)$. Therefore, x is computable in y. ☐

Corollary 4.2.30. *No Silver 3-generic real computes a Sacks 4-generic real.*

Proof. We suppose for a contradiction that there is a Silver 3-generic real $x \in 2^\omega$ such that x computes a Sacks 4-generic real $y \in 2^\omega$. By Corollary 4.2.26, $y <_{\mathrm{T}} x$. But this contradicts Proposition 4.2.29 since y is not computable. ☐

4.2.5 Forcing notions approximating the jump

The goal of this section is to generalize the result of Cholak, Dzhafarov, Hirst, and Slaman (Theorem 4.2.12) that for every $n \geq 3$, every Mathias n-generic real computes a Cohen n-generic real. We show that their proof does not only work for computable Mathias forcing, but also for a certain class of forcing notions, including a computable version of Laver forcing which we shall introduce in

Section 4.2.6. We also show that every n-generic real for these forcing notions computes not only Cohen n-generic reals, but also n-generic reals for any Δ^0_1 forcing notion.

We start with a sketch of Cholak, Dzhafarov, Hirst, and Slaman's proof that every Mathias n-generic real computes a Cohen n-generic real. The goal of the proof is to define a computable function $f : [\omega]^{<\omega} \to \mathbb{C}$ such that if $A \in [\omega]^\omega$ is Mathias n-generic, then $x := \bigcup\{f(A \cap k) : k \in \omega\}$ is Cohen n-generic. The rough idea to define f is to sent a set $F = \{m_0, \dots, m_e\}$ with $m_0 < \cdots < m_e$ to the smallest $s \in \mathbb{C}$ such that $s \leq_{\mathbb{C}} f(F \setminus \{m_e\})$ and $s \in W_e^{\emptyset^{(n-1)}}$ if it exists. However, the question whether such an s exists cannot be answered computably and so f is not necessarily computable. Therefore, Cholak, Dzhafarov, Hirst, and Slaman used computable approximations of Σ^0_n sets to define f. More precisely, they replaced $W_e^{\emptyset^{(n-1)}}$ in the definition of f by

$$W_e^n[\sigma_0, \dots, \sigma_{n-1}] := W_{e,\sigma_{n-1}}^{\emptyset^{(n-1)}[\sigma_0,\dots,\sigma_{n-2}]},$$

where $\sigma_0, \dots, \sigma_{n-1}$ are the n greatest elements of F in decreasing order. Note that for every $e \in \omega$ and every $\sigma_0, \dots, \sigma_{n-1} \in \omega$, $W_e^n[\sigma_0, \dots, \sigma_{n-1}]$ is computable and bounded by σ_{n-1}. Hence, f is computable. Moreover, as in the proof of Lemma 4.2.5, we get that for every $k \in \omega$ and every $E \in [\omega]^\omega$, there are $\sigma_0, \dots, \sigma_{n-1} \in E$ such that $W_e^{\emptyset^{(n-1)}} \cap k = W_e^n[\sigma_0, \dots, \sigma_{n-1}] \cap k$. Thus, for every $e, k \in \omega$, the set of conditions $(F, E) \in \mathbb{R}_c$ such that the n greatest elements of F correctly approximate $W_e^{\emptyset^{(n-1)}} \cap k$ is dense. Cholak, Dzhafarov, Hirst, and Slaman used this to show that f has the desired properties (cf. Theorem 4.2.33 for more details). So far, the argument does not need any special properties of computable Mathias forcing, except that computable Mathias conditions can be extended to approximate Σ^0_n sets. However, there are many other forcing notions which also have this property.

Definition 4.2.31. We say that a forcing notion is $\mathbb{P}$ *flexible* if

(a) for every $p \in \mathbb{P}$, the set $\{m \in \omega : \exists q \leq p(v(q)(\mathrm{lh}(p)) = m)\}$ is infinite and

(b) for every $p' \leq p \in \mathbb{P}$ and every $s \in \omega^{<\omega}$ with $v(p) \subseteq s \subseteq v(p')$, there is some $q \in \mathbb{P}$ such that $p' \leq q \leq p$ and $v(q) = s$.

Lemma 4.2.32. *Let $\mathbb{P}$ be a flexible forcing notion, let $n \geq 1$, and let $e \in \omega$. Then for every $k \in \omega$ and $p \in \mathbb{P}$, there are $p' \leq p$ and $\sigma_0, \dots, \sigma_{n-1} \in \omega$ such that $v(p') = v(p)^\frown\langle\sigma_{n-1}, \dots, \sigma_0\rangle$ and $W_e^{\emptyset^{(n-1)}} \cap k = W_e^n[\sigma_0, \dots, \sigma_{n-1}] \cap k$.*

Proof. Let $k \in \omega$ and let $p \in \mathbb{P}$. Then there is some $\sigma \in \omega$ such that for every $\sigma' \geq \sigma$ and every $k' < k$, $\Phi_e^{\emptyset^{(n-1)}}(k')$ converges if and only if $\Phi_{e,\sigma'}^{\emptyset^{(n-1)}}(k')$ converges. Since $\mathbb{P}$ is flexible, there are $p' \leq p$ and $\sigma_{n-1} \geq \sigma$ such that $v(p') = v(p)^\frown\sigma_{n-1}$. If $n = 1$, then $W_e^n[\sigma_{n-1}] = W_{e,\sigma_{n-1}}$ and so $W_e^{\emptyset^{(n-1)}} \cap k = W_e^n[\sigma_{n-1}] \cap k$. Hence, we can assume that $n > 1$. By convention, for every $k' < k$, $\Phi_{e,\sigma_{n-1}}^{\emptyset^{(n-1)}}(k')$ converges if and only if $\Phi_{e,\sigma_{n-1}}^s(k')$ converges, where $s \in 2^{\sigma_{n-1}}$ such that for every $m < \sigma_{n-1}$, $s(m) = 1$ if and only if $m \in \emptyset^{(n-1)}$. Hence, it is enough to show that for every $n' \geq 1$, there are $p' \leq p$ and $\sigma_0, \dots, \sigma_{n'-1}$ such that $v(p') = v(p)^\frown\langle\sigma_{n'-1}, \dots, \sigma_0\rangle$ and $\emptyset^{(n')} \cap k = \emptyset^{(n')}[\sigma_0, \dots, \sigma_{n'-1}] \cap k$. We show this by induction on n'. It is clear for $n' = 1$. Let $n' > 1$ such that the induction hypothesis is true for $n' - 1$. Then there is some $\sigma \in \omega$ such that for every $\sigma' \geq \sigma$ and every $e' < k$, $\Phi_{e'}^{\emptyset^{(n'-1)}}(e')$ converges if and only if $\Phi_{e',\sigma'}^{\emptyset^{(n'-1)}}(e')$ converges. Since $\mathbb{P}$ is flexible, there are $p' \leq p$ and $\sigma_{n-1} \geq \sigma$ such that $v(p') = v(p)^\frown\sigma_{n-1}$. By induction hypothesis, there are $p'' \leq p'$ and $\sigma_0, \ldots, \sigma_{n'-2} \in \omega$ such that $v(p'') = v(p')^\frown\langle\sigma_{n'-2}, \ldots, \sigma_0\rangle$ and $\emptyset^{(n'-1)} \cap \sigma_{n'-1} = \emptyset^{(n'-1)}[\sigma_0, \ldots, \sigma_{n'-2}] \cap \sigma_{n'-1}$. With a similar argument as in the proof of Lemma 4.2.5, we get that $\emptyset^{(n')} \cap k = \emptyset^{(n')}[\sigma_0, \ldots, \sigma_{n'-1}] \cap k$. $\square$

Next, we show that every n-generic real for a flexible forcing notion computes a Cohen n-generic real. The proof is essentially the same as Cholak, Dzhafarov, Hirst, and Slaman's proof for Mathias n-generic reals. Their proof works not only for Cohen n-generic reals, but also for n-generic reals for any Δ^0_1 forcing notion. We even obtain partial results for forcing notions of higher complexity.

Theorem 4.2.33. *Let $n, n' \in \omega$, let $\mathbb{P}$ be a Δ^0_{n+1} flexible forcing notion, let $\mathbb{Q}$ be $\Delta^0_{n'+1}$ forcing notion, let $m > \max\{n, n'\}$, and let $x \in \omega^\omega$ be m-generic for $\mathbb{P}$. Then $x \oplus \emptyset^{(n')}$ computes an m-generic real for $\mathbb{Q}$.*

Proof. Since $\mathbb{Q}$ is $\Delta^0_{n'+1}$, $\mathbb{Q}$, $\leq_\mathbb{Q}$, and $v_\mathbb{Q}$ are computable in $\emptyset^{(n')}$. We recursively define a function $f : \omega^{<\omega} \to \mathbb{Q}$. Let $1_\mathbb{Q}$ be the weakest condition in $\mathbb{Q}$. We set $f(\emptyset) := 1_\mathbb{Q}$. Let $s \in \omega^{<\omega}$ such that for every $s' \subsetneq s$, $f(s')$ is already defined and let $t \subsetneq s$ be the longest initial segment such that $\mathrm{lh}(t)$ is a multiple of m. Then there is some $q \in \mathbb{Q}$ such that $f(t) = q$. If $\mathrm{lh}(s)$ is not a multiple of m, then we define $f(s) := q$. Otherwise, $\mathrm{lh}(s) = e \cdot m$ for some $e \in \omega$ and there are $\sigma_0, \ldots, \sigma_{m-1} \in \omega$ such that $s = t^\frown\langle\sigma_{m-1}, \ldots, \sigma_0\rangle$. We check whether $\{q' \leq_\mathbb{Q} q : q' \in W^m_e[\sigma_0, \ldots, \sigma_{m-1}]\}$ is non-empty. If that is the case, then we define $f(s) := q'$, where q' is minimal in the canonical enumeration of $\mathbb{Q}$ with $q' \leq_\mathbb{Q} q$ and $q' \in W^m_e[\sigma_0, \ldots, \sigma_{m-1}]$. Otherwise, we set $f(s) := q$.

Then f is a total function which is computable in $\emptyset^{(n')}$. Hence, $y := \bigcup_{k\in\omega} v_\mathbb{Q}(f(x\upharpoonright k))$ is computable in $x \oplus \emptyset^{(n')}$. We show that y is m-generic for $\mathbb{Q}$. Since x is m-generic for $\mathbb{P}$, there is an m-generic filter G for $\mathbb{P}$ such that $x = \bigcup_{p\in G} v_\mathbb{P}(p)$. We define $H := \{q \in \mathbb{Q} : \exists p \in G(q \geq_\mathbb{Q} f(v_\mathbb{P}(p)))\}$. Then H_y is a filter and $\bigcup_{q\in H} v_\mathbb{Q}(q) = y$. It remains to show that H is m-generic for $\mathbb{Q}$. Let $C \subseteq \mathbb{Q}$ be Σ^0_m and let $C' := \{p \in \mathbb{P} : f(v_\mathbb{P}(p)) \in C\}$. Since f is $\Delta^0_{n'+1}$ and $\mathbb{P}$ and $v_\mathbb{P}$ are Δ^0_{n+1}, C' is Σ^0_m. Hence, there is some $p \in G$ such that either $p \in C'$ or p has no extension in C'. We make a case-distinction:

Case 1: $p \in C'$. Let $q := f(v_\mathbb{P}(p))$. Then $q \in C$ and since $p \in G$, $q \in H$. Therefore, q witnesses that H meets C.

Case 2: p has no extension in C'. Let $e \in \omega$ be minimal such that $\mathrm{lh}(v_\mathbb{P}(p)) \leq e \cdot m$ and $C = W^{\emptyset^{(m-1)}}_e$. Then there is some $p' \leq_\mathbb{P} p$ such that $p' \in G$ and $\mathrm{lh}(v_\mathbb{P}(p')) \geq e \cdot m$. Since $\mathbb{P}$ is flexible, we can assume without loss of generality that $\mathrm{lh}(v_\mathbb{P}(p)) = e \cdot m$. We suppose for a contradiction that $q := f(v_\mathbb{P}(p'))$ has an extension in C. Let $q' \leq q$ such that $q' \in C$ and let $k \in \omega$ such that $q' \in C \cap k$. By Lemma 4.2.32, there are $p'' \leq p$ and $\sigma_0, \ldots, \sigma_{m-1} \in \omega$ such that $v(p'') = v(p')^\frown\langle\sigma_{m-1}, \ldots, \sigma_0\rangle$ and $W^{\emptyset^{(m-1)}}_e \cap k = W^m_e[\sigma_0, \ldots, \sigma_{m-1}] \cap k$. Then $q' \in W^m_e[\sigma_0, \ldots, \sigma_{m-1}] \cap k$ and so $f(v_\mathbb{P}(p'')) \in C$. But this is a contradiction since p has no extension in C'. Therefore, q has no extension in C and so q witnesses that H meets the set of conditions having no extension in C. $\square$

Note that computable Mathias forcing is not flexible because the range its valuation function is $2^{<\omega}$ and not $\omega^{<\omega}$. Hence, we cannot directly apply Theorem 4.2.33 to Mathias n-generic reals. However, we with a little extra work we can still apply it.

Corollary 4.2.34. *Let $n \in \omega$, let $\mathbb{P}$ be a Δ^0_{n+1} forcing notion, let $m > \max\{2, n\}$, and let $A \in [\omega]^\omega$ be a Mathias m-generic real. Then $A \oplus \emptyset^{(n)}$ computes an m-generic real for $\mathbb{P}$.*

Proof. Let v be the valuation function for computable Mathias forcing, let $v' : \mathbb{R}_\mathrm{c} \to \omega^{<\omega}$ be the function which maps (F, E) to the finite sequence enumerating F in ascending order, and let $\mathbb{R}'_\mathrm{c}$ be the forcing notion we obtain by replacing v in computable Mathias forcing with v'. Then $\mathbb{R}'_c$ is a Π^0_2 flexible forcing notion. Since A is Mathias m-generic, there is a Mathias m-generic filter G such that $\bigcup\{v(F,E) : (F,E) \in G\}$ is the characteristic function of A. Note that G is also m-generic for $\mathbb{R}'_c$. Let $x := \bigcup\{v'(F,E) : (F,E) \in G\}$. Then x is an m-generic real for $\mathbb{R}'_c$ and $A \equiv_T x$. By Theorem 4.2.33, $x \oplus \emptyset^{(n)}$ computes an m-generic real for $\mathbb{P}$. Since $A \equiv_T x$, $A \oplus \emptyset^{(n)}$ also computes an m-generic real for $\mathbb{P}$. □

Let $n \geq 3$ and let $A \in [\omega]^\omega$ be Mathias n-generic. By Corollary 4.2.34, $A \oplus \emptyset^{(n)}$ computes a Sacks and a Silver n-generic real. Unfortunately, since Mathias 3-generic reals cannot compute $\emptyset'$ (cf. [CDHS14, Proposition 2.8]), this tell us nothing about whether A can compute Sacks or Silver n-generic reals. However, A is high and so $A' \geq_T \emptyset''$. Hence, A' computes a Sacks and a Silver n-generic real. This does not only work for computable Mathias forcing, but for every flexible forcing notion whose n-generic reals compute 1-dominating reals.

Corollary 4.2.35. *Let $n \in \omega$, let $n' \geq 2$, let $\mathbb{P}$ be a Δ^0_{n+1} flexible forcing notion, let $\mathbb{Q}$ be a $\Delta^0_{n'+1}$ forcing notion, let $m > \max\{n,n'\}$, and let $x \in \omega^\omega$ be m-generic for $\mathbb{P}$. If x computes a 1-dominating real, then $x^{(n'-1)}$ computes an m-generic real for $\mathbb{Q}$.*

Proof. By Theorem 4.2.4, $x' \geq_T \emptyset''$. Hence, $x^{(n'-1)} \geq_T \emptyset^{(n)}$ and so $x^{(n'-1)} \geq_T x \oplus \emptyset^{(n')}$. By Theorem 4.2.33, $x^{(n'-1)}$ computes an m-generic real for $\mathbb{Q}$. □

We can also prove a slightly weaker version of Corollary 4.2.35 for flexible forcing notions whose n-generics do not compute 1-dominating reals.

Corollary 4.2.36. *Let $n, n' \in \omega$, let $\mathbb{P}$ be a Δ^0_{n+1} flexible forcing notion, let $\mathbb{Q}$ be $\Delta^0_{n'+1}$ forcing notion, let $m > \max\{n,n'\}$, and let $x \in \omega^\omega$ be m-generic for $\mathbb{P}$. Then $x^{(n')}$ computes an m-generic real for $\mathbb{Q}$.*

Proof. Follows directly from Theorem 4.2.33 and the fact that for every real x, $x^{(n')} \geq_T \emptyset^{(n')}$. □

We conclude this section with a corollary about Cohen forcing. Note that Cohen forcing is not flexible. However, using a similar argument to that for computable Mathias forcing, we can apply Corollary 4.2.36 to Cohen forcing.

Corollary 4.2.37. *Let $n \in \omega$, let $\mathbb{P}$ be a Δ^0_{n+1} forcing notion, let $m > n$, and let $x \in \omega^\omega$ be a Cohen m-generic real. Then $x^{(n)}$ computes an m-generic real for $\mathbb{P}$.*

Proof. Let $v' : \mathbb{C} \to \omega^{<\omega}$ be the function which maps s to the finite sequence which enumerates $\{k : s(k) = 1\}$ in ascending order and let $\mathbb{C}'$ be the Cohen forcing equipped with the valuation function v'. Then $\mathbb{C}'$ is a Δ^0_1 flexible forcing notion. Since x is Cohen m-generic, there is a Cohen m.generic filter G such that $x = \bigcup G$. Note that G is also m-generic for $\mathbb{C}'$. Hence, $y := \bigcup\{v'(s) : s \in G\}$ is a m-generic real for $\mathbb{C}$. It is clear that $x \equiv_T y$. By Corollary 4.2.36, $y^{(n)}$ computes an m-generic real for $\mathbb{P}$. Therefore, $x^{(n)}$ also computes an m-generic real for $\mathbb{P}$. □

4.2.6 Computable Laver forcing

In this section, we introduce a computable version of Laver forcing, which we call computable Laver forcing, and compare its n-generic reals to Cohen, Sacks, and Silver n-generic reals. Recall that in set theory, Laver forcing does not add Cohen reals (cf. Table 4.1). However, the analogue is not true in computability theory. We show that for every $n \geq 3$, every n-generic real for computable Laver forcing computes a Cohen n-generic real. Moreover, we show that no Cohen 2-generic real, Sacks 3-generic real, or Silver 3-generic real computes a 3-generic real for computable Laver forcing (cf. Table 4.3). Note that this is analogous to set theory. We start with the definition of computable Laver forcing.

Definition 4.2.38. *Computable Laver forcing*, denoted by $\mathbb{L}_c$, is the partial order of all computable perfect trees on ω such that every node above the stem splits infinitely often ordered by inclusion and equipped with $\text{stem}(T)$ as the valuation function.

It is clear that computable Laver forcing is a computably arboreal forcing notion. As for the other forcing notions, we write "Laver n-generic" instead of "n-generic for $\mathbb{L}_c$". Before we compare Laver n-generic reals to Cohen, Sacks, and Silver n-generic reals, we first compute the complexity of computable Laver forcing.

Lemma 4.2.39. *Computable Laver forcing is a Π^0_2 forcing notion.*

Proof. By Lemma 4.1.22, it is enough to show that for every computable perfect tree $T \subseteq \omega^{<\omega}$, the statement "every node in T above the stem splits infinitely often" is Π^0_2. Let $e \in \omega$ such that Φ_e is a characteristic function for a tree $T_e \subseteq \omega^{<\omega}$. Then every node in T_e above the stem splits infinitely often if and only if

$$\forall t(t \subsetneq \text{stem}(T_e) \vee \forall k \exists n \geq k(t^\frown n \in T_e)).$$

By Lemma 4.1.22, the statement "$t \subsetneq \text{stem}(T_e)$" is Π^0_2. Hence, the whole statement is Π^0_2 as well. $\square$

Next, we show that for every $n \geq 3$, every Laver n-generic real computes a Cohen n-generic real. It is clear that computable Laver forcing is flexible. Hence, we can simply use Theorem 4.2.33.

Corollary 4.2.40. *Let $n \geq 3$. Then every Laver n-generic real computes a Cohen n-generic real.*

Proof. Follows directly from Theorem 4.2.33 and Lemma 4.2.39. $\square$

In set theory, Laver forcing does not add Cohen reals (cf. Table 4.1). Hence, Laver forcing is in a similar situation to Mathias forcing. Both do not add Cohen reals, but their n-generic reals compute Cohen n-generic reals. Moreover, Gray proved in [Gra80] that Laver reals are minimal. However, this is also different in computability theory. By Corollary 4.2.40, every Laver 3-generic computes a Cohen 3-generic real. Since Cohen 3-generic reals do not have minimal degree, Laver 3-generic reals do not have minimal degree either. By a similar argument, Mathias 3-generic reals also do not have minimal degree. However, Mathias reals are not minimal in set theory (cf. [Mat77, Corollary 8.3]).

Corollary 4.2.41.

(a) Let $n \geq 3$. Then every Laver n-generic real computes an n-unbounded and an n-splitting real.

(b) No Sacks 3-generic real computes a Laver 3-generic real.

(c) No Silver 3-generic real computes a Laver 3-generic real.

Proof. Item (a) follows directly from Proposition 4.2.8 and Corollaries 4.2.9 and 4.2.40, (b) from (a) and Proposition 4.2.17, and (c) from (a) and Proposition 4.2.27. □

Next, we show that no Cohen 2-generic real computes a Laver 3-generic real. By Proposition 4.2.7, it is enough to show that every Laver 3-generic real is 1-dominating.

Proposition 4.2.42. *Every Laver* 3*-generic real is* 1*-dominating.*

Proof. Let $x \in \omega^\omega$ be Laver 3-generic. By Proposition 4.2.3, it is enough to show that x dominates every computable $f \in \omega^\omega$. Let $f \in \omega^\omega$ be computable. Then the set

$$D_f := \{T \in \mathbb{L}_\mathrm{c} : \forall t \in T(t \subseteq \mathrm{stem}(T) \vee t(\mathrm{lh}(t)-1) > f(\mathrm{lh}(t)-1))\}$$

is Π^0_2. We show that D_f is dense. Let $T \in \mathbb{L}_\mathrm{c}$ and let $(sn) \in \omega^{<\omega} \times \omega$ be the least pair such that $s \in T$ and there are $k, k' < n$ with $s^\frown k, s^\frown k' \in T$. We define

$$T' := \{t \in T : t \subseteq s \vee (s \subsetneq t \wedge \forall k < \mathrm{lh}(t)(k < \mathrm{lh}(s) \vee t(k) > f(k)))\}.$$

Then $T' \leq T$ and $T' \in D_f$. Hence, D_f is dense. Since x is Laver 3-generic, $G_x := \{T \in \mathbb{L}_\mathrm{c} : x \in [T]\}$ is Laver 3-generic. Thus, there is some $T \in D_f$ such that $x \in [T]$ and so for every $k > \mathrm{stem}(T)$, $x(k) > f(k)$. Therefore, x dominates f. □

Corollary 4.2.43. *No Cohen* 2*-generic real computes a Laver* 3*-generic real.*

Proof. Follows directly from Propositions 4.2.7 and 4.2.42. □

In set theory, Laver forcing adds dominating reals (cf. Table 4.1). The idea is essentially the same as in Proposition 4.2.42. So one might expect that every Laver n-generic computes an n-dominating real. However, as with computable Mathias forcing, the proof from set theory only works for 1-dominating reals, but not for 2-dominating reals. This is because if $f \in \omega^\omega$ is Δ^0_2 and 1-unbounded, then D_f is empty since otherwise every $T \in D_f$ would compute a 1-unbounded real, and this is not possible. We can even show that no 3-generic real for a Π^0_2 computably arboreal forcing notion can compute a 2-dominating real.

Proposition 4.2.44. *Let $n \geq 1$ and let $\mathbb{P}$ be a Π^0_n computably arboreal forcing notion. Then no $(n+1)$-generic real for $\mathbb{P}$ computes a* 2*-dominating real.*

Proof. Let $x \in \omega^\omega$ be $(n+1)$-generic for $\mathbb{P}$. By Lemma 4.2.13, it is enough to show that x does not compute $\emptyset'$. Let $f \in 2^\omega$ be the characteristic function of $\emptyset'$ and let $A \in [\omega]^\omega$ be computable in x. Then there is an $e \in \omega$ such that Φ^x_e is the characteristic function of A. Let

$$C := \{T \in \mathbb{P} : \exists k(\Phi^{\mathrm{stem}(T)}_e(k)\downarrow \neq f(k))\}.$$

Then C is Σ^0_{n+1}. Since x is $(n+1)$-generic for $\mathbb{P}$, $G_x := \{T \in \mathbb{P} : x \in [T]\}$ meets either C or the set of conditions having no extension in C. We make a case-distinction:

Case 1: G_x meets C. Then there is some $T \in \mathbb{P}$ such that $x \in [T]$ and $T \in C$. Hence, there is some $k \in \omega$ such that $\Phi^x_e(k)\downarrow \neq f(k)$. Therefore, $A \neq \emptyset'$.

Case 2: G_x meets the set of conditions having no extension in C. Then there is some $T \in \mathbb{P}$ such that $x \in [T]$ and for every $T' \leq T$, $T' \notin D$. We define a computable function g by $g(k) := \Phi^t_{e,\mathrm{lh}(t)}(k)$, where t is the least $t \in T$ such that $\Phi^t_{e,\mathrm{lh}(t)}(k)$ converges. Let $k \in \omega$. Since $x \in [T]$, there is some $t \in T$ such that $\Phi^t_{e,\mathrm{lh}(t)}(k)$ converges. Let $t \in T$ be minimal with that property. Then $T' := \{t' \in T : t \subseteq t' \vee t' \subseteq t\} \leq T$. Since T has no extension in C, $T' \notin C$ and so $g(k) = \Phi^t_{e,\mathrm{lh}(t)}(k) = f(k)$. Hence, $f = g$ and so f is computable. But this is a contradiction since $\emptyset'$ is not computable. Therefore, G_x cannot meet the set of conditions having no extension in C. $\square$

It is not not known whether Laver n-generic reals can compute Sacks or Silver n-generic reals. The best we can show is that the jump of a Laver n-generic real always computes a Sacks and a Silver n-generic real.

Corollary 4.2.45. *Let $n \in \omega$, let $\mathbb{P}$ be Δ^0_{n+1} forcing notion, let $m > \max\{2, n\}$, and let $x \in \omega^\omega$ be Laver m-generic for $\mathbb{P}$. Then $x^{(n-1)}$ computes an m-generic real for $\mathbb{P}$.*

Proof. Follows directly from Corollary 4.2.35 and Proposition 4.2.42. $\square$

4.2.7 Computable Miller forcing

In this section, we introduce a computable version of Miller forcing, which we call computable Miller forcing, and compare its n-generic reals to Cohen, Sacks, and Silver n-generic reals. Recall that in set theory, Miller forcing does not add Cohen, Sack, or Silver reals, and vice versa (cf. Table 4.1). We show that neither Sacks nor Silver 3-generic reals compute 3-generic reals for computable Miller forcing (cf. Table 4.3). Moreover, we show that for every $n \geq 3$, every n-generic real for computable Miller forcing computes an n-splitting real. Note that the analogue is not true in set theory. Miller proved in [Mil84] that Miller forcing does not add splitting reals.

We start with the definition of computable Miller forcing. Recall that Miller forcing is the partial order of all super-perfect trees ordered by inclusion and that a tree $T \subseteq \omega^{<\omega}$ is super-perfect if for every $t \in T$, there is some $s \in T$ such that $t \subseteq s$ and s splits infinitely often. Hence, super-perfect trees can contain splitting nodes which only split finitely often. However, this might cause problems for computable super-perfect trees since the question whether a splitting node splits finitely or infinitely often cannot be answered computably. Therefore, we only consider computable super-perfect trees containing no splitting nodes which split finitely often.

Definition 4.2.46. *Computable Miller forcing*, denoted by $\mathbb{M}_c$, is the partial order of all computable super-perfect trees on ω containing no splitting nodes which split finitely often ordered by inclusion and equipped with $\mathrm{stem}(T)$ as the valuation function.

Note that for every super-perfect tree $T \subseteq \omega^{<\omega}$, we can get rid of splitting nodes $t \in T$ which split finitely often by pruning all but one of t's successors. The resulting tree is still super-perfect and does not contain splitting nodes which only split finitely often. Hence, the set $\mathbb{M}'$ of all super-perfect trees containing no splitting nodes which split finitely often is a dense subset of Miller forcing. Therefore, $\mathbb{M}'$ is forcing equivalent to Miller forcing. However, this does not mean that the same is true for computable Miller forcing. If $T \subseteq \omega^{<\omega}$ is a computable super-perfect tree, then there is some $T' \leq T$ such that $T' \in \mathbb{M}'$, but we do not know whether T' is computable or not.

Question 4.2.47. *Does every computable super-perfect tree on ω have a subtree in $\mathbb{M}_c$?*

Another problem with computable perfect trees on ω is that to check that a node does not split, we may have to check all its successors. For computable Laver forcing, this was not a big issue, since in a computable Laver condition every node above the stem splits. Hence, the set of splitting nodes of a computable Laver condition is computable. Note that this does not mean that there is a computable set A such that for every $T \in \mathbb{L}_c$ and every $t \in \omega^{<\omega}$, $(T,t) \in A$ if and only if t splits in T. For computable Miller forcing, we can at least show that the set of all $T \in \mathbb{M}_c$ such that the set of splitting nodes in T is computable is dense.

Lemma 4.2.48. *For every $T \in \mathbb{M}_c$, there is some $T' \leq T$ such that the set of splitting nodes in T' is computable.*

Proof. Let $T \in \mathbb{M}_c$. We recursively define a function $f : \omega^{<\omega} \to 3$. We set $f(\emptyset) := 1$. Let $t \in \omega^{<\omega}$ such that $f(s)$ is already defined for every $s \subsetneq t$ and let $t' := t \restriction \mathrm{lh}(t) - 1$. If $t' \in T$, then let $(s,n) \in \omega^{<\omega} \times \omega$ be the least pair such that $t' \subsetneq s$ and there are $k, k' < n$ such that $s^\frown k, s^\frown k' \in T$. We define

$$f(t) := \begin{cases} 0 & \text{if } t \notin T,\ f(t') = 0, \text{ or } f(t') = 1 \text{ and } t \not\subseteq s, \\ 1 & \text{if } f(t') = 2 \text{ and } t \in T \text{ or } f(t') = 1 \text{ and } t \subsetneq s, \\ 2 & \text{if } f(t') = 1 \text{ and } t = s. \end{cases}$$

Then f is a computable function. Let $T' := \{t \in \omega^{<\omega} : f(t) \neq 0\}$. Since f is computable, T' is a computable subtree of T. Moreover, a node $t \in T'$ is splitting if and only if $f(t) = 2$. Hence, the set $\{t \in T' : t \text{ splits in } T'\}$ is computable. So it remains to show that $T' \in \mathbb{M}_c$. By construction, every splitting node in T' splits infinitely often. Hence, we only have to show that T' is super-perfect. Let $t \in T'$. Since $T' \subseteq T$ and T is super-perfect, there is some splitting node in T which extends t. Let $(s,n) \in \omega^{<\omega} \times \omega$ be the least pair such that $t \subsetneq s$ and there are $k, k' < n$ such that $s^\frown k, s^\frown k' \in T$. Then by construction, $f(s) = 2$ and so s is splitting in T'. Therefore, T' is super-perfect. ☐

It is clear that computable Miller forcing is a computably arboreal forcing notion. Before we study its n-generic reals, we first compute its complexity.

Lemma 4.2.49. *Computable Miller forcing is a Π^0_2 forcing notion.*

Proof. It is clear that computable Miller forcing is a computably arboreal forcing notion. By Lemma 4.1.22, it is enough to show that for every computable perfect tree $T \subseteq \omega^{<\omega}$, the statement "$T$ is super-perfect and every splitting node in T splits infinitely often" is Π^0_2. Note that a perfect tree $T \subseteq \omega^{<\omega}$ which only contains splitting nodes which split infinitely often is super-perfect. Hence, we only have to check that for every computable perfect tree $T \subseteq \omega^{<\omega}$, the statement "every splitting node in T splits infinitely often" is Π^0_2. Let $e \in \omega$ such that Φ_e is a characteristic function for a tree $T_e \subseteq \omega^{<\omega}$. Then every splitting node in T_e splits infinitely often if and only if

$$\forall t(\exists n, n'(t^\frown n \in T_e \wedge t^\frown n' \in T_e) \to \forall k \exists n(t^\frown n \in T_e)).$$

Hence, the statement "every splitting node in T splits infinitely often" is Π^0_2 and so the statement "T is super-perfect and every splitting node in T splits infinitely often" is Π^0_2 as well. ☐

Now we investigate n-generic reals for computable Miller forcing. As usual, we write "Miller n-generic" instead of "n-generic for $\mathbb{M}_c$". In set theory it is well-known that Miller forcing adds unbounded reals. In fact, every Miller real is unbounded (cf. [Hal17, Lemma 25.2]). We can show that the analogue is also true for Miller n-generic reals. The idea is essentially the same as in set theory.

Proposition 4.2.50. *Let $n \geq 3$. Every Miller n-generic real is n-unbounded.*

Proof. Let $x \in \omega^\omega$ be a Miller n-generic real. By Proposition 4.2.3, it is enough to show that no real which is Δ^0_n dominates x. Let $f \in \omega^\omega$ be Δ^0_n, let $k \in \omega$, and let

$$D^k_f := \{T \in \mathbb{M}_c : \exists n \geq k(\mathrm{stem}(T)(n) \geq f(n))\}.$$

Then D^k_f is Σ^0_n. We show that D^k_f is dense. Let $T \in \mathbb{M}_c$, let $s \in T$ such that $\mathrm{lh}(s) \geq k$ and s splits in T and let $m \geq f(\mathrm{lh}(s))$ such that $s' := s^\frown m \in T$. We define $T' := \{t \in T : s' \subseteq t \text{ or } t \subseteq s'\}$. Then $s' \subseteq \mathrm{stem}(T')$ and so $T' \in D^k_f$. Hence, D^k_f is dense. Since x is Miller n-generic, there is some $T \in D^k_f$ such that $x \in [T]$. Therefore, there is some $n \geq k$ such that $x(n) \geq f(n)$. □

Corollary 4.2.51.

(a) No Sacks 3*-generic real computes a Miller* 3*-generic real.*

(b) No Silver 3*-generic real computes a Miller* 3*-generic real.*

Proof. Item (a) follows directly from Propositions 4.2.17 and 4.2.50, and (b) from Propositions 4.2.27 and 4.2.50. □

Corollary 4.2.52. *Let $n \geq 3$. Every Miller n-generic real computes an n-splitting real.*

Proof. Follows directly from Propositions 4.2.6 and 4.2.50. □

The analogue of Corollary 4.2.52 is not true in set theory. Miller proved in [Mil84] that Miller forcing does not add splitting reals. As a consequence, Miller forcing also does not add Cohen, dominating, Laver, or Mathias reals. It is not known whether the analogous result is also true in computability theory (cf. Question 4.3.1). We only know that Miller 3-generic reals cannot compute 2-dominating reals by Proposition 4.2.44. Miller also proved in [Mil84] that Miller reals are minimal. The idea is similar as for Sacks and Silver reals. We have already shown that the proofs for Sacks and Silver reals can be translated into computability theory (cf. Propositions 4.2.22 and 4.2.29). However, the author has not been able to do the same for the proof that Miller reals are minimal or to find another argument that proves or disproves that Miller 3-generic reals have minimal degree.

Question 4.2.53. *Does Miller* 3*-generic reals have minimal degree?*

Note that if Miller 3-generic reals would have minimal degree, we could say much more about the relation of Miller n-generic reals to n-generic reals for other forcing notion: if it exists, let $x \in \omega^\omega$ be a Miller 3-generic real which has minimal degree. Then with a similar argument as in Corollary 4.2.30, x cannot compute Sacks or Silver 3-generic reals. Moreover, since Cohen 1-generic, Laver 3-generic, and Mathias 3-generic reals do not have minimal degree, x cannot compute any of them. So, in particular, x does not compute any Cohen 2-generic real. Hence, by Corollary 4.2.21, Cohen 2-generic reals cannot compute x.

4.2.8 Computable Hechler forcing

In this section, we introduce a computable version of Hechler forcing, which we call computable Hechler forcing, and compare its n-generic reals to Cohen, Sacks, and Silver n-generic reals. Recall that in set theory, Hechler forcing adds Cohen reals, but no Sacks or Silver reals, and Cohen, Sacks, and Silver forcing do not add Hechler reals (cf. Table 4.1). We show that for every $n \geq 3$, every n-generic for computable Hechler forcing computes a Cohen n-generic. Moreover, we show that no Cohen 2-generic real, Sacks 3-generic real, or Silver 3-generic real computes a 3-generic real for computable Hechler forcing (cf. Table 4.3). We start with the definition of computable Hechler forcing.

Definition 4.2.54. *Computable Hechler forcing*, denoted by $\mathbb{D}_c$, is the partial order of all pairs $(n, f) \in \omega \times \omega^\omega$ ordered by

$$(n, f) \leq (m, g) \iff n \geq m,\ f{\restriction}m = g{\restriction}m, \text{ and } \forall k \geq m(f(k) \geq g(k))$$

with the valuation function $v : \mathbb{D}_c \to \omega^{<\omega}$ defined by $v(n, f) := f{\restriction}n$.

Before we investigate n-generic reals for computable Hechler forcing, we first compute its complexity.

Lemma 4.2.55. *Computable Hechler forcing is a Π^0_2 forcing notion.*

Proof. A pair $(n, e) \in \omega^2$ codes a computable Hechler condition if Φ_e is a total function. Hence, the set of code is Π^0_2. Let $(n, e), (n', e') \in \omega^2$ be codes for computable Hechler conditions. Then $(n', \Phi_{e'}) \leq (n, \Phi_e)$ if and only if $n' \geq n$, for every $m < n$, $\Phi_e(m) = \Phi_{e'}(m)$, and for every $m \in \omega$, $\Phi_e(m) \leq \Phi_{e'}(m)$. Hence, the statement "$(n', \Phi_{e'}) \leq (n, \Phi_e)$" is Π^0_1. It remains to show that the valuation function is Π^0_2. Let $(n, e) \in \omega^2$ be a code for a computable Hechler condition and let $s \in \omega^{<\omega}$. Then the statement "$v(n, \Phi_e) = s$" is computable. Hence, the valuation function is Π^0_2. □

Now we study the relationships between n-generics for computable Hechler forcing and Cohen, Sacks, and Silver n-generic reals. As usual, we write "Hechler n-generic" instead of "n-generic for $\mathbb{D}_c$". Unlike Mathias or Laver forcing, Hechler forcing adds Cohen reals in set theory. In fact, it is well-known that for every Hechler real $x \in \omega^\omega$ over V, $y \in 2^\omega$ defined by

$$y(n) := x(n) \pmod 2$$

is a Cohen real over V. We show that the analogue is also true for Hechler n-generic reals.

Proposition 4.2.56. *Let $n \geq 3$, let $x \in \omega^\omega$ be Hechler n-generic, and let $y \in 2^\omega$ be defined by*

$$y(n) := x(n) \pmod 2.$$

Then y is Cohen n-generic.

Proof. Let G be a Hechler n-generic filter such that $x = \bigcup_{(n,f)\in G} v(n, f)$ and let $h : \omega^{<\omega} \to 2^{<\omega}$ be defined by

$$h(s)(m) := s(m) \pmod 2.$$

Then h is computable. We define $H := \{h(f{\restriction}m) : (m, f) \in G\}$. Then H is a filter and since $x = \bigcup_{(m,f)\in G} v(m, f)$, $y = \bigcup H$. So it is enough to show that H is Cohen n-generic. Let $C \subseteq \mathbb{C}$ be Σ^0_n. Then $C' := \{(m, f) \in \mathbb{D}_c : h(f{\restriction}m) \in C\}$ is Σ^0_n as well. Hence, there is some $(m, f) \in G$ such that either $(m, f) \in G$ or there is no $(m', f') \leq (m, f)$ such that $(m', f') \in C'$. In the former case, $v(f{\restriction}m) \in H \cap C$ and in the latter case, $v(f{\restriction}m) \in H$ and there is no $s' \leq v(f{\restriction}m)$ such that $s' \in C'$. Therefore, H is Cohen n-generic and so y is Cohen n-generic as well. □

Corollary 4.2.57.

(a) Let $n \geq 3$. Then every Hechler n-generic real computes an n-unbounded and an n-splitting real.

(b) No Sacks 3-generic real computes a Hechler 3-generic real.

(c) No Silver 3-generic real computes a Hechler 3-generic real.

Proof. Item (a) follows directly from Proposition 4.2.8 and Corollaries 4.2.9 and 4.2.56, (b) from (a) and Proposition 4.2.17, and (c) from (a) and Proposition 4.2.27. $\square$

Next, we show that Cohen 2-generic reals cannot compute Hechler 3-generic reals either. By Proposition 4.2.7, it is enough to show that Hechler 3-generic reals compute 1-dominating reals. Recall that in set theory, Hechler reals are dominating. This follows directly from the fact that for every real $x \in \omega^\omega$, the set of conditions $(n, f) \in \mathbb{D}$ such that f dominates x is dense. We can use the same argument to show that Hechler 3-generic reals are 1-dominating.

Proposition 4.2.58. *Every Hechler 3-generic real is 1-dominating.*

Proof. Let $x \in \omega^\omega$ be Hechler 3-generic and let G be a Hechler 3-generic filter such that $x = \bigcup_{(n,f)\in G} v(n, f)$. By Proposition 4.2.3, it is enough to show that x dominates every computable real. Let $g \in \omega^\omega$ be computable and let

$$D_g := \{(n, f) \in \mathbb{D}_{\mathrm{c}} : f \text{ dominates } g\}.$$

Then D_g is Π^0_2. We show that D_g is dense. Let $(n, f) \in \mathbb{D}_{\mathrm{c}}$. We define $f' \in \omega^\omega$ by

$$f'(m) := \begin{cases} f(m) & \text{if } m < n, \\ f(m) + g(m) + 1 & \text{otherwise.} \end{cases}$$

Then f' is computable and dominates g. Moreover, $(n, f') \leq (n, f)$ and so $(n, f') \in D_g$. Hence, D_g is dense. Let $(n, f) \in G \cap D$. Then f dominates g and for every $m \geq n$, $x(m) \geq f(m)$. Therefore, x dominates g as well. $\square$

Corollary 4.2.59. *No Cohen 2-generic real computes a Hechler 3-generic real.*

Proof. Follows directly from Propositions 4.2.7 and 4.2.58. $\square$

Like with Mathias and Laver 3-generic reals, the proof of Proposition 4.2.58 does not work if g is 1-unbounded. Hence, the same argument does not show that Hechler 3-generic reals are 2-dominating. In fact, we can even prove that no Hechler 3-generic real computes a 2-dominating reals. The idea is essentially the same as for Mathias and Laver 3-generic reals.

Proposition 4.2.60. *Every Hechler 3-generic real does not compute 2-dominating reals.*

Proof. Let $x \in \omega^\omega$ be Hechler 3-generic and let G be a Hechler 3-generic filter such that $x = \bigcup_{(n,f)\in G} v(n, f)$. By Lemma 4.2.13, it is enough to show that x does not compute $\emptyset'$. Let $f \in 2^\omega$ be the characteristic function of $\emptyset'$ and let $A \in [\omega]^\omega$ be computable in x. Then there is an $e \in \omega$ such that Φ_e^x is the characteristic function of A. Let

$$C := \{(n, f) \in \mathbb{D}_{\mathrm{c}} : \exists k(\Phi_e^{f\restriction n}(k)\downarrow \neq f(k))\}.$$

Then C is Σ^0_3. Since G is Hechler 3-generic, G meets either C or the set of conditions having no extension in C. We make a case-distinction:

Case 1: G meets C. Then there is some $(n, f) \in G \cap C$. Hence, there is some $k \in \omega$ such that $\Phi_e^{f \restriction n}(k)\downarrow \neq f(k)$. Since $f \restriction n \subseteq x$, $\Phi_e^x(k)\downarrow \neq f(k)$ and so $A \neq \emptyset'$.

Case 2: G meets the set of conditions having no extension in C. Then there is some $(n, f) \in \mathbb{D}_{\mathrm{c}}$ such that for every $(n', f') \leq (n, f)$, $(n', f') \notin C$. We define a computable function g by $g(k) := \Phi^s_{e,\mathrm{lh}(s)}(k)$, where s is the least $s \in \omega^{<\omega}$ such that $\Phi^s_{e,\mathrm{lh}(s)}(k)$ converges, $f \restriction n \subseteq s$, and for every $n \leq m < \mathrm{lh}(s)$, $s(m) \geq f(m)$. Since $f \restriction n \subseteq x$ and for every $m \geq n$, $s(m) \geq f(m)$, g is total. Let $k \in \omega$, let $s \in \omega^{<\omega}$ be minimal such that $\Phi^s_{e,\mathrm{lh}(s)}(k)$ converges, $f \restriction n \subseteq s$, and for every $n \leq m < \mathrm{lh}(s)$, $s(m) \geq f(m)$, and let $f' \in \omega^\omega$ be defined by

$$f'(m) := \begin{cases} s(m) & \text{if } m < \mathrm{lh}(s), \\ f(m) & \text{otherwise.} \end{cases}$$

Then $(\mathrm{lh}(s), f') \leq (n, f)$. Since (n, f) has no extension in C, $(\mathrm{lh}(s), f') \notin C$ and so $g(k) = \Phi^s_{e,\mathrm{lh}(s)}(k) = f(k)$. Hence, $f = g$ and so f is computable. But this is a contradiction since $\emptyset'$ is not computable. Therefore, G cannot meet the set of conditions having no extension in C. ☐

It is not not known whether Hechler n-generic reals can compute Sacks or Silver n-generic reals. The best we can show is that the jump of a Hechler n-generic real always computes a Sacks and a Silver n-generic real.

Corollary 4.2.61. *Let $n \in \omega$, let $\mathbb{P}$ be Δ^0_{n+1} forcing notion, let $m > \max\{2, n\}$, and let $x \in \omega^\omega$ be Hechler m-generic for $\mathbb{P}$. Then $x^{(n-1)}$ computes an m-generic real for $\mathbb{P}$.*

Proof. Follows directly from Corollary 4.2.35 and Proposition 4.2.58. ☐

4.3 Summary and open questions

In the previous sections, we have investigated whether n-generic reals for computable versions of various set-theoretic forcing notions can compute each other or not. Table 4.3 summarizes the results for $n \geq 4$; the entries have the same meaning as the entries in Table 4.2. Note that all results not involving Hechler, Laver, Miller, or Silver n-generic reals were known before this work.

If we compare Table 4.3 with the situation in set theory (cf. Table 4.1), then there are a few differences: first, Laver and Mathias n-generic reals compute Cohen n-generic reals, second, Miller n-generic reals compute n-splitting reals, and third, no Laver, Mathias, or Hechler 3-generic real computes a 2-dominating real. So far, these are the only known differences. However, there are still many open questions.

Question 4.3.1.

(a) *Is there a Cohen 1-generic real which computes a Hechler, Laver, or Miller 3-generic real?*

(b) *Is there a Hechler 3-generic real which computes a Laver, Mathias, Miller, Sacks, or Silver 3-generic real?*

(c) *Is there a Laver 3-generic real which computes a Hechler, Mathias, Miller, Sacks, or Silver 3-generic real?*

	$\mathbb{C}$ n-gen.	$\mathbb{D}_c$ n-gen.	$\mathbb{L}_c$ n-gen.	$\mathbb{M}_c$ n-gen.	$\mathbb{R}_c$ n-gen.	$\mathbb{S}_c$ n-gen.	$\mathbb{V}_c$ n-gen.
$\mathbb{C}$ n-gen.	✓	✓	✓	?	✓	×	×
$\mathbb{D}_c$ n-gen.	×	✓	?	?	?	×	×
$\mathbb{L}_c$ n-gen.	×	?	✓	?	?	×	×
$\mathbb{M}_c$ n-gen.	?	?	?	✓	?	×	×
$\mathbb{R}_c$ n-gen.	×	?	?	?	✓	×	×
$\mathbb{S}_c$ n-gen.	×	?	?	?	?	✓	×
$\mathbb{V}_c$ n-gen.	×	?	?	?	?	×	✓
1-dom.	×	✓	✓	?	✓	×	×
2-dom.	×	×	×	×	×	×	×
n-unb.	✓	✓	✓	✓	✓	×	×
n-split.	✓	✓	✓	✓	✓	×	✓

Table 4.3: Relationships between Cohen, Hechler, Laver, Mathias, Miller, Sacks, and Silver n-generic reals for $n \geq 4$

(d) Is there a Mathias 3*-generic real which computes a Hechler, Laver, Miller, Sacks, or Silver* 3*-generic real?*

(e) Is there a Miller 3*-generic real which computes a Cohen* 1*-generic real? Is there a Miller* 3*-generic real which computes a Hechler, Laver, Mathias, Sacks, or Silver* 3*-generic real? See also Question 4.2.53.*

(f) Is there a Sacks 3*-generic real which computes a* 1*-splitting real?*

In Section 4.2, we have studied computable versions of set-theoretic forcing notions. However, all conditions of theses computable versions were computable objects. So we could ask the same questions again for versions of set-theoretic forcing notions whose conditions are allowed to have higher complexity. For example, we could define a version of Hechler forcing where the conditions are pairs of natural numbers and Δ^0_m reals.

Question 4.3.2. *Let $m > 1$. What are the relationships between the n-generic reals for the forcing notions from Table 4.3, if we allow conditions of complexity Δ^0_m?*

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Zusammenfassung der Ergebnisse

(German translation of Section 1.1, pp. 2–4)

In Kapitel 2 definieren wir ein allgemeines Framework für Regularitätseigenschaften. Anschließend verwenden wir dieses Framework, um drei verschiedene Regularitätseigenschaften zu untersuchen. Als erstes untersuchen wir die Amöben-Regularität, die von Judah und Repický in [JR95] eingeführt wurde. Unser Hauptergebnis für die Amöben-Regularität ist das folgende Korollar.

Korollar 2.3.20. *Die folgenden Aussagen sind äquivalent:*

(a) jede $\mathbf{\Sigma}^1_2(\mathbf{R})$ Menge ist Amöben-regulär,

(b) für alle $r \in \omega^\omega$ ist die Menge $\{P \in \mathbf{R} : P$ ist keine Amöben-generische reelle Zahl über $\mathrm{L}[r]\}$ $\mathcal{C}_{\mathbb{A}}$-mager und

(c) für alle $r \in \omega^\omega$ ist $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Als zweites definieren wir einen topologischen Raum für das amoeba forcing for category und untersuchen die Baire-Eigenschaft in diesem Raum. Zuletzt machen wir das Gleiche für das localization forcing. Das folgende Korollar bzw. das folgende Theorem sind unsere wichtigsten Ergebnisse für diese Regularitätseigenschaften.

Korollar 2.4.11. *Die folgenden Aussagen sind äquivalent:*

(a) jede $\mathbf{\Sigma}^1_2(\mathbf{U})$ Menge hat die Baire-Eigenschaft in der $\mathbb{UM}$-Topologie,

(b) für alle $r \in \omega^\omega$ ist die Menge $\{x \in \mathbb{UM} : x$ ist keine $\mathbb{UM}$-generische reelle Zahl über $\mathrm{L}[r]\}$ mager in der $\mathbb{UM}$-Topologie und

(c) für alle $r \in \omega^\omega$ ist $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

Theorem 2.5.10. *Die folgenden Aussagen sind äquivalent:*

(a) jede $\mathbf{\Sigma}^1_2(\mathbf{Loc})$ Menge hat Baire-Eigenschaft in der lokalisierenden Topologie,

(b) für alle $r \in \omega^\omega$ ist die Menge $\{f \in \mathbf{Loc} : f$ ist keine $\mathbb{LOC}$-generische reelle Zahl über $\mathrm{L}[r]\}$ mager in der lokalisierenden Topologie,

(c) für alle $r \in \omega^\omega$ ist die Menge $\{f \in \mathbf{Loc} : f$ ist keine lokalosierende reelle Zahl über $\mathrm{L}[r]\}$ mager in der lokalisierenden Topologie und

(d) für alle $r \in \omega^\omega$ ist $\aleph_1^{\mathrm{L}[r]} < \aleph_1$.

In Kapitel 3 vergleichen wir die Konsistenzstärke von deskriptiven Auswahlprinzipien, d.h. von Fragmenten des Auswahlaxioms, die mit Hilfe der deskriptiven Hierarchien definiert werden. Solche Auswahlprinzipien wurden bereits von Kanovei in [Kan79] untersucht. Unser Hauptergebnis ist das folgende Theorem, das ein Separationstheorem von Kanovei verallgemeinert.

Theorem 3.2.10. *Für alle $n \geq 1$ gibt es ein Modell von $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.*

Mit Hilfe eines Kompaktheitsarguments erhalten wir außerdem das folgende Korollar.

Korollar 3.2.11. *Es gibt ein Modell von* $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{ctbl})$.

In Kapitel 4 untersuchen wir mengentheoretische forcing notions in der Berechenbarkeitstheorie und vergleichen ihre n-generischen reellen Zahlen. Obwohl vieles aus der Mengenlehre übertragen werden kann, gibt es auch Unterschiede. Zum Beispiel bewiesen Cholak, Dzhafarov, Hirst und Slaman in [CDHS14], dass alle Mathias n-generischen reellen Zahlen Cohen n-generische reelle Zahlen berechnen. Miller bewies in [Mil81], dass die analoge Aussage in der Mengenlehre nicht gilt. Wir verallgemeinern das Ergebnis von Cholak, Dzhafarov, Hirst und Slaman und zeigen so, dass auch jede Laver n-generische reelle Zahl eine Cohen n-generische reelle Zahl berechnet.

Korollar 4.2.40. *Sei* $n \geq 3$. *Dann berechnet jede Laver* n*-generische reelle Zahl eine Cohen* n*-generische reelle Zahl.*