VIEWS FROM A PEAK

GENERALISATIONS AND DESCRIPTIVE SET THEORY

NED WONTNER

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica *for those who walk a distant land*

N Armstrong

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Chapter 1

Introduction

This dissertation has two main themes. One is *descriptive set theory*, the other is *generalisation in mathematics*. Each of the main chapters (Chapters 3 to 7) develops on these themes. Chapters 3 to 6, and the preliminary Chapter 2 are mathematical, whilst the concluding Chapter 7 is philosophical in approach. We first give a general introduction to the dissertation, including a schematic portrait of each chapter, and state the main mathematical results.

1.1 General introduction

Our mathematical focus is descriptive set theory. The real numbers are one of the key objects of mathematics, and are studied from different perspectives. Real analysis typically takes an axiomatic approach, where the 'real numbers' can mean any model of the (second-order) theory of the real numbers (see e.g. [180]), i.e. any Dedekind-complete ordered field. Meanwhile, descriptive set theory typically takes the reals to be suitable simple sets, e.g. the Baire space (ω^ω), the Cantor space (2^ω), or indeed $\mathscr{P}(\omega)$. These sets do not share all the properties of the ordered field of real numbers (see pages 3 & 116), but are sufficiently close to the real field (see page 11) and are so set-theoretically convenient that they are a good compromise for descriptive set theorists. Ultimately, the roots of descriptive set theory are in real analysis and set-theoretical questions about the real numbers; the connection remains strong (see [16] for a modern introduction to these connections). Descriptive set theory falls under the larger heading of set theory. Set theory itself has two over-arching themes: the study of the infinite, and the foundations of mathematics. Descriptive set theory draws on both of these themes, notably the former. Chapter 3 is closely connected with the second of these themes, the subsequent chapters are more close to the first theme.

We start with Chapter 2, which contains a general set of preliminaries which applies to all the mathematical chapters. This lays down our notation, principal definitions, and some initial basic theory. Each mathematical chapter begins with a sketch of the chapter itself, and a statement of any additional notation required or specific to that chapter. Throughout each chapter, we include open questions as they arise in the development of theory.

In Chapter 3, we draw on the rôle of set theory in the foundation of mathematics: one of the major research directions in set theory distinguishes and compares set *theories*, particularly on the basis of the use of the Axiom of Choice (AC) in the theories in question. We study the effect of AC on descriptive set theory. Among the fragments of AC of the form $\mathsf{AC}_X(Y)$, $\mathsf{AC}_\omega(\omega^\omega)$, has a special position. Most uses of AC in ordinary mathematics are required because we want theorems to hold in general; when specialising the results to concrete mathematical objects, they can often be proved in ZF. However, the uses of AC in the basic foundations of analysis are needed even when working with concrete mathematical objects such as the reals or the complex numbers. The fragment of AC most fundamental in this respect is $\mathsf{AC}_\omega(\omega^\omega)$. Like AC itself, $\mathsf{AC}_\omega(\omega^\omega)$ does not follow from ZF, and there are pathological models of $\mathsf{ZF} + \neg\mathsf{AC}_\omega(\omega^\omega)$. In such models, the behaviour of the reals, real analysis, and topology can vary wildly from the corresponding theory in ZFC (see [5], [66, Chapter 56], [109], [113], [152]). Bizarre happenings include that the reals can be a countable union of countable sets [32, page 146], or all subsets of the reals can be Borel (Theorem 2.1.7).

Chapter 3 looks at fragments of $\mathsf{AC}_\omega(\omega^\omega)$ called *descriptive choice principles.* These restrict $\mathsf{AC}_\omega(\omega^\omega)$ to those sequences of sets of reals which have a particular description, i.e. ensure that these sequences have choice functions. These descriptive choice principles can be compared to *uniformisation* in descriptive set theory (as in [161, Chapter 4]): uniformisation asks whether a particular pointclass has choice functions of a certain low complexity. Here, we ask what it takes for those pointclasses to have a choice function at all. We focus on the descriptions of sequences of sets of reals given by the projective pointclasses. Choice principles like this were studied by Kanovei in [109], who proved a first separation theorem for some of these principles. The main result of this chapter improves on Kanovei's result, with a method based on Jensen forcing. We construct models which separate *uniform projective choice principles* of different levels from one another. We also give something of a flavour of what descriptive set theory is like without AC.

The subsequent mathematical chapters concern generalised descriptive set theory, and its connection to real analysis. (Classical) descriptive set theory views the real numbers as suitable simple sets, like 2^ω and ω^ω. This lends itself to generalisations for cardinals $\kappa > \omega$, which yields the κ-Baire space (κ^κ) and the κ-Cantor space (2^κ). This approach is known as generalised descriptive set theory. Generalised descriptive set theory draws more strongly on the second theme of set theory, the study of the infinite. It is a contemporary research area with strong links to e.g. model theory, determinacy, combinatorial set theory, and the classification of uncountable structures (see, for example, [117]).

As generalised descriptive set theory has progressed, a great deal of classical descriptive set theory has been generalised from 2^ω and ω^ω to 2^κ and κ^κ. Let's give a small taste of the generalisations of Borel and analytic structure (anticipating Chapter 6). Much more can be found in e.g. [6], or [68]. Exactly in analogy to the classical case, if $\kappa^{<\kappa} = \kappa$, then the following hold for 2^κ and κ^κ: the spaces are κ-additive, basic open sets are closed, the spaces have weight κ, the topologies are of size 2^κ, and $(\kappa^\kappa)^n$ is homeomorphic to κ^κ [68, pages 8-9]. Several of the classically equivalent definitions of being a projection of a closed set (i.e. $\mathbf{\Sigma}^1_1$) have also been shown equivalent in the generalised case, e.g. being a projection of a Borel set, a projection of a $\mathbf{\Sigma}^1_1$ set, or a continuous image of a closed set [68, Theorem 19].

In other cases, the structure in the generalised setting is disanalogous to the classical structure of Borel and analytic sets. For example, if $\kappa^{<\kappa} = \kappa$, then the Borel sets are a strict subset of the $\mathbf{\Delta}^1_1$ sets, if $V = \mathrm{L}$ then the codeable Borel sets (i.e. the sets with a 'specified instruction' for how to build them from basic open sets) are exactly the $\mathbf{\Sigma}^1_1$ sets, and it is consistent that the $\mathbf{\Delta}^1_1$ sets are a strict subset of the codeable Borel sets [68, Theorem 18]. Under the same assumption, there is a closed subset of κ^κ which is not a continuous image and of κ^κ, and there is an injective continuous image of κ^κ which is not Borel [135]. All of these examples are false in the classical case with ω^ω.

In Chapters 4 to 6, we focus on the connection between real analysis and generalised descriptive set theory. Our main objective is to develop a theory of *generalised real analysis* on ordered fields of high cardinality ($>2^{\aleph_0}$), which resembles classical real analysis on the real numbers, just as how generalised descriptive set theory on the κ-Baire space is in analogy to (classical) descriptive set theory on the Baire space.

The spaces of generalised descriptive set theory, like κ^κ and 2^κ, are no good for this *generalised real analysis* because these sets have no algebraic (field) structure: just like ω^ω, the set κ^κ has an order structure, but its order has no compatible field structure (see page 116). So, to generalise real analysis, our approach is somewhat more axiomatic: in each generalisation, we specify conditions on a field, $\mathbb{K}$, such that this field generalises the real numbers in the requisite way for the generalisation in question.

Equipped with an algebraic generalisation of the real numbers, generalised real analysis can start in earnest. This is the main project of Chapters 4 and 5. Generalised real analysis is part of the long tradition of generalising the field of real numbers (for a survey, see [52]). We see this in early analysis on the surreal numbers [2, 80] and the hyperreal numbers. More recent developments include the generalisation of $\mathbb{R}$ by Asperó and Tsaprounis [8] and some further analysis on the surreal numbers [39, 62, 179]. The immediate mathematical jumping-off point is Galeotti's introduction of a generalisation of the reals, $\mathbb{R}_\kappa$ [73, 74, 75]. We extend the initial work on the generalised real analysis of $\mathbb{R}_\kappa$ sequences in e.g. [25, 76]. This $\mathbb{R}_\kappa$ will be our principal example of an ordered field of large cardinality (see Theorem 2.3.13 and Section 2.3.5), though we state our results axiomatically where possible.

Chapters 4, 5, and 6 build on this connection in different ways. In Chapter 4, we develop the theory of generalised real analysis of functions. Along with the broader connections that generalised real analysis has with other areas of mathematics, this chapter has close links to the model theory of ordered fields [127, 172, 188], generalisations of continuity (e.g. [166]), and studies of real analysis as completeness [44]. We define a number of generalisations of continuity, both novel and from the literature. The focal notions are κ-continuity, κ-supercontinuity, and sharpness. We then prove generalisations of classical theorems from real analysis for these generalisations of continuity. Some of these generalisations of classical theorems hold independently of the choice of κ, whilst others depend strongly on the properties of κ. A localisation of the inherently global notion of κ-continuity is also mentioned.

In Chapter 5, we sketch the outlook for developing notions of infinite sum on high cardinality fields. We state some natural desiderata that such notions of sum would be expected to satisfy. The outlook is not hopeful: there are no notions of sum which simultaneously satisfy even the most basic properties of a sum (Theorem 5.2.3).

We then analyse notions of infinite sum, including both those from the literature and those that arise naturally, with the intention to develop a robust generalisation of real analysis. When summation and integration have been studied, the perspective has been more model-theoretic [179, Definition 39], or combinatorial (e.g. on the surreals, see [62, page 35]). Our approach is more analytical. Unlike the model-theoretic perspective, we do not restrict ourselves to (an extension of) the language of fields. Instead, we allow ourselves the full power of a background set theory. We also do not in general assume that our field is initial in surreals (*pace* parts of Sections 5.3.2 and 5.3.4), unlike other approaches to infinite sums (e.g. [179]). We aim to have a notion of infinite sum which is somehow correlated to the basenumber of the field in question, rather than countably infinite sums. This makes our approach distinct from e.g. infinite sums of Hahn fields which supervene on countable sums (Section 5.3.3). Our approach connects to work on transfinite generalisation of the Hessenberg sums on ordinals (e.g. in [133] and its bibliography). Here, we approach sums as they are used in real analysis proper. For the notions of sum we analyse, we establish which of these desiderata they satisfy.

On the basis of our discussion of sums, we define a generalised 'long' notion of a polynomial. Our generalisation of polynomials is aimed at a generalisation of Weierstraß' Approximation Theorem, which we show fails for (standard) polynomials.

Chapter 6 is closer in spirit to generalised descriptive set theory proper. It analyses the notion of κ-topology which arises from the generalisation of real analysis in the previous two chapters, particularly for the notion of κ-continuity. This contrasts with the ordinary approach to generalised descriptive set theory, which takes as its starting point the ordinary ('full') topology on the spaces in play. This leads to corresponding κ-topological generalisations of Borel and analytic sets. We are particularly interested in the cases where κ is inaccessible or even weakly compact, which makes the generalised descriptive set theory very well-behaved. We focus on the bounded κ-topology on κ^κ as the canonical topological space of generalised descriptive set theory, including generalised notions of a Borel hierarchy and analytic sets. We also develop the corresponding theory on 2^κ and certain ordered spaces, e.g. $(\kappa^* \sqcup \kappa)^\kappa$ and $\mathbb{R}_\kappa$.

In Chapter 7, we leave the mathematical foreground, take a large step back, and consider the bigger philosophical picture. The previous chapters concern generalisations of descriptive set theory and real analysis, but the question remains of *what a generalisation is* in mathematics. The final chapter is helps remedy this, by giving a philosophical account which explicates generalisation in mathematics. Our approach draws on those of the philosophy of mathematical practice and mathematical change. The account is informed by the mathematical developments of the previous chapters, particularly Chapter 4. We have two core questions: what are mathematical generalisations? Secondly, why do mathematicians generalise at all?

For the first question, like the artist-turned-art critic, we use our experience from Chapters 3 to 6 to give some insight into the mechanics of generalisation as a process. We use this to help provide a typology for these generalisations and others from the literature (the *species*), and of accounts of generalisation. We compare generalisation to other processes of change in mathematical practice, notably abstraction and domain expansion. We then analyse *syntacticist* accounts of the nature of generalisations based on the literature, and suggest instead a *semanticist* account, based on content.

In answering the second question, we assess whether certain traditional accounts of the motivation of mathematical change fit generalisations.

Our core conclusions are that 1) generalisation in mathematics is a *sui generis* process of mathematical change which cannot be reduced to other processes, 2) neither explantoriness nor simplicity is necessary for the success of a generalisation, and 3) a syntacticist account of the nature of generalisation is untenable, we must instead opt for a form of semanticism.

1.2 Main mathematical results

In this section, we state our main mathematical results from Chapters 3 to 6.

In Chapter 3, we study fragments of AC which are defined using descriptive pointclasses. Such choice principles were first considered by Kanovei [109], who proved certain separation theorems. Our main result generalises those of Kanovei:

Theorem 3.2.4 (Separation Theorem)**.** For every $n \geq 1$, there is a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$.

By compactness, we also can construct a model with the following properties:

Corollary 3.2.7. There is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$.

In Chapter 4, we study the generalisations of real analysis on large cardinality fields, particularly on $\mathbb{R}_\kappa$. Many results are independent of the choice of κ, whilst others depend substantially on the properties of κ. We summarise the former results as follows, using $\mathbb{R}_\kappa$ as it satisfies all of the requisite conditions (see page 75). Let $\kappa^{<\kappa} = \kappa$, and let $f : \mathbb{R}_\kappa \to \mathbb{R}_\kappa$:

1. If f is κ-continuous, then f is $\mathsf{IVT}(\mathbb{R}_\kappa)$ and $\mathsf{OMT}(\mathbb{R}_\kappa)$ (Theorem 4.2.3 & Proposition 4.2.13).

2. If f is κ-continuous, then f is $\mathsf{EVT}(\mathbb{R}_\kappa)$ if and only if f is κ-supercontinuous (Corollary 4.2.21).

3. If both f and $x \mapsto f(x) - x$ are κ-continuous, then f is $\mathsf{BFPT}(\mathbb{R}_\kappa)$; however, there are κ-supercontinuous functions that are not $\mathsf{BFPT}(\mathbb{R}_\kappa)$ (Propositions 4.2.14 & 4.2.16).

4. If f is κ-supercontinuous, then f is $\mathsf{BVT}(\mathbb{R}_\kappa)$ and $\mathsf{Rolle}(\mathbb{R}_\kappa)$ (Propositions 4.2.25 & 4.2.19).

5. If both f and $x \mapsto f(x) + a.x + b$ are κ-supercontinuous, then f is $\mathsf{MVT}(\mathbb{K})$, $\mathsf{CVT}(\mathbb{R}_\kappa)$, and $\mathsf{PDT}(\mathbb{R}_\kappa)$ (Proposition 4.2.26).

We also show that, on certain large cardinality fields, the κ-continuous and κ-supercontinuous functions are not closed under uniform convergence (Proposition 4.2.28). In Section 4.3, we give a generalisation turns on a property of the cardinal, κ, extending the work in [25] and [76]:

Theorem 4.3.11. Let $\mathbb{K}$ be a κ-spherically complete ordered field $\mathbb{K}$ where $\mathsf{bn}(\mathbb{K}) = \kappa$. Then the following are equivalent:

1. κ has the tree property,

2. $\mathsf{EVT}(\mathbb{K}, \text{sharp})$, and

3. $\mathsf{BVT}(\mathbb{K}, \text{sharp})$.

In Chapter 5, we characterise and study candidate notions of infinite sums on large cardinality fields. We give full statements of the properties of a sum, S, in Section 5.1. In rough terms: S satisfies **Ext** if it extends finite addition, **Lin** if it is linear in its arguments, **Conc** if the concatenation of two S-summable sequences is itself S-summable, whilst **Comp** generalises Weierstraß' M-test: if a non-negative S-summable sequence s pointwise bounds a non-negative sequence t, then t is S-summable. Then no non-trivial sum satisfies all four conditions (Theorem 5.2.3). Instead, we establish which of the desired properties for a notion of infinite sum our candidates satisfy. This is summarised in Table 1.1.

Table 1.1 includes two further natural properties: **Ind**, where the S-sum of a sequence is invariant under sequential reorderings, and **El**, where a sequence is S-summable if and only if its contraction is. In Table 1.1, ($\exists$) means that there is an infinite support, positive, summable sequence. The grey columns indicate the four incompatible properties from Theorem 5.2.3.

	Section	**Ext**	**Lin**	**Conc**	**Comp**	($\exists$)	**Ind**	**El**
finmod	5.3.1	no	yes	no	no	yes	no	no
infmod	5.3.1	no	yes	no	no	yes	no	no
supfin	5.3.2	yes	yes	yes	yes	no	yes	yes
simple	5.3.2	no	no	no	yes	yes	yes	yes
point	5.3.3	yes	yes	yes	no	yes	no	yes
seq	5.3.4	yes	no	no	no	yes	yes	yes
$\int$	5.3.5	yes	yes	yes	yes	no	yes	yes

Table 1.1: The main properties of the candidate notions of sum

We show that Weierstraß' Approximation Theorem fails in a strong way for (ordinary) polynomials (Proposition 5.4.1):

Proposition 5.4.1. If $\mathbb{K}$ is an rcf with $\mathsf{bn}(\mathbb{K}) = \kappa$, then there is a κ-supercontinuous function that is not uniformly approximated by polynomials.

In Chapter 6, we study the κ-topologies of the previous two chapters, and their resultant generalised descriptive set-theoretic structure. We give several characterisations of the κ-Borel sets, most importantly:

Lemma 6.2.12. Let κ be inaccessible. For every Borel$_\kappa$ set $X \subseteq \kappa^\kappa$, there is a unique family $\mathcal{F}(B) \subseteq \kappa^{<\kappa}$, and a unique conelike partition, $\mathcal{P}(B) = \{P_s : s \in \mathcal{F}(B)\}$, such that $\mathcal{F}(B)$ and $\mathcal{P}(B)$ are optimal for B.

This allows us to prove a further characterisation of the tree property (in the form of the weakly compact cardinals amongst the inaccessibles, see Lemma 6.2.16 and Remark 6.2.18) and the following link to the full topology:

Theorem 6.3.32. Let κ be weakly compact. If A is weakly analytic then $|\mathsf{cl}(A) \setminus A| \leq \kappa$.

We show that Lebesgue was right for projections of κ-Borel sets:

Corollary 6.3.4. If κ is inaccessible, either:

1. Y is κ-additive, in which case $C \subseteq X$ is κ-closed if and only if $\pi_1(D) = C$ for some κ-closed set D in $Y \times X$, or
2. Y is not κ-additive, in which case $\pi_1(D) = C$ for some D which is κ-closed in $Y \times X$ if and only if $C = X \setminus \bigcap_{\alpha\in\lambda} O_\alpha$ where O_α are κ-open and $\lambda < \kappa$.

But the other standard notions of analyticity yield a proper hierarchy of complexity:

Corollary 6.3.51.
1. Let κ be inaccessible. Let X be a dense linear order. Let $Q \subseteq X$ be order dense in X. Suppose that, for all (λ, μ)-gaps in X, $\lambda, \mu \leq \kappa$. Let X have the Q-interval κ-topology. Strong X-analyticity implies, but is not implied by, intermediate X-analyticity, and intermediate X-analyticity implies weak X-analyticity.
2. Let κ be weakly compact. Then strong analyticity implies, but is not implied by, intermediate analyticity, and intermediate analyticity implies, but is not implied by, weak analyticity.

We show the failure of Suslin's theorem for these notions on κ^κ (and $\mathbb{Q}_\kappa$, with a comment on $\mathbb{R}_\kappa$, see Corollaries 6.3.43 and 6.3.47, and Remark 6.3.44):

Theorem 6.3.18. The following proper inclusions hold:

1. $\mathsf{Bor}_\kappa \subsetneq \{A \subseteq \kappa^\kappa : A \text{ is weakly bianalytic}\}$,
2. $\mathsf{Bor}_\kappa \subsetneq \{A \subseteq \kappa^\kappa : A \text{ is intermediately bianalytic}\}$, and
3. $\mathsf{Bor}_\kappa \not\subseteq \{A \subseteq \kappa^\kappa : A \text{ is strongly analytic}\}$ and $\mathsf{Bor}_\kappa \not\subseteq \{A \subseteq \kappa^\kappa : \kappa^\kappa \setminus A \text{ is strongly analytic}\}$.

In analogy to the classical case, we show that we can construct 2^κ many distinct κ-topologies which are nevertheless fully isomorphic:

Corollary 6.4.4. Let κ be inaccessible. There are 2^κ-many non-κ-homeomorphic κ-topologies on κ^κ whose full topology is fully homeomorphic to $(\kappa^\kappa, \langle\tau_b\rangle_\infty)$.

Chapter 2

Preliminaries

Throughout, our basic notation is drawn from [75], [103], [108], and [112].

2.1 Notation and set theory

2.1.1 Sequences, trees, and cardinals

We use the standard notation for cardinals, ordinals, sequences, and trees, from e.g. [75, Chapter 1] or [103]. We use a non-standard notation for sequence variables: blackboard bold letters such as $\mathbb{x}$ and $\mathbb{y}$ are indicating that these objects are sequences, i.e. functions in X^α with ordinal domain α and range in a given set X. In this situation, we write $\mathbb{x} = (x_\beta)_{\beta\in\alpha}$ to specify the coordinate of the sequence and say the *length* of $\mathbb{x}$ is α, written $\mathsf{len}(\mathbb{x}) = \alpha$. For any sets a and b, we write $a^\frown \mathbb{x}^\frown b$ for the sequence $\mathbb{x}' = (x'_j)_{1+\alpha+1}$ where $x'_0 = a$, $x'_{1+j} = x_j$ and $x'_{1+\alpha} = b$. If $\mathbb{y}$ is also a sequence, we write $\mathbb{x}^\frown \mathbb{y}$ for the concatenation of $\mathbb{x}$ and $\mathbb{y}$. If α is an ordinal, and a is a set, we let $\overline{a}_\alpha$ be the constant sequence of length α, where each coordinate is a.

Let X and Y be sets, and let Y have a fixed element, 0_Y (e.g. Y is a monoid). For each $f : X \to Y$, we define the *support of* f by $\mathsf{supp}(f) := \{x \in X : f(x) \neq 0_Y\}$. Similarly, if $\mathbb{y} \in Y^\alpha$, then $\mathsf{supp}(\mathbb{y}) = \{\beta \in \alpha : y_\beta \neq 0_Y\}$.

Definition 2.1.1 (Trees).

1. A *set-theoretic tree* is a partially ordered set, $(T, <)$, such that for all $t \in T$, $\{s \in T : s < t\}$ is well-ordered by $<$, with a unique element, r, such that for all $t \in T$, $r \leq t$.
2. A *descriptive set-theoretic tree* is a set $T \subseteq X^{<\alpha}$, where X is a set, α is an ordinal, and T is closed under taking subsequences.

Every descriptive set-theoretic tree is a set-theoretic tree, with the ordering $\subsetneq$. In Chapter 3, we focus on set-theoretic trees, and descriptive set-theoretic trees included in $\omega^{<\omega}$. Otherwise, our trees are on larger cardinals, e.g. in Chapter 6, our target spaces are κ^κ and 2^κ (defined in Section 2.1.2), which are naturally viewed as descriptive set-theoretic trees. As it is clear from context, we ambiguously refer to both as *trees*.

The subsequent definitions are common to both notions of tree, where descriptive set-theoretic trees are ordered by $\subsetneq$. We say that $b \subseteq T$ is a *branch* if it is a maximal chain of T. We let $[T]$ be the set of branches of T. We say that $s \in T$ is a *leaf* of T if there is no $t \in T$ such that $s \subsetneq t$. If T has no leaves, we say that T is *pruned*. We say a tree is *perfect* if for every $t \in T$ there are s, s' such that $s \neq s'$ and $t^\frown s, t^\frown s' \in T$. For $s, t \in T$, we say that $s \perp t$ if neither $s \subseteq t$ nor $t \subseteq s$, and if $A \subseteq T$, we say that $s \perp A$ if for all $t \in A$, $s \perp t$. If some $s \in T$ has more than one successor, then the *stem* of T, $\mathsf{stem}(T)$, is the unique $s \in T$ of the lowest level which has more than one successor. We define $T_t := \{s \in T : s \subseteq t \vee t \subseteq s\}$.

Now assume T is descriptive set-theoretic. We write $[T] \subseteq X^\alpha$, as each maximal chain can be viewed as an $x \in X^\alpha$. The α^{th} *level* of T is the set $\{s \in T : \mathsf{len}(s) = \alpha\}$, we denote it $\mathsf{Lev}_\alpha(T)$. If κ is a cardinal, we call T a *κ-tree* if $T \subseteq \kappa^{<\kappa}$, T has height κ, and for all $\alpha < \kappa$, $|\mathsf{Lev}_\alpha(T)| < \kappa$. If $A \subseteq T$, we define $\downarrow A := \{t \in T : \exists a \in A(a \leq t)\}$. Define $\delta(T) = \{s \in \kappa^{<\kappa} \setminus \downarrow T : \forall \alpha < \mathsf{len}(s)(s \restriction \alpha \in \downarrow T)\}$, we call $\delta(T)$ the *frontier* of T.

Throughout, we use κ, λ, and μ for cardinals. We call a set $C \subseteq \kappa$ a *club* if it is closed and unbounded [103, page 91]. The intersection of $<\kappa$ many clubs on κ is itself a club [103, Theorem 8.3]. A cardinal, κ, is *regular* if $\mathsf{cof}(\kappa) = \kappa$, otherwise it is *singular*. We say κ is a *strong limit* if whenever $\alpha < \kappa$ is a cardinal, $2^\alpha < \kappa$. If κ is regular and a strong limit, κ is called *inaccessible*. We say that $\kappa^{<\kappa} = \kappa$, if for all $\alpha \in \kappa \setminus \{0\}$, $\kappa^\alpha = \kappa$. Note that if $\kappa^{<\kappa} = \kappa$, then κ is regular (as, for any infinite cardinal, $\kappa^{\mathsf{cof}(\kappa)} > \kappa$). Likewise, if κ is inaccessible, then $\kappa^{<\kappa} = \kappa$ [103, 5.21, page 60].

Definition 2.1.2. A cardinal κ has the *tree property* is every κ-tree has a branch of length κ. If κ is inaccessible and has the tree property, we call κ *weakly compact*.

Definition 2.1.3. A *κ-Aronszajn* is a κ-tree with no branch of length κ.

Hence, κ has the tree property if and only if there are no κ-Aronszajn trees.

Definition 2.1.4. A κ-tree, T, is called *well-pruned* if for all $x \in T$, $|\{y \in T : x \subseteq y\}| = \kappa$.

Lemma 2.1.5 (Kunen, [129, Lemma III.5.27])**.** If κ is regular, then there is a κ-Aronszajn tree if and only if there is a well-pruned κ-Aronszajn tree.

In Chapters 4 to 6, we are mainly concerned with situations where regular $\kappa > \omega$, often also assuming that $\kappa^{<\kappa} = \kappa$. In Chapter 6, we often assume κ is inaccessible, though we remark on when weaker cardinal assumptions are sufficient. Throughout, we note which assumptions on κ are necessary for the results to hold.

2.1.2 Real numbers and other spaces

In this section, we introduce our principal topological spaces, assuming familiarity with basic topological notions, preempting a more systematic and thorough discussion in Section 2.2.1.

We let $\mathbb{Q}$ be the ordered field of *rationals* and $\mathbb{R}$ be the ordered field of the real numbers, defined as the Cauchy completion of $\mathbb{Q}$. These are topological spaces by their order topology.

If X is any set and α any ordinal, we let $N_s := \{\mathbb{x} \in X^\alpha : s \subseteq \mathbb{x}\}$. The collection $\mathcal{B}_\mathrm{b} := \{N_s : s \in X^{<\alpha}\}$ forms a topology basis (cf. Section 2.2.1) and the topology generated by it is called the *bounded topology* on X^α. The sets ω^ω, 2^ω, κ^κ, and 2^κ equipped with their bounded topologies are called the *Baire space*, *Cantor space*, *κ-Baire space*, and *κ-Cantor space*, respectively.

Note that the elements of $\mathcal{B}_\mathrm{b}$ are clopen in the bounded topology, so the Baire and Cantor spaces are *zero-dimensional.* We can identify finite products of the Baire space with the Baire space via Cantor's pairing function $\ulcorner\cdot,\cdot\urcorner : \omega \times \omega \to \omega$: for $x, y \in \omega^\omega$, we let $f(x, y)(n) := \ulcorner x(n), y(n)\urcorner$; then $f : \omega^\omega \times \omega^\omega \to \omega^\omega$ is a homeomorphism. Similarly, if $A = (A_n : n \in \omega)$ where $A_n \subseteq \omega^\omega$, we can identify it with the set $\widehat{A} := \{n^\frown x : x \in A_n\}$ while preserving basic topological features of the set (e.g. if $\widehat{A}$ is open or closed, then all A_n must be open or closed).

The spaces $\mathbb{Q}$, $\mathbb{R}$, ω^ω, and 2^ω are classic examples of *Polish spaces*, i.e., separable and completely metrisable topological spaces. *Par abus de langage*, but as usual in descriptive set theory, we call the elements of ω^ω *real numbers* or *reals* in Chapter 3, even though it is not homeomorphic the ordered field of real numbers; however, the notation $\mathbb{R}$ is reserved for the ordered field.

2.1.3 Axioms of set theory

We mostly work in Zermelo-Fraenkel set theory (ZF) and Zermelo-Fraenkel set theory with the axiom of choice (ZFC). In Chapter 3, our base theory is ZF; in all other chapters, our base theory is ZFC. If we wish to emphasise that a result uses AC, we indicate this by writing "(AC)"; similarly, if we wish to emphasise that a result does not use AC, we indicate this by writing "(ZF)". We write ZF^- for ZF without the power set axiom and the collection scheme instead of the replacement scheme. Likewise, we write ZFC^- for $\mathsf{ZF}^- + \mathsf{AC}$. (See [78] for a discussion of this theory and why collection has to be substituted for replacement.)

The Axiom of Choice, AC, states that every family of non-empty sets has a choice function. A *fragment* of AC is a statement Φ such that $\mathsf{ZFC} \vdash \Phi$ but $\mathsf{ZF} \nvdash \Phi$. There is a huge literature on fragments of AC, and one major industry is showing the *equivalence* or *separation* of pairs of fragments; a classic source for this is [95].

The Axiom of Choice can be stratified into fragments based on the index set of the family and the domain the non-empty sets come from. If we write $\mathsf{AC}_X(Y)$ for "every family indexed by elements of X of non-empty subsets of Y has a choice function", then AC is equivalent to "$\mathsf{AC}_X(Y)$ holds for all sets X and Y". Amongst these fragments of AC, *the countable axiom of choice for reals*, denoted $\mathsf{AC}_\omega(\omega^\omega)$, is particularly important for the purposes of descriptive set theory and real analysis. We note that there are models of ZF where $\mathsf{AC}_\omega(\omega^\omega)$ (and hence AC) fails, e.g. the Feferman-Lévy model (see Theorem 2.1.7). Furthermore, we write AC_ω for the statement "for all sets X, $\mathsf{AC}_\omega(X)$ holds" and call this the *Axiom of Countable Choice.*

For any set, X, a relation $R \subseteq X \times X$ is called *total* if for all $x \in X$ there is a $y \in Y$ such that $x\ R\ y$. For $X \neq \emptyset$, $\mathsf{DC}(X)$ states that for every total relation $R \subseteq X \times X$, there is a sequence $(x_n)_{n\in\omega} \in X^\omega$ such that $x_n\ R\ x_{n+1}$ for all $n \in \omega$.

Then, DC is the statement that for all non-empty X, $\mathsf{DC}(X)$ holds. It is well known that DC implies AC_ω, and indeed $\mathsf{DC}(X)$ implies $\mathsf{AC}_\omega(X)$ (e.g. [205, Proposition 1.2.1]). Once more, in our setting, the most relevant fragment is $\mathsf{DC}(\omega^\omega)$ which is of particular importance in descriptive set theory and real analysis.

The fragments $\mathsf{AC}_\omega(\omega^\omega)$ and $\mathsf{DC}(\omega^\omega)$ can themselves be stratified into a family of smaller fragments that will be the main topic of Chapter 3.

Let Ξ be any collection of countable sequences of sets of reals. Then we write $\mathsf{AC}_\omega(\omega^\omega;\Xi)$ for the statement "every countable sequence of non-empty sets of reals in Ξ has a choice function".

Let Γ be any collection of sets of reals. Using the zero-dimensionality of the Baire space, we identify ω^ω and $\omega^\omega \times \omega^\omega$ via a homeomorphism (see Section 2.1.2, page 11). Thus, if $R \subseteq \omega^\omega \times \omega^\omega$, we say that R is in Γ if the homeomorphic image of R in the Baire space is in Γ. We write $\mathsf{DC}(\omega^\omega;\Gamma)$ for the statement "for any $X \in \Gamma$ and any total relation $R \subseteq X \times X$ that is in Γ, there is a sequence $(x_n)_{n\in\omega} \in X^\omega$ such that for all $n \in \omega$, we have $x_n \; R \; x_{n+1}$", called the *axiom of dependent choices for* Γ *relations*.

We give two examples: let **Ctbl** be the set of countable sets of reals; we say that a sequence of sets is in **Ctbl** if all of its members are countable. Let **Closed** be the set of closed sets of reals (later denoted by $\mathbf{\Pi}^0_1$; see Section 2.1.4); we say that a sequence of sets is in **Closed** if all of its members are closed. Then $\mathsf{AC}_\omega(\omega^\omega;\mathbf{Ctbl})$ is the same principle that is sometimes known as $\mathsf{CAC}_\omega(\omega^\omega)$ in the literature, whilst $\mathsf{AC}_\omega(\omega^\omega;\mathbf{Closed})$ corresponds to "ω^ω is countably Loeb" (cf. [21, page 442], [114, page 15], [115, page 3]).

2.1.4 Descriptive pointclasses

Throughout this dissertation, particularly Chapters 3 and 6, we are concerned largely with definable sets, either of Polish spaces or their large cardinality analogues. In our context, a *pointclass* is a collection of subsets of Polish spaces.

In this section, we define the pointclasses of Borel and projective sets and their definable analogues, being especially sensitive to the choice-free context of Chapter 3. All pointclasses defined will be referred to as *descriptive pointclasses*. In Chapter 6, we define their generalisations to κ-topologies. Our notation is based on [103, §25] and [161, Chapter 3].

Borel sets

Let X be a Polish space. We define the following pointclasses:

1. $\mathbf{\Sigma}^0_1(X) := \{A \subseteq X : A \text{ is open}\}$,
2. $\mathbf{\Pi}^0_\xi(X) := \{X \backslash A : A \in \mathbf{\Sigma}^0_\xi(X)\}$,
3. if $\lambda > 0$, then $\mathbf{\Sigma}^0_\lambda(X) := \{\bigcup_{n\in\omega} A_n : A_n \in \bigcup_{\xi\in\lambda} \mathbf{\Pi}^0_\xi(X)\}$,
4. $\mathcal{B}(X) := \bigcup_{\xi\in\mathsf{Ord}}(\mathbf{\Sigma}^0_\xi(X) \cup \mathbf{\Pi}^0_\xi(X))$, which we call the *Borel sets* of X, and
5. $\mathbf{\Delta}^0_n(X) := \mathbf{\Sigma}^0_n(X) \cap \mathbf{\Pi}^0_n(X)$.

As usual, we code Borel sets by well-founded trees: let $\mathcal{T}$ be the set of well-founded trees $T \subseteq \omega^{<\omega}$; we can define by recursion a function $\phi : \mathcal{T} \to \mathscr{P}(X)$ such that each $\varphi(T)$ is a Borel set (e.g. [103, page 504]). We say that $B \subseteq X$ is *codeable Borel* if there is a Borel code T such that $\phi(T) = B$, and write $\mathcal{B}^*(X)$ for the set of all codeable Borel sets. By a simple induction, using $\mathsf{AC}_\omega(X)$ to pick the codes in the case of countable unions, every Borel set has a Borel code, i.e. $\mathcal{B}(X) = \mathcal{B}^*(X)$. This fails without AC; in the ZF-context, we only have the following result.

Lemma 2.1.6 (ZF; Fremlin, [66, Proposition 562Da])**.** If X is a Polish space, then $\mathbf{\Pi}^0_2(X), \mathbf{\Sigma}^0_2(X) \subseteq \mathcal{B}^*(X) \subseteq \mathcal{B}(X)$.

Without AC, Lemma 2.1.6 is optimal (see Theorem 2.1.7).

The usual inclusion hierarchy between the Borel pointclasses is a ZF-result, as it can be easily checked by observing that the standard proof does not use any choice: for any Polish space X and $0 < \xi < \eta$, we have that:

$$\begin{array}{ccccc} \mathbf{\Pi}^0_\xi(X) & & & & \mathbf{\Pi}^0_\eta(X) \\ & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\ & & \mathbf{\Delta}^0_\eta(X) & & \\ & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\ \mathbf{\Sigma}^0_\xi(X) & & & & \mathbf{\Sigma}^0_\eta(X). \end{array}$$

So, the Borel sets form a (not necessarily strictly) increasing hierarchy. If, for any α, $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Delta}^0_{\alpha+1}(X)$, then all subsequent pointclasses are the same (we say in this case that the hierarchy *collapses at* α). The smallest α where this happens is called the *length* of the Borel hierarchy.

Let X, Y be Polish spaces. If $s \in X$ and $t \in Y$, we define $A_s = \{y \in Y : (s, y) \in A\}$ and $A_t = \{x \in X : (x, t) \in A\}$ and call these sets the *slices of* A. We say that a set $U \in X \times Y$ is *universal for* Γ if $U \in \Gamma(X \times Y)$ and for all $X \in \Gamma(X)$, there is a $y \in Y$ such that $U_y = A$.

In ZFC, we prove that the Borel hierarchy does not collapse by constructing universal sets for each Borel pointclass (as in [103, page 141]). If X is a perfect Polish space, we can recursively define universal sets for all codeable $\mathbf{\Sigma}^0_\alpha(X)$ sets (where $\alpha < \omega_1$); thus, assuming $\mathsf{AC}_\omega(X)$, we have universal sets for $\mathbf{\Sigma}^0_\alpha(X)$, which implies that the hierarchy does not collapse at α (see, e.g. [103, Corollary 11.3]). Using the regularity of ω_1 (which follows from $\mathsf{AC}_\omega(\omega^\omega)$), we obtain that for all perfect Polish spaces, the length of the Borel hierarchy in X is ω_1; in ZF, this cannot be proved; here, the best lower bound is 4, as the next collection of results shows.[1]

Theorem 2.1.7. In ZF, we can prove that $\mathbf{\Pi}^0_3 \neq \mathbf{\Sigma}^0_3$. However, there is a model of ZF where ω^ω is a countable union of countable sets and ω_1 is singular. In particular, in this model, $\mathsf{AC}_\omega(\omega^\omega)$ fails, $\mathcal{B} \neq \mathcal{B}^*$, $\mathbf{\Pi}^0_4 = \mathbf{\Sigma}^0_4 = \mathscr{P}(\omega^\omega)$, and $\mathbf{\Sigma}^0_3 \not\subseteq \mathcal{B}^*$ (hence $\mathbf{\Pi}^0_3 \not\subseteq \mathcal{B}^*$).

[1] The ordinal ω_1 is also not an upper bound in ZF, as Miller has shown [152, 153, 154, 155, 156].

Proof. The first statement can be found in [5, Ex 3.38] or [154, 2.1]. One model witnessing the second statement is known as the Feferman-Lévy model (originally [59], for a detailed construction, see [104, page 142]). □

In fact, in ZF, if $\boldsymbol{\Sigma}^0_3 \subseteq \mathcal{B}^*$, then $\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$ holds [5, Ex 21.24].[2]

Note that $\mathcal{B}^*$ is obviously closed under complementation and contains all open sets, so if it is closed under countable unions, it forms a σ-algebra, and therefore $\mathcal{B} = \mathcal{B}^*$. Thus, in the Feferman-Lévy model, $\mathcal{B}^*$ is not closed under countable unions. This failure to be closed under countable unions is at the heart of many of the pathologies of set theory without $\mathsf{AC}_\omega(\omega^\omega)$ (cf. Section 3.1 for more on this). It is not known to the author whether we can construct a model where $\mathcal{B}$ is proper and $\mathcal{B} \neq \mathcal{B}^*$:

Question 2.1.8. Is there a model of ZF where $\mathcal{B}^* \subsetneq \mathcal{B} \subsetneq \mathscr{P}(\omega^\omega)$?

Projective sets

Let X and Y be Polish spaces. If $A \subseteq X \times Y$, we say that the *projection* of A into the first coordinate is $\pi(A) := \{x \in X : \exists y((x, y) \in A)\}$. We now define the *projective hierarchy*. Let $n \in \omega$. Then a set $A \subseteq X$ is:

1. $\boldsymbol{\Sigma}^1_0(X)$ if it is $\boldsymbol{\Sigma}^0_1(X)$,
2. $\boldsymbol{\Sigma}^1_{n+1}(X)$ if it is the projection of a $\boldsymbol{\Pi}^1_n(X \times \omega^\omega)$ set,
3. $\boldsymbol{\Pi}^1_n(X)$ if it is the complement of a $\boldsymbol{\Sigma}^1_n(X)$ set,
4. $\boldsymbol{\Delta}^1_n(X)$ if it is both $\boldsymbol{\Sigma}^1_n(X)$ and $\boldsymbol{\Pi}^1_n(X)$, and
5. *projective* if it is $\boldsymbol{\Sigma}^1_n(X)$ for some $n \geq 1$; we denote the class of projective sets by $\mathbf{Proj}(X)$.

As before, we obtain in ZF, that for $n < m$, we have:

$$\begin{array}{ccccc} \boldsymbol{\Pi}^1_n(X) & & & & \boldsymbol{\Pi}^1_m(X) \\ & \subseteq & & \subseteq & \\ & & \boldsymbol{\Delta}^1_m(X) & & \\ & \subseteq & & \subseteq & \\ \boldsymbol{\Sigma}^1_n(X) & & & & \boldsymbol{\Sigma}^1_m(X). \end{array}$$

Hence, the projective sets form an increasing hierarchy. However, in this case, the recursive construction of the universal sets does not require $\mathsf{AC}_\omega(\omega^\omega)$ (since no countable unions are involved) and therefore, if X is a perfect Polish space, we can

[2] The proof has two steps: let $\mathsf{Det}(\Gamma)$ mean that for every $X \in \Gamma$, G_X is determined (as defined in [103, page 629]). So, if $\Xi \subseteq \Xi'$, then $\mathsf{Det}(\Xi')$ implies $\mathsf{Det}(\Xi)$. Next, by relativising the classical proof that $\mathsf{Det}(\mathcal{P}(\omega^\omega))$ implies $\mathsf{AC}_\omega(\omega^\omega)$ (i.e. that of [163, Theorem 1]), we get that $\mathsf{Det}(\boldsymbol{\Sigma}^0_3)$ implies $\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$ [5, Ex 21.24]. Andretta observes that the standard ZFC-proof of $\mathsf{Det}(\mathcal{B})$ is a ZF-proof of $\mathsf{Det}(\boldsymbol{\Delta}^1_1)$ ([5, page 205], see [112, Theorem 2.5] for the standard proof). Hence, by Theorem 2.1.9, $\mathsf{Det}(\mathcal{B}^*)$ holds.

prove the existence of universal sets for all $\mathbf{\Sigma}^1_n(X)$ and therefore the strictness of the projective hierarchy in ZF.

In the ZF-context, Suslin's theorem places the codeable Borel sets at the bottom of the projective hierarchy.

Theorem 2.1.9 (ZF; Suslin's Theorem, [66, Theorem 562F]). $\mathcal{B}^*(X) = \mathbf{\Delta}^1_1(X)$.

With $\mathsf{AC}_\omega(\omega^\omega)$, Theorem 2.1.9 yields $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$, but without $\mathsf{AC}_\omega(\omega^\omega)$, the latter can be far from true: the strictness of the hierarchy implies that $\mathbf{\Sigma}^1_1(\omega^\omega) \neq \mathscr{P}(\omega^\omega)$, but in the Feferman-Lévy model, we have that $\mathbf{\Sigma}^0_3(\omega^\omega) \not\subseteq \mathbf{\Sigma}^1_1(\omega^\omega)$ (see Theorem 2.1.7).

In contrast to this, we can show in ZF that the set **Ctbl** of countable subsets of ω^ω is contained in $\mathbf{\Sigma}^1_1$: clearly $\mathbf{Ctbl} \subseteq \mathbf{\Sigma}^0_2$ (as all singletons are closed), then use Lemma 2.1.6 and Theorem 2.1.9. Therefore, in ZF, we obtain the following chain of strict inclusions:

$$\mathbf{Ctbl} \subsetneq \mathbf{\Sigma}^1_1 \subsetneq \ldots \subsetneq \mathbf{\Sigma}^1_n \subsetneq \mathbf{\Sigma}^1_{n+1} \subsetneq \ldots \subsetneq \mathbf{Proj}.$$

Lightface hierarchies

The Borel and projective sets measure complexity from the perspective of topology. We also have a *syntactic* notion of complexity, i.e. a measurement of the complexity of the sentences defining the sets in question. We recall some standard definitions for defining projective sets, for more detail, see [161].

Let $\mathcal{L}^2$ be the language of second-order arithmetic, and

$$\mathcal{A}^2 := (\omega, \omega^\omega, \mathrm{ap}, +, \times, <, 0, 1)$$

be the structure of (full) second-order arithmetic, where $\mathrm{ap} : \omega^\omega \times \omega : \omega$ is the function $\mathrm{ap}(r,k) = r(k)$. Let $\exists^0, \forall^0$ and $\exists^1, \forall^1$ be the first-order and second-order quantifiers respectively.

Definition 2.1.10. Let $n \in \omega$. A formula, φ, in $\mathcal{L}^2$ is called:

1. Σ^0_0 if it contains no second-order quantifiers, and every first-order quantifier is bounded by some term,
2. Π^0_n if it is the formula $\neg\psi$ for some Σ^0_n formula, ψ,
3. Σ^0_{n+1} if it is the formula $\exists^0 k\psi$ for some Π^0_{n+1} formula, ψ,
4. *arithmetical* if it is Σ^0_m or Π^0_m for some m,
5. Σ^1_1 if it is the formula $\exists^1 r\psi$ for some arithmetical ψ,
6. Π^1_n if it is the formula $\neg\psi$ for some Σ^1_n formula, ψ, and
7. Σ^1_{n+1} if it is the formula, $\exists^1 r\psi$ for some Π^1_n formula, ψ.

A formula of $\mathcal{L}^2$ with ℓ free second-order variables defines a subset of $(\omega^\omega)^\ell$, giving rise to the lightface hierarchies of sets of reals. In contrast, we call the Borel and projective classes *boldface pointclasses*.

Definition 2.1.11. Let $r \in \omega^\omega$. We say that $A \subseteq \omega^k \times (\omega^\omega)^\ell$ if $\Sigma^0_n(r)$ if there is φ be a formula of $\mathcal{L}^2$ with k-many free first-order variables and ℓ-many second-order variables which is Σ^0_n, such that $A = \{s \in \omega^k \times (\omega^\omega)^\ell : \mathcal{A}^2 \vDash \varphi(s,r)\}$. Likewise for $\Pi^0_n(r)$, $\Sigma^1_n(r)$, and $\Pi^1_n(r)$. We say that A is $\Delta^0_n(r)$ if it is $\Sigma^0_n(r)$ and $\Pi^0_n(r)$, so too for $\Delta^1_n(r)$. If r is computable, we write Σ^0_n, Π^0_n, Δ^0_n, Σ^1_n, Π^1_n, or Δ^1_n for $\Sigma^0_n(r)$, $\Pi^0_n(r)$, $\Delta^0_n(r)$, $\Sigma^1_n(r)$, $\Pi^1_n(r)$, or $\Delta^1_n(r)$, respectively. If A is Σ^0_n or Π^0_n, we say that A is *arithmetical*. If A is Σ^1_n or Π^1_n, we say that A is *analytical* or *lightface projective*.

The relationship between the boldface and the lightface pointclasses is stated in Addison's theorem.

Theorem 2.1.12 ($\mathsf{AC}_\omega(\omega^\omega)$; Addison's Theorem)**.** For any set $A \subseteq \omega^\omega$, A is $\mathbf{\Sigma}^0_n$ if and only if A is $\Sigma^0_n(r)$ for some $r \in \omega^\omega$, and likewise for $\mathbf{\Pi}^0_n$, $\mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_n$.

Proof. E.g. [108, Proposition 12.6]. □

We remark that the backwards direction of Addison's Theorem holds in ZF.

Analytic sets

Of particular interest in analysis is the first level of the projective hierarchy. The $\mathbf{\Sigma}^1_1$ sets are also called the *analytic* sets. There are a number of equivalent characterisations of analytic sets, some are stated in Proposition 2.1.13.

For any function $f : X \to Y$, we define $\mathsf{Graph}(f) := \{(x, f(x)) \in X \times Y : x \in X\}$. We call a function $f : X \to Y$ between Polish spaces X and Y *Borel** if $\{g^{-1}(U_n) : U_n \in \mathcal{B}\}$ is a family of codeable Borel sets which is codeable (in the sense of [66, 562J]), where $\mathcal{B}$ is an enumerated base for Y.

Proposition 2.1.13 (ZF)**.** Let X be a Polish space. Let $A \subseteq X$ be non-empty. The following are equivalent:

1. A is $\mathbf{\Sigma}^1_1(X)$.
2. there is a continuous $f : \omega^\omega \to X$ such that $A = f(\omega^\omega)$ ("*strongly analytic*"),
3. there is a $B \in \mathcal{B}^*(\omega^\omega)$ and a continuous function $f : \omega^\omega \to X$ such that $A = f(B)$ ("*intermediately analytic*"), and
4. there is a Borel* function $f : \omega^\omega \to X$ such that $A = f(\omega^\omega)$ ("*weakly analytic*").

Proof. Part 1. to Part 2. and Part 2. to Part 3. are trivial. For Part 3. to Part 4., fix an $x_0 \in B$ and define the following function:

$$h(x) := \begin{cases} x & \text{if } x \in B \\ x_0 & \text{otherwise.} \end{cases}$$

This h is obviously Borel*. By [66, 562Md], f is Borel*. Finally, Borel* functions are closed under composition by [66, 562Mb] so $f \circ h : \omega^\omega \to B$ is Borel* and $A = f \circ h(\omega^\omega)$.

Note that, in ZF, if f is continuous, then $\mathsf{Graph}(f) \in \mathbf{\Pi}^0_1(X \times Y)$. Hence, as usual, Part 2. implies Part 1., so it suffices to prove Part 4. implies Part 2.

The proof is a choice-free version of the standard proof (in e.g. [112, Proposition 14.4(ii)]). As projections are obviously continuous, it suffices to check that the graph of a Borel* function is analytic. Using Theorem 2.1.9, this reduces to checking the codeable version of the Borel Graph Theorem [112, Proposition 12.4], i.e. if g is Borel*, $\mathsf{Graph}(g) \in \mathcal{B}^*(X \times Y)$.

For this, let $\{U_n : n \in \omega\}$ be an enumeration of $\mathcal{B}$. Let $(x, y) \notin \mathsf{Graph}(g)$. Then $y \neq g(x)$. Hence, there are disjoint open U_n, U_m in Y such that $y \in U_n$ and $g(x) \in U_m$. Since g is Borel*, $g^{-1}(U_m) \in \mathcal{B}^*(X)$, and contains x. Then $g^{-1}(U_m) \times U_n \in \mathcal{B}^*(X \times Y)$, contains (x, y), and is disjoint from $\mathsf{Graph}(g)$. Let $I \subset \mathbb{N}^2$ be the set of pairs $(n_{x,y}, m_{x,y})$ such that there is a pair $(x, y) \in X \times Y$ such that $y \neq g(x)$, where $(n_{x,y}, m_{x,y})$ are minimal (in the $\mathbb{N}^2$ ordering) so that $y \in U_{n_{x,y}}$ and $g(x) \in U_{m_{x,y}}$. So, $(X \times Y) \setminus \mathsf{Graph}(g) = \bigcup_{(n,m)\in I} g^{-1}(U_m) \times U_n$. As $\{g^{-1}(U_n) : n \in \mathcal{B}\}$ is a codeable family of $\mathcal{B}^*(X)$ sets, $\{g^{-1}(U_n) \times U_m : m, n \in \mathcal{B}\}$ is a codeable family of $\mathcal{B}^*(X \times Y)$ sets. So, $(X \times Y) \setminus \mathsf{Graph}(g)$ is a codeable union of Borel* sets. By [66, 562Ka], $(X \times Y) \setminus \mathsf{Graph}(g)$ is Borel*. Hence, $\mathsf{Graph}(g)$ is Borel*. □

We look at three corresponding definitions of analyticity given in Proposition 2.1.13 in Chapter 6 and observe that in the generalised setting, the equivalence need not hold (see Corollary 6.3.51).

2.1.5 Models of set theory

Here we describe the main models of set theory which we use in Chapter 3. Our notation is based on [103, Chapter 13].

For an infinite regular cardinal, κ, let $\mathbf{H}_\kappa$ be the collection of all sets whose transitive closure has cardinality $<\kappa$. Then each $\mathbf{H}_\kappa$ is a transitive model of ZFC$^-$ [129, Theorem II.2.1]. For Chapter 3, we use $\mathbf{HC} := \mathbf{H}_{\omega_1}$, which we call the *hereditarily countable sets*.

We say that a model of a theory is a *ctm* if it is a countable transitive model. A model of ZFC is called an *inner model* if it is transitive and contains all the ordinals.

In Chapter 3, we use a standard model of ZFC, the *constructible universe*, denoted by L. The constructible universe was first introduced by Gödel [79]) and is an inner model of ZFC which satisfies GCH [103, §13]; its construction can be relativised to a set A.

Definition 2.1.14. Let $\mathcal{L}$ be the language of set theory with an additional unary relation. Let A be any set. A set $X \subseteq M$ is *definable over M relative to A* if there is a formula φ in $\mathcal{L}$, and $p_0, ..., p_n \in M$ such that $X = \{x \in M : (M, \in, A \cap M) \models \varphi(x, p_0, ..., p_n)\}$. Let $\mathsf{def}_A(M)$ be the collection of sets definable over M relative to A.

Definition 2.1.15. Let A be a set. We recursively define $\mathrm{L}[A]$:

1. $\mathrm{L}_0[A] := \emptyset$,
2. for all α, $\mathrm{L}_{\alpha+1}[A] := \mathsf{def}_A(\mathrm{L}_\alpha[A])$,
3. if λ is a limit, $\mathrm{L}_\lambda[A] := \bigcup_{\alpha\in\lambda} \mathrm{L}_\alpha[A]$, and
4. $\mathrm{L}[A] := \bigcup_{\alpha\in\mathsf{Ord}} \mathrm{L}_\alpha[A]$.

If $x \in \mathrm{L}[A]$, we that x is *constructible relative to* A. We write $\mathrm{L}_\alpha := \mathrm{L}_\alpha[\emptyset]$, $\mathrm{L} := \mathrm{L}[\emptyset]$, say that a set is *constructible* if it is constructible relative to $\emptyset$, and we write $V = \mathrm{L}$ for the statement "every set is constructible". Obviously, L satisfies $V = \mathrm{L}$. Gödel also showed that there is a definable global well-order on L, i.e. a well-order $\leq_\mathrm{L}$ such that for all $x, y \in \mathrm{L}$, $x \leq_\mathrm{L} y$ or $y \leq_\mathrm{L} x$. For many more details on L and constructibility, see [45, 103, 108].

In Chapter 3, we use $\mathbf{HC}$ to define projective sets of reals. For this, we define the *Lévy hierarchy* on formulae in the language of set theory, and a corresponding hierarchy of sets of $\mathbf{HC}$:

Definition 2.1.16. Let $n \in \omega$. Let φ be a formula in the language of set theory. We say that φ is:

1. Σ_1 if every quantifier is bounded,
2. Π_n if φ is the formula $\neg\psi$ for a Σ_n formula, ψ, and
3. Σ_{n+1} if φ is the formula $\exists x\psi$ for a Π_n formula, ψ.

Definition 2.1.17. Let $n \in \omega$. If $A \subseteq \mathbf{HC}$, then we say that A is:

1. $\Sigma_n^{\mathbf{HC}}$ (or $\Pi_n^{\mathbf{HC}}$) if there is a Σ_n (or Π_n) formula, φ, with exactly one free variable such that $A = \{x \in \mathbf{HC} : \mathbf{HC} \vDash \varphi(x)\}$, and
2. $\boldsymbol{\Sigma}_n^{\mathbf{HC}}$ (or $\boldsymbol{\Pi}_n^{\mathbf{HC}}$) if there is a Σ_n (or Π_n) formula, φ, with at least one free variable, and some $p_0, ..., p_k \in \mathbf{HC}$ such that $A = \{x \in \mathbf{HC} : \mathbf{HC} \vDash \varphi(x, p_0, ..., p_k)\}$.

We say that A is $\Delta_n^{\mathbf{HC}}$ if it is $\Sigma_n^{\mathbf{HC}}$ and $\Pi_n^{\mathbf{HC}}$, and likewise for $\boldsymbol{\Delta}_n^{\mathbf{HC}}$.

We end this section with the setup for a technical lemma which we use in Lemma 3.3.33.

Lemma 2.1.18. Let $n \geq 1$. A set of reals is Σ_{n+1}^1 if and only if it is $\Sigma_n^{\mathbf{HC}}$.

Proof. By e.g. [103, Lemma 25.25] □

Lemma 2.1.19. The restriction of the global well-order on L, $<_\mathrm{L}\restriction\mathbf{HC}^2$ on $\mathbf{HC}^\mathrm{L}$, is $\Delta_1^{\mathbf{HC}}$.

Proof. See e.g. [111, §4.3, page 19]. □

Lemma 2.1.20. For each $n > 2$, there is a universal $\Sigma_{n-2}^{\mathbf{HC}}$ set $\Gamma \subseteq \omega_1 \times \mathbf{HC}$.

Proof. Recall that there is a set $U \subseteq (\omega^\omega)^2$ which is universal for Σ_{n-1}^1 [161, 3F.6]. In L, $\mathbf{HC} = \mathrm{L}_{\omega_1}$, so $|\mathbf{HC}| = \omega_1 = |\omega^\omega|$ [45, Lemma 1.1vii]. So, there is a bijection $b : \omega_1 \times \mathbf{HC} \to (\omega^\omega)^2$. By Lemma 2.1.19, b can be chosen with low complexity. Then by Lemma 2.1.18, $b(U)$ is universal for $\Sigma_{n-2}^{\mathbf{HC}}$ sets of $\omega_1 \times \mathbf{HC}$. □

2.1.6 Absoluteness

In Chapter 3, we need to 'locate' sets of reals in different models of set theory, i.e. we need to make sure that we have suitably equivalent sets (which satisfy the same relevant properties) in different models, and have a way to describe these sets. For this, we use *absoluteness*, a particular property of formulae. Hence, we require the syntactic characterisation of sets of reals from Section 2.1.4. We first describe absoluteness, and then we state some facts concerning the absoluteness of sentences defining sets of real numbers. Our notation is based on [103, §25] and [161, Chapter 3].

If M is a model of set theory, N is a submodel of N, and φ is a formula in the language of set theory, we say that φ is *absolute between M and N* if for all $x_0, ..., x_n \in N$, $M \vDash \varphi(x_0, ..., x_n)$ if and only if $N \vDash \varphi(x_0, ..., x_n)$. Any formula with only bounded quantifiers is absolute between transitive models of set theory (by, e.g. [103, Lemma 12.9]).

We are interested in absoluteness for formulae of strictly richer complexity. Some sentences of second-order arithmetic are not absolute, indeed there is a Σ^1_3 formula which is true in L but not in L$[r]$, for a non-constructive real r [205, §1.2.8]. But formulae of lower complexity are absolute:

Theorem 2.1.21 (Analytic Absoluteness)**.** Every Σ^1_1 and Π^1_1 formula is absolute between transitive models of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$.

Proof. E.g. [103, Theorem 25.4]. □

As an immediate consequence, Σ^1_2 formulae are upwards absolute for transitive models of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$, and likewise Π^1_2 formulae are downwards absolute for such models. Meanwhile, Σ^1_2 formulae need not be downwards absolute in such models, but are in larger models:

Theorem 2.1.22 (Shoenfield Absoluteness)**.** Every Σ^1_2 and Π^1_2 formula is absolute between inner models of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$.

Proof. E.g. [103, Theorem 25.20]. □

2.1.7 Forcing and symmetric extensions

In Chapter 3, the proof of the main theorem uses a forcing-like argument, which defines a forcing-like relation dependent on the complexity of the sentences involved.

The method of forcing was first introduced by Cohen to prove that there is a model of ZFC which violates the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$), as narrated in [32]. Ever since its introduction, forcing has been one of the central methods of set theory. We assume the reader has a basic familiarity with forcing, so we fix our notation, based on [129]. For an introduction to forcing, see [103] or [129].

Let M be a model of ZF^- or ZF, and let $\mathbb{P}$ be a partial order. In the context of forcing, we call $\mathbb{P}$ the *forcing notion* and the elements $p \in \mathbb{P}$ the *forcing conditions*. If there is an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$, then we say that p and q are *compatible*, and write $p \not\perp q$, otherwise, we say that p, q are *incompatible*, and write $p \perp q$. If $E \subseteq \mathbb{P}$ and $p \in Q$, we write $p \perp E$ if for all $q \in E$, $q \perp p$. Then E is called *predense* if there is no $E \in \mathbb{P}$ such that $p \perp E$. A forcing notion, $\mathbb{P}$, has the *countable chain condition* (c.c.c.) if every antichain in $\mathbb{P}$ is countable.

If G is a $\mathbb{P}$-generic filter over M, we write $M[G]$ for the forcing extension of M by G, and for a $\mathbb{P}$-name, σ, we denote its interpretation by G by σ_G. As usual, for $x \in M$, we let $\check{x}$ be the standard $\mathbb{P}$-name for x [129, Definition IV.2.9].

We work with a variant of product forcing in Chapter 3; here we define standard product forcing. For more details, see [129, Chapter V]. If $(\mathbb{P}_n)_{n\in\omega}$ is a sequence of forcing notions, we say that $\mathbb{P}$ is the *ω-product forcing notion with finite support* (of $(\mathbb{P}_n)_{n\in\omega}$) if $\mathbb{P}$ consists of sequences $(p_n)_{n\in\omega}$ so that for each n, $p_n \in \mathbb{P}_n$ and $\{n \in \omega : p_n \neq 1\}$ is finite. This $\mathbb{P}$ is ordered in the natural way, $(p_n)_{n\in\omega} \leq (q_n)_{n\in\omega}$ if and only if for all n, $p_n \leq q_n$.

Throughout Chapter 3, we only consider products of forcing notions whose generic filters are uniquely determent by a real in the extension. So, we characterise the generics like so:

Remark 2.1.23. Let G be a $\mathbb{P}$-generic filter and let $i \in I$. Then $\{p(i) : p \in G\}$ is a $\mathbb{P}_i$-generic filter. We denote it by G^i and the unique generic real corresponding to G^i by x^i_G. If $J \subseteq I$, we define $G{\restriction}J := G \cap \mathbb{P}{\restriction}J$. Then $G \cap \mathbb{P}{\restriction}J$ is a $\mathbb{P}{\restriction}J$-generic filter over M.

We let $\mathbb{S}$ be the set of all perfect trees of $2^{<\omega}$, ordered by inclusion. We call this *Sacks forcing*. If $T \in \mathbb{S}$ and $t \in T$, then $T_t \in \mathbb{S}$ and every node in T_t is compatible with t.

Let $\mathbb{C} := \{(2^{<\omega})_t : t \in 2^{<\omega}\}$, ordered by inclusion. We call this the (arboreal version of) *Cohen forcing*.

Forcing extensions typically preserve AC from the ground model to the extension; to overcome this we use a variation of a symmetric extension, to ensure that only a fragment of AC is satisfied in the model. Using symmetric extensions, Cohen also showed that there is a model of $\mathsf{ZF} + \neg\mathsf{AC}$. As $\mathrm{L} \vDash \mathsf{ZFC}$, AC is independent of ZF. Roughly, Cohen defined an inner model, N, such that $M \subseteq N \subseteq M[G]$, by evaluating only those names which are symmetric with respect to a group of automorphism, hence the name *symmetric submodel*. For an introduction to symmetric submodels and symmetric extensions, see [46] or [104, §5.2]. We assume the reader has a basic familiarity with classical examples of symmetric extensions, so we fix notation.

Let M be a transitive model of ZFC, and $\mathbb{P}$ be a forcing notion in M. If $\pi : \mathbb{P} \to \mathbb{P}$ is an order-preserving bijection, we say that π is an *automorphism*.

Definition 2.1.24. Let $\pi : \mathbb{P} \to \mathbb{P}$ be an automorphism. We recursively extend π to the set of $\mathbb{P}$-names like so:

1. $\pi(\emptyset) := \emptyset$,
2. $\pi(\sigma) := \{(\pi(\tau), \pi(p)) : (\tau, p) \in \sigma\}$.

Likewise, for every sentence φ in the forcing language, there is a formula $\pi(\varphi)$ formed by replacing every $\mathbb{P}$-name σ by the in φ by $\pi(\sigma)$. As $\mathbb{P}$ is a dense embedding, for every sentence φ in the forcing language, and every forcing condition $p \in \mathbb{P}$, we have that $p \Vdash \varphi$ if and only if $\pi(p) \Vdash \pi(\varphi)$.

A set $F \subseteq \mathscr{P}(S)$ is called a *filter on S* if F is non-empty and closed under supersets and finite intersections. If $\mathcal{G}$ is a group and $F \subseteq \mathscr{P}(\mathcal{G})$ is a filter over $\mathcal{G}$, we say that F is *normal* if for every $K \in F$ and every $g \in \mathcal{G}$, the conjugate $gKg^{-1} \in F$.

Let M be a model, and $\mathbb{P}$ be a forcing notion in M. Let $\mathcal{G}$ be a group of automorphisms of $\mathbb{P}$ in M. Let σ be a $\mathbb{P}$-name. We define $\mathsf{sym}(\sigma) := \{\pi \in \mathcal{G} : \pi(\sigma) = \sigma\}$. Let F be a normal filter of subgroups of $\mathcal{G}$. We call σ *symmetric* if $\mathsf{sym}(\sigma) \in F$, and *hereditarily symmetric* if σ is symmetric and every $\mathbb{P}$-name in σ is hereditarily symmetric. Let G be a $\mathbb{P}$-generic filter over M. The *symmetric submodel*, N, associated to F and G is defined like so $N := \{\sigma_G : \sigma \text{ is a hereditarily symmetric } \mathbb{P}\text{-name in } M\}$.

Theorem 2.1.25. Let M be a transitive model of ZFC, $\mathbb{P}$ a forcing notion in M, $\mathcal{G}$ a group of automorphisms of $\mathbb{P}$ in M, F a normal filter of subgroups of $\mathcal{G}$ in M, and G a $\mathbb{P}$-generic filter over M. Then the symmetric submodel, N, associated to F and G is a transitive model of ZF such that $M \subseteq N \subseteq M[G]$.

Proof. E.g. [104, Theorem 5.14]. □

2.2 Topological and order-theoretic notions

2.2.1 Topologies and κ-topologies

Here we give a very brief outline of the topology necessary for Chapters 4 to 6. Chapter 6 contains a much more sustained development of κ-topologies and their descriptive set theory, so we leave most detail for there. The notion of a κ-topology was first introduced in [2] in studying subfields of the surreal numbers (defined in Section 2.3.5), and has been studied more recently in e.g. [74] and [75]. Our notation for topologies is standard, e.g. [164], whilst our notation for κ-topologies follows [2, Chapter 0] and [74, §3.2].

Let $\mathcal{A} \subseteq \mathscr{P}(X)$ be a family of subsets of a set, X. We say that $\mathcal{A}$ is a *subbasis* on X if $\bigcup \mathcal{A} = X$. It is a *basis* on X if it is a subbasis and closed under finite intersections. We say that $\mathcal{A}$ is a *topology* on X if it is a basis and closed under arbitrary unions. If $\mathcal{A}$ is a basis on X, then $\langle \mathcal{A} \rangle_\infty := \{A \subseteq X \,;\, \exists \mathcal{A}_0 \subseteq \mathcal{A}(A = \bigcup \mathcal{A}_0)\}$ is a topology on X, the *topology generated by* $\mathcal{A}$. By convention, $\bigcup \emptyset = \emptyset$, so $\emptyset$ is an element of all topologies.

If κ is a cardinal, a basis $\mathcal{A}$ is called a *κ-topology* on X if $\mathcal{A}$ is closed under unions of size $<\kappa$, and $X \in \mathcal{A}$. Similarly, if $\mathcal{A}$ is a basis on X, then $\langle \mathcal{A} \rangle_\kappa := \{A \subseteq X \,;\, \exists \mathcal{A}_0 \subseteq \mathcal{A}(|\mathcal{A}_0| < \kappa \text{ and } A = \bigcup \mathcal{A}_0)\} \cup \{X\}$ is a κ-topology on X, the *κ-topology generated by* $\mathcal{A}$. Again, $\emptyset$ is an element of all κ-topologies. Clearly, every topology is a κ-topology and every κ-topology is a basis (that generates a full topology).

For any κ-topology, τ, we define its *weight* by:

$$\mathsf{wei}(\tau) := \min\{|\mathcal{A}| : \mathcal{A} \text{ is a basis and } \tau = \langle \mathcal{A} \rangle_\kappa\}.$$

If $\mathsf{wei}(\tau) < \kappa$, then τ is a topology.

Elements of a (κ-)topology are called *(κ-)open*, and their complements are called *(κ-)closed*. If $A \subseteq X$ is any set, then $\mathsf{cl}(A) := \bigcap\{B \subseteq X : A \subseteq B$ and B is closed$\}$ is closed and called the *closure of* A.

A set $\mathcal{A} \subseteq \mathscr{P}(X)$ is called an *algebra on* X if A is closed under relative complements with X and under finite unions. We call $\mathcal{A}$ a *κ-algebra* if $\mathcal{A}$ is an algebra and closed under unions of size $<\kappa$.

We use the standard notion of a metric d on a set X. If d is a metric on X, then $\{B(x,\varepsilon) : x \in X$ and $\varepsilon > 0\}$ where $B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$ forms a basis of the *d-metric topology*. As usual, in a topological space $(X,\mathcal{A})$, a subset is called *dense* if it meets every non-empty open set; a space is called *separable* if it has a countable dense set and *completely metrisable* if there is a complete metric d on X such that $\mathcal{A}$ is the d-metric topology. A topological space is a *Polish space* if it is separable and completely metrisable.

Following [1, §2.1], a *κ-metric* on X is a function $d : X^2 \to \mathbb{G}^{\geq 0}$ which obeys the standard rules of a distance ($\forall x \in X(d(x,x) = 0)$, symmetry, and the triangle inequality), such that $\mathbb{G}$ is an ordered Abelian group where $\mathsf{bn}(\mathbb{G}) = \kappa$. We say that a topology is *κ-metrisable* if it admits a compatible κ-metric (for more details, see [1] or [75, §2.3.2]). As space, X, is called *κ-Polish* if it is (Cauchy) completely κ-metrisable, and $\mathsf{wei}(X) \leq \kappa$ [1, Definition 2.3].

Continuity and κ-Continuity

In complete generality, if $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of X and Y, respectively, then a function $f : X \to Y$ is called *$\mathcal{A},\mathcal{B}$-continuous* if the f-preimage of sets in $\mathcal{B}$ is in $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are topologies on X and Y and the choice of topologies is clear from the context, we also just say that f is *continuous*. The dual notion is a function such that all f-images of sets in $\mathcal{A}$ are in $\mathcal{B}$. In the case of topological spaces, we say that such a function is *open*.

The notion of κ-continuity is related in the same way to the notion of κ-topology: if $\mathcal{A}$ and $\mathcal{B}$ are κ-topologies on X and Y and the choice of κ-topologies is clear from the context, we also just say that f is *κ-continuous*; similarly, f is called *κ-open* if f-images of κ-open sets are κ-open.

The notion of κ-continuity can be traced to [2, §2.01]; our notation follows [74, 75]. We remark that if $\mathcal{A} \subseteq \mathscr{P}(X)$ and $\mathcal{B} \subseteq \mathscr{P}(Y)$ are bases and $\lambda \leq \lambda'$ are (regular) cardinals and we generate the λ- and λ'-topologies $\langle\mathcal{A}\rangle_\lambda$, $\langle\mathcal{B}\rangle_\lambda$, $\langle\mathcal{A}\rangle_{\lambda'}$, and $\langle\mathcal{B}\rangle_{\lambda'}$, then if a function is λ-continuous, it is λ'-continuous. As a consequence, we obtain the following fact.

Fact 2.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be bases on X and Y respectively. Consider the topologies and κ-topologies generated from $\mathcal{A}$ and $\mathcal{B}$. Then for any function $f : X \to Y$, each of the following properties implies the next:

1. f is $\mathcal{A},\mathcal{B}$-continuous.
2. f is κ-continuous.
3. f is continuous.

The reverse implications do not hold in general (see Proposition 4.1.3 and Example 4.1.4).

2.2.2 Orders

Our notation for orders and ordered fields is based on [42] and [75].

Total orders

All orders are assumed to be total unless otherwise stated. Let $(\mathbb{O}, <)$ be an order. If $A, B \subseteq \mathbb{O}$, we write $A < B$ if every $a \in A$ and $b \in B$ are such that $a < b$. A *convex subset* of $\mathbb{K}$ is a $C \subseteq \mathbb{O}$ such that if $a < c < b$ with $a, b \in C$ and $c \in \mathbb{O}$ then $c \in C$. If $\mathbb{O}$ is a linear order and $a, b \in \mathbb{O}$, then we write $(a, b) := \{x \in \mathbb{O} : a < x < b\}$, $(-\infty, b) := \{x \in \mathbb{O} : x < b\}$, and $(a, \infty) := \{x \in \mathbb{O} : a < x\}$. Likewise, we define $[a, b]$, $(-\infty, b]$, $[a, \infty)$ as usual. If $a, b \in \mathbb{O}$, we call each of (a, b), $[a, b]$, $[a, b)$, and $(a, b]$ an *interval* in $\mathbb{O}$. If $a \in \mathbb{O}$, we call each of $(a, \infty), (-\infty, a), [a, \infty), (-\infty, a]$ an *improper interval* in $\mathbb{O}$.

If $Q \subseteq \mathbb{O}$, we let $\mathcal{B} = \{(p, q) : p, q \in Q \cup \{\pm\infty\}\}$ and observe that this is a basis on $\mathbb{O}$ in the sense of Section 2.2.1. We call it the *Q-interval basis*. The κ-topology $\langle \mathcal{B} \rangle_\kappa$ is called the *Q-interval κ-topology on X*. If this κ-topology is a topology, we call it the *interval topology* or *order topology*.

Let $\mathsf{coi}(\mathbb{O})$ be the smallest cardinal λ such that there is a set $S \subseteq \mathbb{O}$, with $|S| = \lambda$, such that for every $x \in \mathbb{O}$, there is an $s \in S$ with $s < x$. We call $Q \subseteq \mathbb{O}$ *order dense* in $\mathbb{O}$ if for all $a, b \in \mathbb{O}$, where $a < b$, there is a $q \in Q$ such that $a < q < b$. If λ is the smallest cardinal such that there is a set $Q \subseteq \mathbb{O}$ such that Q is order dense in $\mathbb{O}$ and $|S| = \lambda$, we say $\mathsf{wei}(\mathbb{O}) = \lambda$ (or, to disambiguate, $\mathsf{owei}(\mathbb{O}) = \lambda$).[3] If $S \subseteq \mathbb{O}$ is order dense in $\mathbb{O}$, then $\mathsf{wei}(\mathbb{O}) = \mathsf{wei}(S)$.

We say that $\mathbb{O}$ is called η_κ if for all $L, R \subseteq \mathbb{O}$ where $L < R$ and $|L \cup R| < \kappa$, there is an $x \in \mathbb{O}$ such that $L < \{x\} < R$. Being η_κ means there are no κ-clopen sets in the order κ-topology. We say that $\mathbb{O}$ is a *dense linear order* if it is an η_0-order (equivalently, for any $x, y \in \mathbb{O}$ with $x < y$, there is a $z \in \mathbb{O}$ such that $x < z < y$).

Theorem 2.2.2 (Hausdorff, [88], [178, Ex 9.23(2)])**.** Any two η_κ orders of cardinality κ are order isomorphic.

Definition 2.2.3. We say that $\mathbb{O}$ is *κ-spherically complete* if for all sequences of closed nested intervals, $([a_\alpha, b_\alpha])_{\alpha \in \lambda}$ with $\lambda < \kappa$, we have $\bigcap_{\alpha \in \lambda} [a_\alpha, b_\alpha] \neq \emptyset$.

Note that if $\mathbb{O}$ is η_κ then $\mathbb{O}$ is κ-spherically complete.

If $\mathbb{O}$ is any total order and $x_0 < \ldots < x_n$ any finite sequence in $\mathbb{O}$, it splits $\mathbb{O}$ into $n+2$ intervals $I_0 := (-\infty, x_0]$, $I_1 := [x_0, x_1]$, ..., $I_n := [x_{n-1}, x_n]$, $I_{n+1} := [x_n, \infty)$, overlapping in the points of the sequence. We can think of a function $f : \mathbb{O} \to \mathbb{O}$ as split into the $n+2$ many functions $f_k : I_k \to \mathbb{K}$ (for $0 \leq k \leq n_1$) by $f_k := f \restriction I_k$. In this case, we say that the f_k can be *glued together* to yield f. If there is a finite sequence such that all of the f_k have property P, we say that f is *piecewise* P. E.g. we can consider piecewise constant, piecewise monotone, or piecewise strictly monotone functions.

[3]The ambiguity of notation is between $\mathsf{wei}(\tau)$, the weight of the order (κ-)topology, and $\mathsf{wei}(\mathbb{O})$, the weight of the order. For the most part, it is easy to distinguish which notion is intended by context.

2.2.3 Gaps

An order, $\mathbb{O}$, may have order-theoretic gaps. For this, we again follow the notation in [42] and [75].

Definition 2.2.4. Let $\mathbb{O}$ be an order. If $L, R \subseteq \mathbb{O}$ are such that $|\mathbb{O} \setminus (L \cup R)| \leq 1$, $L < R$, L has no least upper bound, and R has no greatest lower bound, then we say that the pair (L, R) is an *almost gap* in $\mathbb{O}$ (also called *quasicuts*, [128, §2]). If, in addition, $L \cup R = \mathbb{O}$, then we call (L, R) a *gap*.

An almost gap, (L, R), in an ordered group, $\mathbb{G}$, is called a (λ, μ)-*(almost)gap* if $\mathsf{cof}(L) = \lambda$ and $\mathsf{cof}(R) = \mu$.

Every $a \in \mathbb{O}$ defines an almost gap, $((-\infty, a), (a, \infty))$, and every almost gap is either defined from an $x \in \mathbb{O}$ or is a gap. Let $\mathsf{Ded}(\mathbb{O})$ be the *Dedekind completion* of $\mathbb{O}$ (i.e. the set $\mathbb{O} \cup \{G : G \text{ is a gap in } \mathbb{O}\}$). Note that the set of almost gaps in $\mathbb{O}$ is order isomorphic to $\mathsf{Ded}(\mathbb{O})$.

If $A \subseteq \mathbb{O}$, let $\mathsf{hull}_{\mathrm{L}}(A) = \{z \in \mathbb{O} : \forall x \in A(z < x)\}$, and $\mathsf{hull}_{\mathrm{R}}(A) = \{z \in \mathbb{O} : \forall x \in A(x < z)\}$. We define the *top* and *bottom* of A like so:

$$\mathsf{top}(A) := \begin{cases} \sup(A) & \text{if } \sup(A) \text{ exists,} \\ (\mathbb{O} \setminus \mathsf{hull}_{\mathrm{R}}(A), \mathsf{hull}_{\mathrm{R}}(A)) & \text{if } A \text{ has no supremum and} \\ & \qquad \text{is bounded above,} \\ \infty & \text{otherwise.} \end{cases}$$

$$\mathsf{bot}(A) := \begin{cases} \inf(A) & \text{if } \inf(A) \text{ exists,} \\ (\mathsf{hull}_{\mathrm{L}}(A), \mathbb{O} \setminus \mathsf{hull}_{\mathrm{L}}(A)) & \text{if } A \text{ has no infimum and} \\ & \qquad \text{is bounded below,} \\ -\infty & \text{otherwise.} \end{cases}$$

If $\mathsf{top}(A)$ is a gap, we call it the *gap defined by* A. Suppose $G = (L, R)$ is a gap in $\mathbb{O}$. If $a < L$, we write $a < G$, and we define $(a, G] := \{x \in \mathbb{O} : a < x < G\}$, likewise for $a < G$ and $[G, a]$. When it is clear that G is a gap, we write $G \in (a, b)$ to means $a < G < b$, and so on.

Note that we can also define a function based on splitting $\mathbb{O}$ at gaps, e.g. into $f : (-\infty, G) \to \mathbb{O}, g : (G, \infty) \to \mathbb{O}$. However, we reserve the notation *piecewise* P for glueing partial functions on closed intervals (i.e. convex which have endpoints, not 'endgaps').

2.3 Fields

2.3.1 Ordered algebraic structures

As usual, algebraic structures are called *ordered*, if they have a total order that is compatible with the algebraic operations. Thus, $\mathbb{G} = (G, <, +, 0)$ is an *ordered group* if $(G, +, 0)$ is a group, $(G, <)$ is a total order and for all $x, y, z \in G$, if $x < y$, then $x + z < y + z$. Similarly, $\mathbb{K} := (K, <, +, \times, 0, 1)$ is an *ordered field* if

$(K, +, \times, 0, 1)$ is a field and $(K, <)$ is a total order such that for all $x, y, z \in K$, if $x < y$, then $x + z < y + z$ and if $x, y > 0$, then $x \times y > 0$. As usual, we also write $x.y$ or xy for $x \times y$. All ordered fields $\mathbb{K}$ contain a copy of the rational numbers, so without loss of generality, we assume that $\mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{K}$. A field $\mathbb{K}$ is *Archimedean* if for all $x, y \in \mathbb{K}^{>0}$, there is an $n \in \mathbb{N}$ such that $x < n.y$. An $x \in \mathbb{K}$ is *finite* if there is an $n \in \mathbb{N}$ such that $-n < x < n$; we denote the set of finite elements by FIN.

Let $\mathbb{G}$ be an ordered group. If $a, b, c \in \mathbb{G}$ are such that $a + b = c$, we write $c - b = a$. The *length* (or *diameter*) of an interval, $[a, b]$, $\mathsf{len}([a, b]) := b - a$, likewise for $(a, b), (a, b]$, and $[a, b)$. We denote $\mathbb{G}^{>0} := \{x \in \mathbb{K} : x > 0\}$ and $\mathbb{G}^{\geq 0} := \{x \in \mathbb{G} : x \geq 0\}$. We call the coinitiality of $\mathbb{G}^{>0}$ the *basenumber* of $\mathbb{G}$, and denote it $\mathsf{bn}(\mathbb{G})$ (as in [75, §1.3.4], [191, Theorem viii]).

If $\mathbb{G}$ is an ordered group, we say that x and y are *equivalent*, in symbols $x \equiv y$, if $|x| \leq n|y|$ and $|y| \leq m|x|$ for some $n, m \in \mathbb{N}$. We call $\Gamma(\mathbb{G}) := (\mathbb{G}\backslash\{0\})/\equiv$ the *value set* of $\mathbb{G}$ and write $O(x) := \{y \in \mathbb{G} : x \equiv y\} \in \Gamma(\mathbb{G})$. If $\Gamma(\mathbb{G})$ is a group, we call it the *value group* of $\mathbb{G}$.

Ordered fields have the interval topology generated by their order; this gives rise to a notion of convergence for sequences in the usual way: if λ is a limit ordinal, a sequence $\mathbb{x} \in \mathbb{K}^{\lambda}$ *tends to* ℓ if, for all $\varepsilon > 0$ (in $\mathbb{K}$, not just in $\mathbb{R}$!), there is an $\beta < \lambda$ such that for all $\gamma > \beta$, $|x_\gamma - \ell| < \varepsilon$. In which case, we write $\lim_{\beta \to \lambda} x_\beta = \ell$. We likewise define convergence on $\mathsf{bn}(\mathbb{K})$-metric spaces, (X, d), where d takes values in $\mathbb{K}^{\geq 0}$. A function $f : \mathbb{K} \to \mathbb{K}$ is *sequentially continuous* if whenever $\mathbb{x}$ tends to ℓ and $f(\mathbb{x})$ is the pointwise image of the sequence $\mathbb{x}$ under f, then $f(\mathbb{x})$ tends to $f(\ell)$. Clearly, all continuous functions are sequentially continuous (e.g. [180, Theorem 4.8]).

If $\mathbb{K}$ is an ordered field where $\mathsf{bn}(\mathbb{K}) = \kappa$, κ-spherical completeness is equivalent to κ-intersections of nested sequences of *open* intervals being empty [25, Lemma 2.4].

If $\mathbb{K}$ is an ordered field, $a, b \in \mathbb{K}$, $\mathbb{x} = (x_\beta)_{\beta \in \alpha}, \mathbb{y} = (y_\beta)_{\beta \in \alpha} \in \mathbb{K}^\alpha$ are sequences, let the pointwise sum and scalar multiplication be defined as usual: $a.\mathbb{x} + \mathbb{y} + b = (a.x_\beta + y_\beta + b)_{\beta \in \alpha}$.

We write $\mathbb{K}[X]$ for the set of polynomials with coefficients in $\mathbb{K}$ and write $p(x) = a.X^n$ etc. for $p \in \mathbb{K}[X]$. We say $\mathbb{K}$ is an rcf if it is a real closed field, i.e. there is no $n < \omega$ and no $a_0, ..., a_n \in \mathbb{K}$ such that $(a_0)^2 + ... + (a_n)^2 = -1$, for all $a \in \mathbb{K}$, there is a $b \in \mathbb{K}$ such that $a = b^2$ or $a = -b^2$, and every polynomial $p \in \mathbb{K}[X]$ with odd degree has a root [144, page 94].

For any $\mathbb{K}$, we define the ($\mathbb{K}$-)rational function as $\mathbb{K}(X) := \{\frac{p}{q} : p, q \in \mathbb{K}[X]$ and $q \neq 0\}$.

The following theorem is a strengthening of Hausdorff's Theorem 2.2.2.

Theorem 2.3.1 (Erdős, Gillman, Henriksen [57, Theorem 2.1])**.** Any two ordered rcfs of cardinality κ that are η_κ orders are ordered field isomorphic.

We use the following version of [25, Lemma 2.12(3)] (for fields which are not necessarily Cauchy complete) in the proof of Theorem 4.3.3.

Lemma 2.3.2. Let $\mathbb{K}$ be an ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$. Let C be a convex subset of $\mathbb{K}$.

1. If $\mathsf{coi}(\mathsf{top}(C)) < k$ and C has no supremum, then there is an $\varepsilon \in \mathbb{K}^{>0}$ such that for all $a \in C$ we have $a + \varepsilon \in C$.

2. If $\mathsf{cof}(\mathsf{bot}(C)) < k$ and C has no infimum, then there is an $\varepsilon \in \mathbb{K}^{>0}$ such that for all $a \in C$ we have $x - \varepsilon \in C$.

3. If $\mathsf{coi}(\mathsf{top}(C)) < \kappa$, $\mathsf{cof}(\mathsf{bot}(C)) < \kappa$, and C has no infimum and supremum, then there is an $\varepsilon \in \mathbb{K}^{>0}$ such that for all $a \in C$ we have $(a - \varepsilon, a + \varepsilon) \subset C$.

Proof. Note that Part 3. follows by Parts 1. and 2., and that Part 2. follows from Part 1. considering $\{-a : a \in C\}$. So, it suffices to prove Part 1.

It is enough to show that there is an $\varepsilon \in \mathbb{K}^{>0}$ such that for all $a \in \mathsf{top}(C)$ we have $a - \varepsilon \in \mathsf{top}(C)$. Indeed, in that case for all $b \in C$ and $a \in \mathsf{top}(C)$, we have that $b < a - \varepsilon$, and therefore $b + \varepsilon < a$ since $\mathbb{K}$ is an ordered field.

Assume that for all $\varepsilon \in \mathbb{K}^{>0}$, there is an $a \in \mathsf{top}(C)$ such that $a - \varepsilon \notin \mathsf{top}(C)$. Since $\mathsf{bn}(\mathbb{K}) = \kappa$ and $\mathsf{coi}(\mathsf{top}(C)) < \kappa$, there is an $a \in \mathsf{top}(C)$ such that, for all $\varepsilon \in \mathbb{K}^{>0}$, we have that $a - \varepsilon \notin \mathsf{top}(C)$. But then, for all $b \in \mathbb{K}$ such that $b < a$, we have that $b \notin \mathsf{top}(C)$. Therefore, $a = \mathsf{top}(C)$, so a is the supremum of C. But this contradicts our assumptions. □

2.3.2 Gaps in fields

Non-Archimedean ordered fields always have gaps: remember from Section 2.2.2 that $\mathsf{FIN} := \{x : x \text{ is finite}\}$ and let $\mathsf{INF} := \{x : x > 0\} \setminus \mathsf{FIN}$. Then $\mathsf{bot}(\mathsf{INF})$ is an (ω, λ)-gap for some λ (as $\mathsf{cof}(\mathsf{FIN}) = \omega$).

Proposition 2.3.3 (Folklore)**.** Let $\mathbb{G}$ be an ordered group. If $\mathbb{G}$ has a (λ, μ)-gap, then it has a (μ, λ)-gap.

Proof. Let (L, R) be a (λ, μ)-gap in $\mathbb{G}$. For any $X \subseteq \mathbb{G}$, let $-X := \{-x : x \in X\}$. Then $(-R, -L)$ is a (μ, λ)-gap. □

As usual, if $X \subseteq \mathbb{K}$ and $a, b \in \mathbb{K}$, we write $a.X + b := \{a.x + b : x \in X\}$ and $\frac{1}{X} := \{\frac{1}{x} : x \in X \text{ and } x \neq 0\}$. Using the field properties, we can also scale and translate gaps, to form an algebra of gaps (expanding the description in [63, §2]). If $G = (L, R)$ is a gap, we write $a.G + b := (a.L + b, a.R + b)$ and $\frac{1}{G} := (\frac{1}{L}, \frac{1}{R})$. Note that this operation may not change the gap at all, e.g. $n.\mathsf{bot}(\mathsf{INF}) + m = \mathsf{bot}(\mathsf{INF})$ for each $n, m \in \mathbb{N}$.

Proposition 2.3.4 (Folklore)**.** Let $\mathbb{K}$ be an ordered field.

1. If (L, R) is a (λ, μ)-gap, G, in $\mathbb{K}$, $d \in \mathbb{K}$, and if $L < c$ for some constant $c \in \mathbb{K} \setminus \{0\}$, then $c.G + d$ is a gap. If $c > 0$ then $c.G + d$ is a (λ, μ)-gap, otherwise $c.G + d$ is a (μ, λ)-gap.

2. If (L, R) is a (λ, μ)-gap, G, then $\frac{1}{G}$ is a (μ, λ)-gap.

3. If (L, R) is a (μ, λ)-gap in $\mathbb{K}$, $x, y \in \mathbb{K}$, and $x < y$, then there is a (μ, λ)-gap between x and y. So, the (μ, λ)-gaps are order dense in $\mathbb{K}$.

Proof. Parts 1. and 2. are clear (a similar result is in [44, Lemma 3.7]). Part 3. follows from Parts 1. and 2.: let G be a (λ,μ)-gap in $\mathbb{K}$. Let $G < r \in \mathbb{K}$, let $y - x < \frac{1}{r'}$ and let $m = \max(r, r')$. By scaling, there is a (λ,μ)-gap in $(0, \frac{1}{m})$. So, by shifting, there is a (λ,μ)-gap in $(x, x + \frac{1}{m}) \subseteq (x, y)$. □

Proposition 2.3.5 (Folklore)**.** Let $\mathbb{G}$ be an η_κ-ordered group where $\mathsf{wei}(\mathbb{G}) = \kappa$. If (L, R) is a (λ,μ)-gap, then at least one of μ, λ equals κ, and $\omega \leq \mu, \lambda \leq \kappa$. If, in addition, $\mathsf{bn}(\mathbb{G}) = \kappa$, then if A is a (μ, λ)-almost gap, then $\omega \leq \mu, \lambda \leq \kappa$.

Proof. First, suppose for contradiction that both $\mu, \lambda < \kappa$. Take a cofinal sequence $(a_\alpha)_{\alpha\in\mu}$ and a coinitial sequence $(b_\beta)_{\beta\in\lambda}$ with G respectively. As $\mathbb{G}$ is an η_κ order, there is an element $r \in \mathbb{G}$ such that $a_\alpha < r < b_\beta$ for all $\alpha \in \mu, \beta \in \lambda$, contradicting that $L \cup R = \mathbb{G}$. Next, we bound λ. As $\mathsf{wei}(\mathbb{G}) = \kappa$, there is some $Q \subseteq \mathbb{G}$ which is order dense, with $|Q| = \kappa$, so there is a Q-sequence which is coinitial in R, hence $\lambda = \mathsf{coi}(R) \leq \kappa$. If $\mathsf{coi}(R) < \omega$ then $\mathsf{coi}(R) = 1$, so R has a maximum, so L, R do not define a gap, a contradiction. The bounds for μ are analogous.

For almost gaps, the last case is that A is defined by $x \in \mathbb{G}$. As $\mathsf{bn}(\mathbb{G}) = \kappa$, A is a (κ, κ)-almost gap, as we can approximate x by a sequence in an order dense subset, Q, where $|Q| = \kappa$. □

A gap G is called *Cauchy* if there is a strictly increasing Cauchy sequence, $\varkappa$, such that $\mathsf{top}(\varkappa) = G$. An ordered field is *Cauchy complete* if every Cauchy sequence converges, i.e. there are no Cauchy gaps.

2.3.3 Integer parts and ordinal embeddings

We discuss various ways to identify analogues of the integers and the ordinals within ordered fields.

Let $\mathbb{K}$ be an ordered field and $Z \subseteq \mathbb{K}$ be a subring of $\mathbb{K}$. We call Z *a set of integer parts* of $\mathbb{K}$ if it is discrete and if for any $x \in A$, there is $z \in Z$ such that $z \leq x < z \oplus 1$. We write $Q(Z) := \{\frac{z}{z'} \in \mathbb{K} : z, z' \in Z, z' \neq 0\}$ for the *set of fractional parts of* $\mathbb{K}$ *associated to* Z.

If $\mathbb{K}$ is a field that has a set of integer parts, then by [125, Lemma 29b], all sets of integer parts are order isomorphic and hence the same cardinality; thus, we can define $\mathsf{ip}(\mathbb{K})$ to be the cardinality of any of these sets of integer parts.

Proposition 2.3.6 (Folklore)**.** Let $\mathbb{K}$ be an ordered field. If Z is a set of integer parts, then $\mathsf{owei}(\mathbb{K}) = \mathsf{ip}(\mathbb{K})$.

Proof. First, we prove that $\mathsf{ip}(\mathbb{K}) \geq \mathsf{owei}(\mathbb{K})$. Note that $Q(Z)$ is dense in $\mathbb{K}$, (e.g. [145, page 499]). As $|Z| = |Z|^2 = |Q(Z)|$, we have that $|Z| \geq \mathsf{owei}(\mathbb{K})$.

Then we prove that $\mathsf{ip}(\mathbb{K}) \leq \mathsf{owei}(\mathbb{K})$. Let Q be order dense in $\mathbb{K}$. Then, for any $z \in Z$, there is a $q \in Q$ such that $q \in (z, z + 1)$. So $|Z| \leq |Q|$, so the result follows. □

Any rcf has a set of integer parts [162]; we shall see very clear examples in Section 2.3.5.

Ordinal embeddings

Let $\mathbb{K}$ be an ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$. We equip κ with the commutative, associative, and cancelative Hessenberg operations (cite[page 5]galeotti2019theory, [189, page 366], written as $\alpha \oplus \beta$ and $\alpha \otimes \beta$. A map $e : \kappa \to \mathbb{K}$ is called an *ordinal embedding* if it is a monomorphism of $(\kappa, \oplus, \otimes)$ into $(\mathbb{K}, +, \times)$ and $\mathsf{Ran}(e)$ is cofinal in $\mathbb{K}$.[4] We sometimes call the elements of $\mathsf{Ran}(e)$ the *almost ordinals* of $\mathbb{K}$. Note that $\omega = \mathbb{N} \subseteq \mathbb{K}$ and that $e \restriction \omega = \mathrm{id}$.

Corollary 2.3.7 (Folklore)**.** If $\mathbb{K}$ is an rcf, then there is an ordinal embedding $e : \mathsf{bn}(\mathbb{K}) \to \mathbb{K}$.

Proof. This is a consequence of results discussed in Section 2.3.5: rcfs are ordered-field isomorphic to initial subfields of the surreal numbers No (by [51, page 1232]), and those have ordinal embeddings by Theorem 2.3.10. □

2.3.4 Exponential fields

The following definitions are taken from [54, §6] and [126, page 22]. A function $\ln : \mathbb{K}^{>0} \to \mathbb{K}$ is called a *logarithm* if it is a group monomorphism from $(\mathbb{K}^{>0}, \times, 1)$ to $(\mathbb{K}, +, 0)$ satisfying $\ln(y) \leq y - 1$ for every $y \in \mathbb{K}^{>0}$. If ln is a logarithm, its functional inverse $\exp := \ln^{-1} : \mathsf{Ran}(\ln) \to \mathbb{K}$ is called an *exponential*.

An ordered field is called *logarithmic* if it has a logarithm; it is called *exponential* if it has a surjective logarithm (equivalently, if it has an exponential with domain $\mathbb{K}$). On an exponential field with an exponential exp, we write $x^y := \exp(y. \ln(x))$.

Proposition 2.3.8. Let $\mathbb{K}$ be an exponential rcf such that for all $x > 1$ and $y > 0$, $x^y > 1$, and $\mathsf{bn}(\mathbb{K}) = \kappa$. Then for any set of integer parts, Z, on $\mathbb{K}$, there is an $m \in \mathbb{K}$ such that $|\{n \in Z^{\geq 0} : n < m\}| = \kappa$.

Proof. If $\mathbb{K}$ is exponential and $x > 1$ and $y > 0$, $x^y > 1$. Then note that if $\alpha < \beta < \kappa$, then $\omega^{1 \oplus \frac{1}{\alpha}} \oplus 1 > \omega^{1 \oplus \frac{1}{\beta}} > \omega$. So, there are κ-many integer parts in $[0, \omega^2]$. □

2.3.5 Famous fields and where to find them

In this section, we provide the definitions and basic properties of some of the fields we shall use in this thesis, in particular, the field $\mathbb{R}_\kappa$. Our notation follows [36, 42, 75, 80].

Surreal numbers

The *surreal numbers*, introduced by Conway and denoted by No, are the class of binary sequences $\bigcup_{\alpha \in \mathbf{On}} \{0, 1\}^\alpha$, ordered lexicographically, so that 0 < 'undefined' < 1 , e.g. 0100 < 010 < 0101. A subfield of No is called *initial* if it is closed under initial segments. We let $\mathsf{No}_{<\kappa} = \bigcup_{\alpha \in \kappa} \{0, 1\}^\alpha$ and $\mathsf{No}_{\leq\kappa} = \bigcup_{\alpha \leq \kappa} \{0, 1\}^\alpha$, both ordered by the order inherited from No.

[4] A similar notion is α-*Archimedeanicity* [53, page 23].

Theorem 2.3.9 (Conway; e.g. [36, page 23])**.** Every non-empty convex set X of surreal numbers has a unique element of minimal length, called the *simplest element* of X.

The class No is a Cauchy-complete class which is satisfies the axioms of an rcf, and is η_κ for every κ [36, Theorem 25]. Moreover, every rcf is isomorphic to an initial subfield of No [51, page 1232], in particular, the reals $\mathbb{R}$ are contained within $\mathsf{No}_{\leq\omega}$.

The ordinals embed into No by $\alpha \mapsto \bar{1}_\alpha$; and No has a canonical set of integer parts: let Oz be the class defined in [36, Definition page 45].

Theorem 2.3.10 (Ehrlich, [51, Theorem 20])**.** If $\mathbb{K}$ is an initial subfield of No, then $\mathbb{K} \cap \mathsf{Oz}$ is a set of integer parts for $\mathbb{K}$.

We can equip $\mathbb{K} \cap \mathsf{Oz}$ with the Hessenberg operations, which yields an ordinal embedding from $\mathsf{bn}(\mathbb{K})$ into $\mathbb{K}$.

We call $\mathsf{No}_{<\kappa}$ the *κ-rationals* and denote it by $\mathbb{Q}_\kappa$ (see [74] and [80, page 29]). The set $\mathbb{Q}_\kappa$ is a field which inherits much of the structure of No, e.g. $\mathbb{Q}_\kappa$ is an rcf [2].

Proposition 2.3.11 (Folklore, e.g. [74, Propositions 3.4.3 & 3.4.4])**.** If $\kappa > \omega$ is $\kappa^{<\kappa} = \kappa$, then $|\,\mathbb{Q}_\kappa\,| = \mathsf{bn}(\mathbb{Q}_\kappa) = \mathsf{wei}(\mathbb{Q}_\kappa) = \kappa$ and $\mathbb{Q}_\kappa$ is an η_κ-order.

Galeotti's field $\mathbb{R}_\kappa$

We call a gap *Veronese* if and for each for all $\varepsilon \in \mathbb{K}^{>0}$, there are $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$ [75, page 37].[5] We define the *Veronese completion* of $\mathbb{K}$, denoted by $\mathsf{VC}(\mathbb{K}) := \mathbb{K} \cup \{(L, R) : (L, R) \text{ is a Venonese gap in } \mathbb{K}\}$. Ordered in the obvious way, $\mathsf{VC}(\mathbb{K})$ is an ordered field [74, Theorem 2.2.7].

Theorem 2.3.12 (Folklore, e.g. [42, Proposition 3.5])**.** A totally ordered field is Veronese complete if and only if it is Cauchy complete.

Galeotti's field $\mathbb{R}_\kappa$ is defined by $\mathbb{R}_\kappa := \mathsf{VC}(\mathbb{Q}_\kappa)$.[6] We have that $\mathbb{R}_\kappa \subsetneq \mathsf{No}_{\leq\kappa}$ and that it is an initial subfield of No. In general, $\mathbb{R}_\kappa$ is only well-behaved under the assumption of $\kappa^{<\kappa} = \kappa$, so this will be our general assumption.

Theorem 2.3.13 (Galeotti [25, Theorem 3.6], [73, Theorem 4])**.** Let $\kappa > \omega$ and suppose $\kappa^{<\kappa} = \kappa$. Up to ordered field isomorphism, $\mathbb{R}_\kappa$ is the unique ordered field such that $\mathsf{bn}(\mathbb{R}_\kappa) = \kappa$, $\mathsf{wei}(\mathbb{R}_\kappa) = \kappa$, $\mathbb{R}_\kappa$ is an η_κ-order, $\mathbb{R}_\kappa$ is an rcf, and $\mathbb{R}_\kappa$ is Cauchy complete.

Under our general assumption $\kappa^{<\kappa} = \kappa$, consider $\mathsf{Oz}_\kappa := \mathsf{Oz} \cap \mathbb{R}_\kappa$. Since $\mathbb{R}_\kappa$ is an initial subfield of No, by Theorem 2.3.10, this is a set of integer parts. By Proposition 2.3.6, we have the following:

Corollary 2.3.14. If $\kappa^{<\kappa} = \kappa$, then $\mathsf{ip}(\mathbb{R}_\kappa) = \kappa$.

[5] Also called *Scott* cuts [188, Definition 2.3].

[6] For more detailed constructions of $\mathbb{R}_\kappa$, see [25] or [75].

In fact, it can be shown that $Q(\mathsf{Oz}_\kappa) \subseteq \mathbb{Q}_\kappa$: a regular κ is is closed under ordinal multiplication, so, by [48, Proposition 4.7(iii)] and [51, Theorem 20], the Conway normal forms of Oz_κ are a strict subset of the Conway normal forms of $\mathbb{Q}_\kappa$. Hence, $\mathsf{Oz}_\kappa \subseteq \mathbb{Q}_\kappa$, and as $\mathbb{Q}_\kappa$ is a field, $Q(\mathsf{Oz}_\kappa) \subseteq \mathbb{Q}_\kappa$. However, we do not know whether $Q(\mathsf{Oz}_\kappa) = \mathbb{Q}_\kappa$.

The field $\mathbb{R}_\kappa$ is exponential

For $\mathbb{R}_\kappa$, if λ is a cardinal (or indeed an ε-number), then $\mathbb{Q}_\lambda$ is exponential and is an elementary extension of $\mathbb{R}$ with real exponentiation ([48, page 174 & Corollary 5.5]; the same holds for No). Hence, Proposition 2.3.8 holds for $\mathbb{Q}_\lambda$. As exp is continuous, if $\mathbb{K}$ is exponential, then $\overline{\mathbb{K}}^C$ is exponential (by the standard argument, as in [187, page 107]). So, $\mathbb{R}_\kappa$ is exponential.

Indeed, by the algebra of limits on $\mathbb{Q}_\kappa$, $\mathbb{R}_\kappa$ is such that for all $x > 1$ and $y > 0$, $x^y > 1$. Hence, by Proposition 2.3.8, there are integer parts of $\mathbb{R}_\kappa$ which have κ-many positive integer parts below them.

Gaps in the field $\mathbb{R}_\kappa$

By Proposition 2.3.5, every $\mathbb{R}_\kappa$ gap is a (κ, λ)- or (κ, λ)-gap for some $\omega \leq \lambda \leq \kappa$, we can show that $\mathbb{R}_\kappa$ has gaps of all of these types, hence is not Dedekind complete. If one of $\mu, \lambda < \kappa$ then these gaps are easily defined:

Example 2.3.15. Let $\omega \leq \lambda < \kappa$ be a cardinal. Define $L_\lambda = \{x \in \mathbb{R}_\kappa : x$ there is an ordinal $\alpha < \lambda$ such that $x < \alpha\}$. Then $L_\lambda, \mathbb{R}_\kappa \setminus L_\lambda$ defines a (κ, λ)-gap, denoted $\neg\lambda$, and $-(\neg\lambda)$ defines a (κ, λ)-gap.

So, $\neg\omega = \mathsf{bot}(\mathsf{INF})$.[7] These $\neg\lambda$ witness the (λ, κ)- and (κ, λ)-gaps for all $\lambda < \kappa$. Only (κ, κ)-gaps remain. To this end, we next generalise Dales & Woodin's version of Sierpiński's construction from ω_1 to κ [42], [189, Remark page 463-464]. This gives a specific construction which shows that $\mathbb{R}_\kappa$ has a (κ, κ)-gap. For this, we recall the notion of a Hahn field.

Hahn fields

To show that $\mathbb{R}_\kappa$ has a (κ, κ)-gap, we use a slightly different construction, based on Dales and Woodin's [42], using the notion of a Hahn field. We return to these in Section 5.3.3.

Definition 2.3.16 (following e.g. [72, Chapter VIII §5])**.** Let $\mathbb{O}$ be any ordered set and κ be a cardinal with $\kappa^{<\kappa} = \kappa$. We define

1. $\mathbb{H}(\mathbb{R}, \mathbb{O}) := \{f \in \mathbb{R}^{\mathbb{O}} : \mathsf{supp}(f)$ is well-ordered$\}$, and

2. $\mathbb{H}_\kappa(\mathbb{R}, \mathbb{O}) := \{f \in \mathbb{R}^{\mathbb{O}} : \mathsf{supp}(f)$ is well-ordered and $|\mathsf{supp}(f)| < \kappa\}$.

[7] Also called '∞' [36, page 37]. However, we use ∞ to mean $> a$ for all $a \in \mathbb{K}$.

For $x \in \mathbb{O}$, we define $(f+g)(x) := f(x) + g(x)$ and $f \leq g$ if $f(x) \leq g(x)$ where $x = \min(\mathsf{supp}(f) \cup \mathsf{supp}(g))$. Then $(\mathbb{H}(\mathbb{R}, \mathbb{O}), +, \leq)$ is an ordered group, called the *Hahn group of* $\mathbb{O}$; similarly, we call $(\mathbb{H}_\kappa(\mathbb{R}, \mathbb{O}), +, \leq)$ the κ*-Hahn group of* $\mathbb{O}$. If $\mathbb{O}$ is a divisible ordered group, we can define a multiplication on $\mathbb{H}(\mathbb{R}, \mathbb{O})$ and $\mathbb{H}_\kappa(\mathbb{R}, \mathbb{O})$ by

$$(f \times g)(x) := \sum_{y+z=x} f(y).g(z)$$

(see [42, page 41] for the argument that this is well-defined and has the desired properties) and $(\mathbb{H}(\mathbb{R}, \mathbb{O}), +, \times, \leq)$ becomes an ordered field, called the *Hahn field of* $\mathbb{O}$ [87]; again similarly, we call $(\mathbb{H}_\kappa(\mathbb{R}, \mathbb{O}), +, \times, \leq)$ the κ*-Hahn field of* $\mathbb{O}$.

We think of these as *formal power series*: if $I \subseteq \mathbb{O}$ and $r : I \to \mathbb{R}$, we think of r as the formal power series $\mathsf{S}_{i \in I} r(i) X^i$. If $f \in \mathbb{H}(\mathbb{R}, \mathbb{O})$, then f represents the formal power series $\mathsf{S}_{i \in \mathsf{supp}(f)} f(i) X^i$.

Every ordered field is isomorphic to a subfield of a Hahn field [35, §4], [72, page 60]. The order used in the construction of a κ-Hahn field can be recovered via the value set.

Proposition 2.3.17. If $\kappa > \omega$, $\kappa^{<\kappa} = \kappa$, and $\mathbb{O}$ is an ordered set, then $\mathbb{O} = \Gamma(\mathbb{H}_\kappa(\mathbb{R}, \mathbb{O}))$.

Proof. A straightforward generalisation of [42, Definition 1.26]. □

We follow the construction in [42, Chapters 1 & 2]: let $S_\kappa := \{0, 1\}^\kappa$ and $Q_\kappa := \{\mathbb{x} \in S_\kappa : \{\alpha < \kappa : x_\alpha = 1\}$ has at least 2 elements and has a maximum$\}$. Ordered lexicographically, S_κ and Q_κ are ordered sets. Then define $G_\kappa := \mathbb{H}_\kappa(\mathbb{R}, Q_\kappa)$ to be the κ-Hahn group of Q_κ and $R_\kappa := \mathbb{H}_\kappa(\mathbb{R}, G_\kappa)$ to be the κ-Hahn field of G_κ.

Proposition 2.3.18 (Folklore)**.** Let $\kappa > \omega$, and suppose $\kappa^{<\kappa} = \kappa$. Then R_κ is an η_κ rcf of size κ.

Proof. By [42, Theorem 2.15(iv)], R_κ is an rcf. Observe that $|Q_\kappa| = \kappa$, so $G_\kappa = \bigcup_{\alpha \in \kappa} \{f \in \mathbb{R}^{Q_\kappa} : |\mathsf{supp}(f)| = \alpha\}$, hence $|G_\kappa| = \kappa^{<\kappa}$. As $\kappa^{<\kappa} = \kappa$, $|G_\kappa| = \kappa$. By exactly the same reasoning $|R_\kappa| = \kappa$.

We show that Q_κ is η_κ (building on [42, Proposition 1.43]): if $L < R$ and $|L \cup R| < \kappa$, then there is some $\alpha < \kappa$ such that for every $(x_\gamma)_{\gamma \in \kappa} \in L \cup R$, for all $\beta > \alpha$ we have $x_\beta = 0$. Fix the least such α. Now we define an x such that $L < x < R$: let $x_\alpha = 1$. For $\beta \in \kappa \backslash \{\alpha\}$, let $x_\beta = 1$ if there is $\ell \in L$ such that $\ell_\beta = 1$ and the first point of disagreement between ℓ and ℓ' is no earlier than β for each smaller $\ell' \in L$, i.e. $\ell' \leq_{Q_\kappa} \ell$. Otherwise, let $x_\beta = 0$. Then x will do [42, Proposition 1.9(iii)]. As Q_κ is η_κ, so G_κ is η_κ: let $L, R \subseteq G_\kappa$ are such that $L < R$ and $|L \cup R| < \kappa$. As Q_κ is η_κ, there is an $x \in Q_\kappa$ such that $\{O(\ell) \in Q_\kappa : \ell \in L\} < x < \{O(r) \in Q_\kappa : r \in R\}$. So, there is some $y \in R_\kappa$ such that $O(y) = x$. Then $L < y < R$. The same argument shows that R_κ is η_κ. □

Corollary 2.3.19. Let $\kappa > \omega$ and suppose $\kappa^{<\kappa} = \kappa$. Then the fields R_κ and $\mathbb{Q}_\kappa$ are isomorphic.

Proof. Follows directly from Proposition 2.3.11 & 2.3.18 and Theorem 2.3.1. □

A (κ,κ)-gap in $\mathbb{R}_\kappa$

We are working under the assumption that $\kappa > \omega$ and $\kappa^{<\kappa} = \kappa$.

Proposition 2.3.20 (Following [42, Proposition 1.11(ii)])**.** There is a (κ,κ)-gap in Q_κ.

Proof. For each $0 < \alpha < \kappa$, let $u_\alpha = 0^\frown \overline{1}_{\alpha+1}{}^\frown \overline{0}_\kappa$, $d_\alpha = 1^\frown \overline{0}_\alpha{}^\frown 1^\frown \overline{0}_\kappa$, so $u_\alpha, d_\alpha \in Q_\kappa$. Then $u_\alpha < d_\beta$ for all $\alpha, \beta \in \kappa$. There is no $x \in Q_\kappa$ such that $u_\alpha < x < d_\beta$ for all $\alpha, \beta \in \kappa$, as the only elements of S_k with this property are $0^\frown \overline{1}_\kappa \notin Q_k$ and $1^\frown \overline{0}_\kappa \notin Q_k$. So, $(u_\alpha)_{\alpha\in\kappa} < (d_\beta)_{\beta\in\kappa}$ defines a (κ,κ)-gap in Q_κ. □

Proposition 2.3.21. If $\mathbb{G}$ is an ordered group with value set $\Gamma(\mathbb{G})$ and $\Gamma(\mathbb{G})$ has a (μ,λ)-gap, then $\mathbb{G}$ has a (μ,λ)-gap.

Proof. Exactly as in [42, Proposition 1.24]. □

Corollary 2.3.22. Both G_κ and R_κ have (κ,κ)-gaps.

Proof. By Proposition 2.3.20, Q_κ has a (κ,κ)-gap. By Proposition 2.3.17 and the definitions of G_κ and R_κ, we have that $Q_\kappa = \Gamma(G_\kappa)$ and $G_\kappa = \Gamma(R_\kappa)$. Then apply Proposition 2.3.21 twice. □

By Corollary 2.3.19, this gives us a (κ,κ)-gap in $\mathbb{Q}_\kappa$; we now just need to show that it is preserved in $\mathbb{R}_\kappa$.

Proposition 2.3.23 (Folklore)**.** If $\mathbb{K}$ is a non-Archimedean ordered field, then $\Gamma(\mathsf{VC}(\mathbb{K})) = \Gamma(\mathbb{K})$.

Proof. By e.g. [42, Definition 2.8], if $O(a) \neq O(b)$ then $O(a+b) = \min(O(a), O(b))$ for any $a, b \in \mathbb{K}$. Let $x \in \mathsf{VC}(\mathbb{K}) \setminus \mathbb{K}$. So, $x = \lim_{\alpha\to\kappa} x_\alpha$ for some Cauchy sequence $(x_\alpha)_{\alpha\in\kappa}$, and for all $\varepsilon > 0$ there is some α such that for all $\beta > \alpha, |x - x_\beta| < \varepsilon$. Hence also, there is some γ such that for all $\delta > \gamma$, $O(x_\delta - x) > O(x)$. So, by the triangle inequality, for all $\zeta > \max\{\alpha,\gamma\}$, $O(x_\zeta) = O(x + (x_\zeta - x)) = O(x)$. Hence, $\Gamma(\mathsf{VC}(\mathbb{K})) \subseteq \Gamma(\mathbb{K})$. The converse inclusion is trivial. □

Now Proposition 2.3.23 shows that $\Gamma(\mathbb{R}_\kappa) = \Gamma(\mathbb{Q}_\kappa) = \Gamma(R_\kappa) = G_\kappa$. Corollary 2.3.19 shows that G_κ has a (κ,κ)-gap; thus, by Proposition 2.3.21, $\mathbb{R}_\kappa$ has a (κ,κ)-gap.

2.3.6 Field embeddings

If $\mathbb{K}$ and $\mathbb{L}$ are ordered fields, we write $\mathbb{L} \hookrightarrow \mathbb{K}$ if there is an ordered-field embedding from $\mathbb{L}$ to $\mathbb{K}$ (see [178, Chapter 8] for some background). Our focus is when $\mathbb{Q} \hookrightarrow \mathbb{K}$, $\mathbb{R} \hookrightarrow \mathbb{K}$, and $Q \hookrightarrow \mathbb{K}$, where Q is a subfield of rational parts of $\mathbb{K}$ (as in Section 2.3.3). If $\mathbb{K}$ is an ordered field, then $\mathbb{Q} \hookrightarrow \mathbb{K}$ (e.g. [96, Exercise 2.5.1]).

Proposition 2.3.24. If $\mathbb{K}$ is an η_1 rcf, then there is an order embedding from $\mathbb{R}$ into $\mathbb{K}$.

Proof. As $\mathbb{K}$ is an rcf, $\mathbb{Q} \hookrightarrow \mathbb{K}$. Let $e : \mathbb{Q} \to \mathbb{K}$ be a witnessing field embedding. So, if $x \in \mathbb{R} \setminus \mathbb{Q}$, there are countable $L, R \subseteq \mathbb{Q}$ such that $\mathbb{Q}$, $L < \{x\} < R$ and if $y \in \mathbb{R}$ and $L < \{y\} < R$, then $x = y$. Then as $\mathbb{K}$ is η_1, $e(L) < z_x < e(R)$ for some $z \in \mathbb{K}$. For each $x \in \mathbb{R} \setminus \mathbb{Q}$, choose some z_x. Then $e \cup \{(x, z_x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ is an order embedding from $\mathbb{R}$ into $\mathbb{K}$. □

Only certain order embeddings are ordered-field embeddngs. If $\mathbb{K}$ is $\eta_{2^{\aleph_0}}$, we can strengthen Proposition 2.3.24 to an ordered-field embedding. This is already known for $\mathbb{R}_\kappa$:

Proposition 2.3.25 (Folklore)**.** Let $\kappa > \omega$ and suppose $\kappa^{<\kappa} = \kappa$. Then $\mathbb{R} \hookrightarrow \mathbb{R}_\kappa$ and $\mathbb{R} \hookrightarrow \mathsf{No}$.

Proof. That $\mathbb{R} \hookrightarrow \mathsf{No}$ is exactly [80, Theorem 4.3]. For $\mathbb{R}_\kappa$, we note $\mathbb{R} \hookrightarrow \mathsf{No}_{\leq\omega} \subseteq \mathbb{R}_\kappa$. □

We can prove the general fact model-theoretically (see [144] for an introduction). If M, N are structures in the same language, $\mathcal{L}$, and there is an embedding $j : M \to N$ such that for every $\mathcal{L}$-formula, $\varphi(x)$, and every $\mathbb{x} \in M^{<\omega}$, $M \vDash \varphi(\mathbb{x}) \iff N \vDash \varphi(j(\mathbb{x}))$, then we say that j is an *elementary embedding* [144, Definition 2.3.1]. A model $M \vDash T$ is called *κ-universal* if for every $N \vDash T$ such that $|N| < \kappa$, there is an elementary embedding from N to M [144, Definition 4.3.16].

If $\lambda \geq 2^{\aleph_0}$, then an rcf, $\mathbb{K}$, is η_λ if and only if it is λ-saturated in the sense of model theory [26, Proposition 5.4.2]. This λ-saturation implies λ^+-universality [144, Lemma 4.3.17]. As $|\mathbb{R}| = 2^{\aleph_0}$, if $\lambda \geq 2^{\aleph_0}$, then there is an elementary embedding from $\mathbb{R}$ to $\mathbb{K}$. But an elementary embedding is automatically an ordered field embedding. We summarise this like so:

Remark 2.3.26. If $\mathbb{K}$ is an rcf and η_λ for some $\lambda \geq 2^{\aleph_0}$, then there is an elementary embedding in the language of ordered fields from $\mathbb{R}$ into $\mathbb{K}$, hence $\mathbb{R} \hookrightarrow \mathbb{K}$.

In models where $\aleph_1 < 2^{\aleph_0}$, we do not know whether η_1-ness suffices.

More generally, we can ordered-field embed a subfield of rational parts into any rcf, $\mathbb{K}$: let $\mathsf{bn}(\mathbb{K})$ have the Hessenberg operations. By Proposition 2.3.7, embed $\mathsf{bn}(\mathbb{K})$ into $\mathbb{K}$. Hence, $Q(\mathsf{bn}(\mathbb{K})) \hookrightarrow \mathbb{K}$. This $Q(\mathsf{bn}(\mathbb{K}))$ is (ordered-field isomorphic to) Asperó and Tsaprounis' $\mathsf{bn}(\mathbb{K})$-$\mathbb{Q}$ [8, page 10]. Subfields of rational parts sometimes play an analogous rôle to that of $\mathbb{Q}$ for the reals (see the cardinality arguments using $\mathbb{Q}_\kappa$ in Sections 6.2.3 and 6.3.4), but it is unclear how much analysis is common to fields with ordered-field isomorphic subfields of rational parts.

Chapter 3

Descriptive Choice Principles and How to Separate Them

This chapter is based on joint work with Lucas Wansner (Universität Hamburg); both contributors contributed equally to this work. All definitions, statements, and proofs of this chapter were jointly written and will form part of a joint publication [206]; the material is similarly included in Wansner's doctoral dissertation [205, Chapter 3] which therefore has significant overlap with this chapter.

In this chapter, we study the interplay between very weak fragments of AC and descriptive set theory. We define *descriptive choice principles*, a family of subfragments that stratify $\mathsf{AC}_\omega(\omega^\omega)$ and prove separation results for them. Kanovei proved the first separation theorem for some of these principles [109]. In this chapter, we improve on his result.

In Section 3.1, we define our *descriptive choice principles* and discuss their basic properties; in Section 3.2, we discuss previous separation theorems and state our main result (the Separation Theorem 3.2.4) and its corollaries. Finally, the technical Section 3.3 provides the proof of the main result. The beginning of Section 3.3 will give a roadmap through the proof of the main theorem and its related concepts (page 38).

3.1 Descriptive choice principles

If Γ is any of the descriptive pointclasses defined in Section 2.1.4, we say that a countable sequence, $A = (A_n)_{n\in\omega}$, of sets of reals is *in* Γ if all of its members are; we say that it is *uniformly in* Γ if $A \in \Gamma$ and the set $\widehat{A} \in \Gamma$ (where $\widehat{A}$ was the set of reals encoding the entire sequence; see page 11). We write $\mathrm{unif}\Gamma$ for the collection of countable sequences that are uniformly in Γ. We remind the reader that without $\mathsf{AC}_\omega(\omega^\omega)$, descriptive pointclasses can fail to be closed under countable unions (see page 14). Uniformity is a remedy for this problem: if Γ is any boldface or lightface projective pointclass and $A = (A_n)_{n\in\omega}$ is uniformly in Γ, then $\bigcup_{n\in\omega} A_n \in \Gamma$.

In Section 2.1.3, we introduced the fragments $\mathsf{AC}_\omega(\omega^\omega;\Xi)$ meaning "every countable sequence of non-empty sets of reals in Ξ has a choice function" and $\mathsf{DC}(\omega^\omega;\Gamma)$ referring to the principle of dependent choices for relations in Γ.

Applying these definitions to our descriptive pointclasses, we obtain the following principles:

$$\mathsf{AC}_\omega(\omega^\omega;\Gamma),\ \mathsf{DC}(\omega^\omega;\Gamma),\ \mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Gamma),\text{ and }\mathsf{DC}(\omega^\omega;\mathrm{unif}\Gamma).$$

We call these fragments *descriptive choice principles* and the latter two *uniform*. Clearly, for any descriptive pointclass Γ, we have that $\mathsf{AC}_\omega(\omega^\omega;\Gamma)$ implies $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Gamma)$ and that $\mathsf{DC}(\omega^\omega;\Gamma)$ implies $\mathsf{DC}(\omega^\omega;\mathrm{unif}\Gamma)$.

Proposition 3.1.1 (ZF)**.** The principle $\mathsf{AC}_\omega(\omega^\omega;\mathbf{\Delta}^0_2)$ holds.

Proof. We show that we can canonically pick an element from any $\mathbf{\Delta}^0_2$ set: by Hausdorff's Difference Lemma, every $\mathbf{\Delta}^0_2$ set of reals, A, can be written as an α-difference of closed sets for some $\alpha < \omega_1$ [5, Theorem 7.16]. So, there is a sequence of closed sets of reals, $(C_\beta)_{\beta\in\alpha}$, such that

$$A = \mathrm{Diff}_{\beta<\alpha}C_\beta := \{x \in \bigcup_{\beta<\alpha} C_\beta :\ \text{the least } \beta < \alpha \text{ such that } x \notin C_\beta \text{ is odd}\}.$$

With extra care, such a sequence can be uniquely specified (for a detailed specification, see [5, 3.E.1, page 57]). Let $A \neq \emptyset$ be a $\mathbf{\Sigma}^0_2 \cap \mathbf{\Pi}^0_2$ set of reals, and let $(C_\beta)_{\beta\in\alpha}$ be the corresponding sequence. It suffices to define a canonical real in A from $(C_\beta)_{\beta\in\alpha}$. There is a minimal $\beta < \alpha$ which is even such that $C_\beta \supsetneq C_{\beta+1}$. Let $\{O_n : n \in \omega\}$ enumerate the basic open sets and let $n \in \omega$ be the minimal element such that $O_n \cap C_{\beta+1} = \emptyset$ and $O_n \cap C_\beta \neq \emptyset$. Since $O_n \cap C_\beta$ is closed, there is a unique pruned tree $T \subseteq \omega^{<\omega}$ such that $[T] = O_n \cap C_\beta$. The left-most branch of T yields a real in A. □

The following relationships between descriptive choice principles hold in ZF.

Theorem 3.1.2 (ZF; [109])**.** Let $n \geq 1$.

1. $\mathsf{DC}(\omega^\omega,\mathbf{\Pi}^1_1)$ holds,
2. $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Pi^1_n)$ is equivalent to $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\Sigma^1_{n+1})$,
3. $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\mathbf{\Pi}^1_n)$ is equivalent to $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\mathbf{\Sigma}^1_{n+1})$,
4. $\mathsf{DC}(\omega^\omega;\mathrm{unif}\Pi^1_n)$ is equivalent to $\mathsf{DC}(\omega^\omega;\mathrm{unif}\Sigma^1_{n+1})$,
5. $\mathsf{DC}(\omega^\omega;\mathrm{unif}\mathbf{\Pi}^1_n)$ is equivalent to $\mathsf{DC}(\omega^\omega;\mathrm{unif}\mathbf{\Sigma}^1_{n+1})$,
6. $\mathsf{DC}(\omega^\omega;\Pi^1_{n+1})$ implies $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$, and
7. $\mathsf{DC}(\omega^\omega;\mathbf{\Pi}^1_n)$ implies $\mathsf{AC}_\omega(\omega^\omega;\mathrm{unif}\mathbf{\Pi}^1_n)$.

Proof. Part 1. is [109, §3.4, Theorem 3] (essentially due to Luzin and Novikov, [136]). Parts 2. to 6. are [109, §2, Theorems 2.5 and 2.6] (a summary in English can be found in [95, Note 61]). The proof of Part 7. follows the pattern of the proof of $\mathsf{AC}_\omega(\omega^\omega)$ from $\mathsf{DC}(\omega^\omega)$: fix an n, and let $(A_k)_{k\in\omega}$ be in $\mathrm{unif}\mathbf{\Pi}^1_n$. Consider the relation $R := \{(x,y) : \exists n(x \in A_k \wedge y \in A_{k+1})\}$. It is obvious that R is $\mathbf{\Sigma}^1_{n+1}$, then use Part 5. □

In Section 2.1.4, we discussed that without $\mathsf{AC}_\omega(\omega^\omega)$, the Borel hierarchy can behave very badly and diverge from the codeable Borel hierarchy. The following result by Ikegami and Schlicht shows that this is not true anymore if we assume the weakest of the (boldface) projective choice principles.

Theorem 3.1.3 (ZF; Ikegami & Schlicht)**.** The following are equivalent:

1. $\mathcal{B}^*$ is closed under countable unions,
2. $\mathcal{B} = \mathbf{\Delta}^1_1 = \mathcal{B}^*$, and
3. $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_1)$.

Proof. Cf. [98, Remark 3.22]. □

Ikegami & Schlicht show that each of these statements implies that ω_1 is regular. This also answers a question of Moore [158, pages 181-182]: his axioms "$\mathcal{B} \subseteq \mathbf{\Sigma}^1_1$" [158, 3.6.5] and "$\mathcal{B} = \mathbf{\Delta}^1_1$" [158, 3.6.6] are equivalent, and so "$\mathcal{B} = \mathbf{\Delta}^1_1$" is not a result of ZF.[1]

The statement "countable unions of countable sets of reals are countable" is usually referred to as the *Countable Union Theorem* denoted by $\mathsf{UT}(\aleph_0)$ [95, Form 6]. Note that there are models of $\mathsf{ZF} + \mathsf{UT}(\aleph_0)$ where $\mathcal{B} = \mathscr{P}(\omega^\omega) \neq \mathcal{B}^*$;[2] Theorem 3.1.3 therefore implies that $\mathsf{UT}(\aleph_0)$ does not imply $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Delta}^1_1)$.

3.2 Separating descriptive choice principles

As descriptive pointclasses need not be closed under countable unions in ZF, we focus on the uniform descriptive choice principles, which avoid the problems of closure under countable unions. Based on what we have already seen, we observe that the uniform and non-uniform principles are not in general equivalent.

Proposition 3.2.1. There is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_1) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_1)$.

Proof. By Theorem 3.1.2, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_1)$ holds in ZF; by Theorem 3.1.3, $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_1)$ does not. □

We show that these are in fact distinct for all $n \geq 1$ (see Figure 3.1). We mention other known separation results.

Theorem 3.2.2 (Kanovei, [109])**.** For each $n \geq 1$, there is a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$.

Theorem 3.2.3 (Friedman, Gitman, & Kanovei, [67])**.** There is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega) + \neg\mathsf{DC}(\omega^\omega; \Pi^1_2)$. Therefore, for $n \geq 2$, $\mathsf{AC}_\omega(\omega^\omega; \Pi^1_n)$ does not imply $\mathsf{DC}(\omega^\omega; \Pi^1_n)$.

Our main result is a strengthening of Kanovei's Theorem 3.2.2.

[1] A revised version of Moore's "all analytic sets are Lebesgue measurable" [158, 3.6.7] can be proved in ZF, using Fremlin's notion of codeable Lebesgue measure [66, Theorem 563I].

[2] E.g. the model called Truss's model or $\mathcal{M}12(\aleph)$ in [95].

Theorem 3.2.4 (Separation Theorem)**.** For every $n \geq 1$, there is a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$.

We can represent this pictorially, with double arrows representing implications, and dashed lines representing the separation between layers due to Theorem 3.2.4. Clearly, if Γ, Γ' are descriptive classes such that $\Gamma' \supseteq \Gamma$, then $\mathsf{AC}_\omega(\omega^\omega; \Gamma')$ implies $\mathsf{AC}_\omega(\omega^\omega; \Gamma)$, so our inclusions of descriptive classes naturally give an implication diagramme for the axiom fragments as displayed in Figure 3.1. The diagramme forms a rectangular solid where the back side consists of the boldface principles, the front side of the lightface principles, the left side of the non-uniform principles, and the right side of the uniform principles.

We obtain the following immediate consequences of the Separation Theorem 3.2.4 (using Lemma 3.1.2).

Corollary 3.2.5. There is a model of $\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$.

Proof. Clear from Theorem 3.2.4 and compactness. □

Corollary 3.2.6. For every $n \geq 1$, there is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$.

Corollary 3.2.7. There is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$.

Corollary 3.2.6 separates the horizontal slices of the diagramme on the uniform side from each other. In the diagramme, this is shown by the dashed line between $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$, indicating that there is no implication from the lower horizontal slice to the higher slice.

Corollary 3.2.7 separates the boldface non-uniform part of the diagramme from the uniform part by showing that the strongest uniform principle does not imply the weakest non-uniform boldface principle. This is shown by the dashed line between $\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$ and $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Pi}^1_1)$ in Figure 3.1. So, in particular, $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^1_n)$ implies, but is not implied by, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Sigma}^1_n)$.

Furthermore, since $\mathbf{Ctbl} \subseteq \mathbf{\Sigma}^0_2$, our results separate all non-trivial non-uniform Borel descriptive choice principles from the uniform descriptive choice principles.[3]

Corollary 3.2.8. There is a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{Proj}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_2)$.

3.3 The proof of the Separation Theorem

Kanovei's original approach to separating descriptive choice principles in [109] involved inner models.

We shall define a notion of an n-slicing forcing (Section 3.3.1) and prove that if there is such an n-slicing forcing notion, the Separation Theorem (Theorem 3.2.4) holds (Section 3.3.2).

Thus, all that remains to be shown is the existence of n-slicing forcing notions. We call this the *Slicing Theorem* (Theorem 3.3.15).

[3] By Proposition 3.1.1, any weaker Borel descriptive choice principles are provable in ZF.

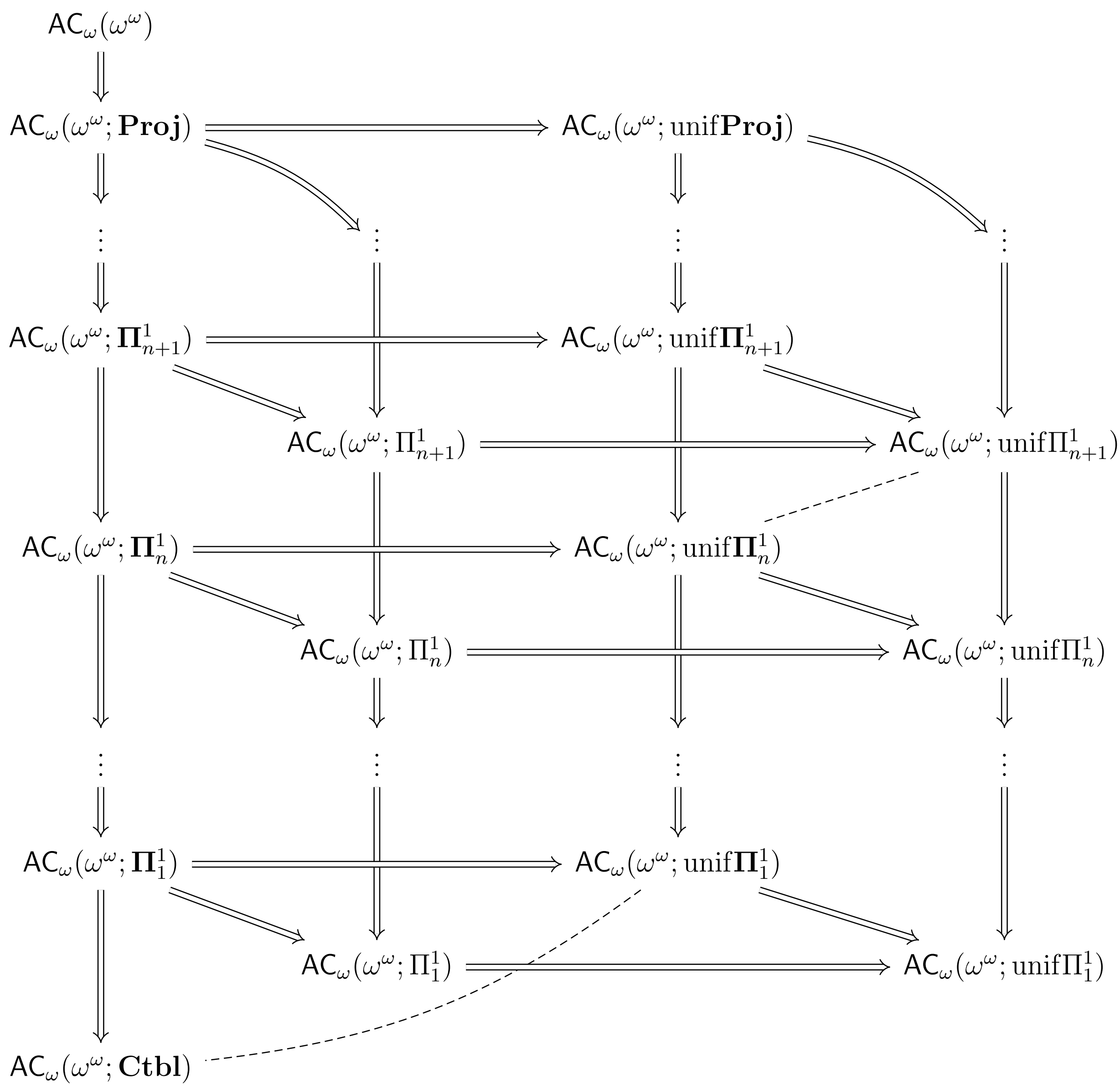

Figure 3.1: Implication diagramme of the fragments of $\mathsf{AC}_\omega(\omega^\omega)$

The proof of the Slicing Theorem is covered in Sections 3.3.3 & 3.3.4: first, we discuss the Kanovei-Lyubetsky Lemma (Theorem 3.3.43), a technical tool. We introduce *Jensen forcing*, a forcing notion that was used to create Π^1_2-singletons, extract a relevant property from it that we call *n-Jensen* (Definition 3.3.65) and then use the Kanovei-Lyubetsky Lemma to prove the existence of n-Jensen forcings (Theorem 3.3.66); finally, we construct an n-slicing forcing notion from an n-Jensen forcing notion (page 65).

3.3.1 Slicing forcing notions

In this section, we define the property of being *n-slicing* which is central for our proof.

As an illustration of the motivation for this, consider the standard construction of a model of $\mathsf{ZF} + \neg\mathsf{AC}_\omega(\omega^\omega, \mathbf{Ctbl})$ in a symmetric extension [103, Example 15.50]: we take an ordinary ω-product of a forcing notion and make sure that the symmetric extension does not have a choice function for some countable collection of countable sets. In this standard construction, the descriptive complexity of this failure of choice is not known.

In our construction, we preserve enough choice by ensuring that the forcing notion is n-slicing (Definition 3.3.13). Note that this notion deals with a product notion of length ω, the ω-slice-product (Definition 3.3.2), but one of the key technical results of our proof, the Kanovei-Lyubetsky Lemma (see Theorem 3.3.43) works for $(\omega_1 \times \omega)$-products (see Question 3.3.44); thus, we later have to modify the standard construction in the following way: we take an $(\omega_1 \times \omega)$-product, then ignore the later reals and only consider the countable $\omega \times \omega$ corner of the $\omega_1 \times \omega$ grid to produce the countable collection of countable sets.

This means that only the initial ω-many slices of the product matter for the model we construct. Hence, we define our filter such that the permutation of the subgroups in the filter can move any finite initial subsequence of ω_1, but fixes every entry $\geq\omega$. Formally: let $\xi \leq \omega_1$ be an ordinal and let $(\mathbb{P}_\nu)_{\nu\in\xi}$ be a sequence of forcing notions.

Definition 3.3.1. We define the *ω-slice-product of $(\mathbb{P}_\nu)_{\nu\in\xi}$ with finite support* as the set $\mathbb{P}$ of all partial functions p such that $\mathsf{Dom}(p) \subseteq \xi \times \omega$ is finite and for every $(\nu, k) \in \xi \times \omega$, $p(\nu, k) \in \mathbb{P}_\nu$ ordered by:

$$p \leq q \iff \forall(\nu, k) \in \mathsf{Dom}(q)(p(\nu, k) \leq q(\nu, k)).$$

We say $\mathbb{P}$ has *length* ξ, denoted $\mathsf{len}(\mathbb{P}) = \xi$.

Definition 3.3.2. If, for every $\nu < \omega_1$, we have that $\mathbb{Q} = \mathbb{P}_\nu$, then we say $\mathbb{P}$ is the *ω-slice-product of $\mathbb{Q}$ with finite support of length ξ*. Let $I \subseteq \xi \times \omega$. For $A \subseteq \mathbb{P}$, we define $A{\upharpoonright}I := \{p{\upharpoonright}I : p \in A\}$.

Then for every $\mathbb{P}$-generic filter G, $G{\upharpoonright}I$ is a $\mathbb{P}{\upharpoonright}I$-generic filter.

Next, let $\xi = \omega_1$. We define $\mathrm{Aut}(\omega_1 \times \omega, \omega)$ as the group of all bijections π of $\omega_1 \times \omega$ such that for every $\nu \in \omega_1$, $\pi(\{\nu\} \times \omega) = \{\nu\} \times \omega$.

Definition 3.3.3. Let $p \in \mathbb{P}$ and let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$. We define $\pi^*(p) := p'$ with $\mathsf{Dom}(p') = \pi(\mathsf{Dom}(p))$ and for $i \in \mathsf{Dom}(p')$, $p'(i) = p(\pi^{-1}(i))$. So, every automorphism $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ induces an automorphism π^* on $\mathbb{P}$. Let $\mathcal{G} := \{\pi^* : \pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)\}$ be the group of all such automorphisms on $\mathbb{P}$.

Definition 3.3.4. We say $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ is a *set of slices* if, for every $s \in \mathcal{S}$, s is of the form $X \times \omega$, where $X \subseteq \omega_1$.

Definition 3.3.5. Let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices and for every $s \in \mathcal{S}$, let $H_s := \{\pi^* : \forall(\nu, k) \in s(\pi(\nu, k) = (\nu, k))\} \subseteq \mathcal{G}$ be the subgroup of all π^* such that π point-wise fixes s. We define $\mathcal{F}_\mathcal{S}$ to be the filter on the subgroups of $\mathcal{G}$ generated by $\{H_s : s \in \mathcal{S}\}$.

Proposition 3.3.6. For any set of slices $\mathcal{S}$, $\mathcal{F}_\mathcal{S}$ is normal.

Proof. A simple algebraic check: let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ and $H \in \mathcal{F}_\mathcal{S}$. Then there is an $s \in \mathcal{S}$ such that $H_s \subseteq H$. It suffices to show that $H_s \subseteq \pi^* H (\pi^*)^{-1}$. Let $\tau \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ be such that $\tau^* \in H_s$. So, every $(\nu, k) \in s$ is fixed by $(\pi^*)^{-1} \circ \tau^* \circ \pi^*$. So, $(\pi^*)^{-1} \circ \tau^* \circ \pi^* \in H_s \subseteq H$. Hence,

$$\pi^* \circ \left((\pi^*)^{-1} \circ \tau^* \circ \pi^*\right) \circ (\pi^*)^{-1} = \tau^* \in \pi^* H (\pi^*)^{-1}.$$

□

Definition 3.3.7. Let G be a $\mathbb{P}$-generic filter and $\mathcal{S}$ be a set of slices. We write $V(G, \mathcal{S})$ for the symmetric submodel associated to $\mathcal{F}_\mathcal{S}$ and G.

Lemma 3.3.8. Let $(\mathbb{P}_\nu)_{\nu \in \omega_1}$ be a sequence of forcing notions, let $\mathbb{P}$ be the ω-slice-product of $(\mathbb{P}_\nu)_{\nu \in \omega_1}$ with finite support, let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices, and let G be a $\mathbb{P}$-generic filter. Let X be a set of ordinals. If $X \in V(G, \mathcal{S})$, then there is an $s \in \mathcal{S}$ such that $X \in V[G \restriction s]$.

Proof. Let σ be a symmetric $\mathbb{P}$-name for X, let $s \in \mathcal{S}$ be such that $H_s \subseteq \mathsf{sym}(\sigma)$, and let $p \in G$ be such that p forces that X is a set of ordinals. We define a $\mathbb{P}\restriction s$-name σ' such that $\sigma'_{G\restriction s} = \sigma_G = X$. Let $\sigma' := \{(\check{\xi}, q\restriction s) : q \in \mathbb{P}$, and $q \leq p$, and $q \Vdash \check{\xi} \in \sigma\}$.

First, we prove $\sigma_G \subseteq \sigma'_{G\restriction s}$. Let $\xi \in \sigma_G$. Then there is a $q \in G$ such that $(\check{\xi}, q) \in \sigma$. As G is a filter, there is an $r \in G$ which witnesses that p and q are compatible. Hence, $r \leq p$ and $r \Vdash \check{\xi} \in \sigma$. Then $(\check{\xi}, r\restriction s) \in \sigma'$, so $\xi \in \sigma'_{G\restriction s}$.

Next, we prove $\sigma'_{G\restriction s} \subseteq \sigma_G$. Let $\xi \in \sigma'_{G\restriction s}$. Then there is a $q \in G\restriction s$ such that $(\check{\xi}, q) \in \sigma'$. Hence, there is an $r \leq p$ such that $q = r\restriction s$ and $r \Vdash \check{\xi} \in \sigma$. We suppose for a contradiction, that $q \nVdash \check{\xi} \in \sigma$. Then there is a $q' \leq q$ such that $q' \Vdash \check{\xi} \notin \sigma$. Hence, q' and r are incompatible, but agree on s as functions. Let $\pi \in H_s$ be such that $\mathsf{Dom}(r) \cap \pi[\mathsf{Dom}(q')] \subseteq s$. Then $\pi^*(q')$ and r are compatible, as they agree on their common domain. By construction $\pi^* \in H_s \subseteq \mathsf{sym}(\sigma)$, so $\pi^*(q') \Vdash \pi^*(\check{\xi}) \notin \pi^*(\sigma)$ and hence $\pi^*(q') \Vdash \check{\xi} \notin \sigma$. But this is a contradiction since r and $\pi^*(q')$ are compatible. Hence, $q \Vdash \check{\xi} \in \sigma$ and so $\xi \in \sigma_G$. □

Lemma 3.3.9. Let $(\mathbb{P}_\nu)_{\nu \in \omega_1}$ be a sequence of forcing notions, let $\mathbb{P}$ be the ω-slice-product of $(\mathbb{P}_\nu)_{\nu \in \omega_1}$ with finite support, let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices, and let G be a $\mathbb{P}$-generic filter. Then:

1. for every $s \in \mathcal{S}$, $V[G{\restriction}s] \subseteq V(G,\mathcal{S})$ and
2. for every set of ordinals X, $X \in V(G,\mathcal{S})$ if and only if there is some $s \in \mathcal{S}$ such that $X \in V[G{\restriction}s]$.

Proof. For Part 1., let $X \in V[G{\restriction}s]$. Then there is a $\mathbb{P}{\restriction}s$-name $\dot{X}$ such that $\dot{X}_{G{\restriction}s} = X$. Since $\mathbb{P}{\restriction}s \subseteq \mathbb{P}$, $\dot{X}$ is also a $\mathbb{P}$-name and $\dot{X}_G = X$. Hence, it is enough to show that every $\mathbb{P}{\restriction}s$-name is hereditarily symmetric. Let σ be a $\mathbb{P}{\restriction}s$-name, let $\pi^* \in H_s$, and let $(\tau, p) \in \sigma$. By assumption, $p \in \mathbb{P}{\restriction}s$ and π^* fixes s, so $\pi^*(p) = p$. Hence, $\pi^*(\sigma) = \{(\pi^*(\tau), \pi^*(p)) : (\tau, p) \in \sigma\} = \{(\pi^*(\tau), p) : (\tau, p) \in \sigma\}$. By induction, σ is hereditarily symmetric.

The backwards direction of Part 2. follows immediately from Part 1., whilst the forwards direction is exactly Lemma 3.3.8. □

So, a real $x \in \omega^\omega$ is in $V(G,\mathcal{S})$ if and only if $x \in V[G{\restriction}s]$ for some $s \in \mathcal{S}$. Then note that a choice function for countable is a countable sequence of reals. So, a countable family, $\mathcal{F}$, of non-empty sets of reals has a choice function in $V(G,\mathcal{S})$ if and only if $\mathcal{F}$ has a choice function in $V[G{\restriction}s]$ for some $s \in \mathcal{S}$. We exploit this fact to show that some descriptive choice principles fail in $V(G,\mathcal{S})$. However, we also need to retain some descriptive choice in $V(G,\mathcal{S})$. For this, we need a different approach. Hence, we define the following property:

Definition 3.3.10. Let $\mathbb{P}$ be an ω-slice-product with finite support of a sequence $(\mathbb{P}_\nu)_{\nu\in\omega_1}$ of forcing notions and let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be a set of slices. We say $\mathbb{P}$ is *n-absolute for $\mathcal{S}$-slices* if, for every $s \in \mathcal{S}$, every Σ^1_n formula with real parameters $V[G{\restriction}s]$ is absolute between $V[G{\restriction}s]$ and $V[G]$.

Definition 3.3.11. A set of slices $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ is *unbounded* if, for every $s \in \mathcal{S}$, $\{\nu : \{\nu\} \times \omega \subseteq s\}$ is unbounded in ω_1. We say $\mathbb{P}$ is *n-absolute for slices* if, for every unbounded set of slices $\mathcal{S}$, $\mathbb{P}$ is n-absolute for $\mathcal{S}$-slices.

By Theorem 2.1.22, any ω-slice-product with finite support of length ω_1 is 2-absolute for slices. However, 3-absoluteness for slices is not automatic:

Example 3.3.12. Let $\mathbb{P}_0$ be random forcing (i.e. $\{A \in \mathcal{B}(\omega^\omega) :$ the Lebesgue measure of A is positive$\}$, ordered by inclusion, as in [103, page 243]). For all $\nu \in \omega_1 \setminus \{0\}$, let $\mathbb{P}_\nu = \mathbb{C}$ (i.e. Cohen forcing). Let $\mathbb{P}$ be the ω-slice-product of $(\mathbb{P}_\nu)_{\nu\in\omega_1}$ with finite support, and let G be $\mathbb{P}$-generic over L. Then L$[G]$ contains a random real over L (from the first coordinate, $\mathbb{P}_0$), but L$[G{\restriction}(\omega_1 \setminus \{0\}) \times \omega]$ does not (as the other coordinates are Cohen forcing, so do not add a random real). Finally, note that the statement "there is a random real over L" is Σ^1_3. So, by definition, $\mathbb{P}$ is not 3-absolute for slices.

Definition 3.3.13. A forcing notion $\mathbb{P}$ is *n-slicing* if there is a sequence of non-atomic forcing notions $(\mathbb{P}_\nu)_{\nu\in\omega_1}$ such that:

1. $\mathbb{P}$ is the ω-slice-product of $(\mathbb{P}_\nu)_{\nu\in\omega_1}$ with finite support,
2. for every $\nu < \omega_1$, every $\mathbb{P}_\nu$-generic filter G is uniquely determined by a real in $V[G] \setminus V$,

3. $\mathbb{P}$ is n-absolute for slices, and

4. for every $\mathbb{P}$-generic filter G, the set $\{(\ell, x_G^{(\ell,k)}) : (\ell, k) \in \omega^2\}$ is Π^1_n in $V[G]$, where $x_G^{(\ell,k)}$ is the generic real defined by $G{\restriction}\{(\ell, k)\}$.

Lemma 3.3.14 (Sandwiching Lemma)**.** Let $n \in \omega$, let $\mathbb{P}$ be n-slicing, let G be a $\mathbb{P}$-generic filter, and let $\mathcal{S} \subseteq \mathcal{P}(\omega_1 \times \omega)$ be an unbounded set of slices. For every $s \in \mathcal{S}$, every Σ^1_n-formula with real parameters in $V[G{\restriction}s]$ is absolute between $V[G{\restriction}s]$, $V[G]$, and $V(G, \mathcal{S})$. So, every Σ^1_n-formula with real parameters in $V(G, \mathcal{S})$ is absolute between $V[G]$ and $V(G, \mathcal{S})$.

Proof. The second part follows from the first and Lemma 3.3.9. The first part is proved by induction. For the base case, let $s \in \mathcal{S}$ and φ be a Σ^1_1 formula with parameters in $V[G{\restriction}s]$. By definition, there is an arithmetical ψ such that $\varphi = \exists x \psi(x)$. Note that for every real $x \in V[G{\restriction}s]$, $\psi(x)$ is absolute between $V[G{\restriction}s]$, $V[G]$, and $V(G, \mathcal{S})$. So, by upwards-absoluteness, if $V(G, \mathcal{S}) \vDash \varphi$, then $V[G] \vDash \varphi$. Likewise, by downwards-absoluteness, $V(G, \mathcal{S}) \vDash \neg\varphi$, then $V[G{\restriction}s] \vDash \neg\varphi$. Since $\mathbb{P}$ is n-absolute for slices, φ is absolute between $V[G{\restriction}s]$ and $V[G]$. So, φ is absolute between $V[G{\restriction}s]$, $V[G]$, and $V(G, \mathcal{S})$. The induction step is exactly similar. $\square$

The following theorem will be proved in Section 3.3.4.

Theorem 3.3.15 (Slicing Theorem)**.** Let $n \geq 2$. In L, there is an n-slicing forcing notion.

3.3.2 The Slicing Theorem implies the main theorem

In this section, we prove Theorem 3.2.4 under the assumption that the Slicing Theorem (Theorem 3.3.15) has been established. In particular, we construct our model from L which then, by Slicing Theorem (Theorem 3.3.15) has an n-slicing forcing notion for every $n \geq 2$.

We fix a natural number $n \geq 1$ and aim to construct a model of the following:

$$\mathsf{ZF} + \mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1}) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl}).$$

As $n + 1 \geq 2$, let $\mathbb{P}$ be $(n + 1)$-slicing, let G be a $\mathbb{P}$-generic filter over L, let $Z := \{(F \cup (\omega_1 \backslash \omega)) \times \omega : F \subseteq \mathbb{N} \text{ is finite}\}$, let $N := L(G, Z)$, for every $\ell \in \omega$, let $A_\ell := \{x_G^{(\ell,k)} : k \in \omega\}$ and let $A := \{A_\ell : \ell \in \omega\}$. Then, in L$[G]$, each A_ℓ is countable and A is in $\mathrm{unif}\Pi^1_{n+1}$. We show that, in N, A has no choice function, each A_ℓ is still countable, and A is still in $\mathrm{unif}\Pi^1_{n+1}$.

Lemma 3.3.16. In N, A is $\mathrm{unif}\Pi^1_{n+1}$, and for every $\ell \in \omega$, A_ℓ is countable.

Proof. For every $\ell \in \omega$, let $z := (\{\ell\} \cup (\omega_1 \backslash \omega)) \times \omega$. Then A_ℓ is countable in L$[G{\restriction}z]$ by the enumeration defined by $G{\restriction}z$, hence A_ℓ is countable in N by Lemma 3.3.9. Since $\mathbb{P}$ is $(n+1)$-slicing, $\widehat{A}$ is Π^1_{n+1} in L$[G]$. Let φ be a Π^1_{n+1} formula defining $\widehat{A}$. By Lemmata 3.3.9 and 3.3.14, N contains each $x_G^{(\ell,k)}$ and φ is absolute between L$[G]$ and N. Therefore, φ defines $\widehat{A}$ in N and so A is in $\mathrm{unif}\Pi^1_{n+1}$. $\square$

Proposition 3.3.17. In N, $\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\Pi^1_{n+1})$ and $\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$ fail.

Proof. By Lemma 3.3.16, it suffices to show that A has no choice function in N. Suppose there is a choice function $f : \omega \to \omega^\omega$ for A in N. Since f is a countable sequence of reals, we can code f as a real, and so as a subset of ω. By Lemma 3.3.9, there is some $z \in Z$ such that $f \in L[G{\restriction}z]$. Hence, there is a $\mathbb{P}{\restriction}z$-name $\dot{f}$ for f. As $z \in Z$, there is an $\ell \in \omega$ such that z does not meet any $\{\ell'\} \times \omega$ with $\ell' \geq \ell$. Let ℓ be minimal with that property and let $k \in \omega$ be such that $f(\ell) = x_G^{(\ell,k)}$. Then there is a $p \in G$ such that $p \Vdash$ "$\dot{f}$ is a choice function for A" $\wedge\, \dot{f}(\ell) = \dot{x}_G^{(\ell,k)}$, where $\dot{x}_G^{(\ell,k)}$ is the canonical $\mathbb{P}$-name for $x_G^{(\ell,k)}$. Let $k' \in \omega \setminus \{k\}$ be such that $(\ell, k') \in (\{\ell\} \times \omega) \setminus \mathsf{Dom}(p)$ and let $\pi \in \mathrm{Aut}(\omega_1 \times \omega, \omega)$ which only swaps (ℓ, k) and (ℓ, k'). Then $\pi^* \in H_z \subseteq \mathsf{sym}(\dot{f})$ and p and $\pi^*(p)$ are compatible. But $\pi^*(p) \Vdash \dot{f}(\ell) = \pi^*(\dot{x}_G^{(\ell,k)}) = \dot{x}_G^{(\ell,k')}$, which is impossible as these are distinct. □

Proposition 3.3.18. In N, $\mathsf{DC}(\omega^\omega; \mathbf{\Pi}^1_n)$ holds.

Proof. Let $X \subseteq \omega^\omega$ be a $\mathbf{\Pi}^1_n$ set of reals in N and let $R \subseteq (\omega^\omega)^2$ be a total relation such that R is $\mathbf{\Pi}^1_n$ in N. Then there are Π^1_n formulae φ and ψ with real parameters a and b in N which define X and R in N, respectively. Without loss of generality, we can assume $a = b$. By Lemma 3.3.9, there is a $z \in Z$ such that $a \in L[G{\restriction}z]$. Let X_φ and R_ψ be the sets defined by $\varphi(a)$ and $\psi(a)$ in $\mathrm{L}[G{\restriction}z]$, and let $\chi(x, a)$ be the formula $\varphi(x, a) \to \exists x' \psi((x, x'), a)$. Then $\chi(x, a)$ is Σ^1_{n+1} and so absolute by Lemma 3.3.14. By downwards-absoluteness, for every x, $\chi(x, a)$ is true in $\mathrm{L}[G{\restriction}z]$. As usual $\mathrm{L}[G{\restriction}z] \models \mathsf{ZFC}$, so there is a sequence $(x_k)_{k\in\omega}$ in $\mathrm{L}[G{\restriction}z]$ such that $x_k \; R_\psi \; x_{k+1}$ for every $k \in \omega$. By Lemma 3.3.9, $(x_k)_{k\in\omega} \in N$, and $\varphi(a)$ and $\psi(a)$ are absolute between $\mathrm{L}[G{\restriction}z]$ and N, so $x_k \; R \; x_{k+1}$ in N for every $k \in \omega$. □

By Propositions 3.3.17 & 3.3.18, the model N witnesses the validity of our Theorem 3.2.4.

3.3.3 The Kanovei-Lyubetsky Lemma

This is a largely technical section, where we generalise Kanovei and Lyubsetsky's framework in [111, §5 & §6], for constructing forcing notions which are n-absolute for slices. In particular, we construct certain ω-slice-products which are n-absolute for slices.

Our central result is the Kanovei-Lyubetsky Lemma (Lemma 3.3.43), which generalises [111, Theorem 13]. We follow the general idea of [111, §5]; this framework is an important tool in proving the Slicing Theorem (Theorem 3.3.15) in the next section.

For the rest of Section 3.3.3, we work in L.

Definition 3.3.19. Let $(x_\beta)_{\beta\in\alpha}$ be a sequence of sets. We say that $(x_\beta)_{\beta\in\alpha}$ is *continuous at limits* if, for every limit $\lambda < \beta$, we have that $x_\lambda = \bigcup_{\beta\in\lambda} x_\beta$.

Definition 3.3.20. A partial order $(\mathcal{M}, \preccurlyeq) \in \mathrm{L}$ is a *storage order* if it satisfies the following 5 conditions:

1. Every element $m \in \mathcal{M}$ is of the form $m = (M, P)$, where $M = L_\gamma$ for some countable $\gamma > \omega$, $M \models \mathsf{ZFC}^-$, and P is an ω-slice forcing notion with finite support of length $<\omega_1$ and is in M.

2. The pair $(\mathcal{M}, \preccurlyeq)$ is $\Delta_1^{\mathbf{HC}}$ in L.

3. If $(M, P) \preccurlyeq (N, Q)$, then $M \subseteq N$, $P \subseteq Q$, every predense set $D \subseteq P$ in M remains predense in Q, and conditions which are incompatible in P are also incompatible in Q.

A pair $(M, P) \in \mathcal{M}$ is *strictly $\preccurlyeq$-less* than another pair $(N, Q) \in \mathcal{M}$ if $M \subsetneq N$, $\mathsf{len}(P) < \mathsf{len}(Q)$, and for every $(\nu, k) \in \mathsf{Dom}(p)$, $P \restriction \{(\nu, k)\} \subsetneq Q \restriction \{(\nu, k)\}$.

4. For every $(M, P) \in \mathcal{M})$, there is an $(N, Q) \in \mathcal{M}$ such that (M, P) is strictly $\preccurlyeq$-less than (N, Q).

We say that strictly $\preccurlyeq$-increasing sequence, $((M_\xi, P_\xi))_{\xi \in \zeta}$, is a *storage sequence* if $(P_\xi)_{\xi \in \zeta}$ is continuous at limits.

5. Let $\zeta \leq \omega_1$, let $((M_\xi, P_\xi))_{\xi \in \zeta}$ be s a storage sequence, and let $\mathbb{P} := \bigcup_{\xi \in \zeta} P_\xi$.

   (a) If $\zeta < \omega_1$, then there is an M such that $(M, P) \in \mathcal{M}$ and $(M_\xi, P_\xi) \preccurlyeq (M, \mathbb{P})$ for every $\xi < \zeta$.[4]

   (b) If $\zeta = \omega_1$, then every predense set $D \subseteq P_\xi$ in M_ξ remains predense in $\mathbb{P}$ for every $\xi < \omega_1$, P is c.c.c. in L, and all conditions which are incompatible in P_ξ are incompatible in $\mathbb{P}$.

If $((M_\xi, P_\xi))_{\xi \in \omega_1}$ is a storage sequence, then $\bigcup_{\xi \in \omega_1} P_\xi$ is an ω-slice-product of length ω_1. For the rest of this section, fix a storage order, $(\mathcal{M}, \preccurlyeq)$.

The next step is to define a forcing-like relation for a certain language of second-order arithmetic. Firstly, we define the language itself. The main difference is having constants for names of reals, and bounds on certain quantifiers. Formally, for any forcing notion, $\mathbb{P}$, we say that σ is a $\mathbb{P}$-*name for a real* if $1_{\mathbb{P}} \Vdash$ "σ is a real".

Definition 3.3.21. For $(M, P) \in \mathcal{M}$, let $\mathcal{FL}^2(M, P)$ be the language of second-order arithmetic augmented with a constant symbol, c_n, for every $n \in \omega$ and a new constant symbol, c_σ, for every P-name for a real in M, σ, and such that second-order quantifiers may be bounded by a countable set $B \subseteq \omega_1$ in M, which we denote by $\forall^B$ and $\exists^B$.

Definition 3.3.22. Let $(M, P) \in \mathcal{M}$, G be a P-generic filter over M and φ be a formula in $\mathcal{FL}^2(M, P)$. The *valuation* of φ by G, denoted φ_G, is the formula produced by replacing every c_n in ϕ by n, every c_σ in φ by σ, and every $\forall^B$ in φ by $\forall x \in \omega^\omega \cap M[G \restriction B \times \omega]$, so too for $\exists^B$.

The language, $\mathcal{FL}^2(M, P)$, is a variant of the standard forcing language in having some quantifiers which are bounded (i.e. $\forall^B$ and $\exists^B$). This bounding helps control the fragment of the product we need. Clearly, every formula in $\mathcal{FL}^2(M, P)$ can be seen as a formula in the standard forcing language. The names and formulae in the order of the forcing languages of a storage order behave in the natural way:

[4] In other words, $(\mathcal{M}, \preccurlyeq)$ is $<\omega_1$-closed.

Remark 3.3.23. If $(M, P), (N, Q) \in \mathcal{M}$, where $(M, P) \preccurlyeq (N, Q)$, and G is a Q-generic filter over N, then:

1. if $\sigma \in M$ is a P-name for a real, then σ is also a Q-name for a real, and $\sigma_G = \sigma_{G \cap P}$, and

2. if φ is an $\mathcal{FL}^2(M, P)$ formula, then φ is also a $\mathcal{FL}^2(N, Q)$ formula, and $\varphi_G = \varphi_{G \cap P}$.

We likewise define a variation of the forcing relation. We define it only for *sentences* of $\mathcal{FL}^2(M, P)$, and only sentences of a certain form. As usual, we can treat $\vee$ as abbreviations for formulae containing $\neg$, $\wedge$. Our definition is in the style of the *strong* forcing relation for $\mathbb{P}$, that is where $p \Vdash_s \exists x \varphi(x)$ if there is $\mathbb{P}$-name, σ, such that $p \Vdash_s \sigma$.

Definition 3.3.24 (The **forc** relation)**.** Let $(M, P) \in \mathcal{M}$, let $p \in P$, let $k > 0$, and let φ be a $\Sigma^1_k(M, P)$ or $\Pi^1_k(M, P)$ sentence. We say that p $\mathbf{forc}^M_P \varphi$ if one of the following holds:

1. φ is $\Sigma^1_1(M, P)$ and $p \Vdash^M_P \varphi$,

2. $\varphi = \exists x \psi(x)$ where ψ is $\Pi^1_k(M, P)$ for $k > 1$ and there is a P-name $\sigma \in M$ for a real such that p $\mathbf{forc}^M_P \psi(c_\sigma)$,

3. $\varphi = \exists^B x \psi(x)$ where ψ is $\Pi^1_k(M, P)$ for $k > 1$ and there is a $(P \restriction B \times \omega)$-name $\sigma \in M$ for a real such that p $\mathbf{forc}^M_P \psi(c_\sigma)$, or

4. φ is $\Pi^1_k(M, P)$ and there is no $(M', P') \in \mathcal{M}$ extending (M, P) and no $q \in P'$ such that $q \leq p$ and q $\mathbf{forc}^{M'}_{P'} \neg\varphi$.

Remark 3.3.25. Strictly speaking, the **forc** relation is not defined for sentences of the form $\neg\neg\varphi$. However, throughout, for each $\Pi^1_n(M, P)$ sentence φ, when we refer to $\neg\varphi$, we mean the $\Sigma^1_n(M, P)$ formula ψ such that $\neg\psi$ is the formula φ.

Just as we do for the ordinary forcing language, we need to check that the **forc** relation obeys all of the properties we expect, such as monotonicity, consistency, and negation completeness. This we now prove.

Lemma 3.3.26 (Monotonicity)**.** Let (M, P), φ, and p be as in Definition 3.3.24. If p $\mathbf{forc}^M_P \varphi$, $(M, P) \preccurlyeq (N, Q) \in \mathcal{M}$, and $q \in Q$ with $q \leq p$, then q $\mathbf{forc}^N_Q \varphi$.

Proof. By definition, every predense set in P in M remains predense in Q and so every P-name in φ is also a Q-name for a real. The proof is by induction on φ.

If φ is $\Sigma^1_1(M, P)$, we have to show that $q \Vdash^N_Q \varphi$. Let G be a Q-generic filter over N containing q. Then $G \cap P$ is P-generic over M containing p. By assumption, p $\mathbf{forc}^M_P \varphi$, so $p \Vdash^M_P \varphi$ and so $M[G \cap P] \models \varphi[G \cap P]$. By analytic absoluteness, $N[G] \models \varphi[G \cap P] = \varphi[G]$. Hence, $q \Vdash^N_Q \varphi$.

If φ is $\Sigma^1_{k+1}(M, P)$, then there is a $\Pi^1_k(P, M)$ formula ψ such that $\varphi = \exists x \psi(x)$ or $\varphi = \exists^B x \psi(x)$. As both cases are similar, we prove the former. Since p $\mathbf{forc}^M_P \varphi$, there is a P-name $\sigma \in M$ for a real such that p $\mathbf{forc}^M_P \psi(c_\sigma)$. Then σ is also Q-name for a real in N. By the induction hypothesis, q $\mathbf{forc}^N_Q \psi(c_\sigma)$ and so q $\mathbf{forc}^N_Q \varphi$.

Finally, if φ is $\Pi^1_k(M,P)$ then $p\ \mathbf{forc}^M_P\varphi$. So, there is no $(M',P') \in \mathcal{M}$ extending (M,P) and no $p' \in P'$ such that $p' \leq p$ and $p'\ \mathbf{forc}^{M'}_{P'}\neg\varphi$. Hence, there is no $(N',Q') \in \mathcal{M}$ extending (N,Q) and $q' \in Q'$ such that $q' \leq q$ and $q'\ \mathbf{forc}^{N'}_{Q'}\neg\varphi$. So, $q\ \mathbf{forc}^N_Q\varphi$. □

Lemma 3.3.27 (Consistency)**.** Let (M,P) and p be as in Definition 3.3.24 and let $k \geq 1$. Then no $\Pi^1_k(M,P)$ sentence φ is such that $p\ \mathbf{forc}^M_P\varphi$ and $p\ \mathbf{forc}^M_P\neg\varphi$.

Proof. Suppose that $p\ \mathbf{forc}^M_P\varphi$. Then there is no $(M',P') \in \mathcal{M}$ extending (M,P) and no $q \in P'$ such that $q \leq p$ and $q\ \mathbf{forc}^{M'}_{P'}\neg\varphi$. So, $p\ \mathbf{forc}^M_P\neg\varphi$ does not hold. □

Note that the proofs of Lemmata 3.3.26 and 3.3.27 are similar to the proofs of the same properties for ordinary syntactic forcing relation, $\Vdash$ [129, IV.2.22]. Negation completeness also holds for **forc**, but this proof proceeds differently (Lemma 3.3.37).

Next, we extend **forc** to storage sequences, so that the forcing notion $\mathbb{P}$ it generates is such that **forc** 'approximates' $\Vdash_{\mathbb{P}}$ for sentences of $\mathcal{FL}^2(L,\mathbb{P})$ with of a certain complexity (Corollary 3.3.39).

Definition 3.3.28. Let $((M_\xi,P_\xi))_{\xi\in\omega_1}$ be a storage sequence and let $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$. We define $\mathcal{FL}^2(\mathbb{P}) := \bigcup_{\xi\in\omega_1}\mathcal{FL}^2(M_\xi,P_\xi)$.

Definition 3.3.29. Let $k \geq 1$, $((M_\xi,P_\xi))_{\xi\in\omega_1}$ be a storage sequence let $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$. We say a formula φ is $\Sigma^1_k(\mathbb{P})$ (or $\Pi^1_k(\mathbb{P})$) if there is some $\xi < \omega_1$ such that φ is $\Sigma^1_k(M_\xi,P_\xi)$ (or $\Pi^1_k(M_\xi,P_\xi)$).

By Lemma 3.3.26, if φ is a $\mathcal{FL}^2(\mathbb{P})$ sentence such that there is a $\xi < \omega_1$ where $p\ \mathbf{forc}^{P_\xi}_{M_\xi}\varphi$, then $p\ \mathbf{forc}^{P_{\xi'}}_{M_{\xi'}}\varphi$ for every $\xi' > \xi$. So, we can extend **forc** to $\mathbb{P}$ in a well-defined way:

Definition 3.3.30. Let $((M_\xi,P_\xi))_{\xi\in\omega_1}$ be a storage sequence and let $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$. Let $p \in \mathbb{P}$ and let φ be a $\Pi^1_k(\mathbb{P})$ or $\Sigma^1_{k+1}(\mathbb{P})$ sentence. We write $p\ \mathbf{forc}_\xi\varphi$ if $p\ \mathbf{forc}^{M_\xi}_{P_\xi}\varphi$ and $p\ \mathbf{forc}_\infty\varphi$ if there is a $\xi < \omega_1$ such that $p\ \mathbf{forc}_\xi\varphi$.

It is this $\mathbf{forc}_\infty$ that we show approximates the standard forcing relation. We begin with some basic properties:

Lemma 3.3.31. Let $((M_\xi,P_\xi))_{\xi\in\omega_1}$ be a storage sequence, $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$, and let $k \geq 1$.

1. Let $p \in P_\xi$ and let φ be a $\Pi^1_k(M_\xi,P_\xi)$ or $\Sigma^1_{k+1}(M_\xi,P_\xi)$ sentence. If $p\ \mathbf{forc}_\xi\varphi$, $\xi < \zeta < \omega_1$, and $q \in P_\zeta$ with $q \leq p$, then $q\ \mathbf{forc}_\zeta\varphi$.

2. Let $p \in \mathbb{P}$. There is no $\Pi^1_k(\mathbb{P})$ or $\Sigma^1_{k+1}(\mathbb{P})$ sentence φ such that $p\ \mathbf{forc}_\infty\varphi$ and $p\ \mathbf{forc}_\infty\neg\varphi$.

Proof. Parts 1. and 2. follow directly from Lemmata 3.3.26 and 3.3.27, respectively. □

Only certain storage sequences suffice for generating suitable forcing notions whose products are n-absolute for slices. To this end, we define a kind of sequence which will suffice, in analogy to [111, Definition 15].

Definition 3.3.32. Let $n > 2$. A storage sequence $((M_\xi, P_\xi))_{\xi\in\omega_1}$ is called *n-complete* if, for every $\Sigma^{\mathbf{HC}}_{n-2}$ set $D \subseteq \mathcal{M}$, there is a $\xi < \omega_1$ such that either $(M_\xi, P_\xi) \in D$ or there is no $(N, Q) \in D$ extending (M_ξ, P_ξ).

Lemma 3.3.33. For every $n > 2$, there is a $\Delta^{\mathbf{HC}}_{n-1}$, n-complete storage sequence from $\mathcal{M}$ in L .

Proof. Let $n > 2$. By Lemma 2.1.20, let $\Gamma \subset \omega_1 \times \mathbf{HC}$ be a universal $\Sigma^{\mathbf{HC}}_{n-2}$ set. We define the required sequence recursively. Let (M_0, P_0) be the $<_L$-least pair such that $(M_0, P_0) \in \mathcal{M}$. Suppose that $((M_{\xi'}, P_{\xi'}))_{\xi'\in\xi}$ is already defined. If ξ is a limit, then we set $P_\xi := \bigcup_{\xi'\in\xi} P_{\xi'}$ and let M_ξ be the $<_L$-least ctm of ZFC such that $(M_\xi, P_\xi) \in \mathcal{M}$ and M_ξ contains $((M_{\xi'}, P_{\xi'}))_{\xi'\in\xi}$. If $\xi = \xi' + 1$ is a successor, let (M_ξ, P_ξ) be the $<_L$-least pair such that:

1. $(M_{\xi'}, P_{\xi'})$ is strictly-$\preccurlyeq$ (M_ξ, P_ξ), and
2. either $(M_\xi, P_\xi) \in D_{\xi'} := \{m \in \mathcal{M} : (\xi', m) \in \Gamma\}$ or there is no $(N, Q) \in D_{\xi'}$ extending (M_ξ, P_ξ).

Recall that $<_L \restriction \mathbf{HC}^2$ is $\Delta^{\mathbf{HC}}_1$ (Lemma 2.1.19). By definition, $\mathcal{M}$ and $\preccurlyeq$ are $\Delta^{\mathbf{HC}}_1$, and Γ is $\Sigma^{\mathbf{HC}}_{n-2}$, so $((M_\xi, P_\xi))_{\xi\in\omega_1}$ is $\Delta^{\mathbf{HC}}_{n-1}$. By Property 2., the sequence is n-complete. □

Notice that $\mathcal{M}$ is an arbitrary storage order here. We return to the issue of the complexity of the storage order itself in Section 3.3.4, where we use the complexity of the sequence to bound the complexity of the set of generics. For now, the exact complexity of the storage sequence is not important.

Our next task is to show that $\mathbf{forc}_\infty$ is negation complete for sentences of complexity $\Sigma^1_k(\mathbb{P})$, for a suitable k. For this, we define the class of all forced statements of various complexities, and prove a small lemma, to the effect of negation completeness for Π^1_1 sentences:

Lemma 3.3.34. Let φ be a $\Pi^1_1(M, P)$ sentence of the form $\forall x\psi(x)$ or $\forall^B x\psi(x)$, for some bounded formula ψ. For every $p \in P$, p $\mathbf{forc}^M_P\varphi$ if and only if there is no $p' \leq p$ such that $p' \Vdash^M_P \exists x\neg\psi(x)$.

Proof. Both cases are similar, so we prove the first case. The right-to-left direction is trivial, so we prove the left-to-right direction. For a contradiction, let $(N, Q) \succcurlyeq (M, P)$ be such that $q \in Q$, $q \leq p$ and q $\mathbf{forc}^N_Q\exists x\neg\psi(x)$. Then $q \Vdash^N_Q \exists x\neg\psi(x)$. We can assume that no $p' \leq p$ is such that $p' \Vdash^M_P \exists x\neg\psi(x)$. So, $D := \{p' \leq p : p' \Vdash^M_P \psi\}$ is dense below P. As $(M, P) \preccurlyeq (N, Q)$, D is still predense in Q. So, some $r \in D$ is compatible with q. Let G be a Q-generic filter over N where $r \in G$. Then $G \cap P$ is P-generic over M. Since $r \Vdash^M_P \varphi$, $M[G \cap P] \vDash \varphi_{G\cap P}$. By analytic absoluteness, $N[G] \vDash \varphi_G$ and so $r \Vdash^N_Q \varphi$, which contradicts that $q \Vdash^N_G \exists x\neg\psi(x)$. □

Definition 3.3.35. For all $k > 1$, we define the following:

$\mathbf{Forc}(\Sigma^1_k) := \{(M, P, p, \varphi) : (M, P) \in \mathcal{M} \wedge p \in P \wedge \varphi$ is a $\Sigma^1_k(M, P)$ sentence$\wedge p\, \mathbf{forc}^M_P \varphi\}$.

We define $\mathbf{Forc}(\Pi^1_k)$ analogously.

Theorem 3.3.36. For all $k > 1$:

1. $\mathbf{Forc}(\Sigma^1_1)$, $\mathbf{Forc}(\Pi^1_1)$, and $\mathbf{Forc}(\Sigma^1_2)$ are $\Delta^{\mathbf{HC}}_1$; and
2. $\mathbf{Forc}(\Pi^1_k)$ and $\mathbf{Forc}(\Sigma^1_{k+1})$ are $\Pi^{\mathbf{HC}}_{k-1}$.

Proof. For each $k \geq 1$, we define the following two classes of sentences:

$$\Sigma^1_k(\mathcal{M}) := \{(M, P, \varphi) : (M, P) \in M \wedge \varphi \text{ is a } \Sigma^1_k(M, P) \text{ sentence}\},$$
$$\Pi^1_k(\mathcal{M}) := \{(M, P, \varphi) : (M, P) \in M \wedge \varphi \text{ is a } \Pi^1_k(M, P) \text{ sentence}\}.$$

As $\mathcal{M}$ is a storage order, $\mathcal{M}$ is $\Delta^{\mathbf{HC}}_1$. Fix some $k \geq 1$. Note that "φ is $\Sigma^1_k(M, P)$" can be checked in M. So, $\Sigma^1_k(\mathcal{M})$ is $\Delta^{\mathbf{HC}}_1$, and exactly similarly for $\Pi^1_k(\mathcal{M})$. With this, we begin the proof in earnest.

We start with Part 1. As $\Sigma^1_1(\mathcal{M})$ is $\Delta^{\mathbf{HC}}_1$, it suffices to check that "$p\, \mathbf{forc}^M_P \varphi$" is $\Delta^{\mathbf{HC}}_1$. Note that $p\, \mathbf{forc}^M_P \varphi$ if and only if $p \Vdash^M_P \varphi$, the latter of which can also be checked in M. So, "$p\, \mathbf{forc}^M_P \varphi$" is also $\Delta^{\mathbf{HC}}_1$.

Next, we check that $\mathbf{Forc}(\Pi^1_1)$ is $\Delta^{\mathbf{HC}}_1$. Again, it suffices to check that "$p\, \mathbf{forc}^M_P \varphi$" is $\Delta^{\mathbf{HC}}_1$. By assumption, φ is of the form $\forall x \psi(x)$ or $\forall^B x \psi(x)$, for some bounded formula ψ. By Lemma 3.3.34, for each $p \in P$, $p\, \mathbf{forc}^M_P \varphi$ if and only if there is no $p' \leq p$ such that $p' \Vdash^M_P \exists x \neg\psi(x)$. The latter fact can be checked in M, so $\mathbf{Forc}(\Pi^1_1)$ is $\Delta^{\mathbf{HC}}_1$.

Lastly, we show that $\mathbf{Forc}(\Sigma^1_2)$ is $\Delta^{\mathbf{HC}}_1$. Again, we only have to check that "$p\, \mathbf{forc}^M_P \varphi$" is $\Delta^{\mathbf{HC}}_1$. Since φ is $\Sigma^1_2(M, P)$, there is some $\Pi^1_1(M, P)$ formula ψ such that either $\varphi = \exists x \psi(x)$ or $\varphi = \exists^B x \psi(x)$. Both cases are similar, so we can assume the former. By definition, $p\, \mathbf{forc}^M_P \varphi$ if and only if there is a P-name $\sigma \in M$ for a real such that $p\, \mathbf{forc}^M_P \psi(c_\sigma)$. Since $\mathbf{Forc}(\Pi^1_1)$ is $\Delta^{\mathbf{HC}}_1$, "$p\, \mathbf{forc}^M_P \psi(c_\sigma)$" is $\Delta^{\mathbf{HC}}_1$. The existence quantifier for the P-name can be bounded by M, which is in $\mathbf{HC}$. Therefore, "$p\, \mathbf{forc}^M_P \varphi$" is $\Delta^{\mathbf{HC}}_1$.

We prove Part 2. by induction on k. We have to show that $\mathbf{Forc}(\Pi^1_2)$ and $\mathbf{Forc}(\Sigma^1_3)$ are $\Pi^{\mathbf{HC}}_1$. We start with the former. By definition, $p\, \mathbf{forc}^M_P \varphi$ if and only if for every $(M', P') \in \mathcal{M}$ extending (M, P) and every $q \in P'$, $q \not\leq p$ or $q\, \mathbf{forc}^{M'}_{P'} \neg\varphi$ is false. By the previous part, $\mathbf{Forc}(\Sigma^1_2)$ is $\Delta^{\mathbf{HC}}_1$, so $q\, \mathbf{forc}^{M'}_{P'} \neg\varphi$ is also $\Pi^{\mathbf{HC}}_1$. Hence, $p\, \mathbf{forc}^M_P \varphi$ is $\Pi^{\mathbf{HC}}_1$ too. By a similar argument to that of $\mathbf{Forc}(\Sigma^1_2)$, we can show that $\mathbf{Forc}(\Sigma^1_3)$ is $\Pi^{\mathbf{HC}}_1$. The induction step is similar to the base case. □

With this, we can prove negation completeness for $\Pi^1_k(\mathbb{P})$. Careful complexity tracking in Theorem 3.3.36 shows that an n-complete sequence intersects enough extensions to decide whether a negation of a sentence is eventually **forc**ed:

Lemma 3.3.37 (Negation Completeness)**.** Let $n > 2$, $((M_\xi, P_\xi))_{\xi \in \omega_1}$ be an n-complete storage sequence, and $\mathbb{P} := \bigcup_{\xi \in \omega_1} P_\xi$. Let $p \in \mathbb{P}$, $2 \leq k < n$, and φ be a $\Sigma^1_k(\mathbb{P})$ sentence. Then:

1. there is a $q \leq p$ such that either q $\mathbf{forc}_\infty \varphi$ or q $\mathbf{forc}_\infty \neg\varphi$, and
2. p $\mathbf{forc}_\infty \varphi$ if and only if there is no $q \leq p$ such that q $\mathbf{forc}_\infty \neg\varphi$.

Proof. For Part 1., let $\xi < \omega_1$ be such that φ is $\Sigma^1_k(M_\xi, P_\xi)$. Without loss of generality, we assume that $p \in P_\xi$. We define the following sets:

$$D := \{(N,Q) \in \mathcal{M} : (M_\xi, P_\xi) \preccurlyeq (N,Q) \wedge \exists q \in Q(q \leq p \wedge q \ \mathbf{forc}^N_Q \varphi)\}.$$

By Theorem 3.3.36, D is $\Sigma^{\mathbf{HC}}_{k-1}$. Since $((M_\zeta, P_\zeta))_{\zeta\in\omega_1}$ is n-complete, there is some $\xi \leq \zeta < \omega_1$ such that either $(M_\zeta, P_\zeta) \in D$ or there isno pair $(N,Q) \in D$ extending (M_ζ, P_ζ). Then there are two cases.

a. If $(M_\zeta, P_\zeta) \in D$, then there is a condition $q \in P_\zeta$ such that $q \leq p$ and q $\mathbf{forc}_\xi \varphi$. Hence, q $\mathbf{forc}_\infty \varphi$.

b. Otherwise, there is no pair $(N,Q) \in D$ extending (M_ζ, P_ζ). Then there is no pair $(N,Q) \succcurlyeq (M_\zeta, P_\zeta)$ and condition $q \in Q$ such that $q \leq p$ and q $\mathbf{forc}^N_Q \varphi$. Hence, p $\mathbf{forc}_\zeta \neg\varphi$ and so p $\mathbf{forc}_\infty \neg\varphi$.

For Part 2., suppose there is no $q \leq p$ such that q $\mathbf{forc}_\infty \neg\varphi$. By Part 1., $D := \{q \in \mathbb{P} : q \leq p \wedge q \ \mathbf{forc}_\infty \varphi\}$ is dense in $\mathbb{P}$ below p. Let $\mathcal{A} \subseteq D$ be a maximal antichain below p. As $\mathbb{P}$ is c.c.c., $\mathcal{A}$ is countable, so $\mathcal{A} \in \mathbf{HC}$. As $\mathbf{HC} = L_{\omega_1} = \bigcup_{\xi\in\omega_1} M_\xi$, there is some $\xi < \omega_1$ such that $p \in \mathbb{P}_\xi$, $\mathcal{A}$ is a maximal antichain in P_ξ below p, $\mathcal{A} \in M_\xi$, φ is $\Sigma^1_k(M_\xi, P_\xi)$, and for every $q \in \mathcal{A}$, q $\mathbf{forc}_\xi \varphi$. If $k = 1$, then by properties of the ordinary forcing relation, p $\mathbf{forc}_\xi \varphi$. So, we can assume that $k \geq 2$. We suppose for a contradiction that p $\mathbf{forc}_\xi \varphi$ fails. Then there is a pair $(N,Q) \succcurlyeq (M_\xi, P_\xi)$ and some $q \in Q$ such that $q \leq p$ and q $\mathbf{forc}^N_Q \neg\varphi$. Since $\mathcal{A}$ remains a maximal antichain below p in Q, there is some $r \in \mathcal{A}$ which is compatible with q. But this contradicts r $\mathbf{forc}_\xi \varphi$. Therefore, p $\mathbf{forc}_\infty \varphi$. The other direction follows directly from Lemma 3.3.31. □

Next, we prove a version of the Truth Lemma for $\mathbf{forc}_\infty$ (i.e. the analogue of [129, Chapter VII, Theorem 3.5]). For this, we define the valuation in the language $\mathcal{FL}^2(\mathbb{P})$. We do this in the natural way: if $((M_\xi, P_\xi))_{\xi\in\omega_1}$ is a storage sequence, $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$, φ is a $\mathcal{FL}^2(\mathbb{P})$ sentence, and G is $\mathbb{P}$-generic over L, then there is some $\xi < \omega_1$ such that φ is in $\mathcal{FL}^2(P_\xi, M_\xi)$. By the definition of a storage order, $G_\xi := G \cap P_\xi$ is P_ξ-generic over M_ξ. So, we let the *valuation of* φ *by* G, φ_G, be just φ_{G_ξ}. It is simple to check that this is well-defined (see Remark 3.3.23).

Theorem 3.3.38. Let $n > 2$, $((M_\xi, P_\xi))_{\xi\in\omega_1}$ be an n-complete storage sequence, and $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$. Let $1 \leq k < n$, $1 \leq \ell \leq n$, φ be a $\mathcal{FL}^2(\mathbb{P})$ sentence, and G be $\mathbb{P}$-generic over L.

1. If $p \in G$, φ is a $\Pi^1_k(\mathbb{P})$ or $\Sigma^1_\ell(\mathbb{P})$ sentence for $k < n$, and p $\mathbf{forc}_\infty \varphi$, then $\mathrm{L}[G] \models \varphi_G$.
2. If φ is a $\Pi^1_k(\mathbb{P})$ or $\Sigma^1_k(\mathbb{P})$, and $\mathrm{L}[G] \models \varphi_G$, then there is some $p \in G$ such that p $\mathbf{forc}_\infty \varphi$.

Proof. We prove both items simultaneously by induction on φ:

For the base case, assume φ is $\Sigma^1_1(\mathbb{P})$. Suppose that $p \in G$ and p $\mathbf{forc}_\infty\varphi$. Then there is a $\xi < \omega_1$ such that $p \in P_\xi$, φ is $\Sigma^1_1(M_\xi, P_\xi)$, and p $\mathbf{forc}_\xi\varphi$. By definition, we have $p \Vdash^{M_\xi}_{P_\xi} \varphi$. As $G \cap P_\xi$ is P_ξ-generic over M_ξ, $M_\xi[G \cap P_\xi] \models \varphi_{G\cap P_\xi}$. By analytic absoluteness, $\mathrm{L}[G] \models \varphi_{G\cap P_\xi}$. As φ is $\Sigma^1_1(M_\xi, P_\xi)$, $\varphi_G = \varphi_{G\cap P_\xi}$. Hence $\mathrm{L}[G] \models \varphi_G$.

If conversely, $\mathrm{L}[G] \models \varphi_G$, then there is a $\xi < \omega_1$ such that φ is $\Sigma^1_1(P_\xi, M_\xi)$. By analytic absoluteness, $M_\xi[G \cap P_\xi] \models \varphi_{G\cap P_\xi}$. Hence, there is some $p \in G \cap P_\xi$ such that $p \Vdash^{M_\xi}_{P_\xi} \varphi$. Then p $\mathbf{forc}_\xi\varphi$ and so p $\mathbf{forc}_\infty\varphi$.

For the successor case, first assume φ is $\Sigma^1_{k+1}(\mathbb{P})$ and $k < n$. Then there is a $\psi \in \Pi^1_k(\mathbb{P})$ such that either $\varphi = \exists x\psi(x)$ or $\varphi = \exists^B x\psi(x)$. Without loss of generality, we can assume the former. If $p \in G$ and p $\mathbf{forc}_\infty\varphi$, then there is a $\xi < \omega_1$ such that $p \in P_\xi$, ψ is $\Pi^1_k(M_\xi, P_\xi)$, and p $\mathbf{forc}_\xi\exists x\psi(x)$. By definition, there is a P_ξ-name $\sigma \in M_\xi$ for a real such that p $\mathbf{forc}_\xi\psi(c_\sigma)$. Then, by the induction hypothesis, $\mathrm{L}[G] \models (\psi(c_\sigma))_G$. Hence, $\mathrm{L}[G] \models (\psi(c_\sigma))_G$, so $\mathrm{L}[G] \models \varphi_G$. (This also works for $k = n$.)

Conversely, if $\mathrm{L}[G] \models \varphi_G$ then there is some $x \in L[G]$ such that $\mathrm{L}[G] \models \psi_G(x)$. By the definition of a storage order, $\mathbb{P}$ is c.c.c., so there is a countable $\mathbb{P}$-name $\tau \in L$ for a real such that $\tau_G = x$. Then $\mathrm{L}[G] \models (\psi(c_\tau))_G$. As $\tau \in \mathbf{HC} = L_{\omega_1} = \bigcup_{\xi\in\omega_1} M_\xi$, there is some $\xi < \omega_1$ such that $\psi(c_\tau)$ is $\Pi^1_k(M_\zeta, P_\zeta)$. So, by the induction hypothesis, there is some $p \in G$ such that p $\mathbf{forc}_\infty\psi(c_\tau)$. Therefore, p $\mathbf{forc}_\infty\varphi$.

For the second part of the successor case, assume φ is $\Pi^1_k(\mathbb{P})$ and $1 \le k < n$. If $p \in G$ and p $\mathbf{forc}_\infty\varphi$, then by Lemma 3.3.31, there is no $q \le p$ such that q $\mathbf{forc}_\infty\neg\varphi$. By induction, $\mathrm{L}[G] \not\models (\neg\varphi)_G$. Hence, $\mathrm{L}[G] \models \varphi_G$. (This also works for $k = n$.)

Conversely, if $\mathrm{L}[G] \models \varphi_G$ then $\mathrm{L}[G] \not\models (\neg\varphi)_G$. By the induction hypothesis, there is no $p \in G$ such that p $\mathbf{forc}_\infty\neg\varphi$. Note that the set $D := \{p \in \mathbb{P} : p\ \mathbf{forc}_\infty\varphi \vee p\ \mathbf{forc}_\infty\neg\varphi\}$ is dense, so there is some $p \in G$ such that p $\mathbf{forc}_\infty\varphi$. □

Putting the pieces together, $\mathbf{forc}_\infty$ approximates $\mathbb{P}$-forcing along the sequence $((M_\xi, P_\xi))_{\xi\in\omega_1}$, i.e. the relations coincide for sentences of bounded complexity:

Corollary 3.3.39. Let $((M_\xi, P_\xi))_{\xi\in\omega_1}$, $\mathbb{P}$, k, ℓ, and φ be as in Theorem 3.3.38 and let $p \in \mathbb{P}$.

1. If p $\mathbf{forc}_\infty\varphi$, then $p \Vdash_{\mathbb{P}} \varphi$.
2. If φ is $\Pi^1_k(\mathbb{P})$, then $p \Vdash_{\mathbb{P}} \varphi$ if and only if there is no $q \le p$ such that q $\mathbf{forc}_\infty\neg\varphi$.
3. If $\Pi^1_k(\mathbb{P})$ or $\Sigma^1_\ell(\mathbb{P})$, and $p \Vdash_{\mathbb{P}} \varphi$, then there is some $q \le p$ such that q $\mathbf{forc}_\infty\varphi$.
4. If $k < n$, φ is $\Pi^1_k(\mathbb{P})$, and $p \Vdash_{\mathbb{P}} \varphi$, then p $\mathbf{forc}_\infty\varphi$.

Proof. Part 1. follows directly from Theorem 3.3.38. For Part 3., let G be a $\mathbb{P}$-generic filter over L with $p \in G$. Then $\mathrm{L}[G] \models \varphi_G$. By Theorem 3.3.38, there is a $q \in G$ such that q $\mathbf{forc}_\infty\varphi$. As G is a filter, there is some $r \le p, q$. Then $r \le p$ and r $\mathbf{forc}_\infty\varphi$.

For Part 4., let $q \leq p$. Then $q \Vdash_{\mathbb{P}} \varphi$. Suppose that q $\mathbf{forc}_\infty \neg\varphi$. Then, by Part 1., $q \Vdash_{\mathbb{P}} \neg\varphi$, which contradicts the consistency of $q \Vdash_{\mathbb{P}}$. Hence, there is no $q \leq p$ such that q $\mathbf{forc}_\infty \neg\varphi$. By Lemma 3.3.37, p $\mathbf{forc}_\infty \varphi$.

For Part 2., suppose that there is some $q \leq p$ such that q $\mathbf{forc}_\infty \neg\varphi$. By Part 1., $q \Vdash_{\mathbb{P}} \neg\varphi$. Hence, $p \Vdash_{\mathbb{P}} \varphi$ cannot hold. If conversely, $p \Vdash_{\mathbb{P}} \varphi$ does not hold, then there is some $q \leq p$ such that $q \Vdash_{\mathbb{P}} \neg\varphi$. By Part 3., there is some $r \leq q$ such that r $\mathbf{forc}_\infty \neg\varphi$. Hence, $r \leq p$ and r $\mathbf{forc}_\infty \neg\varphi$. □

The coincidence of $\mathbf{forc}_\infty$ and $\Vdash_{\mathbb{P}}$ for such sentences means that in extensions of L by a countable component of G, the exact choice of which component is not so important (in the sense of Lemma 3.3.42, as in [111, page 26]).

Definition 3.3.40. We call a storage order, $(\mathcal{M}, \preccurlyeq)$, *repetitive* if, for every $(M, P) \in \mathcal{M}$, P is an ω-slice-product of a single forcing notion.[5]

Roughly speaking, repetitiveness is used to 'swap rows' in the matrix of conditions. If $\mathbb{P}$ is an ω-slice-product of one forcing notion, then we can swap around the rows in a matrix of conditions, and we still obtain a condition in $\mathbb{P}$. But this need not be true in general for arbitrary products. We assume repetitiveness to guarantee we can swap rows as required.

More formally, if $(\mathcal{M}, \preccurlyeq)$ is repetitive, and $p \in P$ where $(M, P) \in \mathcal{M}$, $\mathsf{len}(p) = \zeta$, and $f : \zeta \to \zeta$ is a bijection in M, then $f(p) := \{((f(\nu), k), q) : ((\nu, k), q) \in p\}$ is in P, so f induces an automorphism on P. Likewise, if $\sigma \in M$ is a P-name and ϕ is a $\mathcal{FL}^2(M, P)$ formula, then we write $f(\sigma)$ for the P-name in M produced by replacing each occurrence of a condition, p, in σ by $f(p)$, and $f(\varphi)$ by replacing each c_σ in φ by $c_{f(\sigma)}$ and each Q^B by $Q^{f(B)}$ for each second-order quantifier Q in φ.

Lemma 3.3.41. Let $(\mathcal{M}, \preccurlyeq)$ be a repetitive storage order, let $(M, P) \in \mathcal{M}$, let $p \in P$, let $\zeta = \mathsf{len}(p)$, let $f : \zeta \to \zeta$ be a bijection in M, let $k \geq 1$, and let φ be a $\Sigma^1_k(M, P)$ or $\Pi^1_n(M, P)$ sentence. If p $\mathbf{forc}^M_P \varphi$ then $f(p)$ $\mathbf{forc}^M_P f(\varphi)$.

Proof. This proof follows from induction, by unpacking the definition. Fix a P-generic filter, G, containing $f(p)$. Then $G' := f^{-1}(G)$ is a P-generic filter containing p and $f(G') = G$.

For the base case, let φ be $\Sigma^1_1(M, P)$. In which case, $M[G] = M[G']$ and so $M[G] \vDash \varphi_{G'}$. Moreover, for every P-name $\sigma \in M$ for a real, $\sigma_G = \sigma_{G'}$. Hence, $M[G] \vDash (f(\varphi))_G$. So, the implication holds.

Next, suppose φ is $\Pi^1_k(M, P)$. Then there is an $\mathcal{FL}^2(M, P)$ formula, ψ, such that either $\varphi = \forall x \psi(x)$ or $\varphi = \forall^B x \psi(x)$. Both cases are similar, so we prove the former. For a contradiction, suppose that $f(p)$ $\mathbf{forc}^M_P f(\varphi)$ fails. Then there is a $(N, Q) \in \mathcal{M}$ and a $q \in Q$ such that $(M, P) \preccurlyeq (N, Q)$, $q \leq f(p)$, and q $\mathbf{forc}^N_Q f(\exists x \neg\psi(x))$. By induction, $f^{-1}(q)$ $\mathbf{forc}^N_Q f^{-1}(f(\exists x \neg\psi(x)))$. Hence, we have that $f^{-1}(q) \leq p$ and $f^{-1}(q)$ $\mathbf{forc}^N_Q \exists x \neg\psi(x)$. But this contradicts p $\mathbf{forc}^M_P \varphi$. So, the implication holds.

[5] Repetitiveness corresponds to simplicity in [205, Definition 3.3.17]. As we reserve simplicity for certain properties of ordered fields, we use an alternative name.

Finally, suppose φ is $\Sigma^1_{k+1}(M,P)$. Then there is a $\Pi^1_k(M,P)$ formula ψ such that either $\varphi = \exists x\psi(x)$ or $\varphi = \exists^B x\psi(x)$. Again, both are similar, so we assume the latter. As $p\ \mathbf{forc}^M_P\varphi$, there is a $(P\restriction(B\times\omega))$-name $\sigma\in M$ such that $p\ \mathbf{forc}^M_P\psi(c_\sigma)$. By induction, $f(p)\ \mathbf{forc}^M_P f(\psi(c_\sigma))$. So, $f(\sigma)$ is a $(P\restriction(f(B)\times\omega))$-name for a real and $f(p)\ \mathbf{forc}^M_P(f(\psi))(c_{f(\sigma)})$. So, the implication holds. □

Lemma 3.3.42. Let $(\mathcal{M},\preccurlyeq)$ be a repetitive storage order, and fix $n>2$. Suppose that $((M_\xi,P_\xi))_{\xi\in\omega_1}$ is an n-complete storage sequence, $\mathbb{P}:=\bigcup_{\xi\in\omega_1}P_\xi$, $d\subseteq\omega_1$ is countable and $d\in\mathrm{L}$, $b,c\subseteq\omega_1\backslash d$ are countably infinite and $b,c\in\mathrm{L}$, G is $\mathbb{P}$-generic over L, and $\psi(y)$ is a Π^1_{n-1} formula with parameters in $\mathrm{L}[G\restriction(d\times\omega)]$. If there is a real $y\in\mathrm{L}[\restriction((d\cup b)\times\omega)]$ such that $\mathrm{L}[G]\models\psi(y)$, then there is a real $y'\in\mathrm{L}[G\restriction((d\cup c)\times\omega)]$ such that $\mathrm{L}[G]\models\psi(y')$.

Proof. Suppose not. Let $\mathbb{Q}=\mathbb{P}\restriction(d\times\omega)$. By assumption, there is a $y\in\mathrm{L}[G\restriction((d\cup b)\times\omega)]$ such that $\mathrm{L}[G]\vDash\phi(y,r)$, but no $y'\in\mathrm{L}[G\restriction((d\cup c)\times\omega)]$ is such that $\mathrm{L}[G]\vDash\phi(y',r)$. Let $\dot r$ be a $\mathbb{Q}$-name for r. As $\mathbb{P}$ is c.c.c., so is $\mathbb{Q}$, so we assume that $\dot r$ is countable. By assumption, $\mathrm{L}[G]\vDash(\exists^{d\cup b}x\psi(x,c_{\dot r}))_G$. By Theorem 3.3.38, there is a $p\in G$ such that $p\ \mathbf{forc}_\infty\exists^{d\cup b}x\psi(x,c_{\dot r})$. We assumed that there is no $y'\in\mathrm{L}[G\restriction((d\cup c)\times\omega)]$ such that $\mathrm{L}[G]\models\psi(y',r)$, so without loss of generality, $p\Vdash_{\mathbb{P}}\neg\exists^{d\cup c}x\psi(x,c_{\dot r})$.

Note that b,c,d are all countable and in $\mathbf{HC}$. So, by unpacking the definition of $\mathbf{forc}_\infty$ (and increasing ξ if necessary to include b,c,d), there is some $\xi<\omega_1$ such that $b,c,d\in M_\xi$, $p\in P_\xi$, $\psi(x)$ is a $\Pi^1_{n-1}(M_\xi,P_\xi)$ formula, and $p\ \mathbf{forc}_\xi\exists^{d\cup b}x\psi(x,c_{\dot r})$. Let $\zeta=\mathsf{len}(p)$. By increasing ξ if necessary, we can assume $\xi\setminus(d\cup c)$ is the same size as $\xi\setminus(d\cup b)$. Let $f:\zeta\to\zeta$ be a bijection in M_ξ such that $f\restriction d$ is the identity, $f[b]=c$, and $f(\nu)=\nu$ for every $\nu\in b\cap c$. By Lemma 3.3.41, $f(p)\ \mathbf{forc}_\xi\exists^{d\cup b}x\psi(x,c_{\dot r})$. As p and $f(p)$ agree on their common domains, they are compatible. Then there is a $q\in P_\xi$ such that $q\le p$ and $q\le f(p)$. Then $q\ \mathbf{forc}_\xi\exists^{d\cup c}x\psi(x,c_{\dot r})$, hence, $q\ \mathbf{forc}_\infty\exists^{d\cup c}x\psi(x,c_{\dot r})$. By Corollary 3.3.39, $q\Vdash_{\mathbb{P}}\exists^{d\cup c}x\psi(x,c_{\dot r})$. But this contradicts that $p\Vdash_{\mathbb{P}}\neg\exists^{d\cup c}x\psi(x,c_{\dot r})$. □

In particular, this yields the Kanovei-Lyubetsky Lemma:

Lemma 3.3.43 (Kanovei-Lyubetsky Lemma)**.** Let $(\mathcal{M},\preccurlyeq)$ be a repetitive storage order, $n>2$, let $((M_\xi,P_\xi))_{\xi\in\omega_1}$ be an n-complete storage sequence, let $\mathbb{P}:=\bigcup_{\xi\in\omega_1}P_\xi$, and $e\subseteq\omega_1$ be unbounded and in L. Then every Σ^1_n formula with parameters in $\mathrm{L}[G\restriction e\times\omega]$ is absolute between $\mathrm{L}[G]$ and $\mathrm{L}[G\restriction e\times\omega]$. So, $\mathbb{P}$ is n-absolute for slices.

Proof. Let $1\le k\le n$ and $\varphi\in\Sigma^1_k$ with parameters in $\mathrm{L}[G\restriction e\times\omega]$. We show that φ is absolute by induction on k.

The case $k=1$ is just analytic absoluteness.

If $\varphi\in\Sigma^1_{k+1}$, then there is a $\psi\in\Pi^1_n$ such that $\varphi=\exists x\psi(x)$. Furthermore, there is a countable set $d\subseteq e$ in L such that every parameter of φ is in $\mathrm{L}[G\restriction d\times\omega]$. Let $b\subseteq e\backslash d$ be countable infinite and in L.

Suppose that $\mathrm{L}[G]\models\varphi$. Then there is some $y\in L[G]$ such that $\mathrm{L}[G]\models\psi(y)$. We can find a countable infinite set c such that $y\in L[G\restriction(d\cup c)\times\omega]$. By Lemma 3.3.42, there is a $y'\in L[G\restriction(d\cup b)\times\omega]\subseteq L[G\restriction e\times\omega]$ such that $\mathrm{L}[G]\models\psi(y')$ and so

$\mathrm{L}[G{\restriction}e \times \omega] \models \psi(y')$ by induction. Therefore, $\mathrm{L}[G{\restriction}e \times \omega] \models \varphi$. The other direction follows directly from upwards-absoluteness.

For the last point, recall that $\mathbb{P}$ is n-absolute for slices if and only if for every unbounded set of slices $\mathcal{S}$ and every $s \in \mathcal{S}$, every Σ^1_n-formula with $\mathrm{L}[G{\restriction}s]$-real parameters is absolute between $\mathrm{L}[G{\restriction}s]$ and $\mathrm{L}[G]$. But certainly, for every unbounded set of slices $\mathcal{S}$, every $s \in \mathcal{S}$ is of the form $e \times \omega$ where $e \subseteq \omega_1$ is unbounded. So, the relevant absoluteness holds, and hence $\mathbb{P}$ is n-absolute for slices. □

The Kanovei-Lyubetsky Lemma gives conditions for a forcing notion in this framework to be n-absolute for slices. Kanovei & Lyubetsky focused on almost disjoint forcing; our generalisation applies to other forcing notions; in particular, we apply the Kanovei-Lyubetsky Lemma in Theorem 3.3.66 to a generalisation of Jensen forcing.

The principal model in Section 3.3.2 is built from an ω-slice-product of length ω_1, but we only use the first ω-many pieces of this product. The remaining ω_1-many pieces are included for the Kanovei-Lyubetsky Lemma, to ensure that $\mathbb{P}$ is n-absolute for slices. It is not known whether there is an analogous construction for forcing notions which are n-absolute for products of length ω:

Question 3.3.44. Is there a variant of the storage-order construction for constructing an ω-slice-product of length ω which is n-absolute for slices?

3.3.4 Jensen forcing and n-Jensen forcing

In this section, we finally prove the Slicing Theorem 3.3.15 based on a generalisation of Jensen forcing. Our presentation and notation are based on that of [67].

Jensen forcing

Here, we introduce Jensen's original construction from [105] (for a modern exposition, see [103, §28]). Jensen forcing was originally used to construct a model with the simplest possible non-constructible real: by Shoenfield's absoluteness theorem all Σ^1_2 reals are constructible; Jensen forcing gives a non-constructible Δ^1_3 real.

For us, three key properties of Jensen forcing are the following:

Theorem 3.3.45 (Jensen, [105, Lemmata 6 & 10, & Corollary 9])**.** In L, there is a c.c.c. forcing notion, $\mathbb{J}$, adding reals such that there is a unique $\mathbb{J}$-generic real in every generic extension by $\mathbb{J}$, and in every model the set of $\mathbb{J}$-generic reals is Π^1_2.

Our generalisation is based on Jensen's original construction from [105] rather than the modern construction as given in [103, Theorem 28.1]).

Following [67, page 5], we define the *meet* of two perfect trees, S and T, denoted $S \wedge T$, to be the largest perfect tree contained in $S \cap T$, if it exists.

Definition 3.3.46. A forcing notion $\mathbb{P} \subseteq \mathbb{S}$, is called *structured* if $\mathbb{P}$ is closed under meets, under finite unions, and for all $s \in 2^\omega$, $(2^{<\omega})_s \in \mathbb{P}$.[6]

[6] In [205], structured forcing notions are called *sufficiently closed*. We save closure for properties of topologies, hence we use different terminology.

For any structured forcing notion, $\mathbb{P}$, each generic filter is uniquely determined by a real, in the standard way: $x_G := \bigcup\{\mathsf{stem}(T) : T \in G\}$ is a real in $V[G]$. Conversely, $G = \{T \in \mathbb{P} : x_G \in [T]\}$ and so $V[G] = V[x_G]$. We call these the $\mathbb{P}$-*generic reals*.

If $\mathbb{P}$ is structured, we order $\mathbb{P} \times \omega$ by:

$$(S, n) \leq (T, m) :\iff S \leq T \wedge m \leq n \wedge 2^m \cap S = 2^m \cap T.$$

We denote the poset $(\mathbb{P} \times \omega, \leq)$ by $\mathbb{Q}(P)$, and call it the *fusion order* (for details on fusion orders, see e.g. [102, pages 15-17]).

If $\mathbb{P}$ is a forcing notion, let $\mathbb{P}^{<\omega}$ be the ω-product of $\mathbb{P}$ with finite support.

Definition 3.3.47 (Jensen's Operation, P^H)**.** Let M be a ctm of ZFC^- + "$\mathcal{P}(\omega)$ exists", $P \in M$ be structured, H be a $\mathbb{Q}(P)^{<\omega}$-generic filter over M, and $(T_k)_{k\in\omega}$ be the generic trees added by H. We define:

$$U := \{T_k \wedge S : S \in P,\ k \in \omega, \text{ and } T_k \wedge S \neq \emptyset\}.$$

Let P^H be the closure of $P \cup U$ under finite unions, and order P^H by inclusion.

This operation expands P to a new poset P^H, which has many nice properties. Crucially, predense sets of P are predense in P^H:

Lemma 3.3.48. Let M, P, H, and $(T_k)_{k\in\omega}$ be as in Definition 3.3.47. Then:

1. P^H is structured,
2. U is dense in P^H and $(T_k)_{k\in\omega}$ is a maximal antichain in P^H, and
3. every predense set $D \subseteq P$ in M remains predense in P^H.
4. every predense set $D \subseteq P^{<\omega}$ in M remains predense in $(P^H)^{<\omega}$.

Proof. Exactly as in [67, Propositions 2.4 & 2.5]. □

The forcing notion, $\mathbb{J}$, which we call *Jensen forcing*, can now be constructed in these terms as follows: in L, we recursively define a sequence $((\gamma_\xi, P_\xi))_{\xi\in\omega_1}$ of pairs of limit ordinals and forcing notion.

1. let P_0 be the closure of $\mathbb{C}$ under finite unions and let γ_0 be the least limit ordinal such that $P_0 \in \mathrm{L}_{\gamma_0}$,
2. if $\xi = \xi' + 1$, then we set $P_\xi := (P_{\xi'})^H$, where H is the $<_{\mathrm{L}}$-least $\mathbb{Q}(P_{\xi'})^{<\omega}$-generic filter over $\mathrm{L}_{\gamma_{\xi'}}$, and
3. if ξ is a limit, then we set $P_\xi := \bigcup_{\xi'\in\xi} P_{\xi'}$.

In the last two cases, let γ_ξ be the least limit ordinal such that $P_\xi \in \mathrm{L}_{\gamma_\xi}$ and $\omega^\omega \cap \mathrm{L}_{\gamma_{\xi+1}} \not\subseteq \mathrm{L}_{\gamma_\xi}$. Finally, we set $\mathbb{J} := \bigcup_{\xi\in\omega_1} P_\xi$. If a real x is $\mathbb{J}$-generic, we call it a *Jensen real*.

Jensen-like forcing

We now iterate Jensen's operation from Definition 3.3.47 to obtain a *Jensen-sequence* in order to define *Jensen-like forcing notions*.

The key to the generalisation is understanding why the set of Jensen reals has complexity Π^1_2. Roughly, a real is Jensen over L if and only if for all $\xi < \omega_1$, there is a $k \in \omega$ such that $x \in [T^\xi_n]$, where $(T^\xi_n)_{n\in\xi}$ is the set of $\mathbb{Q}(P_\xi)$-generic trees used in the construction $\mathbb{J}$ to define $P_{\xi+1}$. This $(T^\xi_n)_{n\in\xi}$ was constructed by always picking the $<_{\mathrm{L}}$-least $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter. So, by Lemma 2.1.19, $(T^\xi_n)_{n\in\xi}$ is $\Delta^{\mathbf{HC}}_1$, so the set of Jensen reals in L is $\Pi^{\mathbf{HC}}_1$. We generalise Jensen's construction by increasing the complexity of the sequence.

Definition 3.3.49. Let $\theta \leq \omega_1$. We say $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\theta}$ is a *Jensen-sequence* if, for every $\xi < \theta$:

1. P_0 is the closure of $\mathbb{C}$ under finite unions,
2. γ_ξ is a countable ordinal such that $\mathrm{L}_{\gamma_\xi} \models \mathsf{ZFC}^- +$ "$\mathscr{P}(\omega^\omega)$ exists" and $\omega^\omega \cap \mathrm{L}_{\gamma_\xi+1} \nsubseteq \mathrm{L}_{\gamma_\xi}$,
3. P_ξ is structured and in L_{γ_ξ},
4. if $\xi = \zeta + 1$, then there is a $\mathbb{Q}(P_\zeta)^{<\omega}$-generic filter $H_\zeta \in \mathrm{L}_{\gamma_\xi}$ over $\mathrm{L}_{\gamma_\zeta}$ such that $P_\xi = (P_\zeta)^{H_\zeta}$, and
5. $(P_\xi)_{\xi\in\theta}$ is continuous at limits.

As in the construction of $\mathbb{J}$ in Section 3.3.4, we can construct Jensen-sequences of length ω_1 by recursion. Whilst Jensen uses the $<_L$-least generic filter, we consider *all* possible forcing notions which are defined by Jensen-sequences, using any generic filter.

Definition 3.3.50. We call a forcing notion, $\mathbb{P}$, *Jensen-like* if there is ω_1-length Jensen-sequence, $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$, such that $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$.

The next major task is to show that Jensen-like forcing notions exhibit generalisations of the properties of $\mathbb{J}$ in Theorem 3.3.45. Obviously, predense sets remain predense at successor steps along a Jensen-sequence as those steps are instances of Jensen's operation; moreover, predense sets remain predense at limit steps and for the union of the sequence, and suitable products along a Jensen-sequence are c.c.c.:

Proposition 3.3.51. Let $\mathbb{P}$ be a Jensen-like forcing notion constructed from the sequence $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$, then:

1. for every $\xi < \omega_1$, every predense set $D \subseteq P_\xi$ in L_{γ_ξ} is predense in $\mathbb{P}$, and
2. for every $\xi < \omega_1$, every predense set $D \subseteq P_\xi^{<\omega}$ in L_{γ_ξ} is predense in $\mathbb{P}^{<\omega}$.

Proof. Suppose that there is some $p \in \mathbb{P}$ such that $p \perp D$. By definition, there is some $\zeta < \omega_1$ such that $p \in P_\zeta$. Let ζ_0 be minimal such that $\xi < \zeta_0$ and $p \in P_{\zeta_0}$. Then D is not predense in P_{ζ_0} and ζ_0 is a successor. Since $\xi < \zeta_0$, $D \in \mathrm{L}_{\gamma_\xi} \subseteq \mathrm{L}_{\gamma_{\zeta_0-1}}$. But this contradicts Lemma 3.3.48. The case where $D \subseteq P_\xi^{<\omega}$ is similar. □

Proposition 3.3.52. If $\mathbb{P}$ is a Jensen-like forcing notion, then $\mathbb{P}$ and $\mathbb{P}^{<\omega}$, are both c.c.c. in L.

Proof. It suffices to prove the result for $\mathbb{P}^{<\omega}$. By a condensation argument (as in [45, Theorem 5.2]), we can reduce this to Proposition 3.3.51 (following the method in [105, Lemma 6]). $\square$

Corollary 3.3.53. Let $\mathbb{P}$ be a Jensen-like forcing notion, and $\mathbb{Q}$ be the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1. Then $\mathbb{Q}$ is c.c.c. in L.

Proof. This follows from Proposition 3.3.52 and a Δ-system argument (as in, e.g. [103, page 188]). $\square$

So, we have generalised one of the three key properties of Jensen forcing, c.c.c.; next, we show that the complexity of the set of generics is bounded. For this, we use a preliminary result:

Lemma 3.3.54. Let $\mathbb{P}$ be a Jensen-like forcing notion, let $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$ be a Jensen-sequence with $\mathbb{P} := \bigcup_{\xi\in\omega_1} P_\xi$, and let $(T^\xi_k)_{k\in\omega}$ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic sequence used to construct $P_{\xi+1}$. The following are equivalent:

1. a real x is $\mathbb{P}$-generic over L,
2. for every $\xi < \omega_1$, x is P_ξ-generic over L_{γ_ξ}, and
3. for every $\xi < \omega_1$, there is some $k \in \omega$ such that $x \in [T^\xi_k]$.

Proof. We start with the argument from Part 1. to Part 3. Let $\xi < \omega_1$. By Lemma 3.3.48, $\{T^\xi_k : k \in \omega\}$ is a maximal anichain in $P_{\xi+1}$ and by Proposition 3.3.51, it remains predense in $\mathbb{P}$. Hence, there is some $k \in \omega$ such that $x \in [T^\xi_k]$.

From Part 3. to Part 2., let $\xi < \omega_1$ and let $D \subseteq P_\xi$ be dense in L_{γ_ξ}. By assumption, there is some $k \in \omega$ such that $x \in [T^\xi_k]$. We define

$$E := \{q \in \mathbb{Q}(P_\xi)^{<\omega} : q(k) = (T, n) \wedge \forall s \in 2^n \cap T(T_s \in D)\}.$$

First, we prove that E is dense in $\mathbb{Q}(P_\xi)^{<\omega}$: let $q \in \mathbb{Q}(P_\xi)^{<\omega}$ and let $q(k) = (T, n)$. For every $s \in 2^n \cap T$, $T_s \in P_\xi$, so there is an $S_s \in D$ with $S_s \leq T_s$. Let $S := \bigcup_{s\in 2^n\cap T} S_s$ and $q' \in \mathbb{Q}(P_\xi)^{<\omega}$ by $q'(k) := (S, n)$ and for $\ell \neq k$, $q'(\ell) := q(\ell)$. Then $(S, n) \leq (T, n)$ and so $q' \leq q$ and $q' \in E$. Hence, E is dense in $\mathbb{Q}(P_\xi)^{<\omega}$. Let H_ξ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter corresponding to $(T^\xi_k)_{k\in\omega}$. Then E meets H_ξ, so there is a pair $(T, n) \in \mathbb{Q}(P_\xi)$ such that for every $s \in 2^n \cap T$, $T_s \in D$ and $T^\xi_k \subseteq T$. Let $s \in 2^n \cap T$ such that $s \subseteq x$. Then $x \in [T_s]$ and $T_s \in D$. Thus, x is P_ξ-generic over L_{γ_ξ}.

Finally, from Part 2. to Part 1., let $\mathcal{A} \subseteq \mathbb{P}$ be a maximal antichain. Since $\mathbb{P}$ is c.c.c., $\mathcal{A}$ is countable, so let $\xi < \omega_1$ be such that $\mathcal{A} \subseteq P_\xi$ and $\mathcal{A} \in \mathrm{L}_{\gamma_\xi}$. By assumption, x is P_ξ-generic over L_{γ_ξ}, so there is a $T \in P_\xi \subseteq P$ with $x \in [T]$. $\square$

So, checking the complexity of the set of generics reduces to checking complexity of the Jensen-sequence itself:

Corollary 3.3.55. Let $\mathbb{P}$ be Jensen-like, and $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi\in\omega_1} P_\xi$. For each $n \geq 2$, if $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$ is $\Delta^{\mathbf{HC}}_{n-1}$, then the set of $\mathbb{P}$-generics is Π^1_n in every transitive model of ZFC.

Proof. Let $M \models \mathsf{ZFC}$ and contain L. By Lemma 3.3.54, a real is $\mathbb{P}$-generic over L if and only if x is P_ξ-generic over L_{γ_ξ}. As ω_1^{L} and $\mathrm{L}_{\omega_1^L}$ are in M, $(\mathrm{L}_{\gamma_\xi}, P_\xi)_{\xi\in(\omega_1)^L)}$ is $\Delta^{\mathbf{HC}}_{n-1}$ in M. So, the set of $\mathbb{P}$-generics over L, G_R, is $\Delta^{\mathbf{HC}}_{n-1}$ in M, and so by Lemma 2.1.18, G_R is Π^1_n. □

The final of the three important properties to generalise is that the generics are suitably unique. Then, we calculate the complexity of the set of generics in analogy to Theorem 3.3.45. This yields the requisite failure of descriptive choice in our final model. We begin by showing the uniqueness of reals in a Jensen-like forcing notion, from which we prove the uniqueness of reals for ω-slice-products of Jensen-like forcing notions.

Lemma 3.3.56. Let $\mathbb{P}$ be a Jensen-like forcing notion. Then for every $\mathbb{P}$-generic filter, G, over L, the set of $\mathbb{P}$-generics over L is a singleton in $\mathrm{L}[G]$.

Proof. Let x and y be $\mathbb{P}$-generic reals over L, where x corresponds to G and $y \neq x$. We use a product argument to show that (x, y) is $(\mathbb{P}\times\mathbb{P})$-generic over L, and hence $y \notin \mathrm{L}[G]$.

Let $((\mathrm{L}_{\gamma_\xi}, \mathbb{P}_\xi))_{\xi\in\omega_1}$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi\in\omega_1} \mathbb{P}_\xi$. As in Lemma 3.3.54, it suffices to show that for every $\xi < \omega_1$, (x, y) is $(\mathbb{P}_\xi \times \mathbb{P}_\xi)$-generic over L_{γ_ξ}.

Let $\xi < \omega_1$, $G_\xi := \{(S, S') \in \mathbb{P}_\xi \times \mathbb{P}_\xi : x \in [S] \wedge y \in [S']\}$, and $D \subseteq \mathbb{P}_\xi \times \mathbb{P}_\xi$ be open dense in L_{γ_ξ}. We first show that $G_\xi \cap D \neq \emptyset$. Let $m \in \omega$ be such that $x{\restriction}m \neq y{\restriction}m$, and let E be the set of all $q \in \mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$ such that:

1. for every $\ell \in \mathsf{Dom}(q)$, if $q(\ell) = (S, n)$, then $n > m$, and
2. for every $\ell, \ell' \in \mathsf{Dom}(q)$, if $q(\ell) = (S, n)$ and $q(\ell') = (S', n')$, then for every $s \in 2^n \cap S$ and $s' \in 2^{n'} \cap S'$, either $(S_s, S'_{s'}) \in D$ or both $\ell = \ell'$ and $s = s'$.

Since D is open dense in $\mathbb{P}_\xi\times\mathbb{P}_\xi$, E is dense in $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$. Let H_ξ be the $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$-generic filter over L_{γ_ξ} which was used to construct $\mathbb{P}_{\xi+1}$, and let $(T^\xi_k)_{k\in\omega}$ be its corresponding $\mathbb{Q}(\mathbb{P}_\xi)^{<\omega}$-generic sequence. So, there is a $q \in H_\xi \cap E$. By Lemma 3.3.54, there are $k, k' \in \omega$ such that $x \in [T^\xi_k]$ and $y \in [T^\xi_{k'}]$. Let $(S, n), (S', n') \in \mathbb{Q}(\mathbb{P}_\xi)$ be such that $q(k) = (S, n)$ and $q(k') = (S', n')$. We define $s := x{\restriction}n$ and $s' := y{\restriction}n'$. Then $x \in [S_s], y \in [S'_{s'}]$, and $(S_s, S'_{s'}) \in D$. So, G_ξ meets D, so (x, y) is $(\mathbb{P} \times \mathbb{P})$-generic over L, as required. Hence y is $\mathbb{P}$-generic over $\mathrm{L}[x]$, and $y \notin \mathrm{L}[x]$. Hence, in particular, $y \notin \mathrm{L}[G]$. □

We use this to prove that the ω-slice-product with finite support of length ω_1 for such a Jensen-like forcing notion also has a unique generic real. Our approach follows that of Kanovei and Lyubetsky [110], who show that for every $\mathbb{J}^{<\omega}$-generic filter G over L, a real $y \in \mathrm{L}[G]$ is Jensen over L if and only if there is some $k \in \omega$ such that $y = x^k_G$, where x^k_G is the Jensen real added by the k^{th} coordinate of G. We use the corresponding result for Jensen-like forcing notions:

Lemma 3.3.57 (Kanovei-Lyubetsky)**.** Let M be a ctm of $\mathsf{ZFC}^- +$ "$\mathcal{P}(\omega)$ exists", and let $P \in M$ be structured. For every $k \in \omega$, let $\dot{x}^k_G$ be the canonical $P^{<\omega}$-name for the P-generic real which is added by the k^{th}-coordinate of the $P^{<\omega}$-generic filter. Let $\dot{y} \in M$ be a $P^{<\omega}$-name for a real such that for every $k \in \omega$, $1_{P^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$. Then for every $\mathbb{Q}(P)^{<\omega}$-filter H over M and every generic tree T_k added by H, in $M[H]$ the set of conditions forcing that $\dot{y} \notin [T_k]$ is dense in $(P^H)^{<\omega}$.

Proof. Let $p \in (P^H)^{<\omega}$ and let $m \in \omega$. We prove the density by constructing a condition $r_q \leq p$ such that $r_q \Vdash \dot{y} \notin [T_m]$. By Lemma 3.3.48, $(T_k)_{k\in\omega}$ is dense in P^H. So, without loss of generality, we assume that for every $k \in \mathsf{Dom}(p)$, there are $\ell_k \in \omega$ and $S_k \in P$ such that $p(k) = T_{\ell_k} \wedge S_k$, and that there is some $k_0 \in \mathsf{Dom}(p)$ such that $\ell_{k_0} = m$.

Claim 3.3.58. There is a $q \in H$ such that $\{\ell_k : k \in \mathsf{Dom}(p)\} \subseteq \mathsf{Dom}(q)$ and for every $k \in \mathsf{Dom}(p)$, there is some $s_k \in R_{\ell_k}$ such that:

- A1. $\mathsf{len}(s_k) = n_{\ell_k}$,
- A2. if $k' \in \mathsf{Dom}(p)$ with $k \neq k'$, then $s_k \neq s'_k$, and
- A3. $(R_{\ell_k})_{s_k} \leq S_k$,

where $R_{\ell_k} \in P$ and $n_{\ell_k} \in \omega$ such that $q(\ell_k) = (R_{\ell_k}, n_{\ell_k})$.

Proof. The proof follows essentially the shape of [67, Theorem 3.1]. Let $\dot{T}_{\ell_k}$ be the canonical $\mathbb{Q}(P)^{<\omega}$-name for T_{ℓ_k}. For all $k \in \omega$, $T_{\ell_k} \wedge S_k \neq \emptyset$. So, there is a $q' \in H$ with $\{\ell_k : k \in \mathsf{Dom}(p)\} \subseteq \mathsf{Dom}(q')$, such that for every $k \in \mathsf{Dom}(p), q' \Vdash \dot{T}_{\ell_k} \wedge \check{S}_k \neq \emptyset$. It suffices to show that $\{s_k : s_k$ satisfies Properties A1., A2., and A3.$\}$ is dense below q'.

Let $q'' \leq q'$, $\ell \in \mathsf{Dom}(q'')$ and $A_\ell := \{k \in \mathsf{Dom}(p) : \ell_k = \ell\}$. Then there are $R_\ell \in P$ and $n_\ell \in \omega$ such that $q''(\ell) = (R_\ell, n_\ell)$. For every $k \in \mathsf{Dom}(p)$, we have that $q'' \Vdash \dot{T}_{\ell_k} \wedge \check{S}_k \neq \emptyset$. So, for every $k \in A_\ell$, there is an $s_k \in R_\ell$ such that $\mathsf{len}(s_k) \geq n_\ell$ and $(R_\ell)_{s_k} \wedge S_k \neq \emptyset$. Without loss of generality, we can assume that for every $k \neq k' \in A_\ell, \mathsf{len}(s_k) = \mathsf{len}(s'_k)$ and $s_k \neq s'_k$. Let R'_ℓ be the tree resulting from replacing $(R_\ell)_{s_k}$ with $(R_\ell)_{s_k} \wedge S_k \in R_\ell$ for every $k \in A_\ell$. We define $q \leq q''$ like so:

$$q(\ell) := \begin{cases} (R'_\ell, \mathsf{len}(s_k)) & \text{if } A_\ell \neq \emptyset, \\ q''(\ell) & \text{otherwise.} \end{cases}$$

Then q satisfies Properties A1., A2., and A3. □

Claim 3.3.59. Let $q \in H$ be the condition from Claim 3.3.58. There is a condition $r_q \in P^{<\omega}$ such that:

- B1. for every $k \in \mathsf{Dom}(p), r_q(k) \leq (R_{\ell_k})_{s_k}$,
- B2. there is a finite set $A_q \subseteq \mathsf{Dom}(r_q)$ such that for every $s \in R_m \cap 2^{n_m}$, there is some $k \in A_q$ such that $r_q(k) \leq (R_m)_s$, and
- B3. for every $k \in \mathsf{Dom}(p) \cup A_q, r_q \Vdash \dot{y} \notin [r_q(k)]$

Proof. Let $B := (R_m \cap 2^{n_m}) \setminus \{s_k : \ell_k = m\}$, $A \subseteq \omega \setminus \mathsf{Dom}(p)$ be such that $|A| = |B|$, and $\{t_k : k \in A\}$ enumerate B. We define $r \in P^{<\omega}$ like so:

$$r(k) := \begin{cases} (R_{\ell_k})_{s_k} & \text{if } k \in \mathsf{Dom}(p), \\ (R_m)_{t_k} & \text{if } k \in A. \end{cases}$$

For every $k \in \omega$, we have that $1_{P^{<\omega}} \Vdash \dot{y} \neq \dot{x}_k^G$. So, there is a condition $r_q \leq r$ such that for every $k \in \mathsf{Dom}(p) \cup A$, $r_q \Vdash \dot{y} \notin [r_q(k)]$. Then r_q satisfies Properties B1., B2., and B3., with $A_q := A \cup \{k \in \mathsf{Dom}(p) : \ell_k = m\}$. □

We use r_q to define a condition $c(q) \leq q$. Let $\ell \in \mathsf{Dom}(q)$ be such that $A_\ell := \{k \in \mathsf{Dom}(p) : \ell_k = l\} \neq \emptyset$. Let R'_ℓ be the tree resulting from replacing $(R_\ell)_{s_k}$ with $r_q(k) \in R_\ell$ for every $k \in A_\ell$, and if $\ell = m$, also replacing $(R_\ell)_{t_k}$ with $r_q(k)$ for every $k \in A$. We define $c(q) \leq q$ like so:

$$c(q)(\ell) := \begin{cases} ((R'_\ell, n_\ell) & \text{if } A_\ell \neq \emptyset, \\ q(\ell) & \text{otherwise.} \end{cases}$$

Similarly, we can construct such a $c(r)$ for every $r \leq q$. Let $D := \{c(r) : r \leq q\} \subseteq \mathbb{Q}(P)^{<\omega}$. Then D is dense below q. Since H is $\mathbb{Q}(P)^{<\omega}$-generic, $G \cap D \neq \emptyset$. Without loss of generality, we can assume that $c(q) \in H$. So, for every $k \in \mathsf{Dom}(p)$, $(T_{\ell_k})_{s_k} \leq r_q(k) \wedge S_n$. Hence p and r_q are compatible.

To complete the proof, it suffices to show that $r_q \Vdash \dot{y} \notin [T_m]$. Let G be a $(P^H)^{<\omega}$-generic filter over $M[H]$ containing r_q. By Lemma 3.3.48, every predense subset of $P^{<\omega}$ in M remains predense in $(P^H)^{<\omega}$. Hence, $G' := G \cap P^{<\omega}$ is a $P^{<\omega}$-generic filter over M containing r_q. Then, in $M[G']$, for every $k \in \mathsf{Dom}(p) \cup A_q$, $\dot{y}_{G'} \notin [r_q(k)]$. By absoluteness, this is also true in $M[H][G]$. Since $c(q) \in H$, for every $s \in T_m \cap 2^{n_m}$, there is some $k \in A_q$ such that $(T_m)_s \leq r_q(k)$. Therefore, $\dot{y}_G \notin [T_m]$ and so $r_q \Vdash \dot{y} \notin [T_m]$. □

We generalise this to the products of Jensen-like forcing notions, such that the only generics in the extension are those defined from the coordinates.

Proposition 3.3.60. Let $\mathbb{P}$ be a Jensen-like forcing notion, G be a $\mathbb{P}^{<\omega}$-generic filter over L, and let $(x_G^k)_{k\in\omega}$ be the sequence of $\mathbb{P}$-generic reals corresponding to G. Then for every $\mathbb{P}$-generic real $y \in \mathrm{L}[G]$ over L, there is some $k \in \omega$ such that $y = x_G^k$.

Proof. We use a condensation argument [45, Theorem 5.2], with an appeal to Lemma 3.3.57.

For a contradiction, suppose that there is a $y \in \mathrm{L}[G]$ such that y is $\mathbb{P}$-generic over L and for every $k \in \omega$, $y \neq x_G^k$. By Proposition 3.3.52, P satisfies the c.c.c., so there is a countable $P^{<\omega}$-name, $\dot{y}$, for y. Let $\dot{x}_G^k$ be the canonical $\mathbb{P}^{<\omega}$-name for the $\mathbb{P}$-generic real which is added by the k^{th} coordinate of G. Without loss of generality, we can assume that for every $k \in \omega$, $1_{\mathbb{P}^{<\omega}} \Vdash \dot{y} \neq \dot{x}_G^k$. Let $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi\in\omega_1}$ be a Jensen-sequence such that $\mathbb{P} = \bigcup_{\xi\in\omega_1} P_\xi$. Let X be a countable elementary submodel of L_{ω_2} which contains the following: $(P_\xi)_{\xi\in\omega_1}$, $\mathbb{P}$, $\mathbb{P}^{<\omega}$, $\dot{y}$, and $\dot{x}_G^k$ for every $k \in \omega$. Let M be the transitive collapse of X, and $\pi : X \to M$ be the collapsing isomorphism.

By condensation [45, Theorem 5.2], there is a $\zeta < \omega_1$ such that $M = \mathrm{L}_\zeta$, and a $\xi < \zeta$ such that $X \cap \mathrm{L}_{\omega_1} = \mathrm{L}_\xi$. So, $\pi(\omega_1) = \xi$ and for every $x \in X \cap \mathrm{L}_{\omega_1}$, $\pi(x) = x$. Moreover, if $x \subseteq \mathrm{L}_{\omega_2}$, then $\pi(x) = x \cap \mathrm{L}_\xi$. So, $\pi(\mathbb{P}) = P_\xi$, $\pi(\dot{y}) = \dot{y}$, and for every $k \in \omega$, $\pi(\dot{x}^k_G) = \dot{x}^k_G \cap (P_\xi)^{<\omega}$. Hence, for every $k \in \omega$, $\pi(\dot{x}^k_G)$ is the canonical $(P_\xi)^{<\omega}$-name for the P_ξ-generic real added by the k^{th} coordinate of G. Let $k \in \omega$. By elementarity, $1_{\mathbb{P}^{<\omega}} \Vdash \dot{y} \neq \dot{x}^k_G$ in X and so $1_{(P_\xi)^{<\omega}} \Vdash \dot{y} \neq \pi(\dot{x}^k_G)$ in L_ζ as well. By definition, $\gamma_\xi \geq \xi$, and $\omega^\omega \cap \mathrm{L}_{\gamma_{\xi+1}} \subsetneq \mathrm{L}_{\gamma_\xi}$. So, $\gamma_\xi \geq \zeta$. Thus, $1_{(P_\xi)^{<\omega}} \Vdash \dot{y} \neq \pi(\dot{x}^k_G)$ in L_{γ_ξ}.

Let $H \in \mathrm{L}_{\gamma_\xi+1}$ be the $\mathbb{Q}(P_\xi)^{<\omega}$-generic filter over L_{γ_ξ} such that $(P_\xi)^H = P_{\xi+1}$ and let $(T_k)_{k\in\omega}$ be the sequence of $\mathbb{Q}(P_\xi)$-generic trees corresponding to H. By Lemma 3.3.54, there is a $k \in \omega$ such that $y \in [T_k]$ in $\mathrm{L}[G]$. So, by Lemma 3.3.57, for every $k \in \omega$, the set $\{p \in P^{<\omega}_{\xi+1} : p \Vdash \dot{y} \notin [T_k]\}$ is dense in $P^{<\omega}_{\xi+1}$. Hence, there is some $p \in G_{\xi+1}$ such that $p \Vdash \dot{y} \notin [T_k]$ and so $\mathrm{L}_{\gamma_\xi+1}[G_{\xi+1}] \vDash y \notin [T_k]$. By analytic absoluteness, $\mathrm{L}[G] \vDash y \notin [T_k]$. But this contradicts the fact that $y \in [T_k]$ in $L[G]$. So, there is a $k \in \omega$ such that $y = x^k_G$. □

Corollary 3.3.61. Let $\mathbb{P}$ be a Jensen-like forcing notion and let $\mathbb{Q}$ be the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1, let $I \subseteq \omega_1 \times \omega$. Let G be a $\mathbb{Q}{\restriction}I$-generic filter over L. Finally, let x^i_g be the $\mathbb{P}$-generic real corresponding to $G{\restriction}\{i\}$. Then for every $\mathbb{P}$-generic real $y \in \mathrm{L}[G]$ over L, there is some $i \in I$ such that $y = x^i_G$.

Proof. Let $y \in L[G]$ be $\mathbb{P}$-generic over L and let $\dot{y}$ be a $\mathbb{Q}{\restriction}I$-name for y. By Corollary 3.3.53, $\mathbb{Q}{\restriction}I$ is c.c.c. is and so $\dot{y}$ is countable. Hence, there is a countably infinite $I' \subseteq I$ such that $\dot{y}$ is a $\mathbb{Q}{\restriction}I'$-name. Since I' is countable, $\mathbb{Q}{\restriction}I'$ is order isomorphic to $\mathbb{P}^{<\omega}$. Hence, we can apply Proposition 3.3.60 and so there is an $i \in I'$ such that $y = x^i_G$. □

n-Jensen Forcing

In our terminology, Kanovei and Lyubetsky proved that ω-slice-products of almost disjoint forcing are n-absolute for slices [111, §4]). We apply this to Jensen-like forcing notions.

Definition 3.3.62. Let $\mathcal{M}_\mathcal{J}$ be the set of all (M, P, μ) with the following 4 properties:

1. there is a countable ordinal γ such that $M = L_\gamma$ and $M \vDash \mathsf{ZFC}^- +$ "$\mathscr{P}(\omega)$ exists",
2. $\mu > 0$ is a countable ordinal in M,
3. P is structured, and
4. there is a $\theta < \omega_1$ and a Jensen-sequence $J \in M$ such that $J^\frown(M, P)$ is a Jensen-sequence.

We order the $(M, P), (N, Q)$ such that $(M, P, \mu), (N, Q, \nu) \in \mathcal{M}_\mathcal{J}$ by $(N, Q) \preccurlyeq (M, P)$ if and only if $\nu \leq \mu$, $N \subseteq M$, and either $Q = P$ or else there is a Jensen-sequence $J \in M$ such that:

5. there is some $\xi \in \mathsf{Dom}(J)$ such that $J(\xi) = (N, Q)$ and

6. $J^\frown(M, P)$ is a Jensen-sequence.

Let $\theta \leq \omega_1$. A $\preccurlyeq$-increasing sequence $((M_\xi, P_\xi))_{\xi\in\theta}$ in $\mathcal{M}_\mathcal{J}$ is *strict* if $M_\xi \subsetneq M_{\xi+1}$ and $P_\xi \subsetneq P_{\xi+1}$ for every $\xi < \theta$.

Lemma 3.3.63. Let $0 < \theta \leq \omega_1$ and let $((M_\xi, P_\xi))_{\xi\in\theta}$ be a strictly $\preccurlyeq$-increasing sequence which is continuous at limits. Then $\bigcup_{\xi\in\theta} P_\xi$ is a Jensen-like forcing notion.

Proof. It is enough to show that there is a Jensen-sequence which contains $((M_\xi, P_\xi))_{\xi\in\theta}$ as an unbounded subsequence. We build a sequence $(J_\zeta)_{\zeta\in\theta}$ of Jensen-sequences such that for every $0 < \zeta \leq \theta$, the following hold:

1. $((M_\xi, P_\xi))_{\xi\in\theta}$ is an unbounded subsequence of J_ζ,

2. if $\zeta' < \zeta < \theta$, then J_ζ extends $J_{\zeta'}$, and

3. $(J_\zeta)_{\zeta\in\theta}$ is continuous at limits.

Then J_θ is the desired Jensen-sequence. We define the J_ζ recursively. By definition, there is a Jensen-sequence $J_0 \in M_0$, such that $J_0{}^\frown(M_0, P_0)$ is also a Jensen-sequence. We set $J_1 := J_0{}^\frown(M_0, P_0)$. Suppose $J_{\zeta'}$ is already defined for $\zeta' < \zeta$. Then there are three cases.

1. If $\zeta = \zeta' + 1$ and $\zeta' = \zeta'' + 1$, then $\mathsf{Dom}(J_{\zeta'})$ is a successor ordinal $\eta = \eta' + 1$ and $J_{\zeta'}(\eta') = (M_{\zeta''}, P_{\zeta''})$. By construction $(M_{\zeta''}, P_{\zeta''}) \preccurlyeq (M_{\zeta'}, P_{\zeta'})$ and the sequence is strict, so there is a Jensen-sequence $J_{\zeta_0} \in M_{\zeta'}$ such that there is some $\xi \in \mathsf{Dom}(J_{\zeta_0})$ with $J_{\zeta_0}(\xi) = (M_{\zeta''}, P_{\zeta''})$ where $J_{\zeta_0}{}^\frown(M_{\zeta'}, P_{\zeta'})$ is a Jensen-sequence. Let $J_{\zeta''}$ be the restriction of J_{ζ_0} to $\mathsf{Dom}(J_{\zeta_0})\setminus(\xi + 1)$. We set $J_\zeta := J_{\zeta_0}{}^\frown J_{\zeta''}{}^\frown(M_{\zeta'}, P_{\zeta'})$.

2. If $\zeta = \zeta' + 1$ and ζ' is a limit, then we set $J_\zeta := J_{\zeta'}{}^\frown(M_{\zeta'}, P_{\zeta'})$. Since $(M_\xi, P_\xi))_{\xi\in\zeta'}$ is an unbounded subsequence of $J_{\zeta'}$ and $P_{\zeta'} = \bigcup_{\xi\in\zeta'} P_\xi$, J_ζ is a Jensen-sequence.

3. Otherwise ζ is a limit. Then, for every $\zeta' < \zeta$, there is a Jensen-sequence $J_{\zeta'}$ which contains $((M_\xi, P_\xi))_{\xi\in\zeta'}$ as an unbounded subsequence. Let $\alpha < \zeta$ be the largest limit ordinal and for every $\alpha \leq \zeta' < \zeta$, and let $\xi_{\zeta'} < \mathsf{len}(J_{\zeta'+1})$ be such that $J_{\zeta'+1}(\xi_{\zeta'}) = (M_{\zeta'}, P_{\zeta'})$. Then:

$$J_\alpha^\frown(J_{\alpha+1}\restriction(\mathsf{len}(J_{\alpha+1}) \setminus \xi_\alpha))^\frown ...^\frown(J_{\alpha+n+1}\restriction(\mathsf{len}(J_{\alpha+n+1}) \setminus \xi_{\alpha+n}))^\frown ...$$

is a Jensen-sequence which contains $((M_\xi, P_\xi))_{\xi\in\zeta}$ as an unbounded subsequence.

□

Recall that if (M, P) is in a repetitive storage order, P is an ω-slice-product of a unique Q_P. Hence for every such P, there is a unique pair (Q_P, μ_P) such that P is the ω-slice-product of Q_P with finite support. We now identify each pair (M, P) in a repetitive storage order with a triple (M, Q_P, μ_P). This makes $\mathcal{M}_\mathcal{J}$ repetitive:

Lemma 3.3.64. The set $(\mathcal{M}_{\mathcal{J}}, \preccurlyeq)$ is a repetitive storage order.

Proof. We first check that $(\mathcal{M}_{\mathcal{J}}, \preccurlyeq)$ is a partial order, then check that it is a repetitive storage order. Reflexivity is clear. For antisymmetry, suppose $(M, P, \mu) \preccurlyeq (N, Q, \nu) \preccurlyeq (M, P, \mu)$. Automatically, $M = N$ and $\nu = \mu$. Then note that for each Jensen-sequence, $((\mathrm{L}_{\gamma_\xi}, P_\xi))_{\xi \in \zeta}$, and for each $\xi' < \xi < \zeta$, we have that $\mathrm{L}_{\gamma_{\xi'}} \subsetneq \mathrm{L}_{\gamma_\xi}$, so the only possibility is $P = Q$. Finally we check transitivity. Let $(M, P, \mu) \preccurlyeq (M', P', \mu') \preccurlyeq (M'', P'', \mu'')$. By definition, $M \subseteq M' \subseteq M''$ and $\mu \leq \mu' \leq \mu''$. Suppose without loss of generality that P, P', and P'' are pairwise distinct. There is Jensen-sequence, J, with $\mathsf{len}(J) < \omega_1$ and an $\xi < \mathsf{len}(J)$ where $J(\xi) = (M, P)$ and $J^\frown(M', P')$ is a Jensen-sequence. Likewise, there is a J' such that $J(\xi') = (M', P')$ and $J'^\frown(M'', P'')$ is a Jensen-sequence. So, let $J'' := J^\frown(J' \restriction (\mathsf{len}(J') \setminus \xi'))$. Then J'' is a Jensen-sequence in M'' and $J''^\frown(M'', P'')$ is a Jensen-sequence, so $(M, P, \mu) \preccurlyeq (M'', P'', \mu'')$.

We need to check each condition of the definition of a storage order. Definition 3.3.20 Parts 1, 2, and 4 are clear. Definition 3.3.20 Part 3 follows from Proposition 3.3.51. So, we prove Definition 3.3.20 Part 5. Let $\zeta \leq \omega_1$ be a limit ordinal, and let $S = ((M_\xi, P_\xi, \mu_\xi))_{\xi \in \zeta}$ be a $\preccurlyeq$-strictly increasing sequence which is continuous at limits. By Lemma 3.3.63, there is a Jensen-sequence, J, which contains S as an unbounded subsequence. Then there are two cases:

1. If $\zeta < \omega_1$, we can find a Jensen-sequence, J' which properly extends J. We define $(M_\zeta, P_\zeta) := J'(\mathsf{len}(J))$. As ζ is a limit, $\mathsf{len}(J)$ is also a limit. So, $P_\zeta = \bigcup_{\xi \in \zeta} P_\xi$. Let μ_ζ be such that for all $\xi < \zeta$, $\mu_\xi < \mu_\zeta$. Without loss of generality, we assume that $\mu_\zeta \in M_\zeta$. So, $(M_\zeta, P_\zeta, \mu_\zeta) \in \mathcal{M}_{\mathcal{J}}$, and $(M_\zeta, P_\zeta, \mu_\zeta) \succcurlyeq (M_\xi, P_\xi, \mu_\xi)$ for each $\xi < \zeta$.

2. If $\zeta = \omega_1$, then $\mathbb{P} = \bigcup_{\xi \in \omega_1} P_\xi$ is a Jensen-like forcing notion, so it suffices to check the predense sets remain predense. Let $\mathbb{Q}$ be the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1, and for each $\xi < \omega_1$, let Q_ξ be the ω-slice-product of P_ξ with finite support of length μ_ξ. Let $D \subseteq Q_\xi$ be dense in M_ξ. Then, arguing as before, D is predense in $\mathbb{Q} \restriction (\omega \times \mu_\xi)$, so is predense in $\mathbb{Q}$.

So, $\mathcal{M}_{\mathcal{J}}$ is a storage order. But repetitiveness is clear, which completes the proof. □

Definition 3.3.65. Let $n \geq 2$. We say that $\mathbb{P}$ is *n-Jensen* if:

1. $\mathbb{P}$ is Jensen-like,

2. the set of $\mathbb{P}$-generic reals over L is Σ^1_n for every transitive model of ZFC, and

3. the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1 is n-absolute for slices.

So, the Jensen forcing notion is 2-Jensen. Note that not all $(n+1)$-Jensen forcing notions are n-Jensen, for example an $(n+1)$-Jensen forcing notion based on an n-complete sequence has a set of generics of exactly complexity Π^1_{n+1} (the exactness holds as, e.g. N satisfies $\mathsf{DC}(\omega^\omega; \Pi^1_n)$). An example of such a forcing notion is in the proof of the Slicing Theorem (Theorem 3.3.15). It is not known whether, up to forcing equivalence, there is exactly one n-Jensen forcing notion. We expect not.

From the Kanovei-Lyubetsky Lemma, we can prove that ω-slice-products defined from n-complete sequences are n-absolute for slices. We provide a forcing-equivalence argument to show exactly this, i.e. that Lemma 3.3.43 can be used to prove n-absoluteness for slices of an ω-slice-product.

Theorem 3.3.66. For all $n \geq 2$, there is an n-Jensen forcing notion.

Proof. The Jensen forcing notion is 2-Jensen, so we prove the case where $n > 2$. By Lemma 3.3.33, there is a storage sequence, $((M_\xi, P_\xi))_{\xi \in \omega_1}$, in $\mathcal{M}_{\mathcal{J}}$ which is $\Delta^{\mathbf{HC}}_{n-1}$ and n-complete. By Lemma 3.3.63, $\mathbb{P} = \bigcup_{\xi \in \omega_1} P_\xi$ is Jensen-like, so by Corollary 3.3.63, the set of $\mathbb{P}$-generic reals over L is Π^1_n in every transitive model of ZFC. By the Kanovei-Lyubetsky Lemma (Lemma 3.3.43), the ω-slice-product of $\mathbb{P}$ with finite support of length ω_1 is n-absolute for slices. So, this slice product is an n-Jensen forcing notion. □

Our final flourish is to prove the Slicing Theorem (Theorem 3.3.15). Recall that we use n-absolute for slices to ensure some choice, and a Π^1_n countable set of countable sets of generics, A, to witness the required failure of choice. If $\mathbb{P}$ is n-Jensen, then we only know that the set of $\mathbb{P}$-generic reals over L is Π^1_n in $\mathrm{L}[G]$. However, we show that every ω-slice-product of an n-Jensen forcing notion with finite support of length ω_1 includes some n-slicing forcing notion. We first show that an n-Jensen forcing notion is suitably equivalent to a forcing notion whose generics use their first bit to 'track' which slice they are from. We then build the ω-slice-product, and show that it is n-slicing.

Theorem 3.3.15 (Slicing Theorem)**.** Let $n \geq 2$. In L, there is an n-slicing forcing notion.

Proof. Fix an $n \geq 2$. By Theorem 3.3.66, there is an n-Jensen forcing notion, $\mathbb{P}$. For each $\nu < \omega_1$, we define:

$$\mathbb{P}_\nu := \begin{cases} \{T \in \mathbb{P} : \mathsf{stem}(T_0) = \nu + 1\} & \text{if } \nu \in \omega \\ \{T \in \mathbb{P} : \mathsf{stem}(T_0) = 0\} & \text{otherwise} \end{cases}$$

Let $k \in \omega$ and $x \in 2^\omega$ with $x(0) = k + 1$.

Claim 3.3.67. The real x is $\mathbb{P}$-generic over L if and only if x is $\mathbb{P}_k$-generic.

Proof. Let $G_x := \{T \in \mathbb{P}_k : x \in [T]\}$, and $H_x = \{T \in \mathbb{P} : x \in [T]\}$. It suffices to show that G_x is $\mathbb{P}_k$-generic over L if and only if H_x is $\mathbb{P}$-generic over L. For the reverse direction, suppose $D \subseteq \mathbb{P}_k$ is dense. Let $D' := D \cup \{T \in \mathbb{P} : \mathsf{len}(\mathrm{stem}(T)) \geq 1 \text{ and } T \notin \mathbb{P}_k\}$. Then D' is dense in $\mathbb{P}$, so there is some $T \in H_x \cap D'$. As $x \in T$ and $x(0) = k + 1$, $T \in \mathbb{P}_k$. So, $T \in G_x \cap D'$, so G_x is $\mathbb{P}_k$-generic over L. The converse is similar. □

Now let $\mathbb{Q}$ be the product of $\mathbb{P}$ with finite support of length ω_1, and let $\mathbb{Q}'$ ω-slice-product of $(P_\nu)_{\nu\in\omega_1}$ with finite support of length ω_1. We show that $\mathbb{Q}'$ is n-slicing.

Let G be a $\mathbb{Q}'$-generic filter over L. If D is dense in $\mathbb{Q}$, then $D\cap\mathbb{Q}'$ is dense in $\mathbb{Q}'$. So, $H := \{p \in \mathbb{Q} : \exists q \in H(q \leq p)\}$ is $\mathbb{Q}$-generic over L. As H can be constructed in $\mathrm{L}[G]$ and vice versa, $\mathrm{L}[H] = \mathrm{L}[G]$. Exactly similarly, for every $I \subseteq \omega_1 \times \omega$, $\mathrm{L}[G{\restriction}I] = \mathrm{L}[H{\restriction}I]$. So, $\mathbb{Q}'$ is n-absolute for slices.

It remains to show that $A := \{(\ell, x_G^{(\ell,k)}) : (\ell, k) \in \omega^2\}$ is Π^1_n in $\mathrm{L}[G]$. In $\mathrm{L}[G]$, let B be the set of all pairs (x, ℓ) where x is $\mathbb{P}$-generic and $x(0) = \ell+1$. Then B is Π^1_n. So, it suffices to show that $A = B$. Let $(\ell, x) \in A$. So, x is $\mathbb{P}_\ell$-generic over L and $x(0) = \ell+1$. By Claim 3.3.67, x is $\mathbb{P}$-generic over L, hence $(\ell, x) \in B$. Conversely, if $(\ell, x) \in B$, then x is $\mathbb{P}$-generic over L and $x(0) = \ell + 1$, so by Corollary 3.3.61, there is some $k \in \omega$ such that $x = x_H^{(\nu,k)}$, where $x_H^{(\nu,k)}$ is the $\mathbb{P}$-generic added by $H{\restriction}\{(\ell, k)\}$. So, for every $T \in G{\restriction}\{(\ell, k)\}$, $x \in [T]$. Hence $x = x_G^{(\nu,k)}$, so $(\ell, x) \in A$. So, $A = B$. □

In particular, a subset of the Jensen forcing notion is 2-slicing.

Returning to the original problem, we could use Kanovei and Lyubetsky's variant of almost disjoint forcing from [111] to prove Corollary 3.2.5, by adapting the proof of Theorem 3.2.4 to almost disjoint rather than Jensen forcing. This yields a model of $\mathsf{ZF} + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{Ctbl})$. As the product of this variant is n-absolute for slices (essentially by [111, Theorem 9]), $\mathsf{DC}(\omega^\omega; \Pi^1_n)$ also holds in this model. A compactness argument then yields Corollary 3.2.5. However, our proof of Theorem 3.2.4 does not work for almost disjoint forcing, as the generics are not unique: we can swap finitely many bits in a generic to generate another generic (as in [111, Lemma 9vi)]). In which case, we do not know the complexity of the set corresponding to A.

3.4 Non-uniform descriptive choice principles

We have so far only considered the hierarchy of uniform choice principles for '$\mathbf{\Pi}$' projective point classes, thanks to Theorem 3.1.2. By contrast, for the non-uniform choice principles, we know only that $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_{n+1})$ implies $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_n)$ and $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^1_n)$. Hence we ask:

Question 3.4.1. Is $\mathsf{AC}_\omega(\omega^\omega; \Pi^1_n)$ ZF-equivalent to $\mathsf{AC}_\omega(\omega^\omega; \Sigma^1_{n+1})$?

Question 3.4.2. Is $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_n)$ ZF-equivalent to $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^1_{n+1})$?

As well as the equivalences of non-uniform descriptive choice principles, we can ask about separating non-uniform descriptive choice principles. By Corollary 3.2.7, we can separate uniform choice for all projective pointclasses from the minimal non-uniform descriptive choice principle beyond ZF (namely $\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_2)$). It is not known whether the non-uniform projective choice principles can be separated from one another:

Question 3.4.3. Is there a model of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Pi}^1_n) + \neg\mathsf{AC}_\omega(\omega^\omega; \Pi^1_{n+1})$?

Finally, our principles are defined for arbitrary pointclasses, so we can ask about Borel choice principles.

Question 3.4.4. Is there a model if $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_\alpha) + \neg\mathsf{AC}_\omega(\omega^\omega; \mathbf{\Sigma}^0_{\alpha+1})$?

Question 3.4.5. Is there a model of $\mathsf{ZF}+\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Sigma}^0_\alpha)+\neg\mathsf{AC}_\omega(\omega^\omega; \mathrm{unif}\mathbf{\Sigma}^0_{\alpha+1})$?

Chapter 4

Elements of Generalised Real Analysis

This chapter is based on joint work with Lorenzo Galeotti

In this chapter, we investigate generalisations of real analysis to fields with large cardinality, focusing on the generalisation of the analysis of real functions.

In Section 4.1, we describe the generalisations and strengthenings of continuity and their relationships; Sections 4.2.1 and 4.3 contain the main development of the generalised real analysis of functions. In Section 4.2.1, we consider results that can be proved independently of the choice of the uncountable κ; in Section 4.3, we consider a result that can only be proved under the assumption that κ has the tree property. We end with an open question about the $\mathbb{R}_\kappa$-analogy of the Suslin Hypothesis (Section 4.4).

Throughout this chapter, we assume κ is a regular uncountable cardinal and that $\mathbb{K}$ is an ordered field, and work with the $\mathbb{K}$-interval $\mathsf{bn}(\mathbb{K})$-topology on $\mathbb{K}$.

4.1 Generalising continuity

Up to isomorphism, the real line is the only Dedekind complete ordered field, so in general, our ordered fields will not be Dedekind complete. In fact, Dedekind completeness is equivalent to many theorems of real analysis.

The failure to be Dedekind complete means that continuity does not necessarily imply all of its classical consequences; e.g., continuous functions need not satisfy the Intermediate Value Theorem (Example 4.2.2). In this section, we discuss strengthenings of the notion of continuity that are a better fit for non-Dedekind complete fields. The notions of κ-continuity and κ-supercontinuity come from the (pre)history of generalised real analysis [2, 74]; in this section, we introduce another notion, called *sharpness*, of intermediate strength between continuity and κ-supercontinuity. This notion is central to Section 4.3.

Definition 4.1.1. A κ-continuous function $f : \mathbb{K} \to \mathbb{K}$ is called *κ-supercontinuous* if for any $a, b \in \mathbb{K}$, $\mathsf{cof}(f([a,b])) < \kappa$ and $\mathsf{coi}(f([a,b])) < \kappa$.

For the next generalised notion of continuity, we use a technical auxiliary property from [25, Definition 4.1.6]: a sequence, $(a_\alpha)_{\alpha\in\kappa}$, is called *interval witnessed* if for every bounded convex $C \subseteq \mathbb{K}$ with $|C \cap \{a_\alpha : \alpha \in \kappa\}| = \kappa$, and every $\varepsilon \in \mathbb{K}^{>0}$, there is a $\mu < \kappa$ and some pairwise disjoint intervals $(I_\beta)_{\beta\in\mu}$ where:

1. for all $\beta < \kappa$, $\mathsf{len}(I_\beta) < \varepsilon$, and
2. $|(\{a_\alpha : \alpha \in \kappa\} \cap C) \setminus \bigcup_{\beta \in \mu} I_\beta| < \kappa$.

Definition 4.1.2. A continuous function $f : \mathbb{K} \to \mathbb{K}$ is called *κ-sharp* if for every $a, b \in \mathbb{K}$, if f has no maximum on $[a, b]$ then there is an interval witnessed sequence $(a_\alpha)_{\alpha\in\kappa}$, where each $a_\alpha \in [a, b]$, such that $(f(a_\alpha))_{\alpha\in\kappa}$ is increasing and cofinal in $f([a, b])$, and so too for $-f$.

We say that f is *sharp* if it is $\mathsf{bn}(\mathbb{K})$-sharp.

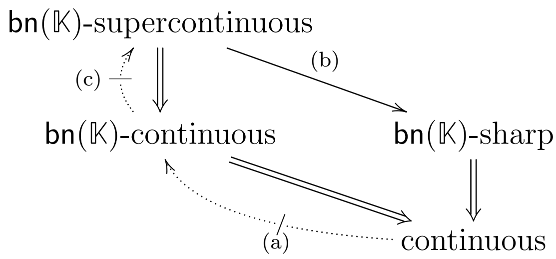

Figure 4.1: Implications between strengthenings of continuity. (a) Continuity does not imply $\mathsf{bn}(\mathbb{K})$-continuity on $\mathbb{R}$ (Proposition 4.1.3) and η_κ-fields (Example 4.1.4). (b) On η_κ fields, $\mathsf{bn}(\mathbb{K})$-supercontinuity implies sharpness (Corollary 4.2.20). (c) Under the conditions of Proposition 4.1.14, $\mathsf{bn}(\mathbb{K})$-continuity does not imply $\mathsf{bn}(\mathbb{K})$-supercontinuity.

By definition, every κ-supercontinuous function is κ-continuous and every sharp function is continuous; furthermore, every κ-continuous function is continuous (Corollary 2.2.1). Later, we see that on η_κ orders, every κ-supercontinuous function is sharp (Corollary 4.2.20). So, in this case, sharpness forms an intermediate generalised notion of continuity.

In the classical case, the notions of continuity and $\mathsf{bn}(\mathbb{R})$-continuity do not coincide: since $\mathsf{bn}(\mathbb{R}) = \omega$, we are considering ω-open sets, i.e. finite unions of open intervals. Clearly, $\sin : \mathbb{R} \to \mathbb{R}$ is continuous, but $\sin^{-1}((0, \frac{1}{2}))$ is not a finite union of open intervals, so it is not ω-continuous (and thus, by definition, not ω-supercontinuous). However, some of the distinctions disappear in the classical case; a summary of the implications and non-implications can be found in Figure 4.1.

Proposition 4.1.3. Let $f : \mathbb{R} \to \mathbb{R}$, then:

1. f is ω-continuous if and only if f is ω-supercontinuous and
2. f is sharp if and only if f is continuous.

Proof. For Part 1., all ω-continuous functions are continuous, hence, by the classical Extreme Value Theorem, they reach their maxima and minima, so are ω-supercontinuous. For Part 2., all continuous functions are bounded on any $[a, b]$, so are trivially sharp. $\square$

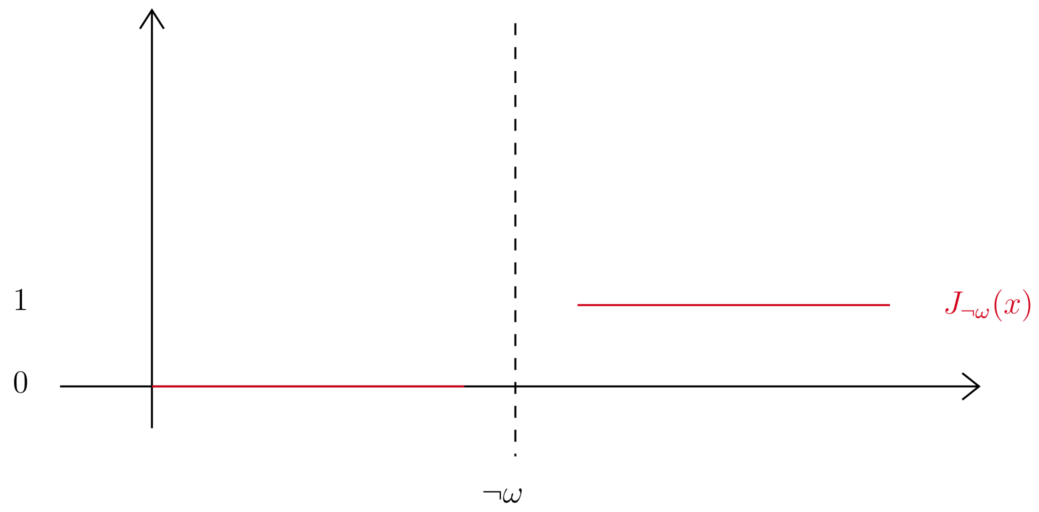

Figure 4.2: The $\neg\omega$-jump function, $J_{\neg\omega}(x)$

We see later (Proposition 4.1.14, Figure 4.1) that under certain conditions, the generalisation of Proposition 4.1.3, Part 1. fails. The following example shows that on non-Archimedean fields, continuity is blind to a change in output at a gap; furthermore, a function that jumps at a gap is an example of a continuous function that is not κ-continuous for η_κ fields.

Example 4.1.4. Let $\mathbb{K}$ be non-Archimedean. As discussed in Section 2.2.3, $\mathsf{bot}(\mathsf{INF})$ is an (ω, λ)-gap for some λ denoted by $\neg\omega$. Let

$$J_{\neg\omega} := \begin{cases} 0 & \text{if there is an } n \in \mathbb{N} \text{ such that } x < n, \\ 1 & \text{otherwise.} \end{cases}$$

be the $\neg\omega$-*jump function* (Figure 4.2). Then $J_{\neg\omega}$ is continuous on $\mathbb{K}$. If $\mathbb{K}$ is η_κ, then $J_{\neg\omega}$ is sharp but not κ-continuous.

This is because the set INF is not κ-open: if it were, then there would be a sequence of fewer than κ nested intervals whose union is INF; by the η_κ property, we could find an element between $\mathbb{N}$ and the set of lower bounds of these intervals, contradicting that $\neg\omega$ is a gap. In order to check that $J_{\neg\omega}$ is sharp, just observe that the image of any nonempty set is either $\{0\}$, $\{1\}$, or $\{0, 1\}$.

Homogeneity

Now, we show the failure of Proposition 4.1.3, Part 1. under certain conditions. To show this, we introduce a homogeneity property for orders, based on model-theoretic homogeneity (as in [144, Definition 4.2.12]).

Definition 4.1.5. An order, $\mathbb{O}$, is called *κ-homogeneous* if for all convex $I, J \subseteq \mathbb{O}$ such that $\mathsf{cof}(I) = \mathsf{cof}(J) \leq \kappa$ and $\mathsf{coi}(I) = \mathsf{coi}(J) \leq \kappa$, there is an order-preserving bijection $b : I \to J$.

If $\mathbb{G}$ is an ordered group, the map $x \mapsto -x$ is a order-reversing bijection with $\mathsf{coi}(I) = \mathsf{cof}(-I)$ and $\mathsf{cof}(I) = \mathsf{coi}(-I)$ for any set I. Thus, if $\mathbb{G}$ is κ-homogeneous and $\mathsf{cof}(I) = \mathsf{coi}(J) \leq \kappa$ and $\mathsf{coi}(I) = \mathsf{cof}(J) \leq \kappa$, then there is an order-reversing bijection $r : I \to J$.

We show that the ordered field $\mathbb{Q}_\kappa$ is κ-homogeneous, but do not give any further general conditions which ensure κ-homogeneity. Any ordered field is ω-homogeneous, by a gluing argument:

Remark 4.1.6. If $\mathbb{K}$ is an ordered field, 1-homogeneity follows immediately from scaling and shifting (i.e. double transitivity, see [178, page 126]): let $I, J \subseteq \mathbb{K}$ be such that $\mathsf{cof}(I) = \mathsf{cof}(J) = \mathsf{coi}(I) = \mathsf{coi}(I) = 1$. Then $I = [a, b]$ and $J = [c, d]$ for some $a, b, c, d \in \mathbb{K}$. So the map $f(x) = c + \frac{c-d}{b-a}(x - a)$ is an order isomorphism from I to J. Then, ω-homogeneity amounts to gluing these partial order isomorphisms along an ω-length coinitial (or cofinal) sequence.

Lemma 4.1.7. Suppose $|\kappa^{<\kappa}| = \kappa$. If $I, J \subseteq \mathbb{Q}_\kappa$ are such that $\mathsf{cof}(I) = \mathsf{cof}(J) = 1$ and $\mathsf{coi}(I) = \mathsf{coi}(J) = \kappa$, then there is an order-preserving isomorphism $b : I \to J$.

Proof. Without loss of generality, we can assume that $\mathsf{top}(I)$ and $\mathsf{top}(J)$ are gaps (note that if $\mathsf{top}(I), \mathsf{top}(J) \in \mathbb{Q}_\kappa$, then we can use the technique in Remark 4.1.6). Let $I = [a, A]$ and $J = [b, B]$ for some $a, b \in \mathbb{Q}_\kappa$, and gaps A, B. Then $(a, A]$ and $(b, B]$ are η_κ orders of size κ, which are dense in I and J respectively. By Theorem 2.2.2, there is an order isomorphism $f : \mathbb{Q}_\kappa \to (a, A]$ and an order isomorphism $g : \mathbb{Q}_\kappa \to (b, B]$. Hence, the map $(g \circ f) \cup \{(a, b)\}$ is an order isomorphism from $[a, A]$ to $[b, B]$. ☐

Proposition 4.1.8. If $|\kappa^{<\kappa}| = \kappa$, then $\mathbb{Q}_\kappa$ is κ-homogeneous.

Proof. By splitting I at some $a \in I$, it suffices to assume that $\mathsf{cof}\, I = \mathsf{cof}\, J = 1$. As in Lemma 4.1.7, we may assume that $I = (A, a)$ and $J = (B, b)$ for some almost gaps A, B and some $a, b \in \mathbb{Q}_\kappa$.

Then there are two cases. Firstly, suppose $\mathsf{coi}(A) = \mathsf{coi}(B) = \kappa$, then use Lemma 4.1.7. Otherwise, $\mathsf{coi}(A) = \mathsf{coi}(B) = \lambda < \kappa$. In which case, let $(a_\alpha)_{\alpha \in \lambda}$ be strictly decreasing and coinitial in A, and likewise $(b_\alpha)_{\alpha \in \lambda}$ in B. By Remark 4.1.6, there is a (linear) order isomorphism $f_\alpha : (a_{\alpha+1}, a_\alpha) \to (b_{\alpha+1}, b_\alpha)$. We inductively define a sequence of order isomorphisms $F_\alpha : (a_\alpha, a) \to (b_\alpha, b)$, for each $\alpha < \lambda$, such that if $\alpha < \beta$, then $F_\alpha \subsetneq F_\beta$.

Firstly, let $F_0 = f_0$. Suppose that F_α has been defined. Then let $F_{\alpha+1}$: $F_\alpha \cup f_{\alpha+1} \cup \{(a_{\alpha+1}, b_{\alpha+1})\}$ (i.e. $F_{\alpha+1}(a_{\alpha+1}) = b_{\alpha+1}$). Clearly, $F_{\alpha+1}$ is an order isomorphism and extends F_α.

Next, suppose $\mu < \lambda$ is a limit and that F_α has been defined for all $\alpha < \mu$. As $\mu < \kappa$, $(a_\alpha)_{\alpha \in \mu}$ is a short sequence, so this sequence has no $\mathbb{Q}_\kappa$-limit (as in [75, Corollary 1.11(ii)]). Hence $(a_\alpha)_{\alpha \in \mu}$ defines a (μ, κ)-gap G in $\mathbb{Q}_\kappa$,[1] likewise $(b_\alpha)_{\alpha \in \mu}$ defines a (μ, κ)-gap G'. So, it suffices to define an F_μ on $[a_\mu, G] = [a_\mu, a] \setminus \bigcup_{\alpha \in \mu} [a_0, a_\alpha]$. We assumed that $(a_\alpha)_{\alpha \in \lambda}$ is strictly increasing, so $a_\mu < G$. Similarly, $b_\mu < G'$. Hence, let $F_\mu : [a_\mu, G] \to [b_\mu, G']$ be the order isomorphism from Lemma 4.1.7.

Finally, let $F := \bigcup_{\alpha \in \lambda} F_\alpha$. By construction, $\mathsf{Dom}(F) = (A, a)$, and $\mathrm{Im}(F) = (B, b)$. Moreover, $F : (A, a) \to (B, b)$ is an order isomorphism, as required. ☐

[1] Suppose $(a_\alpha)_{\alpha \in \mu}$ is not eventually constant. As $\mathsf{bn}(\mathbb{Q}_\kappa) = \kappa$, $(a_\alpha)_{\alpha \in \mu}$ is not cofinal or coinitial $\mathbb{Q}_\kappa$. Hence $(a_\alpha)_{\alpha \in \mu}$ defines a gap.

This actually allows us to fix cofinally-many points of the order isomorphism: if $I, J \subseteq \mathbb{Q}_\kappa$ such that $\mathsf{cof}(I) = \mathsf{cof}(J) = \lambda$, then for any cofinal sequence $(a_\alpha)_{\alpha\in\lambda}$ in I, and cofinal $(b_\alpha)_{\alpha\in\lambda}$ in J, there is an order isomorphism $f : I \to J$ such that $f(a_\alpha) = b_\alpha$. The analogous result holds if $\mathsf{coi}(I) = \mathsf{coi}(J)$, and for the order-reversing case.

This also shows that $\mathsf{Ded}(\mathbb{Q}_\kappa)(= \mathsf{Ded}(\mathbb{R}_\kappa))$ is κ-homogeneous: if $I, J \subseteq \mathsf{Ded}(\mathbb{Q}_\kappa)$ have the same cofinality and coinitiality, let $f : I \cap \mathbb{Q}_\kappa \to J \cap \mathbb{Q}_\kappa$ be the order isomorphism. Then the natural extension, $\hat{f} : I \to J$, where if G is a gap is $I \cap \mathbb{Q}_\kappa$, it is sent to the corresponding gap in $J \cap \mathbb{Q}_\kappa$, is an order isomorphism.

So, $\mathbb{Q}_\kappa \subsetneq \mathbb{R}_\kappa \subsetneq \mathsf{Ded}(\mathbb{Q}_\kappa)$, where $\mathbb{Q}_\kappa$ and $\mathsf{Ded}(\mathbb{Q}_\kappa)$ are κ-homogeneous. However, this natural restriction of this method does not show that $\mathbb{R}_\kappa$ is κ-homogeneous: the f from Lemma 4.1.7 and Proposition 4.1.8 need not be Cauchy continuous (i.e. the image of a Cauchy sequence need not be a Cauchy sequence). Hence, the completion of f need not map Cauchy gaps to Cauchy gaps, so need not be an order isomorphism from I to J (even though it *is* an order isomorphism from $I \cap \mathbb{Q}_\kappa$ to $J \cap \mathbb{Q}_\kappa$), as we see in the following example.

Example 4.1.9. Let G be a (κ, κ)-gap in $\mathbb{R}_\kappa$ (hence also a (κ, κ)-gap in $\mathbb{Q}_\kappa$) such that $0 < G < 1$, and let $r \in (0, 1) \setminus \mathbb{Q}_\kappa$. By the κ-homogeneity of $\mathbb{Q}_\kappa$, there is an order isomorphism $i : [0, r) \cap \mathbb{Q}_\kappa \to [0, G] \cap \mathbb{Q}_\kappa$ and an order isomorphism $j : (r, 0] \cap \mathbb{Q}_\kappa \to [G, 0] \cap \mathbb{Q}_\kappa$. Then, $(i \cup j) : [0, 1] \cap \mathbb{Q}_\kappa \to [0, 1] \cap \mathbb{Q}_\kappa$ is an order isomorphism, but is not Cauchy continuous: by definition, there is a strictly increasing cofinal sequence in $[0, r)$ which is Cauchy, but the image of this sequence is not Cauchy. So $(i \cup j)$ does not extend to an order isomorphism from $[0, 1] \subseteq \mathbb{R}_\kappa$ into itself.

The problem of finding a Cauchy continuous order isomorphism from $I \cap \mathbb{Q}_\kappa \to J \cap \mathbb{Q}_\kappa$ seems fairly rigid.[2] Hence, we ask the following question:

Question 4.1.10. Let $|\kappa^{<\kappa}| = \kappa$. Is $\mathbb{R}_\kappa$ κ-homogeneous?

Separating κ-continuity and κ-supercontinuity

A key fact which we use in separating κ-continuity from κ-supercontinuity is that order isomorphisms are κ-supercontinuous.

Lemma 4.1.11. Let $\mathbb{K}$ be an ordered field, $C, D \subseteq \mathbb{K}$ be convex sets, and $f : C \to D$ be strictly monotone. If f is surjective on D, then f is κ-supercontinuous. If $\mathbb{K}$ is also η_κ, then the converse holds.

Proof. First, suppose f is a surjection. As f is strictly monotone, $f(a, b) = (f(a), f(b))$ for all $a, b \in C$, so f is κ-continuous (exactly as in [74, Lemma 3.2.5]). Similarly, $f([a, b]) = [f(a), f(b)]$. But $\mathsf{cof}(f([a, b])) = \mathsf{coi}(f([a, b])) = 1 < \kappa$. So, f is κ-supercontinuous.

[2] For example, an obvious alternative method is to use the order isomorphisms $f : \mathbb{Q}_\kappa \to I \cap \mathbb{Q}_\kappa$ and $g : \mathbb{Q}_\kappa \to J \cap \mathbb{Q}_\kappa$, let Oz_κ be the set of integers parts in $\mathbb{Q}_\kappa$, let $b_z : [f(z), f(z+1)) \to [g(z), g(z+1))$ be the order isomorphism from Remark 4.1.6, and let $b := \bigcup_{z\in\mathsf{Oz}_\kappa} b_z$. But this also fails, as $\bigcup_{z\in\mathsf{Oz}_\kappa} [f(z), f(z+1)) = \mathsf{Dom}(b)$ need not be the set I, as possibly there is an $x \in I$ such that there is a (κ, κ)-gap, G in Oz_κ where x is in the gap defined by $f(G)$.

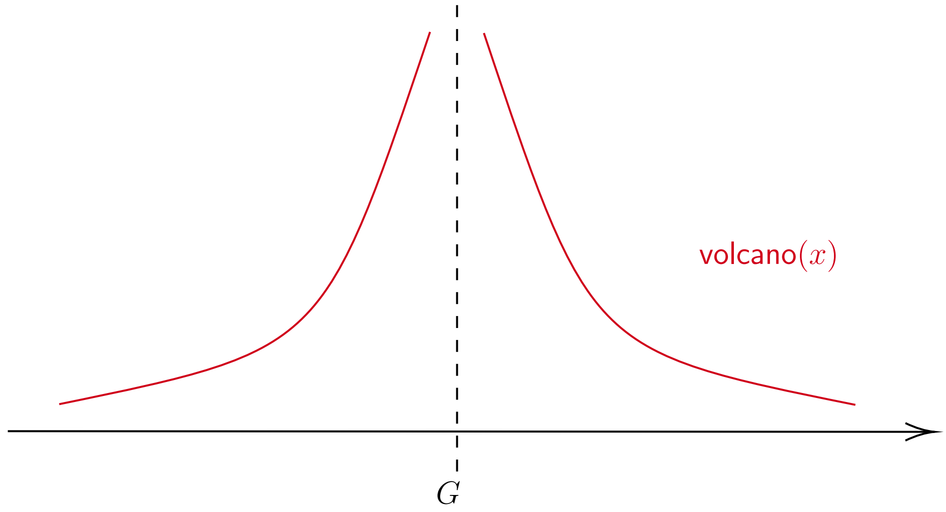

Figure 4.3: The volcano function at G, $\mathsf{volcano}(x)$

For the converse, note that κ-supercontinuity implies κ-continuity. Then the result follows from [74, Theorem 3.3.1] (i.e. Theorem 4.2.3). □

Corollary 4.1.12. Let $\mathbb{K}$ be an ordered field, and let $I, J \subseteq \mathbb{K}$. An order-preserving or reversing bijection from I to J is κ-supercontinuous.

The union of two suitably chosen functions of this type is κ-continuous and unbounded on a bounded interval, hence not κ-supercontinuous (see Lemma 4.2.4).

Example 4.1.13 (Volcano function)**.** Let $\mathbb{K}$ be a κ-homogeneous ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$ and let (L, R) be a (κ, κ)-gap in $\mathbb{K}$. Fix any $\ell \in L$ and $r \in R$.

Since $\mathsf{cof}(\mathbb{K}) = \kappa$, we have that $\{x \in L \colon x \geq \ell\}$ and $[0, \infty)$ have the same cofinality and coinitiality and therefore, using κ-homogeneity, are order isomorphic by an isomorphism i; similarly, $\{x \in R \colon x \leq r\}$ and $[0, \infty)$ have an order-reversing bijection, b. We define the *volcano function* (see Figure 4.3) by:

$$\mathsf{volcano}(x) := \begin{cases} 0 & \text{if } x < \ell, \\ i(x) & \text{if } x \in L \text{ and } x \geq \ell, \\ b(x) & \text{if } x \in R \text{ and } x \leq r, \text{ and} \\ 0 & \text{if } x > r. \end{cases}$$

Similarly, if, for any (κ, κ)-almost gap $A > 0$ (so for any $a \in \mathbb{K}^{>0}$), we can define an *A-bounded volcano function*, instead using an order-isomorphism $i_A : \{x \in L : x \geq \ell\} \to \{x \in \mathbb{K}^{\geq 0} : x < A\}$, and an order-reversing bijection $b_A : \{x \in R : x \leq r\} \to \{x \in \mathbb{K}^{\geq 0} : x < A\}$. This function is bounded, but has no maximum.

Proposition 4.1.14. If $\mathbb{K}$ is a κ-homogeneous ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$ which has a (κ, κ)-gap, then there are functions which are κ-continuous but not κ-supercontinuous.

Proof. Example 4.1.13 suffices: $\mathsf{wei}(\mathbb{K}) = \mathsf{cof}(\mathsf{volcano}([a_0, b_0])) = \kappa$, so $\mathsf{volcano}$ is not κ-supercontinuous. For κ-continuity, it suffices to show that $\mathsf{volcano}^{-1}(a, b) \in \tau_\kappa$ for any $a, b \in \mathbb{K} \cup \{\pm\infty\}$. First, suppose $a \in \mathbb{K}$ and $b \in \mathbb{K} \cup \{-\infty\}$, then by construction $\mathsf{volcano}^{-1}((a, b))$ is the union of two open intervals, one for each side of G. Otherwise $(a, b) = (a, \infty)$, where $\mathsf{volcano}^{-1}(a, \infty)$ is an open interval. □

By Propositions 4.1.14 and 4.1.8, we know that κ-continuity does not imply κ-supercontinuity on $\mathbb{Q}_\kappa$. We do not know whether all assumptions of Proposition 4.1.14 are necessary.

Localising κ-Continuity

The properties of κ-supercontinuity and κ-continuity are 'global' properties and the naïve attempt to localise them fails badly as the following result shows.

Theorem 4.1.15 (Folklore, e.g. [74, Theorem 3.3.6])**.** If $x \in \mathbb{K}$, we say that $f : \mathbb{K} \to \mathbb{K}$ is *κ-continuous at a point x* if for all $V \in \tau_\kappa$ with $f(x) \in V$, there is a $U \in \tau_\kappa$ where $x \in U$ and $f(U) \subseteq V$. If $|\kappa^{<\kappa}| = \kappa$, then $J_{\neg\omega} : \mathbb{R}_\kappa \to \mathbb{R}_\kappa$ is κ-continuous at every point $x \in \mathbb{R}_\kappa$ but not κ-continuous.

One can define a local notion of κ-continuity by requiring several technical properties to hold at each almost gap. This has the advantage that, on $\mathbb{R}$, a function is continuous if and only if it is locally ω-continuous. We do not give the full details here since there are no immediate applications of this notion (however, we briefly mention it on pages 79 and 83, and in Footnote 5). The core idea is to generalise convergence to gaps:

Definition 4.1.16. Let $f : [a, b] \to \mathbb{K}$, A be an almost gap in $[a, b]$, and B be an (almost) gap in $\mathbb{K}$. We call B the *image almost gap* of f at A if, for any sequence, $(x_\alpha)_{\alpha\in\lambda}$, from $[a, b]$, which is cofinal or coinitial with A, we have that $(f(x_\alpha))_{\alpha\in\lambda}$ is cofinal or coinitial with B.[3] If B is a gap, we call it the *image gap* of f at A.

We call f *locally κ-continuous* if, at every almost gap, A, either: (1) f has an image almost gap; or (2) A is a (κ, κ)-gap and f approaches ∞ (or $-\infty$) on both sides of A (as in Example 4.1.13). In other words, f has a well-defined continuous extension $\hat{f} : \mathsf{Ded}(\mathbb{K}) \to \mathsf{Ded}(\mathbb{K}) \cup \{\pm\infty\}$. (To ensure that f is IVT, as on page 79, we also require that if f is not constant on either side of A, and f takes suitable values above and below B on opposites sides of A, then A is not a gap.) On η_κ ordered fields with $\mathsf{wei}(\mathbb{K}) = \mathsf{bn}(\mathbb{K}) = \kappa$, κ-continuity implies local κ-continuity.[4]

4.2 Generalising properties of functions

In this section, we discuss theorems that hold in classical analysis for all continuous functions. In each of the cases, we observe that the theorem does not directly generalise to non-Archimedean fields and discuss whether there is a generalised version and if so, for which class of functions it holds. In this section, we focus on results that hold for arbitrary uncountable κ, and reserve the discussion of results that need some large cardinal property of κ for Section 4.3.

[3] Note that if f is continuous, and $x \in \mathbb{K}$, then $f(x)$ is the image almost gap of f at x.

[4] One proof examines the behaviour of a κ-continuous f at an almost gap, A, with image almost gap, B. The key technique checks the *oscillations* of f, i.e. whether f can cofinally approach, then move away from, B as $x \to A$ (informally: f 'touches/passes through the line $y = B$'). One can check that if f is bounded in some interval, I, around A, then f can only oscillate around B if these oscillations are cofinally/coinitially decrease towards B inside I. This 'dampened oscillation' argument is the gap-analogue of Proposition 4.2.9 (f can 'pass through' B κ-often as $x \to A$, but only if the oscillations are dampening towards B).

4.2.1 Properties of functions

On the real line $\mathbb{R}$, continuous functions satisfy a plethora of theorems: the intermediate value theorem, IVT, the open mapping theorem, OMT, Brouwer's fixed point theorem, BFPT, the extreme value theorem, EVT, and the bounded value theorem, BVT. Furthermore, differentiable functions satisfy Rolle's Theorem, the mean value theorem, MVT, the positive derivative theorem, PDT, and the constant value theorem, CVT.

Definition 4.2.1. Let $\mathbb{K}$ be an ordered field and $f\colon \mathsf{Dom}(f) \to \mathsf{Ran}(f)$ where $\mathsf{Dom}(f), \mathsf{Ran}(f) \subseteq \mathbb{K}$. A function f is called *differentiable at* x_0 if there is an $\ell \in \mathbb{K}$ such that $\frac{f(x)-f(x_0)}{x-x_0} \to \ell$ as $x \to x_0$ from above and below; we write $f'(x_0) := \ell$. We call f *differentiable* if it is differentiable at every $x_0 \in \mathbb{K}$. As usual, if f, g are differentiable, then $f + g$, $f.g$, and $f \circ g$ are differentiable.

We say that f is:

1. $\mathsf{IVT}(\mathbb{K})$ (for "intermediate value theorem") if for any $a, b, c \in \mathbb{K}$, where $f(a) < c < f(b)$, there is $x \in (a, b)$ such that $f(x) = c$;
2. $\mathsf{OMT}(\mathbb{K})$ (for "open mapping theorem") if $I := \mathsf{Dom}(f) \subseteq \mathbb{K}$ is a convex bounded set, f is injective, and for every κ-open set, O, the image of O under f, $f(O \cap I)$, is κ-open;
3. $\mathsf{BFPT}(\mathbb{K})$ (for "Brouwer's fixed point theorem") if $\mathsf{Dom}(f) = \mathsf{Ran}(f) = [a, b]$ and there is some $x \in [a, b]$ such that $f(x) = x$;
4. $\mathsf{EVT}(\mathbb{K})$ (for "extreme value theorem") if for all $a, b \in \mathbb{K}$, f has a maximum and minimum on $[a, b]$;
5. $\mathsf{BVT}(\mathbb{K})$ (for "bounded value theorem") if for any $a, b \in \mathbb{K}$, there are bounds $m, M \in \mathbb{K}$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$;
6. $\mathsf{Rolle}(\mathbb{K})$ (for "Rolle's theorem") if f is differentiable, $\mathsf{Dom}(f)$ is convex, and for all $a, b \in \mathsf{Dom}(f)$ with $f(a) = f(b)$, there is some $c \in (a, b)$ such that $f'(c) = 0$;
7. $\mathsf{MVT}(\mathbb{K})$ (for "mean value theorem") if f is differentiable, $\mathsf{Dom}(f)$ is convex, and for all $a, b \in \mathsf{Dom}(f)$, there is a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$;
8. $\mathsf{PDT}(\mathbb{K})$ (for "positive derivative theorem") if f is differentiable, $\mathsf{Dom}(f)$ is convex, and if $f'(x)$ is non-negative for every $x \in \mathsf{Dom}(f)$, then f is non-decreasing on $\mathsf{Dom}(f)$; and
9. $\mathsf{CVT}(\mathbb{K})$ (for "constant value theorem") if f is differentiable, $\mathsf{Dom}(f)$ is convex, and if $f'(x) = 0$ for every $x \in \mathsf{Dom}(f)$, then f is constant on $\mathsf{Dom}(f)$.

Let P be any of the above properties and $\mathcal{C} \subseteq \mathbb{K}^{\mathbb{K}}$. We say that $\mathbb{K}$ is $\mathsf{P}(\mathbb{K}, \mathcal{C})$ if all $f \in \mathcal{C}$ are $\mathsf{P}(\mathbb{K})$. For each of the properties P, the statement $\mathsf{P}(\mathbb{R}, \text{continuous})$ is a classical theorem. Furthermore, it turns out that $\mathsf{P}(\mathbb{K}, \text{continuous})$ is equivalent to "$\mathbb{K}$ is Dedekind complete", i.e. $\mathbb{K} = \mathbb{R}$, as is systematically laid out by Deveau and Teismann in their [44], as listed in the following table.

Property	Reference
IVT	[44, CA19]
OMT	[44, CA28]
BFPT	[44, CA45]
EVT	[44, CA24]
BVT	[44, CA26]
Rolle	[44, CA21]
MVT	[44, CA22]
PDT	[44, CA38]
CVT	[44, CA39]

In some cases, the classical theorem generalises to some class of functions generalising the continuous functions, in others, it does not. This process of generalisation is later analysed from a philosophical perspective (cf. Section 7.2.2).

We summarise the results of the following sections for the special field $\mathbb{R}_\kappa$ (under the assumption of $|\kappa^{<\kappa}| = \kappa$) as follows:

1. If f is κ-continuous, then f is $\mathsf{IVT}(\mathbb{R}_\kappa)$ and $\mathsf{OMT}(\mathbb{R}_\kappa)$ (Theorem 4.2.3 & Proposition 4.2.13).

2. If f is κ-continuous, then f is $\mathsf{EVT}(\mathbb{R}_\kappa)$ if and only if f is κ-supercontinuous (Corollary 4.2.21).

3. If both f and $x \mapsto f(x) - x$ are κ-continuous, then f is $\mathsf{BFPT}(\mathbb{R}_\kappa)$; however, there are κ-supercontinuous functions that are not $\mathsf{BFPT}(\mathbb{R}_\kappa)$ (Propositions 4.2.14 & 4.2.16).

4. If f is κ-supercontinuous, then f is $\mathsf{BVT}(\mathbb{R}_\kappa)$ and $\mathsf{Rolle}(\mathbb{R}_\kappa)$ (Propositions 4.2.25 & 4.2.19).

5. If both f and $x \mapsto f(x) + a.x + b$ are κ-supercontinuous, then f is $\mathsf{MVT}(\mathbb{K})$, $\mathsf{CVT}(\mathbb{R}_\kappa)$, and $\mathsf{PDT}(\mathbb{R}_\kappa)$ (Proposition 4.2.26).

4.2.2 The Intermediate Value Theorem and the Open Mapping Theorem

We build on Galeotti's study of the Intermediate Value Theorem on $\mathbb{R}_\kappa$ from [74, Theorem 3.3.1].

Example 4.2.2. Let $\mathbb{K}$ be an η_κ ordered field with $\mathsf{wei}(\mathbb{K}) = \kappa$. Then $J_{\neg\omega} : \mathbb{K} \to \mathbb{K}$ from Example 4.1.4 is continuous, but not $\mathsf{IVT}(\mathbb{K})$, so $\mathsf{IVT}(\mathbb{K}, \text{continuous})$ fails.

Theorem 4.2.3 (Galeotti, [74, Theorem 3.3.1])**.** If $\mathbb{K}$ is an η_κ ordered field, then $\mathsf{IVT}(\mathbb{K}, \kappa\text{-continuous})$ holds.

Proof. The proof in [74, Theorem 3.3.1] works for all η_κ ordered fields. □

Some examples

Given a function, f, the assumption that f is $\mathsf{IVT}(\mathbb{K})$ can give back some continuity properties of f; this can be called a *partial converse of* IVT. The following examples are motivated by this phenomenon.

Lemma 4.2.4 (Glueing Lemma)**.** Let $\mathbb{K}$ be an ordered field. For any function $f : \mathbb{K} \to \mathbb{K}$, the following hold.

1. If f is piecewise κ-continuous, then f is κ-continuous; and
2. if f is piecewise κ-supercontinuous, then f is κ-supercontinuous.

Proof. Part 1. is exactly as in [74, Theorem 3.4.24]; for Part 2., observe that $\mathsf{cof}(f([a, b])) = \mathsf{cof}(f([a, b]) \cap I_k)$ and $\mathsf{coi}(f([a, b])) = \mathsf{coi}(f([a, b]) \cap I_\ell)$ for some k and ℓ. □

Proposition 4.2.5. Let $\mathbb{K}$ be an ordered field, and $f : [a, b] \to \mathbb{K}$ be continuous.

1. If f is $\mathsf{IVT}(\mathbb{K})$ and injective, then f is κ-supercontinuous.
2. If f is $\mathsf{IVT}(\mathbb{K})$ and piecewise strictly monotone, then f is κ-supercontinuous. So too if f is $\mathsf{IVT}(\mathbb{K})$ and is either strictly monotone or constant on each piece.

Proof. First, suppose $C \subseteq \mathbb{K}$ is convex, and $f : C \to \mathbb{K}$ is a strictly monotone, $\mathsf{IVT}(\mathbb{K})$ function. Let $D := \{x \in \mathbb{K} : \exists z, y \in C(f(z) \leq x \leq f(y))\}$. We show that f is a bijection from C to D, and hence κ-supercontinuous. Without loss of generality, assume f is strictly increasing. By strict monotonicity, $f(C) \subseteq D$. By $\mathsf{IVT}(\mathbb{K})$, $f(C) = D$. So, it suffices to show that f is injective. But this follows immediately from monotonicity. So, f is a bijection from C to D. Hence, such an f is κ-supercontinuous as in Lemma 4.1.11.

For Part 1., if f is injective and $\mathsf{IVT}(\mathbb{K})$. We show that f is strictly monotone. Suppose f is not strictly increasing. So, there are $a, b, c \in \mathsf{Dom}(f)$ such that $a < b < c$ and $f(b) = \max\{f(a), f(b), f(c)\}$, or $f(b) = \min\{f(a), f(b), f(c)\}$. Without loss of generality, we assume the former. By injectivity, $f(a) < f(b)$, $f(c) < f(b)$, and $f(a) \neq f(b)$. So, without loss of generality, assume $f(a) = \min\{f(a), f(b), f(c)\}$. Then $f(c) \in (f(a), f(b))$. So, by $\mathsf{IVT}(\mathbb{K})$, there is an $x \in (a, b)$ such that $f(x) = f(c)$. But this contradicts injectivity. So, f is strictly monotone. Then, argue as in the first paragraph.

As constant functions are obviously κ-supercontinuous, Part 2. follows from Lemmata 4.1.11 and 4.2.4, and Part 1. □

With this, we can provide a strengthening of one direction of the classical theorem that an ordered field, $\mathbb{K}$, is an rcf if and only if $\mathsf{IVT}(\mathbb{K}, \mathbb{K}[X])$ holds [144, Theorem 3.3.9]:

Corollary 4.2.6. Let $\mathbb{K}$ be an rcf. If $p \in \mathbb{K}[X]$, then p is κ-supercontinuous.

Proof. If p is a polynomial, then p is either constant (hence κ-supercontinuous), or is not constant on any interval $I \subseteq \mathbb{K}$. Without loss of generality, assume p is non-constant. By differentiating as usual, p' is a polynomial. As $\mathbb{K}$ is an rcf, p' has finitely many roots, $a_0, ..., a_n$. Hence, p has finitely many local extrema. A simple check shows that on each of $(-\infty, a_0], [a_m, a_{m+1}]$, and (a_n, ∞), p is a monotone function. As p is non-constant, it is not constant on any subintervals of these intervals. So, in fact, p is piecewise strictly monotone.

It is clear that p is surjective on each of these intervals, hence by Lemma 4.1.11, p is κ-supercontinuous on each of these intervals. So, p is piecewise κ-supercontinuous. Hence, by Proposition 4.2.4 Part 2., p is κ-supercontinuous. □

The assumptions in Proposition 4.2.5, Part 1. is not necessary, as we can define a κ-supercontinuous function that is not piecewise strictly monotone shows (Example 4.2.8). For this, we prove a gap-analogue of the Glueing Lemma (Lemma 4.2.4) at image gaps:

Lemma 4.2.7. Let $\mathbb{K}$ be an ordered field, let $[a, b] \subseteq \mathbb{K}$, let $f : [a, b] \to \mathbb{K}$, let G be a gap in $[a, b]$, and let f have an image gap, H, at G. If $f \restriction [a, G]$ and $f \restriction [G, b]$ are κ-continuous, then f is κ-continuous.

Proof. Let $O \subseteq \mathbb{K}$ be κ-open. If H is not a gap in O, then $f^{-1}(O)$ is obviously κ-open. So suppose H is a gap in O. As $f \restriction [a, G]$ is κ-continuous, $(f \restriction [a, G])^{-1}(O) = (\bigcup_{\alpha \in \lambda} O_\alpha) \cap [a, G]$ for some $\lambda < \kappa$, and some open intervals, $O_\alpha \subseteq \mathbb{K}$. Without loss of generality, we can assume that there is a unique α_0 such that G is a gap in O_{α_0}. Likewise, $(f \restriction [G, b])^{-1}(O) = (\bigcup_{\beta \in \mu} O'_\beta) \cap [G, b]$, where O'_β are open intervals, $\mu < \kappa$, and there is a unique β_0 such that G is a gap in O'_{β_0}. Then,

$$f^{-1}(O) = \bigcup_{\alpha \in \mu \setminus \{\alpha_0\}} O_\alpha \cup \bigcup_{\beta \in \lambda \setminus \{\beta_0\}} O'_\beta \cup (O_{\alpha_0} \cap O'_{\beta_0}).$$

But this is a $<\kappa$-sized union of open intervals. So $f^{-1}(O)$ is κ-open. Hence, f is κ-continuous. □

The same is true if $f(x) \to \infty$ (or $-\infty$) as x tends to G from above G, and from below G (this generalises the fact that volcano and bounded volcano function are κ-continuous, see Example 4.1.13).[5]

Example 4.2.8. Let $\mathbb{K}$ be an ordered field, where $\mathsf{bn}(\mathbb{K}) > \omega$. We define the following function (see Figure 4.4):

$$\mathsf{upwardsstumble}(x) := \begin{cases} 2n + 1 - (x - 2n) & \text{if } x \in [2n, 2n + 1] \text{ with } n \in \mathbb{Z}, \\ 2n + 2(x - (2n + 1)) & \text{if } x \in (2n + 1, 2n + 2) \text{ with } n \in \mathbb{Z}, \\ x & \text{otherwise.} \end{cases}$$

The image gap of $\mathsf{upwardsstumble}$ at $\neg\omega$ is $\neg\omega$, likewise for $-(\neg\omega)$. So, by Lemma 4.2.7, f is κ-continuous. Indeed, $\mathsf{upwardsstumble}$ is κ-supercontinuous, but not piecewise strictly monotone.

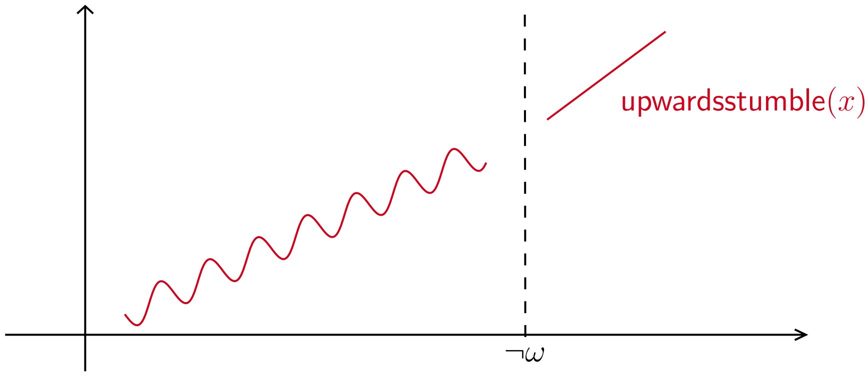

Figure 4.4: Example 4.2.8: a κ-continuous function with ω-many turning points

Let Z be a set of integer parts for $\mathbb{K}$ (from Section 2.3.3). The Z-analogue of Example 4.2.8, using a small zigzag at each $z \in Z$, has exactly $\mathsf{ip}(\mathbb{K})$-many turning points,[6] and is κ-supercontinuous (the preimage of an image of an interval is at most a union of three intervals). As usual, a function $f : \mathbb{K} \to \mathbb{K}$ is called *periodic* if there is a $P \in \mathbb{K}$ such that $f(x) = f(x + P)$ for all $x \in \mathbb{K}$, P is called the *period* of f. For the next example, we need a result by Galeotti and Nobrega.

Proposition 4.2.9 (Galeotti & Nobrega; [77, Lemma 13])**.** Let $\mathbb{K}$ be an ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$, $C \subseteq \mathbb{K}$ be convex, and $f : C \to \mathbb{K}$. Let $a \in \mathbb{K}$ and $(x_\alpha)_{\alpha\in\kappa}$ be a strictly monotone sequence in C such that $f(x_{2\alpha}) = a$ and $f(x_{2\alpha+1}) \neq a$. Then f is not κ-continuous.

Example 4.2.10. Let Z be a set of integer parts in a field, $\mathbb{K}$, where $\mathsf{ip}(\mathbb{K}) = \kappa$. We define the following function:

$$\mathsf{sawtooth}(x) = \begin{cases} 4(x-z) & \text{if } x \in [z, z+\frac{1}{4}] \text{ for some } z \in Z, \\ 2 - 4(x-z) & \text{if } x \in (z+\frac{1}{4}, z+\frac{3}{4}] \text{ for some } z \in Z, \\ -4 + 4(x-z) & \text{if } x \in (z+\frac{3}{4}, z) \text{ for some } z \in Z. \end{cases}$$

This function is periodic with period 1, and is κ-supercontinuous on any period. However, Proposition 4.2.9 shows that it is not κ-continuous.

A classical converse to IVT

Another classical converse to IVT is the following. If f is $\mathsf{IVT}(\mathbb{R})$ and $f^{-1}(\{x\})$ is closed for all $x \in \mathbb{R}$ then f is continuous [180, §4, Ex 19]. This fails in $\mathbb{R}_\kappa$.

Proposition 4.2.11. Let $|\kappa^{<\kappa}| = \kappa$. Then there is a continuous function $f\colon \mathbb{R}_\kappa \to \mathbb{R}_\kappa$ such that $\mathsf{IVT}(\mathbb{R}_\kappa)$ holds, $f^{-1}(\{x\})$ is closed for all $x \in \mathbb{R}_\kappa$, and f is not κ-continuous.

[5] A similar glueing result holds for κ-supercontinuity, assuming $f([a, G]) < H < f([G, b])$ and suitable boundedness, using the local κ-continuity property from page 73.

[6] There are clearly at most $\mathsf{wei}(\mathbb{K})$-many turning points (or turning gaps, i.e. where f has an image almost gap, A, at some gap, G, and there is some interval, I, including G where $f(I) \geq A$ or $f(I) \leq A$) for any continuous $f : \mathbb{K} \to \mathbb{K}$.

Proof. Define the following function:

$$f(x) := \begin{cases} x.\mathsf{sawtooth}(\frac{1}{x}) & \text{if } x \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 4.2.9, $f(x)$ is not κ-continuous. If $x \in [-1,1]\backslash\{0\}$, then we have that $|f^{-1}(\{x\})| < \kappa$, hence $f^{-1}(\{x\})$ is (κ-)closed. Next, $f^{-1}(\{0\})$ has one limit point, 0. As $f^{-1}(\{0\}) \ni 0$, $f^{-1}(\{0\})$ is closed, but $f^{-1}(\{0\})$ is not κ-closed. Finally if $|x| > 1$, then $f^{-1}(\{x\}) = \emptyset$ is (κ-)closed. □

Question 4.2.12. Does Proposition 4.2.11 hold if we replace "closed" by "κ-closed"?

We close this section by remarking that OMT remains a consequence of IVT.

Proposition 4.2.13. Let $\mathbb{K}$ be an ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$. Then, $\mathsf{OMT}(\mathbb{K}, \mathsf{IVT}(\mathbb{K}))$ holds. So, if $\mathbb{K}$ is η_κ, then $\mathsf{OMT}(\mathbb{K}, \kappa\text{-continuous})$ holds.

Proof. Let f be injective and $\mathsf{IVT}(\mathbb{K})$. Then f is not constant on any subinterval, so f is strictly monotone bijective and hence $f((a,b)) = (f(a), f(b))$ for any $a,b \in I$ (as in [74, Lemma 3.2.5]). If $I = [c,d]$ then, $f([c,d]) = [f(c), f(d)]$ which is κ-open in the subspace κ-topology on $f(I)$, and the other cases are similar. So, f maps sets of the form (a,b) to (κ-open) intervals, and hence κ-open sets to κ-open sets. The final part follows from Theorem 4.2.3. □

We remark that a tedious check shows that $\mathsf{IVT}(\mathbb{K}, \text{locally } \kappa\text{-continuous})$ holds for the property of local κ-continuity mentioned, but not precisely defined on page 73.

4.2.3 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem (BFPT) says that for any continuous real function $f : [a,b] \to [a,b]$, there is an $x \in [a,b]$ such that $f(x) = x$. On $\mathbb{R}$, BFPT is typically proved from IVT as follows: if f is continuous, then so is $x \mapsto f(x) - x$, so the latter function satisfies IVT and thus has to have a zero between a and b which is a fixed point of f. We show that the generalisations of these arguments fail.[7] Remember from Example 4.1.4 that for any $r \in \mathsf{INF}$, the function

$$J_{\neg\omega}{\restriction}[0,r] \colon [0,r] \to [0,r]$$

is continuous, not κ-continuous, and does not satisfy $\mathsf{IVT}(\mathbb{K})$. However, it clearly satisfies $\mathsf{BFPT}(\mathbb{K})$ since $J_{\neg\omega}(0) = 0$.

The classical proof of BFPT from IVT mentioned above yields the following generalisation.

Proposition 4.2.14. Let $\mathbb{K}$ be an ordered field. Let $p \subseteq \{f \in \mathbb{K}^X : X \subseteq \mathbb{K}\}$ be such that if $f \in p$, then $f(x) - x \in p$. If $\mathsf{IVT}(\mathbb{K}, p)$, then $\mathsf{BFPT}(\mathbb{K}, p)$. If, in addition, $\mathbb{K}$ is η_κ and $\mathsf{bn}(\mathbb{K}) = \kappa$, then if $\mathcal{F} := \{f \in \mathbb{K}^{\mathbb{K}} \colon \text{ both } f \text{ and } f(x) - x \text{ are } \kappa\text{-continuous}\}$, then $\mathsf{BFPT}(\mathbb{K}, \mathcal{F})$.

[7]The analysis of BFPT is a key example for our philosophical analysis of generalisations in Section 7.2.2.

Proof. Let $f : [a,b] \to [a,b]$ be such that $f \in p$. There are $c, d \in [a,b]$ such that $f(c) - c \leq 0 \leq f(d) - d$. If $f(c) - c = 0$ or $f(d) - d = 0$ we are done. Otherwise $0 \in \mathsf{Ran}(f(x) - x)$. If $f \in p$ then $f(x) - x \in p$, so f is $\mathsf{IVT}(\mathbb{K})$. Hence there is an $e \in [c,d]$ such that $f(e) - e = 0$. □

Examples of classes of functions which satisfy the condition in Proposition 4.2.14 include the polynomials, decreasing bijections, and increasing bijections which are 'fast increasing', i.e. such that for all $a \in [0, f(x_{k+1}) - f(x_k)]$, and all $m \in \mathbb{N}$, we have that $f(x_k + a) > f(x_k) + m.a$ (and likewise $f(x_0 - a) < f(x_0) - m.a$)). Indeed, any piecewise combination of these three cases (with finitely many pieces) also suffices.

However, the κ-continuity of f does not in general imply the κ-continuity of $x \mapsto f(x) - x$.

Proposition 4.2.15. Let $\mathbb{K}$ be a η_κ ordered field. Let $\mathcal{C}$ be either the κ-continuous or the κ-supercontinuous functions. Then the following hold.

1. If $a \in \mathbb{K} \setminus \{0\}$, there is an $f \in \mathcal{C}$ such that $f(x) + ax \notin \mathcal{C}$. Hence $\mathcal{C}$ is not closed under addition of linear polynomials.

2. If $a \in \mathbb{K} \setminus \{0\}$, there are $f, g \in \mathcal{C}$ such that $f(x)/ax, g(x).ax \notin \mathcal{C}$. Hence $\mathcal{C}$ is not closed under multiplication or division by linear polynomials.

Proof. In each case, it suffices to construct a κ-supercontinuous f such that the resulting function ($f(x) + x$, etc.) is not κ-continuous.

For Part 1., let $a \neq 0$, and

$$f_a(x) := \begin{cases} -a(x-1) & \text{if } x < \neg\omega \\ -a(x+1) & \text{if } x > \neg\omega. \end{cases}$$

Clearly, f_a is monotone. For surjectivity, note that $(-a(n-1))_{n\in\omega}$ is cofinal with $(-a(n+1))_{n\in\omega}$. Hence, by Lemma 4.1.11, f_a is κ-supercontinuous. As $f_a(x)+ax \neq 0$, $f_a(x) + ax$ violates $\mathsf{IVT}(\mathbb{K})$, so by Theorem 4.2.3, $f_a(x) + ax$ is not κ-continuous (see also Figure 4.5).

For Part 2., we define functions g, h which are κ-supercontinuous such that $g(x)/x$ and $h(x).x$ are not κ-continuous. For g, we can use f_{-1} from Part 1.: $f_{-1}(x)/x$ is not $\mathsf{IVT}(\mathbb{K})$ as it omits 1. The function h is similar:

$$h(x) := \begin{cases} 1 & \text{if } x < 2 \\ \frac{1}{x-1} & \text{if } 2 < x < \neg\omega \\ \frac{1}{x+1} & \text{if } x > \neg\omega. \end{cases}$$

So $h : \mathbb{K} \to \mathbb{K}$, is κ-supercontinuous (above 2, f is a monotone surjection, below 2 f is constant, so use Proposition 4.2.4). So, $x.h(x)$ omits 1 on $[3, \infty)$, so is not κ-continuous. Elaborations with $a \neq 1$ are similar. As $g(x) = x$ is a polynomial, the 'hence' parts follows. □

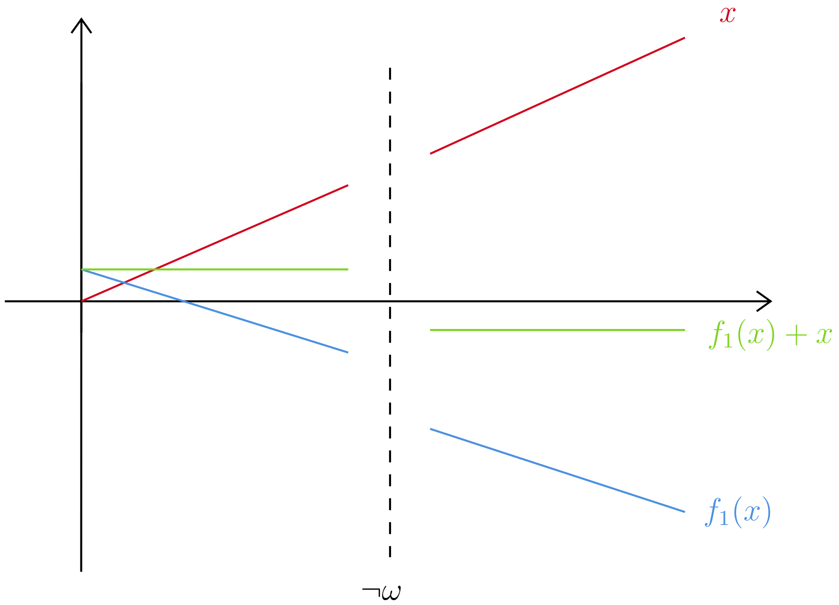

Figure 4.5: The κ-supercontinuous functions are not closed under addition

By Corollary 4.2.6, polynomials are κ-supercontinuous on rcfs. Hence, if $\mathbb{K}$ is also an rcf, Proposition 4.2.15 shows that neither the κ-continuous functions nor the κ-supercontinuous functions are closed under algebraic operations. Not only does the classical proof of BFPT break down, but we can give a concrete counterexample.

Proposition 4.2.16. Let $\mathbb{K}$ be a κ-homogeneous ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$ and a (κ, κ)-gap. Then $\mathsf{BFPT}(\mathbb{K}, \kappa\text{-supercontinuous})$ fails.

Proof. Let G be a (κ, κ)-gap. Without loss of generality, let $0 < G < 1$. We construct a strictly decreasing $f : [0, 1] \to [0, 1]$ such that $f : [0, G] \to [G, 1]$ and $f : [G, 1] \to [0, G]$. By κ-homogeneity and $\mathsf{bn}(\mathbb{K}) = \kappa$, there is a κ-continuous reverse order-isomorphism $g : [0, G] \to [G, 1]$. As $[1, G] \cap [G, 1] = \emptyset$, g has no fixed points. Similarly, let $h : [G, 1] \to [0, G]$ be a reverse order-isomorphism. Then let $f = g \cup h : [0, 1] \to [0, 1]$. Clearly, f has no fixed points. By Lemma 4.1.11, f is κ-supercontinuous. The result follows. □

We close this section with a remark about crossing and intersecting of functions. If $f, g\colon \mathbb{K} \to \mathbb{K}$ are functions, we say (as usual) that they *intersect* if there is some x such that $f(x) = g(x)$. We say that they *cross* if there is an interval $[a, b]$, partitioned into two non-empty convex sets $L \cup R = [a, b]$ such that $L < R$ and $f < g$ on L and $f > g$ on R (see Figure 4.6). Clearly, on $\mathbb{R}$, intersecting and crossing are equivalent since there are no gaps. However, if (L, R) is an almost gap, it is possible to cross without intersecting.[8]

Proposition 4.2.17. Let $\mathbb{K}$ be an ordered field with $\mathsf{bn}(\mathbb{K}) > \omega$. There are functions $f, g : \mathbb{K} \to \mathbb{K}$ which are strictly monotone, $\mathsf{bn}(\mathbb{K})$-supercontinuous which cross but do not intersect.

[8]Note that Proposition 4.2.14 uses that id and f intersect if they cross.

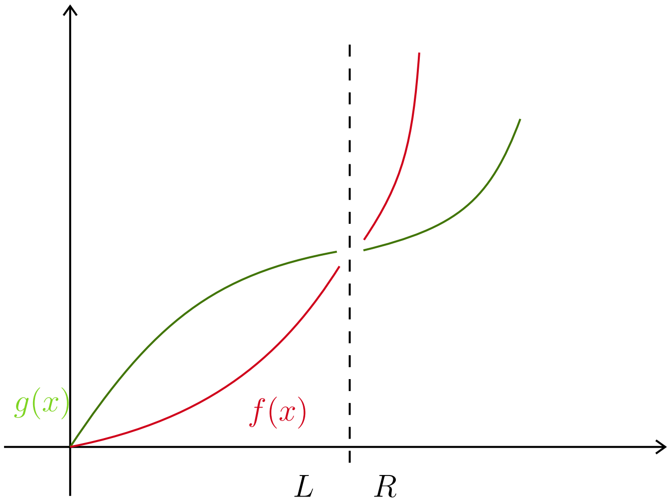

Figure 4.6: Functions which cross at a gap (L, R) but do not intersect

Proof. We define two functions:

$$f(x) := \begin{cases} x & \text{if } x < \neg\omega \\ 2x & \text{if } x > \neg\omega; \end{cases}$$

$$g(x) := \begin{cases} 2x & \text{if } x < \neg\omega \\ x & \text{if } x > \neg\omega. \end{cases}$$

By Lemma 4.1.11, both f and g are κ-supercontinuous (surjectivity amounts to checking that $(n)_{n\in\omega}$ and $(2n)_{n\in\omega}$ are both cofinal with $\neg\omega$). Clearly, f and g do not intersect. However, they do cross at $\neg\omega$ (again as $(2n)_{n\in\omega}$ is cofinal with $\neg\omega$). □

4.2.4 The Extreme Value Theorem and the Bounded Value Theorem

In this section, we extend Galeotti's analysis of the Extreme Value Theorem from [74] and get a characterisation of the κ-supercontinuous functions in terms of EVT.

Remark 4.2.18. For any ordered field $\mathbb{K}$, if $f : \mathbb{K} \to \mathbb{K}$ is $\mathsf{EVT}(\mathbb{K})$ then f is $\mathsf{BVT}(\mathbb{K})$.

We can construct explicit continuous, EVT functions which are not κ-continuous using the gaps in non-Archimedean fields (Corollary 4.1.4). Galeotti initially proved $\mathsf{EVT}(\mathbb{R}_\kappa, \kappa\text{-supercontinuity})$ [74, Theorem 3.3.3]. In fact, this is an exact characterisation for η_κ ordered fields:

Theorem 4.2.19 (Galeotti)**.** Let $\mathbb{K}$ be an η_κ ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$. Then both $\mathsf{EVT}(\mathbb{K}, \kappa\text{-supercontinuous})$ and $\mathsf{BVT}(\mathbb{K}, \kappa\text{-supercontinuous})$ hold.

Proof. Galeotti proved $\mathsf{EVT}(\mathbb{K}, \kappa\text{-supercontinuous})$ in [74, Theorem 3.3.3]. The second claim follows from Remark 4.2.18. □

As EVT functions always have maxima and minima, we obtain our missing implication from Section 4.1.

Corollary 4.2.20. Let $\mathbb{K}$ be an η_κ ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$. If $f : \mathbb{K} \to \mathbb{K}$ is κ-supercontinuous, then f is sharp.

Corollary 4.2.21. Let $\mathbb{K}$ be an η_κ ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$. Suppose $f : \mathbb{K} \to \mathbb{K}$ is κ-continuous. Then the following are equivalent:

1. f is κ-supercontinuous,
2. f is $\mathsf{EVT}(\mathbb{K})$, i.e. for all $a, b \in \mathbb{K}$, $\mathsf{cof}(f([a, b])) = \mathsf{coi}(f([a, b])) = 1$.

Proof. From Part 1. to Part 2. is Theorem 4.2.19. For Part 2. to Part 1., if f is $\mathsf{EVT}(\mathbb{K})$, then for every interval $[a, b] \in \mathbb{K}$, $\mathsf{cof}(f([a, b])) = \mathsf{coi}(f([a, b])) = 1$, as if $f(y) = M$ is maximal on $[a, b]$ then $f(y)$ is cofinal in $f([a, b])$. So, too for coinitiality. □

So, by e.g. Example 4.1.13, we have the following:

Corollary 4.2.22. Let $\mathbb{K}$ be κ-homogeneous, such that $\mathsf{bn}(\mathbb{K}) = \kappa$ and $\mathbb{K}$ has a (κ, κ)-gap. Then $\mathsf{EVT}(\mathbb{K}, \kappa\text{-continuous})$ and $\mathsf{BVT}(\mathbb{K}, \kappa\text{-continuous})$ fail.

On non-Archimedean fields, not every continuous $\mathsf{BVT}(\mathbb{K})$ function is $\mathsf{EVT}(\mathbb{K})$ as the following example shows.

Example 4.2.23. Let $\mathbb{K}$ be non-Archimedean, let $a \in \mathsf{INF}$, and define

$$f(x) := \begin{cases} x - a & \text{if } x < \neg\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is continuous and $\mathsf{BVT}(\mathbb{K})$, but not $\mathsf{EVT}(\mathbb{K})$.

If we weaken κ-continuity to continuity, then Corollary 4.2.21 fails as the following example shows.

Example 4.2.24. Let $|\kappa^{<\kappa}| = \kappa$. We define a continuous function, J^ω, such that for all $a, b \in \mathbb{R}_\kappa$, $\mathsf{coi}(J^\omega([a, b])) < \kappa$ and $\mathsf{cof}(J^\omega([a, b])) < \kappa$, which is not $\mathsf{EVT}(\mathbb{R}_\kappa)$. Let $I(n) := \{x \in \mathbb{R}_\kappa : (\forall k \in \omega(|x - n| < \frac{1}{k}) \text{ or } x > n)\}$ (i.e. x which are greater than or infinitely close to n). Let $\neg n$ be the (ω, κ)-gap defined by $\mathbb{R}_\kappa \setminus I(n)$ and $I(n)$. The following function will do (i.e. Figure 4.7):

$$J^\omega(x) := \begin{cases} 0 & \text{if } x \in (-\infty, \neg 1] \cup (\neg\omega, \infty), \\ n & \text{if } x \in [\neg n, \neg(n + 1)]. \end{cases}$$

Local κ-continuity (page 73) suffices for $\mathsf{IVT}(\mathbb{K})$. We do not know whether $\mathsf{EVT}(\mathbb{K})$ holds for every local κ-continuous function, f, such that for all $a, b \in \mathbb{K}$, $\mathsf{cof}(f([a, b])) < \kappa$ and $\mathsf{coi}(f([a, b])) < \kappa$.

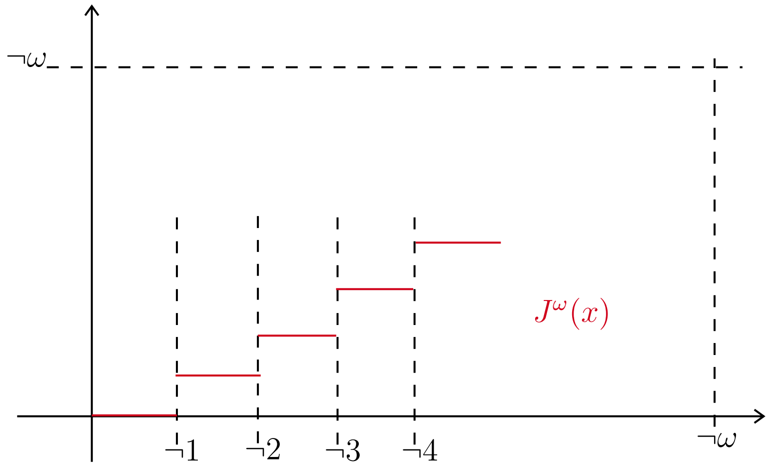

Figure 4.7: The ω-jumping function, J^ω

4.2.5 Rudiments of calculus: Differentiable functions

In this section, we deal with the properties Rolle, MVT, PDT, and CVT which concern differentiable functions, building on [8] and [179]. Like continuity, differentiability is a local property and therefore is blind to jumps at gaps. As mentioned, all of the properties for the class of continuous functions only hold if $\mathbb{K} = \mathbb{R}$. Assuming $|\kappa^{<\kappa}| = \kappa$, we can construct specific counterexamples to each of the four properties on $\mathbb{R}_\kappa$: the function $J_{\neg\omega}$ clearly fails $\mathsf{CVT}(\mathbb{R}_\kappa)$, but also fails $\mathsf{MVT}(\mathbb{R}_\kappa)$: $J'_{\neg\omega}(x) = 0$ for all x, but $\frac{f(\omega)-f(1)}{\omega-0} = \frac{1}{\omega} > 0$, contradicting $\mathsf{MVT}(\mathbb{R}_\kappa)$. For $\mathsf{Rolle}(\mathbb{R}_\kappa)$ and $\mathsf{PDT}(\mathbb{R}_\kappa)$, we define:

$$\mathsf{drop}(x) := \begin{cases} x & \text{if } x > \neg\omega \\ x - \omega & \text{if } x < \neg\omega. \end{cases}$$

Then $\mathsf{drop}(0) = \mathsf{drop}(\omega)$, but $\mathsf{drop}'(x) = 1$ on $[0, \omega]$, so fails $\mathsf{Rolle}(\mathbb{R}_\kappa)$. As drop has a positive derivative at every point, but is not increasing on $[0, \omega]$ (e.g. $\mathsf{drop}(1) > \mathsf{drop}(\frac{\omega}{2})$), it fails $\mathsf{PDT}(\mathbb{R}_\kappa)$.

Note that classically, the proof of Rolle and MVT uses EVT and MVT implies both PDT and CVT. This results in the following observations.

Proposition 4.2.25. Let $\mathbb{K}$ be any ordered field. Then the following hold.

1. $\mathsf{Rolle}(\mathbb{K}, \mathsf{EVT}(\mathbb{K}))$ and
2. if f is $\mathsf{MVT}(\mathbb{K})$, then f is both $\mathsf{PDT}(\mathbb{K})$ and $\mathsf{CVT}(\mathbb{K})$.

Proof. The classical proof of Rolle from EVT gives Part 1. (see, e.g. [50, Theorem 32]). The classical proofs of PDT and CVT from MVT give Part 2. (see, e.g. [50, page 105] and [180, Theorem 5.11]). □

Note that Propositions 4.2.21 and 4.2.25 Part 1. yield $\mathsf{Rolle}(\mathbb{K}, \kappa\text{-supercontinuous})$ if $\mathbb{K}$ is an η_κ ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$.

As with BFPT, the classical proofs of our current target properties require some closure under algebraic operations that is not in general true for the class of κ-supercontinuous functions. A class of functions, $\mathcal{C}$, is *closed under reflection* if, for every $f \in \mathcal{C}$, the function $-f$ is also in $\mathcal{C}$; it is *closed under the addition of linear functions* if for every $f \in \mathcal{C}$, and for every $a, b \in \mathbb{K}$, we have that $x \mapsto f(x) + a.x + b \in \mathcal{C}$.

Proposition 4.2.26. Let $\mathbb{K}$ be an ordered field and $\mathcal{C}$ be a class of functions.

1. If $\mathcal{C}$ is closed under reflection and $\mathsf{PDT}(\mathbb{K}, \mathcal{C})$ holds, then $\mathsf{CVT}(\mathbb{K}, \mathcal{C})$ holds.
2. If $\mathcal{C}$ is closed under the addition of linear functions and $\mathsf{Rolle}(\mathbb{K}, \mathcal{C})$ holds, then $\mathsf{MVT}(\mathbb{K}, \mathcal{C})$ holds.

Proof. For Part 1, let $f : C \to \mathbb{K}$ with $f \in \mathcal{C}$ such that $f'(C) = \{0\}$. By the algebra of limits, $(-f)' = -f'$. So, $(-f)'(C) = \{0\}$. So, $-f$ and f are both non-decreasing on C. Hence f is constant on C. Part 2. is essentially the classical proof (e.g. [50, Theorem 33]). □

Obviously, the class of κ-supercontinuous function is closed under reflection. Therefore, if $\mathbb{K}$ is an η_κ ordered field, and $\mathsf{bn}(\mathbb{K}) = \kappa$, then every function, f, such that f and $x \mapsto f(x)+a.x+b$ are κ-supercontinuous is $\mathsf{Rolle}(\mathbb{K})$, $\mathsf{MVT}(\mathbb{K})$, $\mathsf{PDT}(\mathbb{K})$, and $\mathsf{CVT}(\mathbb{K})$ (use Propositions 4.2.19 & 4.2.25 and the proof of Proposition 4.2.26). One example is $\mathbb{K}[X]$, the polynomials. If $\mathbb{K}$ is also an rcf, then the rational functions satisfy these conditions too.

Note that by Proposition 4.2.15, the set of all κ-supercontinuous functions does not have these closure properties. At the moment, we do not know whether any of $\mathsf{MVT}(\mathbb{R}_\kappa, \kappa\text{-supercontinuous})$, $\mathsf{PDT}(\mathbb{R}_\kappa, \kappa\text{-supercontinuous})$, or $\mathsf{CVT}(\mathbb{R}_\kappa, \kappa\text{-supercontinuous})$ hold (or, more strongly, whether $\mathsf{PDT}(\mathbb{R}_\kappa, \kappa\text{-continuous})$, and hence $\mathsf{CVT}(\mathbb{R}_\kappa, \kappa\text{-continuous})$, holds). A differentiable version of the volcano function (Example 4.1.13) could provide a concrete counterexample. It would also provide a counterexample to *Darboux's property* (cf. [44, CA27]), the remaining *pillar of calculus* [200, §3].

4.2.6 The Uniform Limit Theorem

We close this section with a property that differs from the previous ones, as it is not a property of individual functions, but a closure property of a class of functions. The classical Uniform Limit Theorem (ULT) says that the uniform limit of an ω-sequence of continuous real functions is continuous.

Definition 4.2.27. Let $\mathbb{K}$ be an ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$, and $(a_\alpha)_{\alpha\in\kappa}$ be coinitial in $\mathbb{K}^{>0}$. The sequence of functions $(f_\alpha)_{\alpha\in\kappa} \in (\mathbb{K}^{\mathbb{K}})^\kappa$ *converges uniformly* to $f : \mathbb{K} \to \mathbb{K}$ if for each a_α there is an $\alpha < \kappa$ such that for every $\beta > \alpha$ and $x \in \mathbb{K}$, $|f_\beta(x) - f(x)| < a_\alpha$. Let $p \subseteq \mathbb{K}^{\mathbb{K}}$ and $q \subseteq (\mathbb{K}^{\mathbb{K}})^\kappa$. We say that a class of functions, $\mathcal{C}$, satisfies $\mathsf{ULT}(\mathbb{K})$ if, for any uniformly convergent sequence $(f_\alpha)_{\alpha\in\kappa} \subseteq \mathcal{C}$, the uniform limit is in $\mathcal{C}$.

It is easy to check that the classical proof of ULT (e.g. [180, Theorem 7.12]) immediately generalises to show that the class of continuous functions satisfies $\mathsf{ULT}(\mathbb{K})$ for every ordered field $\mathbb{K}$.

Proposition 4.2.28. Let $\mathbb{K}$ be a κ-homogeneous η_κ ordered field with a (κ,κ)-gap such that $\mathsf{bn}(\mathbb{K})=\kappa$. Then neither the class of κ-continuous functions nor the class of κ-supercontinuous functions satisfies $\mathsf{ULT}(\mathbb{K})$.

Proof. We construct a κ-sequence of κ-supercontinuous functions which uniformly approximate a function, f, which is not $\mathsf{IVT}(\mathbb{K})$, so f is not κ-continuous. Let G be a (κ,κ)-gap. By κ-homogeneity, let $f\restriction(-\infty,G]$ be a κ-continuous strictly increasing surjection from $(-\infty,G]$ to $(-\infty,0)$ and $f\restriction[G,\infty)$ be a κ-continuous strictly increasing surjection from $[G,\infty)$ to $(0,\infty)$. Obviously, for some $x,y\in\mathbb{K}$, $f(x)<0<f(y)$, but $0\notin f(\mathbb{K})$, so f is not $\mathsf{IVT}(\mathbb{K})$.

Next, we construct a sequence of κ-continuous functions which converge uniformly to f. Let $(a_\alpha)_{\alpha\in\kappa}$ be a strictly decreasing coinitial sequence in $\mathbb{K}^{>0}$. We define the following sequence of functions:

$$f_\alpha(x):=\begin{cases} f(x) & \text{if } x\notin[-a_\alpha,a_\alpha],\\ f(-a_\alpha)+\frac{(f(a_\alpha)-f(-a_\alpha))(x+a_\alpha)}{2a_\alpha} & \text{if } x\in[-a_\alpha,a_\alpha].\end{cases}$$

As f_α is linear between $-a_\alpha$ and a_α, f_α is κ-continuous by Lemma 4.2.4. Moreover $(f_\alpha)_{\alpha\in\kappa}$ tends uniformly to f: fix an $\varepsilon\in\mathbb{K}^{>0}$, then pick an α such that $a_\alpha<\varepsilon$. By construction, for all $\beta\geq\alpha$, for all x, $|f_\beta(x)-f(x)|<a_\alpha$. $\square$

4.3 Generalisations and large cardinals

As opposed to the results of Section 4.2.1, which do not depend on the choice of κ, the results in this section require κ to have some property associated with large cardinals: they are either weakly compact or have the tree property.

In [25] & [76], Carl, Galeotti, Hanafi, and Löwe, showed that generalisations of the Bolzano-Weierstraß and Heine-Borel theorems are each equivalent to κ having the tree property.[9]

Theorem 4.3.1 (Carl, Galeotti, Hanafi, & Löwe [76, Theorem 7.1])**.** Let $\mathbb{K}$ be a κ-spherically complete ordered field, with $\mathsf{bn}(\mathbb{K})=\kappa$. Then the following are equivalent:

1. κ has the tree property, and
2. w$\mathsf{BWT}_{\mathbb{K}}$: every bounded interval witnessed κ-sequence has a convergent κ-subsequence, i.e. no gap in $\mathbb{K}$ has a cofinal or cointial interval witnessed κ-sequence.

In this section, we obtain a further connection: we prove that sharp functions are $\mathsf{EVT}(\mathbb{K})$ if and only if $\mathsf{bn}(\mathbb{K})$ has the tree property (Theorem 4.3.11). We return to this from a philosophical perspective in Section 7.2.2.

[9] A connection between Bolzano-Weierstraß and the tree property was anticipated in reverse mathematics where Weak Kőnig's Lemma, the relevant analogue of the tree property, is equivalent to the Bolzano-Weierstraß theorem for $\mathbb{R}$ [69, Theorem 1.1]. Using [25, Theorem 5.8] and [76, Lemma 4.7], the equivalence with the Heine-Borel theorem, [76, Corollary 5.9] does not require κ to be strongly inaccessible or $\mathbb{K}$ to be Cauchy complete.

Lemma 4.3.2. If $\mathbb{K}$ is a κ-spherically complete ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$, and κ satisfies the tree property, then $\mathsf{EVT}(\mathbb{K}, \text{sharp})$ holds.

Proof. By Theorem 4.3.1, we argue from w$\mathsf{BWT}(\mathbb{K})$ that all sharp functions are $\mathsf{EVT}(\mathbb{K})$. Suppose $f : [a, b] \to \mathbb{K}$ is sharp. Maxima and minima are exactly dual, so we only consider maxima. Either f has a maximum on $[a, b]$, in which case we are done, or f has no maximum on $[a, b]$. So, suppose f has no maximum.

As f is sharp, let $(a_\alpha)_{\alpha\in\kappa}$ be an interval witnessed sequence in $[a, b]$ such that $(f(a_\alpha))_{\alpha\in\kappa}$ is increasing and cofinal in $f([a, b])$. By w$\mathsf{BWT}(\mathbb{K})$, $(a_\alpha)_{\alpha\in\kappa}$ has a convergent subsequence with length λ, $(a_{\alpha_\beta})_{\beta\in\lambda}$, such that the limit of $(a_{\alpha_\beta})_{\beta\in\lambda}$ is $x \in \mathbb{K}$. As $[a, b]$ is closed, $x \in [a, b]$. As $\mathsf{bn}(\mathbb{K}) = \kappa$, $\lambda = \kappa$ [75, Corollary 1.11(ii)]. As $(f(a_{\alpha_\beta}))_{\beta\in\kappa}$ is increasing, $(f(a_{\alpha_\beta}))_{\beta\in\kappa}$ is also cofinal in $f([a, b])$. As f is sharp, it is continuous, so $f(x) = \lim_{\beta\to\kappa} f(a_{\alpha_\beta})$. As $(f(a_{\alpha_\beta}))_{\beta\in\kappa}$ is cofinal, $f(x)$ must be a maximum, which contradicts that f has no maximum on $[a, b]$. □

Theorem 4.3.3. Let $\mathbb{K}$ be a κ-spherically complete ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$. If $\mathsf{EVT}(\mathbb{K}, \text{sharp})$ then κ has the tree property.

Proof. We prove the contrapositive. Suppose κ does not have the tree property. By Lemma 2.1.5, let T be a well-pruned κ-Aronszajn tree. We define a sharp function, $f : [0, 1] \to \mathbb{K}$, which violates $\mathsf{EVT}(\mathbb{K})$. We do this by labelling the tree, T, with certain convex subsets of $[0, 1]$, and using this labelling to define $f(x)$ according to the least limit level, $\mathsf{Lev}_\lambda(T)$, such that x does not appear in a $\mathsf{Lev}_\lambda(T)$ label. We prove that f is sharp and violates $\mathsf{EVT}(\mathbb{K})$ using a series of claims. We first define the labelling and the function f, then we show that f is continuous, unbounded, and sharp.

Fix a sequence, $(\varepsilon_\alpha)_{\alpha\in\kappa}$, which is coinitial in $\mathbb{K}^{>0}$. We modify the labelling construction in [25, Theorem 4.17], by labelling each node $t \in T$ with a convex set $L(x) \subseteq [0, 1]$, so that:

1. if t is the root, i.e. $t \in \mathsf{Lev}_0(T)$, then $L(t) = [0, 1]$,
2. if $\alpha < \kappa$ is a successor, if $t \in \mathsf{Lev}_\alpha(T)$, then $L(t)$ is an open interval in $[0, 1]$, and $\mathsf{len}(L(t)) < \varepsilon_\alpha$,
3. for every limit $\lambda < \kappa$, then $L(t) = \bigcap_{s\subsetneq t} L(s)$, and
4. for all $\alpha < \kappa$, there is a $\gamma_\alpha < \kappa$ such that for all $s, t \in \mathsf{Lev}_\alpha(t)$, for all $a \in L(t)$ and $b \in L(s)$, $|a - b| > \varepsilon_{\gamma_\alpha}$, i.e. the "distance"[10] between $L(s)$ and $L(t)$ is at least $\varepsilon_{\gamma_\alpha}$.

By κ-spherical completeness, these labels can be chosen to be non-empty (for a detailed argument, see [25, Lemma 2.9]). The only modification of the labelling in [25, §4.4] is that in the limit case, we take $L(t) := \bigcap_{s\subsetneq t} L(s)$, rather than a subset of this intersection.

Claim 4.3.4. For each $t \in T$, $L(t)$ is open in $[0, 1]$.

[10]Note that, in the limit case, there will be no minimal such $\varepsilon_{\gamma_\alpha}$.

Proof. The proof is by induction. At successor levels, this holds by definition. For a limit γ, suppose that, for all $t \in \bigcup_{\beta\in\gamma} \mathsf{Lev}_\beta(T)$, $L(t)$ is open in $[0,1]$. Let $O := \bigcap_{\beta\in\gamma} L(t_\beta)$, for some strictly decreasing nested sequence of labels, $(L(t_\beta))_{\beta\in\gamma}$ (note for any $t \in \mathsf{Lev}_\gamma(t)$, we even have that $L(t) = \bigcap_{\beta\in\gamma} L(t\restriction\beta)$). As $\mathsf{bn}(\mathbb{K}) = \kappa$, the order topology on $\mathbb{K}$ is κ-additive, i.e. the intersection of $<\kappa$-many open sets is open (this follows from [190, Theorem viii] or [75, Lemma 2.19], noting that $\mathbb{K}$ κ-metrises itself, see Chapter 6 for more on additivity). By induction, each $L(t_\beta)$ is open. So, by κ-additivity, O is also open. □

Note that for all $a \in [0,1]$, there is a smallest $\beta(a) < \kappa$ such that $a \notin \bigcup_{t\in\mathsf{Lev}_{\beta(a)}(T)} L(t)$, otherwise $\{t \in T : a \in L(t)\}$ would be a branch of length κ. Hence, there is a smallest *limit* ordinal, $\alpha(a)$, such that $a \notin \bigcup_{t\in\mathsf{Lev}_{\alpha(a)}(T)} L(t)$.

Let $(m_\alpha)_{\alpha\in\kappa}$ be strictly increasing and cofinal in $\mathbb{K}$.[11] Then we define the function f as follows:

$$f(a) := m_{\alpha(a)}.$$

At limit levels, $L(t)$ is a convex set with "endgaps" (rather than endpoints); so f only changes value at gaps, and is hence continuous. Formally:

Claim 4.3.5. The function $f : [0,1] \to \mathbb{K}$ is continuous.

Proof. Note that $\{\alpha(a) : a \in [0,1]\}$ is a collection of limit ordinals, so its induced topology is discrete. As $(m_\alpha)_{\alpha\in\kappa}$ is strictly increasing, the induced topology on $\{m_{\alpha(a)} : a \in [0,1]\} = \mathrm{Im}(f)$ is also discrete. Therefore, it suffices to show that, if $m \in \mathrm{Im}(f)$, then $f^{-1}(m)$ is open.

Let $m \in \mathrm{Im}(f)$ and $a \in [0,1]$ be such that $f(a) = m_{\alpha(a)} = m$. We will prove that there is $\varepsilon_a \in \mathbb{K}^{>0}$ such that for all $b \in B(a,\varepsilon_a)$ we have $f(b) = m$.

Note that by definition, for all $t \in \mathsf{Lev}_{\alpha(a)}(T)$, we have that $a \notin L(t)$. Since T is a κ-tree and $\mathsf{bn}(\mathbb{K}) = \kappa$, we can show that there is $\varepsilon \in \mathbb{K}^{>0}$ such that for all $b \in B(a,\varepsilon)$ and all $t \in \mathsf{Lev}_{\alpha(a)}(T)$ we have $b \notin L(t)$. Indeed, let $t \in \mathsf{Lev}_{\alpha(a)}(T)$. By construction, $L(t)$ is a convex set with no supremum and infimum such that $\mathsf{coi}(\mathsf{top}(L(t))) = \mathsf{cof}(\mathsf{bot}(L(t))) < \kappa$. By Lemma 2.3.2 Part 3., there is $\varepsilon_t \in \mathbb{K}^{>0}$ such that for all $b \in L(t)$, we have that $b \notin B(a,\varepsilon_t)$. As T is a κ-tree, $|\mathsf{Lev}_{\alpha(a)}(T)| < \kappa$. Therefore, there is a $\varepsilon \in \mathbb{K}^{>0}$ such that for all $t \in \mathsf{Lev}_{\alpha(a)}(T)$, $\varepsilon < \varepsilon_t$. So, in particular, for all $t \in \mathsf{Lev}_{\alpha(a)}(T)$ and $b \in L(t)$, $a \notin L(t) \supseteq B(b,\varepsilon)$.

We now distinguish two cases. First, suppose that $\beta(a)$ is a limit ordinal. Then, by definition, $\beta(a) = \alpha(a)$ and there is a strictly increasing sequence, $(t_\beta)_{\beta\in\alpha(a)}$, in T such that, for all $\beta < \alpha(a)$, $a \in L(t_\beta)$. Then, again by Lemma 2.3.2 Part 3., for each $\beta < \alpha(a)$, there is an $\varepsilon'_\beta \in \mathbb{K}^{>0}$ such that $B(a,\varepsilon'_\beta) \subseteq L(t_\beta)$. So, as $\alpha(a) < \kappa = \mathsf{bn}(\mathbb{K})$, there is an $\varepsilon' \in \mathbb{K}^{>0}$ such that for all $\beta < \alpha(a)$, $\varepsilon' < \varepsilon'_\beta$. In particular, for all $\beta < \alpha(a)$, we have $B(a,\varepsilon') \subseteq L(t_\beta)$. Let $\varepsilon_a = \min\{\varepsilon',\varepsilon\}$. Then, by construction, for all $b \in B(a,\varepsilon_a)$ and for every $\beta < \alpha(a)$, there is a $t \in \mathsf{Lev}_\beta(T)$ such that $b \in L(t)$, and for every $s \in \mathsf{Lev}_{\alpha(a)}(T)$ we have that $b \notin L(s)$. Therefore, for all $b \in B(a,\varepsilon_a)$ we have $\alpha(b) = \alpha(a)$ and $f(b) = m_{\alpha(b)} = m_{\alpha(a)} = m$ as desired.

Otherwise, suppose that $\beta(a)$ is a successor. Then $\beta(a) = \gamma + n$, where γ is a limit ordinal, and n is a natural number greater than 0. So, by definition, $\gamma < \beta(a)$

[11] If $\mathbb{K}$ has a set of almost ordinals, we can let $m_\alpha = e(\alpha)$ for each $\alpha \in \mathsf{bn}(\mathbb{K})$.

and there is a $t \in \mathsf{Lev}_{\gamma+(n-1)}(T)$ such that $a \in L(t)$. By Claim 4.3.4, $L(t)$ is open in $[0,1]$. Therefore, there is an $\varepsilon' \in \mathbb{K}^{>0}$ such that $B(a, \varepsilon') \subseteq L(t)$. As before, let $\varepsilon_a = \min\{\varepsilon', \varepsilon\}$ and, as in the limit case, note that for all $b \in B(a, \varepsilon_a)$, we have that $\alpha(b) = \alpha(a)$ and $f(b) = m_{\alpha(b)} = m_{\alpha(a)} = m$ as desired. □

Claim 4.3.6. For every $t \in T$, the set $f(L(t))$ is unbounded in $\mathbb{K}$, hence f is not $\mathsf{EVT}(\mathbb{K})$.

Proof. As T is a well-pruned κ-Aronszajn tree, for any $t \in T$, T_t is a κ-Aronszajn tree. In particular, T_t has κ-many levels, and every branch of T_t is of length $<\kappa$. So, there are cofinally many $\alpha \in \kappa$ such that $m_\alpha \in f(L(t))$, i.e. $f(L(t))$ is unbounded in $\mathbb{K}$. □

Claim 4.3.7. The function f is sharp.

Proof. We prove this with two claims, one of which shows that f has a maximum or is unbounded on any interval $[c, d]$, the other of which ensures that, in the case where f is unbounded on $[c, d]$, there is a suitable sequence which is interval witnessing.

Let $[c, d] \subseteq [0, 1]$. Clearly, f achieves a minimum, as $\mathrm{Im}(f) \subseteq \{m_\gamma : \gamma \in \kappa\}$, where $(m_\gamma)_{\gamma\in\kappa}$ is strictly increasing. To prove the upper bound, we need a further claim:

Claim 4.3.8. Either f has a maximum on $[c, d]$, or the set $f([c, d])$ is unbounded in $\mathbb{K}$.

Proof. Suppose that f has no maximum on $[c, d]$. As $[c, d]$ is convex, and $L(t)$ is convex for all $t \in T$, $L(t) \cap [c, d] \neq \emptyset$ implies one of four possible cases:

1. $L(t) \subseteq [c, d]$,
2. $[c, d] \subseteq L(t)$,
3. $L(t) \cap [c, d] = [c, A)$ where A is an almost gap, or
4. $L(t) \cap [c, d] = (B, d]$ where B is an almost gap.

In Case 2, we show that either f has a maximum on $[c, d]$, or we are in one Cases 1, 3, and 4. Suppose we are not in one of Cases 1, 3, and 4. Then there is a $\delta \leq \kappa$ such that for all $\eta < \delta$, there is a $t_\eta \in \mathsf{Lev}_\eta(T)$ such that $[c, d] \subseteq L(t_\eta)$, but no $t_\delta \in \mathsf{Lev}_\delta(T)$ such that either $L(t_\delta) \subseteq [c, d]$, $[c, d] \cap L(t_\delta) = [c, A)$, or $[c, d] \cap L(t_\delta) = (B, d]$, where A, B are almost gaps. As T is a κ-tree, $\delta < \kappa$. Hence, for all $x \in [c, d]$, the first level at which x is not included by in a label is level δ. So, by definition, $f(x) = m_\delta$. Hence, f is constant on $[c, d]$, and so trivially has a maximum.

If Case 1 holds for some t, unboundedness follows directly from Claim 4.3.6.

So, we consider Cases 3 and 4. Note that it is possible that both happen simultaneously, i.e. there are t, t' such that $L(t) \cap [c, d]$ is a non-empty convex set with minimum value c, and $L(t') \cap [c, d]$ is a non-empty convex set with maximum value d. By labelling Condition 4, we know that these two sets must be separated.

So, we can assume without loss of generality that we are in Cases 3 and 4 (simultaneously), as being in at most one of Cases 3 or 4 is simpler. That is, there is no $t \in T$ such that $L(t) \subseteq [c, d]$, and there is a $t \in T$ such that $L(t) \cap [c, d] \neq \emptyset$. Note that there is some γ such that there are at most two distinct $s, t \in \mathsf{Lev}_\gamma(T)$ with $L(s) = (\ell_s, r_s)$ and $L(t) = (\ell_t, r_t)$ such that $\ell_s < c < r_s$ and $\ell_t < d < r_t$ (at most two follows from convexity: otherwise we are in Case 1). In which case $r_s < \ell_t$. Let γ_0 be the minimal such γ, then γ_0 is a successor, and there is an $r \in \mathsf{Lev}_{\gamma_0 - 1}(T)$ such that $[c, d] \subseteq L(r)$.

We prove that $f(c) \geq f((c, r_s))$. An analogous proof shows that $f(d) \geq f((\ell_t, d))$. Suppose, for a contradiction, that there is an $e \in (c, r_s)$ such that $f(e) > f(c)$. Suppose $f(c) = m_{\alpha_c}$. Then, by the definition of f, there is a $t' \in \mathsf{Lev}_{\alpha_c}(T)$ such that $L(t') \subseteq (\ell_s, r_s)$ where $e \in L(t')$, such that $c < L(t')$. But this contradicts our assumption that there is no $t'' \in T$ such that $L(t'') \subseteq [c, d]$. So, $f(c) \geq f((c, r_s))$, and likewise $f(d) \geq f((\ell_t, d))$.

Finally note that f is bounded by $\max\{f(c), f(d)\}$ on $[r_s, \ell_t]$, since for all $t \in \mathsf{Lev}_{\gamma_0}(T)$, the interval $[r_s, \ell_t]$ has empty intersection with $L(t)$, and there is a $t' \in \mathsf{Lev}_{\gamma_0 - 1}(T)$ such that $[r_s, t_\ell] \subseteq L(t')$. So, on $[c, d]$, f has a maximum, namely $\max\{f(c), f(d)\}$. □

By Claim 4.3.8, let $(b_\delta)_{\delta \in \kappa}$ be any sequence in $[c, d]$ such that $(f(b_\delta))_{\delta \in \kappa}$ is strictly increasing and unbounded in $\mathbb{K}$. The proof that $(b_\delta)_{\delta \in \kappa}$ is interval witnessed in $\mathbb{K}$ has two steps. First, we show that $(b_\delta)_{\delta \in \kappa}$ is interval witnessed in the Cauchy closure of $\mathbb{K}$, $\mathsf{VC}(\mathbb{K})$. We use this to prove that $(b_\delta)_{\delta \in \kappa}$ is interval witnessed in $\mathbb{K}$.

The proof that $(b_\delta)_{\delta \in \kappa}$ is interval witnessed in the Cauchy closure of $\mathbb{K}$ follows exactly as in [75, Claim 3.19], noting that this claim shows that *any* sequence which is cofinal in the tree of labels is interval witnessing.

Claim 4.3.9. The sequence $(b_\delta)_{\delta \in \kappa}$ is interval witnessed in the Cauchy completion of $\mathbb{K}$, $\mathsf{VC}(\mathbb{K})$.

Proof. Throughout the proof of this claim, we work in $\mathsf{VC}(\mathbb{K})$. For any $t \in T$, let $L_{\mathsf{VC}}(t) := \{x \in \mathsf{VC}(\mathbb{K}) : \exists a, b \in L(t)(a \leq x \leq b\}$, the convex subset of $\mathsf{VC}(\mathbb{K})$ defined by $L(t)$. Note that $\mathsf{len}(L(t)) = \mathsf{len}(L_{\mathsf{VC}}(t))$.

Let $B := \{b_\delta : \delta \in \kappa\}$ and $C \subseteq (0, 1)$ be a bounded convex subset of $\mathsf{VC}(\mathbb{K})$ such that $|C \cap B| = \kappa$. Fix an $\hat{\varepsilon}_0 \in \mathbb{K}^{>0}$. Without loss of generality, assume that C has neither a supremum nor an infimum. By [25, Lemma 2.12(3)], there is an $\hat{\varepsilon}_1 \in \mathbb{K}^{>0}$ such that for all $x \in C, (x - \hat{\varepsilon}_1, x + \hat{\varepsilon}_1) \subseteq C$. Let $\hat{\varepsilon} := \min\{\hat{\varepsilon}_0, \hat{\varepsilon}_1\}$. Recall that $(\varepsilon_\delta)_{\delta \in \kappa}$ is coinitial in $\mathbb{K}^{>0}$ (and hence in $\mathsf{VC}(\mathbb{K})^{>0}$). So, there is a limit ordinal $\lambda < \kappa$ such that $\varepsilon_\lambda < \hat{\varepsilon}$. By Condition 2 of the labelling, if $t \in \bigcup_{\lambda \leq \eta < \kappa} \mathsf{Lev}_\eta(T)$ has successor length, then the interval $L_{\mathsf{VC}}(t)$ has diameter $<\varepsilon_\lambda < \hat{\varepsilon}$.

Let $t \in \mathsf{Lev}_{\lambda+1}(T)$, and $c \in L_{\mathsf{VC}}(t)$. We claim that the following are equivalent:

1. $L_{\mathsf{VC}}(t) \subseteq C$,

2. $c \in C$, and

3. $L_{\mathsf{VC}}(t) \cap C \neq \emptyset$.

The implications from Condition 1 to Condition 2 to Condition 3 are obvious. As $L_{\mathsf{VC}}(t)$ has diameter $<\hat{\varepsilon}$, and $(c-\hat{\varepsilon}, c+\hat{\varepsilon}) \subseteq C$, and we have that Condition 3 implies Condition 1.

Let $T\restriction C := \{t \in \mathsf{Lev}_{\lambda+1}(T) : \exists\delta \in \kappa(b_\delta \in C \cap L_{\mathsf{VC}}(t))\}$. Since T is a κ-tree, we have that $|\mathsf{Lev}_{\lambda+1}(T)| < \kappa$. So, we can write $T\restriction C = \{t_\gamma : \gamma < \mu\}$ for some $\mu < \kappa$. By construction, for each γ, $L_{\mathsf{VC}}(t_\gamma) \subseteq C$ and the diameter of $L_{\mathsf{VC}}(t_\gamma)$ is less than $\hat{\varepsilon}$.

It remains to prove Condition 2 in the definition of interval witnessing (page 68). By the previous equivalence, if $t' \in \bigcup_{\lambda\leq\eta<\kappa} \mathsf{Lev}_\eta(T)$ and $t \subseteq t'$ where $t \in \mathsf{Lev}_{\lambda+1}(T)$, then $L_{\mathsf{VC}}(t') \subseteq C$ if and only if there is a γ such that $t = t_\gamma$. So, $L_{\mathsf{VC}}(t') \subseteq L_{\mathsf{VC}}(t_\gamma)$. Hence:

$$I := (B\cap C)\setminus\bigcup_{\gamma\in\mu} L_{\mathsf{VC}}(t_\gamma) \subseteq \{t_\gamma : \exists\eta < \lambda(t_\gamma \in \mathsf{Lev}_\eta(T))\} = \bigcup_{\eta\in\lambda+1} \{t_\gamma : t_\gamma \in \mathsf{Lev}_\eta(T)\}.$$

As T is a κ-tree and κ is regular, this shows $|I| < \kappa$. Hence, $(b_\delta)_{\delta\in\kappa}$ is interval witnessed in $\mathsf{VC}(\mathbb{K})$. □

Claim 4.3.10. The sequence $(b_\delta)_{\delta\in\kappa}$ is interval witnessed in $\mathbb{K}$.

Proof. By Claim 4.3.9 and [76, Lemma 4.7], either $(b_\delta)_{\delta\in\kappa}$ has a convergent subsequence in $\mathsf{VC}(\mathbb{K})$, or $(b_\delta)_{\delta\in\kappa}$ is interval witnessed in $\mathbb{K}$.

Suppose, for a contraction, that $(b_\delta)_{\delta\in\kappa}$ has a convergent subsequence, $(b_{\delta_\alpha})_{\alpha\in\kappa}$ (as $\mathsf{bn}(\mathbb{K}) = \kappa$, we know that this subsequence must have length κ). Let $\ell \in \mathsf{VC}(\mathbb{K})$ be its limit. Note that $(b_{\delta_\alpha})_{\alpha\in\kappa}$ is cofinal with $(b_\delta)_{\delta\in\kappa}$. So, in particular, $(f(b_\delta))_{\delta\in\kappa}$ is cofinal with $f([0,1])$, so is unbounded.

We prove that, in this case, f has a continuous extension $\hat{f} : \mathsf{VC}([0,1]) \to \mathsf{VC}([0,1])$ (note that, in general, a continuous f on $\mathbb{K}$ need not have a continuous extension on $\mathsf{VC}(\mathbb{K})$, see page 71). Recall the definition of $L_{\mathsf{VC}}(t)$ from the proof of Claim 4.3.9. Exactly as with f, for any $c \in \mathsf{VC}([0,1])$, let $\beta(c)$ be the least ordinal such that $c \notin \bigcup_{t\in\mathsf{Lev}_{\beta(c)}(T)} L_{\mathsf{VC}}(t)$, let $\alpha(c)$ be the least limit ordinal greater than or equal to $\beta(c)$, and let $\hat{f}(c) = m_{\alpha(c)}$. Clearly, $\hat{f}$ extends f. So, it suffices to prove that $\hat{f}$ is ($\mathsf{VC}([0,1])$-)continuous. But, the proof of Claim 4.3.5 also works for $\hat{f}$ on $\mathsf{VC}([0,1])$, so $\hat{f}$ is $\mathsf{VC}([0,1])$-continuous.

Finally, note that by construction, $\hat{f} \to \infty$ as $x \to \ell$. So, $\hat{f}$ is not continuous at ℓ, contradicting our initial assumption that $(b_\delta)_{\delta\in\kappa}$ had a convergent subsequence in $\mathsf{VC}(\mathbb{K})$. □

Hence, every strictly increasing sequence, $(f(b_\delta))_{\delta\in\gamma}$, which is unbounded in $\mathbb{K}$, is such that $(b_\delta)_{\delta\in\gamma}$ is interval witnessed in $\mathsf{VC}(\mathbb{K})$, and so is interval witnessed in $\mathbb{K}$. □

So, by Claims 4.3.6 and 4.3.7, f is sharp but not $\mathsf{EVT}(\mathbb{K})$. □

This yields the main theorem.

Theorem 4.3.11. Let $\mathbb{K}$ be a κ-spherically complete ordered field $\mathbb{K}$ where $\mathsf{bn}(\mathbb{K}) = \kappa$. Then the following are equivalent:

1. κ has the tree property,

2. $\mathsf{EVT}(\mathbb{K}, \text{sharp})$, and

3. $\mathsf{BVT}(\mathbb{K}, \text{sharp})$.[12]

Proof. By Lemma 4.3.2 and Theorem 4.3.3, it suffices to prove the equivalence of Part 2. and Part 3. If $\mathsf{EVT}(\mathbb{K}, \text{sharp})$ then clearly $\mathsf{BVT}(\mathbb{K}, \text{sharp})$. Conversely, it suffices to prove that if κ does not have the tree property, then there is an unbounded sharp function. The f from Theorem 4.3.3 will do. □

Theorem 4.3.11 reveals some independence in generalised real analysis, unlike the above absolute generalisations. This and Proposition 4.2.19 immediately imply:

Corollary 4.3.12. Let $\mathbb{K}$ be a κ-spherically complete, η_κ ordered field with $\mathsf{bn}(\mathbb{K}) = \kappa$, where κ does not have the tree property. Then some sharp functions on $\mathbb{K}$ are not κ-supercontinuous.

Corollary 4.3.13. Let $\mathbb{K}$ be a κ-homogeneous η_κ ordered field $\mathbb{K}$ where $\mathsf{bn}(\mathbb{K}) = \mathsf{wei}(\mathbb{K}) = \kappa$, which has a (κ, κ)-gap, and suppose κ has the tree property. Then:

1. $f : \mathbb{K} \to \mathbb{K}$ is κ-supercontinuous if and only if f is sharp and κ-continuous,

2. not all sharp functions are κ-continuous, and

3. not all κ-continuous functions are sharp.

Proof. Part 1. is by Proposition 4.2.20, Corollary 4.2.21, and Theorem 4.3.11. Part 2. is by Part 1. and Proposition 4.1.14. Part 3. is by Corollary 4.1.4. □

Question 4.3.14. If κ does not have the tree property, does κ-continuity imply sharpness?

Remark on weak compactness and symmetrical completeness

An order is called *symmetrically complete* if none of its gaps is symmetric (i.e. a (λ, λ)-gap for some λ, following [128, page 263]). In 1908, Hausdorff showed how to construct orders with prescribed types of gaps [88]. Later, Kuhlmann, Kuhlmann, & Shelah proved that any field can be extended to a symmetrically complete field [128, 188]. We do not know whether one can construct symmetrically complete fields with additional properties (a possible first step would be to generalise [128, Theorem 7]). However, if they exist, they would provide another connection between weak compactness and EVT.

If there is a symmetrically complete η_κ ordered field, $\mathbb{K}$, with $\mathsf{bn}(\mathbb{K}) = \mathsf{wei}(\mathbb{K}) = \kappa$ where κ is weakly compact, then κ-continuity and κ-supercontinuity over $\mathbb{K}$ coincide. One can prove this using local κ-continuity (page 73): on $\mathbb{K}$, a (local) κ-continuous function has an image almost gap at every gap (i.e. the case where

[12] Reverse mathematics predicts this: over RCA_0, the tree property of ω is equivalent to $\mathsf{BVT}(\mathbb{R})$ [193, IV.2.3].

f approaches $\pm\infty$ is excluded). Weak compactness implies that every bounded κ-sequence of $\mathbb{K}$ has a strictly monotone κ-subsequence (by [108, Theorem 7.8]; see [40, page 134], [76, Theorem 3.4], and [185, page 148] for more on this property). The coincidence of κ-continuity and κ-supercontinuity then follows easily. Hence, by Theorem 4.2.19, we have that $\mathsf{EVT}(\mathbb{K}, \kappa\text{-continuous})$ holds.

4.4 The κ-Suslin Hypothesis for ordered fields

Finally, we pose an $\mathbb{R}_\kappa$-analogue of Suslin's Question. In 1920, Suslin asked whether any Dedekind complete dense linear order where every collection of pairwise disjoint open intervals is countable was isomorphic to $\mathbb{R}$ [195]. This turns out to be independent of ZFC [194]. On a space, X, we define[13] the *Suslin number* to be the smallest cardinal such that any family of open sets of X of size $\mathfrak{sn}(X)$ is not pairwise disjoint [34].[14] We can generalise this to κ-topologies:

Definition 4.4.1. For a κ-topological space, (X, τ_κ), the *κ-Suslin number*, $\mathfrak{sn}_\kappa(X)$, is the least cardinal such that any family $\mathcal{F} \subseteq \tau_\kappa$ of size $\mathfrak{sn}_\kappa(X)$ is not pairwise disjoint.

Proposition 4.4.2. If $\mathbb{O}$ is an order, $\mathfrak{sn}_\kappa(\mathbb{O}) \leq \mathfrak{sn}(\mathbb{O}) \leq \mathsf{owei}(\mathbb{O})^+$. If $\mathbb{K}$ is an ordered field with a set of integer parts, then $\mathfrak{sn}_\kappa(\mathbb{O}) = \mathfrak{sn}(\mathbb{O}) = \mathsf{owei}(\mathbb{O})^+$

Proof. For the first part, suppose Q is order dense in an order, $\mathbb{O}$, $|Q| = \mathsf{owei}(\mathbb{O})$, and $\mathcal{P}$ partitions $\mathbb{O}$. Then, if $P \in \mathcal{P}$, then $P \cap Q$ is nonempty. So, the maximum size of a family of disjoint non-empty open intervals in $\mathbb{O}$ is bounded by $|Q| = \mathsf{owei}(\mathbb{O})$ (as if $\mathcal{Q}$ partitions Q, then $|\mathcal{Q}| \leq |Q|$). Also, every family of κ-open sets is a family of open sets. So the result holds.

For the second part, we show that $\mathsf{ip}(\mathbb{K})^+ \leq \mathfrak{sn}_\kappa(\mathbb{K})$. Let Z be a set of integer parts for $\mathbb{K}$. If $z \neq z'$ are elements of Z, then $(z - \frac{1}{3}, z + \frac{1}{3}) \cap (z' - \frac{1}{3}, z' + \frac{1}{3}) = \emptyset$. So the lower bound holds. The equality follows from Proposition 2.3.6. □

Hence, in particular, $\mathfrak{sn}(\mathbb{R}_\kappa) = \mathfrak{sn}_\kappa(\mathbb{R}_\kappa) = \kappa^+$.

Much ink has been spilled on the generalised tree-version of Suslin's problem (e.g. [131, 176]), i.e. the consistency of the existence of *κ-Suslin trees* (κ-Aronszajn trees where every antichain is of size $<\kappa$). A κ-Suslin tree exists if and only if a *κ-Suslin order* exists, that is a Dedekind complete order, $\mathbb{O}$, such that $\mathfrak{sn}(\mathbb{O}) \leq \kappa^+$ and $\kappa < \mathsf{wei}(\mathbb{O})$ [106, page 292]. Suslin's Question is also generalised like so:[15]

(SH$_\kappa$) If $\mathbb{O}$ is a symmetrically complete, Dedekind complete dense linear order with $\mathsf{cof}(\mathbb{O}) = \mathsf{coi}(\mathbb{O}) = \kappa$, $\mathfrak{sn}(\mathbb{O}) = \kappa^+$, then $\mathbb{O}$ is order isomorphic to $\mathsf{Ded}(\mathbb{Q}_\kappa)$.

[13] Suslin's Question yields a second cardinal function, *calibre* [7, page 150], [107, Definition 1.19, pages 114-123, 147-149]. These coincide for order topologies [34, Theorem 2.7]. Calibre can also be κ-topologised; we do not know whether the κ-version coincides with $\mathfrak{sn}_\kappa$.

[14] A.k.a. *cellularity* [107, Definition 1.7]. See also [168, Theorem 3.8].

[15] By [49, Theorem 1], $\mathsf{cof}(\mathsf{Ded}(\mathbb{Q}_\kappa)) = \mathsf{coi}(\mathsf{Ded}(\mathbb{Q}_\kappa)) = \kappa$, $\mathsf{wei}(\mathsf{Ded}(\mathbb{Q}_\kappa)) = \kappa$, $\mathfrak{sn}(\mathsf{Ded}(\mathbb{Q}_\kappa)) = \kappa^+$ and $\mathsf{Ded}(\mathbb{Q}_\kappa)$ is symmetrically complete.

Eda proved that (SH_κ) holds if and only if there is no κ-Suslin tree, and that (SH_κ) fails in L for all regular κ [49, Theorems 3 and 4].

However, both (SH_κ) and being a κ-Suslin order are too strong for fields for generalised real analysis: no large cardinality rcf is Dedekind complete (and $\mathsf{wei}(\mathbb{R}_\kappa) = \kappa$). But, we can state a question for Cauchy completeness:

Question 4.4.3. Let κ be an uncountable regular cardinal such that $|\kappa^{<\kappa}| = \kappa$. Is (it consistent with ZFC that) any Cauchy complete η_κ-ordered field, $\mathbb{K}$, such that $\mathfrak{sn}(\mathbb{K}) = \kappa^+$, ordered-field isomorphic to $\mathbb{R}_\kappa$? What about with $\mathfrak{sn}_\kappa(\mathbb{K}) = \kappa^+$?

By Theorem 2.3.13 and Proposition 4.4.2, if $\mathbb{K}$ is a Cauchy complete η_κ-ordered rcf with $\mathsf{bn}(\mathbb{K}) = \kappa$, $\mathfrak{sn} = \kappa^+$ (or $\mathfrak{sn}_\kappa(\kappa) = \kappa^+$), then $\mathbb{K}$ is ordered-field isomorphic to $\mathbb{R}_\kappa$.

Chapter 5

Transfinite Sums and Polynomials

This chapter focuses on the question of whether the theory of infinite sums and series can be extended to large cardinality fields. Overall, the answer is mostly negative.

In Section 5.1, we state the basic desiderata on possible sums; in Section 5.2, we prove that these properties are incompatible (Theorem 5.2.3); in Section 5.3 we analyse some candidate notions of sum from the literature and novel ones (these results are summarised in Table 1.1); finally, Section 5.4 concerns the implication of this difficulty in defining infinite sums on generalised polynomials and the Weierstraß' Approximation Theorem.

Throughout this chapter, we assume that κ is an uncountable regular cardinal. We reserve $\sum$ for *finite* sums on arbitrary fields, or for the standard *countable* summation on $\mathbb{R}$ (i.e. either finite sums on $\mathbb{R}$ or convergence of $\mathbb{R}$-valued series).

5.1 Desiderata for sums

In this section, we stipulate some desiderata that should be satisfied by a notion of sum on an ordered field $\mathbb{K}$. The section includes a list of the desiderata.

A *sum* on $\mathbb{K}$ is a function $S : \mathsf{Dom}(S) \to \mathbb{K}$ where $\mathsf{Dom}(S) \subseteq \bigcup_{\mu\in\mathsf{On}} \mathbb{K}^\mu$. The elements of $\mathsf{Dom}(S)$ are called *S-summable*. If $\mu \in \mathsf{On}$, we also write $S_\mu(x_\alpha)$ for $S((x_\alpha)_{\alpha\in\mu})$.

We remind the reader of our notation for sequences from Section 2.1.1. Recall that if $\mathbb{x}$ and $\mathbb{y}$ are sequences of the same length, then $\mathbb{x}+\mathbb{y}$ is the sequence define by their pontwise sum (i.e. $(x_\alpha)_{\alpha\in\mu} + (y_\alpha)_{\alpha\in\mu} = (x_\alpha + y_\alpha)_{\alpha\in\mu}$). If $\mathbb{x} \in \mathbb{K}^\mu$, we call $\mathsf{supp}(\mathbb{x}) := \{\alpha \in \mu \colon x_\alpha \neq 0\}$. Clearly, $\mathsf{supp}(\mathbb{x})$ is a well-ordered set of order type $\xi \leq \mu$. If $i : \xi \to \mathsf{supp}(\mathbb{x})$ is the order isomorphism, then we define the *contraction* of $\mathbb{x}$, denoted by $\mathsf{contract}(\mathbb{x}) \in \mathbb{K}^\xi$, as $\alpha \mapsto x_{i(\alpha)}$. A sequence $\mathbb{x}$ is called *positive* if all elements of the sequence are positive. A sum S is called *positive* if, for all positive $\mathbb{x} \in \mathsf{Dom}(S)$, we have that $S(\mathbb{x}) > 0$.

If S is a sum, we say that $\mathsf{spect}(S) := \{\mu \in \mathsf{On}\colon$ there is some $\mathbb{x} \in \mathsf{Dom}(S_\mu)$ with $\mathsf{supp}(\mathbb{x}) = \mu\}$ is the *spectrum* of S, i.e., the class of ordinals μ for which there is a non-trivial S-summable sequence of length μ.

There are a number of sequences for which classical analysis gives us a clear indication that they should be summable and what their sum should be. E.g. for all $\mathbb{x} \in \mathbb{K}^{<\omega}$, the algebraic operation $+$ on $\mathbb{K}$ provides a natural candidate for $S(\mathbb{x})$, viz. $x_0 + \ldots + x_n$ where $\mathsf{len}(\mathbb{x}) = n + 1$ (this corresponds to **Ext**, stated formally in the list below).

Since $\mathbb{Q} \subseteq \mathbb{K}$ (Section 2.3.6), and some infinite $\mathbb{Q}$-sequences are $\sum$-summable, e.g. $\sum_\omega(\frac{1}{2^n}) = 1$, we might similarly expect that if $\mathbb{x} \in \mathbb{Q}^\omega$ is a convergent series with value $X := \sum_{i=0}^{\infty} x_i$, then $\mathbb{x}$ is S-summable and $S(\mathbb{x}) = X$. In which case, say that S satisfies $\mathbf{Ext}_\mathbb{Q}$ (this strengthens **Ext**).

We may also expect S to depend on $\mathsf{bn}(\mathbb{K})$. The most elementary connection generalises the sum of the geometric series, i.e. $S(\frac{1}{2^\alpha})_{\alpha\in\mathsf{bn}(\mathbb{K})}) = 1$. This connection is captured by **Or** in the list below. Similarly, we might expect the summation of constants to be multiplication, i.e. (for a set of ordinal ordinals, $\mathsf{bn}(\mathbb{K})$, and a $\lambda < \mathsf{bn}(\mathbb{K})$) we might expect $S(\overline{1}_\lambda) = \lambda$. This corresponds to $\mathbf{Cons}_e$ in the list below.

More cautiously, there are some properties of (countable) sums which are equivalent to the Dedekind completeness of $\mathbb{K}$ (i.e. they only hold if $\mathbb{K} = \mathbb{R}$), e.g. if for every sequence $(x_n)_{n\in\omega}$ from $\mathbb{K}$, such that $\lim_{n\to\omega} |\frac{x_{n+1}}{x_n}| < 1$, the series $(\sum_{n=1}^{n=N} x_n)_{N\in\omega}$ converges (see also [44, CA47-CA50]).

For our purposes, we focus on the following properties of sums:

1. **Ext** (for "extending finite sums"). If $\mathsf{supp}(\mathbb{x}) = \{x_{\alpha_0}, ..., x_{\alpha_n}\}$ is finite, then $S(\mathbb{x}) = x_{\alpha_0} + ... + x_{\alpha_n}$.

2. **Lin** (for "linearity"). If $\mathbb{x}, \mathbb{y} \in \mathsf{Dom}(S) \cap \mathbb{K}^\mu$ and $a \in \mathbb{K}$, then the sequence $\mathbb{z} := \alpha \mapsto x_\alpha + a.y_\alpha$ is S-summable and $S(\mathbb{x}) + a.S(\mathbb{y}) = S(\mathbb{z})$.

3. **Conc** (for "concatenation"). If $\mathbb{x}$ and $\mathbb{y}$ are S-summable, then so is $\mathbb{x}^\frown \mathbb{y}$ and $S(\mathbb{x}) + S(\mathbb{y}) = S(\mathbb{x}^\frown \mathbb{y})$.

4. **Comp** (for "comparison test"). If $\mathbb{x} \in \mathsf{Dom}(S_\mu)$ and $\mathbb{y}$ is any sequence of length μ such that for all $\alpha \in \mu$, we have that $0 \leq y_\alpha \leq x_\alpha$, then $\mathbb{y}$ is S-summable and $S_\mu(\mathbb{y}) \leq S_\mu(\mathbb{x})$.

5. $\mathbf{Cons}_e$ (for "constant"). The map $e : \mathsf{bn}(\mathbb{K}) \to \mathbb{K}$ is an ordinal embedding (see Section 2.3.3) and for all $\lambda < \mathsf{bn}(\mathbb{K})$, we have that $S(\overline{1}_\lambda) = e(\lambda)$.

6. **Ind** (for "independence of order"). Let $f : \mu \to \mu$ be a bijection. Let $\mathbb{x} \in \mathsf{Dom}(S_\mu)$ be non-negative and let $x^f_\alpha := x_{f(\alpha)}$. Then $\mathbb{x}^f$ is S-summable and $S(\mathbb{x}) = S(\mathbb{x}^f)$.

7. **El** (for "elimination of zeroes"). A sequence $\mathbb{x}$ is S-summable if and only if $\mathsf{contract}(\mathbb{x})$ is S-summable; furthermore, $S(\mathbb{x}) = S(\mathsf{contract}(\mathbb{x}))$.

8. $\mathbf{Ext}_\mathbb{Q}$ (for "extending series on $\mathbb{Q}$"). If $\mathbb{x} \in \mathbb{Q}^\omega$ is a series converging (in $\mathbb{Q}$) to some $\ell \in \mathbb{K}$, then $S(\mathbb{x}) = \ell$.[1]

[1] By Remark 2.3.26, if $\mathbb{K}$ is an rcf and an $\eta_{2^{\aleph_0}}$ order, then $\mathbb{R} \hookrightarrow \mathbb{K}$. In which case, we could also consider the strengthening "$\mathbf{Ext}_\mathbb{R}$", requiring converging $\mathbb{R}^\omega$-series to have an S-sum. As $\mathbf{Ext}_\mathbb{Q}$ is already difficult to satisfy (e.g. Theorem 5.2.3), we focus on this.

9. **Or** (after Nicole Oresme, [18, page 241]). For all $x \in \mathbb{K}$ such that $0 < |x| < 1$, the $\mathsf{bn}(\mathbb{K})$-sequence $\mathbb{g}^x$ defined by $\mathbb{g}^x_\alpha := \frac{1}{x^\alpha}$ is S-summable and $S(\mathbb{g}^x) = \frac{1}{1-x}$; a sum satisfying **Or** will be called *Oresmic*.

10. **Geo** (for "geometric"). If I with $\mathsf{len}(I) = x$ is an interval, partitioned into a λ-sequence of disjoint interval I_α with $\mathsf{len}(I_\alpha) = x_\alpha$, then $(x_\alpha)_{\alpha\in\lambda}$ is S-summable and $S((x_\alpha)_{\alpha\in\lambda}) = x$; a sum satisfying **Geo** will be called *geometric*.

11. **Tr** (for "truncation"). If $\mathbb{x} \in \mathsf{Dom}(S)$, then for all μ, $\mathbb{x}\restriction\mu \in \mathsf{Dom}(S)$.

Fact 5.1.1. Let S be a sum; we observe a number of immediate consequences of the properties listed:

1. If S satisfies **Ext**, then $\mathbb{K} = \mathsf{Ran}(S)$ and $\mathbb{K}^{<\omega} \subseteq \mathsf{Dom}(S)$.

2. If S satisfies **Lin**, and $\mu \in \mathsf{spect}(S)$, then $\overline{0}_\mu$ is S-summable and $S_\mu(\overline{0}_\mu) = 0$.

3. If S satisfies **Ext** and **Conc**, then $\mathsf{spect}(S)$ is closed under successors: if $\mathbb{x}$ is positive and S-summable, then $\mathbb{x}^\frown 1$ is too.

4. If S satisfies **Lin** and **Conc**, $\mathbb{x} \in \mathsf{Dom}(S_\mu)$, and $\lambda \in \mathsf{spect}(S)$, then $\mathbb{x}^\frown\overline{0}_\lambda \in \mathsf{Dom}(S)$ and $S(\mathbb{x}) = S(\mathbb{x}^\frown\overline{0}_\lambda)$.

5. If S satisfies **Ext** and **Comp**, then S is positive: given any positive $\mathbb{x}$, let $\mathbb{y}$ be the sequence with $y_0 := x_0$ and $y_\alpha := 0$ (for $\alpha > 0$); then $0 < x_0 = S(\mathbb{y}) \leq S(\mathbb{x})$.

6. If S satisfies **Comp** and **El**, then S satisfies **Tr**.

Example 5.1.2. Let S^{finite} be defined by $\mathsf{Dom}(S^{\mathsf{finite}}) := \{\mathbb{x} \colon \mathsf{supp}(\mathbb{x}) \text{ is finite}\}$ and $S^{\mathsf{finite}}(\mathbb{x}) := \sum_{\alpha\in\mathsf{supp}(\mathbb{x})} x_\alpha$. Clearly, S^{finite} satisfies **Ext**, **Lin**, **Conc**, **Comp**, **Ind**, **El**, **Tr** (all inherited from finite sums), but not $\textbf{Ext}_{\mathbb{Q}}$.

Example 5.1.3. For any $c \in \mathbb{K}$, let S^c be the following sum:

$$S^c(\mathbb{x}) = \begin{cases} x_{\alpha_0} + \ldots + x_{\alpha_n} & \text{if } \mathsf{supp}(\mathbb{x}) = \{x_{\alpha_0}, \ldots, x_{\alpha_n}\}, \\ c & \text{otherwise.} \end{cases}$$

Then for all $c \in \mathbb{K}$, S^c satisfies **Ext**, but not **Lin** as $S^c(\overline{1}_1 + \overline{1}_\omega) = c \neq 1 + c$.

Fairly weak conditions are enough to ensure that infinite sequences can have unbounded sums:

Remark 5.1.4. If S satisfies **Ext** and **Conc**, and there is a $\mu \geq \omega$ and an $\mathbb{x} \in (\mathbb{K}^{>0})^\mu$ which is S-summable, then we have that for any $c \in \mathbb{K}$, there is an S-summable $\mathbb{y} \in (\mathbb{K}^{>0})^\mu$ such that $S(\mathbb{y}) > c$: if $S(\mathbb{x}) < 0$, let $m = c - S(\mathbb{x})$, otherwise let $m = c$. By **Ext** and **Conc**, $(m+1)^\frown\mathbb{x}$ is S-summable and $S((m+1)^\frown\mathbb{x}) = m + 1 + S(\mathbb{x})$, so $m + 1 + S(\mathbb{x}) > c$.

Combinations of the desiderata mean that certain sequences are not summable. For example, if S satisfies **Lin**, **Conc**, **Comp**, and **Cons**$_e$, then $\overline{1}_{\mathsf{bn}(\mathbb{K})}$ is not S-summable. We can prove this by comparing $\overline{1}_{\mathsf{bn}(\mathbb{K})}$ against each $\overline{1}_\mu$ for all $\mu < \mathsf{bn}(\mathbb{K})$, which are S-summable and are such that $S_\mu(\overline{1}_\mu) = e(\mu)$, by **Cons**$_e$. We can strengthen this by extending an argument for transfinite sums on ordinals from [133, page 523]:

Lemma 5.1.5. If S satisfies **Ext**, **Lin**, and **Conc**, then $\overline{1}_\alpha$ is not S-summable, for any $\alpha \geq \omega$.

Proof. The ω case is representative, so we assume $\alpha = \omega$. Assume towards a contradiction that $\overline{1}_\omega \in \mathsf{Dom}(S)$. By **Ext** and Fact 5.1.1 Part 2, $S(\overline{0}_1) = 0$ and $S(\overline{1}_1 {}^\frown \overline{0}_\omega) = 1$; by **Conc**, $S(\overline{0}_1 {}^\frown \overline{1}_\omega) = S(\overline{1}_\omega)$. By **Lin**,

$$\begin{aligned} S(\overline{1}_\omega) &= S(\overline{0}_1 {}^\frown \overline{1}_\omega + \overline{1}_1 {}^\frown \overline{0}_\omega) \quad \text{(i.e. } S \text{ of the pointwise sum of } \overline{0}_1 {}^\frown \overline{1}_\omega \text{ and } \overline{1}_1 {}^\frown \overline{0}_\omega) \\ &= S(\overline{0}_1 {}^\frown \overline{1}_\omega) + S(\overline{1}_1 {}^\frown \overline{0}_\omega) \\ &= S(\overline{1}_\omega) + 1, \end{aligned}$$

a contradiction. □

Proposition 5.1.6. If S satisfies **El** and **Lin**, then S satisfies **Conc**.

Proof. Using Fact 5.1.1 Part 2, by **Lin**, $S(\overline{0}_\lambda) = 0$ for all $\lambda \in \mathsf{spect}(S)$. Let $\mathbb{x} \in \mathbb{K}^\lambda$ and $\mathbb{y} \in \mathbb{K}^\mu$ be S-summable. By **El**, $\overline{0}_\lambda {}^\frown \mathbb{y}$ and $\mathbb{x} {}^\frown \overline{0}_\mu$ are S-summable, and $S(\overline{0}_\lambda {}^\frown \mathbb{y}) = S(\mathbb{y})$ and $S(\mathbb{x} {}^\frown \overline{0}_\mu) = S(\mathbb{x})$. Then by **Lin**, $\mathbb{x} {}^\frown \mathbb{y} = \mathbb{x} {}^\frown \overline{0}_\mu + \overline{0}_\lambda {}^\frown \mathbb{y}$ (the pointwise sum) is S-summable and $S(\mathbb{x} {}^\frown \mathbb{y}) = S(\mathbb{x}) + S(\mathbb{y})$. □

Hence, the combination of **Ext**, **Lin**, and **El** also shows that $\overline{1}_\alpha$ is not summable for any infinite α.

Proposition 5.1.7. If S satisfies **Ext**, **Lin**, **Comp**, and **El**, and $\mathbb{x}$ is infinite, positive, and S-summable, then $\mathbb{x}$ tends to 0.

Proof. By Proposition 5.1.6, S satisfies **Conc**. Suppose that $\mathbb{x}$ does not tend to 0. Then, there is some $\varepsilon > 0$ and some subset $X \subseteq \mathsf{bn}(\mathbb{K})$ of cardinality $\mathsf{bn}(\mathbb{K})$ such that $x_\xi > \varepsilon$ for all $\xi \in X$. Define $\mathbb{y}$ by

$$y_\gamma := \begin{cases} x_\gamma & \text{if } \gamma \in X, \\ 0 & \text{otherwise.} \end{cases}$$

By **Comp**, we have that $\mathbb{y}$ is S-summable and therefore, by **El**, $\mathsf{contract}(\mathbb{y}) = \overline{\varepsilon}_{\mathsf{bn}(\mathbb{K})}$ is S-summable. Hence, by **Lin**, $\overline{1}_\alpha$ is S-summable, in contradiction to Lemma 5.1.5. □

By the comments on page 98, if S satisfies **Lin**, **Comp**, **El**, and **Cons**$_e$ and $\mathbb{x}$ is a positive, and S-summable, $\mathsf{bn}(\mathbb{K})$-sequence, then $\mathbb{x}$ tends to 0.

5.2 Incompatibility of desiderata

Here, we show that combinations of desiderata are incompatible. In particular, a combination of the other desiderata will imply that $\mathbf{Ext}_{\mathbb{Q}}$ cannot hold. Any sum operation on non-Archimedean fields therefore cannot both extend the sums on $\mathbb{Q}$ and satisfy all of the desiderata. Approaches from the literature either turn on the problems in summing a geometric series (i.e. being Oresmic), or in summing a divided interval (i.e. **Geo**). We state versions of these arguments in Remark 5.2.1 and Proposition 5.2.2. In fact, an underlying argument strengthens both of these approaches, namely Theorem 5.2.3.

Remark 5.2.1. A standard method of showing the failure of desiderata on non-Archimedean fields is to consider geometric series, e.g. $(\frac{1}{2^n})_{n\in\omega}$ (see, for example, [44, AP38]). The key move is to show that $S((\frac{1}{2^\alpha})_{\alpha\in\omega}) = S((\frac{1}{2^\alpha})_{\alpha\in\kappa}) - S(\overline{0}_\omega{}^\frown(\frac{1}{2^\alpha})_{\alpha\in\kappa}) = 1-\varepsilon < 1$, for some $\varepsilon > 0$, by showing that all of the necessary sequences are summable. Using our terminology, we can state the result like so: if $\mathbb{K}$ is an exponential rcf, where $\mathsf{bn}(\mathbb{K}) = \kappa$, with a sum S satisfying **Ext**, **Lin**, **Comp**, **El**, and **Or**, and $\mathbb{x}$ is defined by $x_n := \frac{1}{2^n}$, then $S(\mathbb{x}) < 1$ (so S does not satisfy $\mathbf{Ext}_{\mathbb{Q}}$).

Proposition 5.2.2 (Folklore)**.** Let $\mathsf{bn}(\mathbb{K}) > \omega$ and $S((\frac{1}{2^n})_{n\in\omega}) = 1$. Then S is not geometric. So, any sum satisfying **Geo** cannot satisfy $\mathbf{Ext}_{\mathbb{Q}}$.

Proof. For all $n < \omega$, let $I_n := [1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$. Then $(\mathsf{len}(I_n))_{n\in\omega} = (\frac{1}{2^n})_{n\in\omega}$. So, $S(\mathsf{len}([1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})) = 1$. However $\bigcup_{n\in\omega}[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}) \neq [0,1]$, because e.g. $\bigcup_{n\in\omega}[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}) \subsetneq [0, 1-\frac{1}{\omega})$. So, $\mathsf{len}(\bigcup_{n\in\omega}[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}) \leq 1-\frac{1}{\omega}$. □

Our principal incompatibility result strengthens Lemma 5.1.5 to *all* infinite positive sequences, assuming **Comp**:

Theorem 5.2.3 (Folklore)**.** Suppose that $\mathsf{bn}(\mathbb{K}) > \omega$ and that S satisfies **Ext**, **Lin**, **Conc**, and **Comp**. Then no positive $\mathbb{x}$ of infinite length is S-summable.

Proof. Suppose towards a contradiction that $\mathbb{x}$ is positive, has length $\mu \geq \omega$, and is S-summable.

By **Comp**, $\mathbb{x}\restriction\omega^\frown\overline{0}_\mu$ is S-summable. As $\omega < \mathsf{bn}(\mathbb{K})$, and $\mathbb{x}\restriction\omega$ is positive, $\mathbb{x}\restriction\omega$ has a lower bound $b > 0$. By **Comp** again, $\overline{b}_\omega{}^\frown\overline{0}_\mu$ is S-summable, and hence, by **Lin**, $\overline{1}_\omega{}^\frown\overline{0}_\mu$ is S-summable. The rest is a variation of the proof of Lemma 5.1.5; without loss of generality, we can assume that $\mu \geq \omega^2$, so sequences of length $\omega+\mu$ or $1+\omega+\mu$ can be considered sequences of length μ.

By **Ext**, $S(\overline{0}_1) = 0$ and $S(\overline{1}_1{}^\frown\overline{0}_\mu) = 1$; by **Conc**, $S(\overline{0}_1{}^\frown\overline{1}_\omega{}^\frown\overline{0}_\mu) = S(\overline{1}_\omega{}^\frown\overline{0}_\mu)$. By **Lin**, noting that the following sequences have the same length, we have that:

$$\begin{aligned} S(\overline{1}_\omega{}^\frown\overline{0}_\mu) &= S(\overline{0}_1{}^\frown\overline{1}_\omega{}^\frown\overline{0}_\mu + \overline{1}_1{}^\frown\overline{0}_\mu) \\ &= S(\overline{0}_1{}^\frown\overline{1}_\omega{}^\frown\overline{0}_\mu) + S(\overline{1}_1{}^\frown\overline{0}_\mu) \\ &= S(\overline{1}_\omega{}^\frown\overline{0}_\mu) + 1, \end{aligned}$$

a contradiction. □

Corollary 5.2.4. Suppose that $\mathsf{bn}(\mathbb{K}) > \omega$ and that S satisfies **Ext**, **Lin**, **Conc**, and **Comp**. Then S fails $\mathbf{Ext}_{\mathbb{Q}}$.

The reasoning in Theorem 5.2.3 is fairly flexible, e.g. if S satisfies **Ext**, **Lin**, **Concat**, and **Tr**, then no positive infinite sequence is S-summable. Note that, unlike in Remark 5.2.1 and Proposition 5.2.2, the value of $S(\overline{1}_\omega)$ in Lemma 5.1.5 is not important, summability alone suffices.

5.3 Candidate notions of sum

In this section, we define and analyse several notions of sum from the literature, and some further natural definitions. As we have seen in the previous section, notably in Theorem 5.2.3, each notion of sum must violate some desiderata and for each candidate, we identify which desiderata are satisfied or violated. The results are summarised in Table 1.1.

5.3.1 Model-theoretic sums

Finite Model-theoretic Sum

Let $\mathbb{K}$ be an rcf. By Proposition 2.3.7, we can fix an ordinal embedding $e\colon \mathsf{bn}(\kappa) \to \kappa$. For the remainder of this section, we write λ for $e(\lambda)$ for any ordinal $\lambda < \mathsf{bn}(\mathbb{K})$ where the ordinal embedding e is fixed. The following definition generalises one of Rubinstein-Salzedo and Swaminathan's [179, Definition 39]:[2]

Definition 5.3.1. Let $\mathsf{len}(\mathbb{x}) = \lambda < \mathsf{bn}(\mathbb{K})$. If there is a unique rational function $f \in \mathbb{K}(X)$ such that $f(n) = x_0 + \ldots + x_n$ (note that $\mathbb{N} \subseteq \mathbb{K}$), then we say that f *sums* $\mathbb{x}$. In that case, define $S^{\mathsf{finmod}}(\mathbb{x}) := f(\lambda)$.

Proposition 5.3.2. Let $\mathbb{K}$ be an rcf. Then S^{finmod} satisfies **Lin** and $\mathbf{Cons}_e$.

Proof. For **Lin**, let $\mathbb{x} = (x_\alpha)_{\alpha\in\mu}$ and $\mathbb{y} = (y_\alpha)_{\alpha\in\mu}$ be S^{finmod}-summable and $a \in \mathbb{K}$. Then there are unique $f, g \in \mathbb{K}(X)$ which sum $\mathbb{x}, \mathbb{y}$ respectively. So, $a.f + g \in \mathbb{K}(X)$ and $a.f + g(n) = \sum_{\alpha\leq n}((a \times x_\alpha) + y_\alpha)$. If a distinct h also sums $a.\mathbb{x} + \mathbb{y}$ then $h - a.f$ sums $\mathbb{y}$, contradicting that $\mathbb{y}$ is S^{finmod} summable. So, $a.f + g$ uniquely sums $a.\mathbb{x} + \mathbb{y}$. For **Cons**, use the identity function. ☐

Remark 5.3.3. On rcfs, S^{finmod} satisfies a weakening of **Ind**, namely constancy under any bijection $f : \alpha \to \alpha$ such that $f(n) = n$ for all $n \in \omega$ (as if f sums $(x_\alpha)_{\alpha\in\lambda}$ then f sums $(x_{f(\alpha)})_{\alpha\in\lambda}$).

To prove the desiderata of S^{finmod}, we use a standard fact about polynomials:

Remark 5.3.4. Let $\mathbb{x} \in \mathbb{K}^\omega$. Suppose $f, g \in \mathbb{K}(X)$ sum $\mathbb{x}$, where $g = \frac{p^g}{q^g}$ for some unique coprime $p^g, q^g \in \mathbb{K}[X]$, likewise for f. Let P_n be the sum of the first n terms of $\mathbb{x}$. Suppose $p^g = \sum_{i=1}^{N_g} p_i^g x^i$ and $q^g = \sum_{j=1}^{M_g} q_j^g x^j$ for $p_i^g, q_j^g \neq 0$. For every $n \in \omega$, $g(n) = P_n$, so $p^g(n) = P_n.q^g(n)$, i.e. $\sum_{i=1}^{N_g} p_i^g n^i = P_n. \sum_{j=1}^{M_g} q_j^g n^j$, likewise for f. By

[2] Their definition is in the context of No; which we generalise to arbitrary ordered fields.

a simple algebraic argument, we can solve $\max\{M_g, M_f\} + \max\{N_g, N_f\} + 1$-many of these simultaneous equations, which then shows that $M_g = M_f$, $N_g = N_f$, $p_i^g = p_i^f$, and $q_j^g = q_j^f$. So, $f = g$.

Proposition 5.3.5. If $\mathbb{K}$ is an rcf, then S^{finmod} violates **Ext**, **Conc**, **El**, **Comp**, and **Ind**.

Proof. For **Ext**, let $\alpha \geq \omega$ and $\mathbb{x} = \overline{0}_\alpha{}^\frown 1$. Let $c_0 : \mathbb{K} \to \{0\}$ be the constant 0 function. Then c_0 sums $\mathbb{x}$. By Remark 5.3.4, c_0 uniquely sums $\mathbb{x}$. Hence $S^{\mathsf{finmod}}(\mathbb{x}) = 0$. But assuming **Ext**, $S^{\mathsf{finmod}}(\mathbb{x}) = 1$, a contradiction. The same example shows that S^{finmod} fails **El**.

The identity $f \in \mathbb{K}(X)$ sums the sequence $\overline{1}_\omega$ and thus for any sequence $\mathbb{x}$ with $\mathbb{x}\restriction\omega = \overline{1}_\omega$, we get $S^{\mathsf{finmod}}(\mathbb{x}) = \mathrm{id}(\omega) = \omega$. Similarly, the constant zero function sums the sequence $\overline{0}_\omega$, and thus any sequence $\mathbb{x}$ starting with ω many zeros, $S^{\mathsf{finmod}}(\mathbb{x}) = 0$. Let $\mathbb{y}$ be any sequence such that $S^{\mathsf{finmod}}(\mathbb{y}) > 0$. This gives immediate refutations of **Conc** and **Ind**: if **Conc** holds, then $\omega = S^{\mathsf{finmod}}(\overline{1}_\omega{}^\frown \mathbb{y}) = \omega + S^{\mathsf{finmod}}(\mathbb{y}) > \omega$. If **Ind** holds, then $0 = \overline{0}_\omega \frown \overline{1}_\omega = \overline{1}_\omega{}^\frown\overline{0}_\omega = \omega$.

For **Comp**, let $\lambda \geq \omega$. The sequence $\overline{1}_\lambda$ is S^{finmod}-summable, and summed by the identify, $f(x) = x$. We define the 'zero-one' sequence like so:

$$x_\alpha := \begin{cases} 0 & \text{if } \alpha = 2\beta, \\ 1 & \text{if } \alpha = 2\beta + 1. \end{cases}$$

Suppose, for a contradiction, that S^{finmod} satisfies **Comp**. Then, $\overline{1}_\lambda$ point-wise bounds $(x_\alpha)_{\alpha\in\lambda}$, so $(x_\alpha)_{\alpha\in\lambda}$ is S^{finmod}-summable, as summed by some $\frac{p}{q} \in \mathbb{K}(X)$. As $x_0 = 0$, $p(0) = 0$, so $p = x.p'$ for some $p \in \mathbb{K}[X]$. For all $n < \omega \leq \lambda$, $(x.\frac{p'}{q})(2n) = n$, i.e. $2n.\frac{p'(2n)}{q(2n)} = n$. So, $2.p'(2n) = q(2n)$. By Remark 5.3.4, we know that $2p' = q$. Hence $\frac{p}{q} = \frac{1}{2}x$. So $\frac{p}{q}$ does not sum $(x_\alpha)_{\alpha\in\lambda}$: it is incorrect at all odd $m < \omega$. □

The underlying reason is that S^{finmod} ignores all infinite entries in a summable sequence. Any sum which depends on only a $<\kappa$-length subsequence of an S-summable x has the same fate: by a variation of the proof of Proposition 5.3.5, it will fail **Ext**, **Conc**, **Ind**, and **El**.

Infinite model-theoretic sum

The natural modification of S^{finmod} to avoid Proposition 5.3.5 is to require that the sum takes into account infinite coordinates of a sequence. As before, we assume that $\mathbb{K}$ is an rcf with an ordinal embedding $e : \mathsf{bn}(\mathbb{K}) \to \mathbb{K}$ and we identify $\mathsf{bn}(\kappa)$ with the $\mathsf{Ran}(e)$.

Definition 5.3.6. Let $\mathsf{len}(\mathbb{x}) = \lambda < \mathsf{bn}(\mathbb{K})$. If there is a unique rational function $f \in \mathbb{K}(X)$ such that $f(0) = x_0$ and for all $\alpha < \lambda$, $f(\alpha + 1) = f(\alpha) + x_{\alpha+1}$, then we say that f *well-sums* $\mathbb{x}$. In that case, define $S^{\mathsf{infmod}}(\mathbb{x}) := f(\lambda)$.

Proposition 5.3.7. The sum S^{infmod} satisfies **Lin**, but not **Ext**, **El**, **Ind**, **Conc**, and **Comp**.

Proof. The proof of **Lin** is exactly as in Proposition 5.3.2.

Next we prove that S^{infmod} violates **Ext**. Suppose for a contradiction that it satisfies S^{infmod}. Let $\lambda < \mathsf{bn}(\mathbb{K})$ and $\varkappa := \overline{0}_\omega {}^\frown 1 {}^\frown \overline{0}_\lambda$. If S^{infmod} satisfies **Ext**, then $\varkappa$ is S^{infmod}-summable as it has finite support, and $S^{\mathsf{infmod}}(\varkappa) = 1$. So, there are unique $p, q \in \mathbb{K}[X]$ such that $\frac{p}{q}$ well-sums $\varkappa$. Hence $\frac{p(n)}{q(n)} = 0$ for all $n \in \omega$, and $\frac{p(\alpha)}{q(\alpha)} = 1$ for all $\alpha \in \lambda \setminus \omega$. So, $p(n) = 0$ for all $n \in \omega$. As p is a polynomial, p has only finitely many turning points (as in Corollary 4.2.6, since we assume $\mathbb{K}$ is an rcf). So, if $p(n) = 0$ for all $n \in \omega$, p is the constant 0 function. Hence $\frac{p(\alpha)}{q(\alpha)} = 0$ for all $\alpha \in \lambda \setminus \omega$, a contradiction.

Moreover, the constant 1 function clearly uniquely represents the sequence $\overline{1}_1$ and the sequence $1 {}^\frown \overline{0}_\lambda$. So, by comparing with $\varkappa$, S^{infmod} fails **El** and **Ind** respectively.

Meanwhile, the constant 0 function uniquely sums $\overline{0}_\omega$ and $\overline{0}_\lambda$, so $\overline{0}_\omega$ and $\overline{0}_\lambda$ are S^{infmod}-summable, and so is $\overline{1}_1$, but $\varkappa$ is not, hence S^{infmod} violates **Conc**.

Lastly, note that $\overline{1}_\lambda$ pointwise-bounds $\varkappa$, and is S^{infmod}-summable, as it is uniquely summed by the identity function. So, S^{infmod} fails **Comp**. □

Finally, we can expand the permissible representations of sums, beyond $\mathbb{K}(X)$. Rubinstein-Salzedo and Swaminathan's sum on No includes $\exp, \log$, and arctan. For an arbitrary exponential field, $\mathbb{K}$, there are two natural expansions of $\mathbb{K}(X)$:

1. Let $G(\mathbb{K}) \subseteq \mathbb{K}^{\mathbb{K}}$ be the closure of $\mathbb{K}(X) \cup \{\exp, \log\}$ under addition, subtraction, multiplication, and composition of functions, and by division by non-zero functions.

2. Let $H(\mathbb{K}) := \{f : \mathbb{K}^{\geq 0} \to \mathbb{K} : f$ is defined using a formula in the language of exponential rcfs with a single free variable$\}$.

Then $\mathbb{K}(X) \subsetneq G(\mathbb{K}) \subsetneq H(\mathbb{K})$, e.g. $h = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{otherwise} \end{cases}$ is in $H(\mathbb{K}) \setminus G(\mathbb{K})$.

If S^G and S^H are the sums defined as in Definition 5.3.6, but with $\mathbb{K}(X)$ replaced by $G(\mathbb{K})$ and $H(\mathbb{K})$, respectively, then Proposition 5.3.7 still holds for S^G, using the same argument. A similar result holds for S^H with a more careful argument that there is no unique $h \in H(\mathbb{K})$ which represents the sum of $\varkappa$. The key observation is that ω is not definable in the language of exponential rcfs (as the theory is o-minimal, which follows in much the same way as [144, Corollary 3.3.23]), so we cannot define $J_{\neg\omega}$ (from Example 4.1.4), as is required to well-sum $\overline{0}_\omega {}^\frown 1 {}^\frown \overline{0}_\lambda$.

5.3.2 Suprema of finite sums

In this section, we analyse the definition of sums as the supremum of the collection of finite subsums (Theorem 5.3.12 and a version of its proof were communicated by Galeotti). As usual, if X is a set and μ is a cardinal, let $[X]^{<\mu} = \{A \subseteq X : |X| < \mu\}$.

The basic idea of this definition is to consider a sequence $\mathbb{x}$ of length α, its finite partial sums, $\sum_{\gamma\in I} x_\gamma$, for $I \in [\alpha]^{<\omega}$ and the sets

$$\mathsf{FS}(\mathbb{x}) := \{\sum_{\gamma\in I} x_\gamma : I \in [\alpha]^{<\omega}\} \text{ and}$$
$$\mathsf{FB}(\mathbb{x}) := \{b \in \mathbb{K} : \forall x \in \mathsf{FS}(\mathbb{x})(x \leq b)\}.$$

Note that the definition of $\mathsf{FS}(\mathbb{x})$ ignores all negative values of the sequence $\mathbb{x}$, so we need to handle these separately. For a sequence $\mathbb{x}$, let $x^+_\gamma := \min(0, x_\gamma)$ and $x^-_\gamma := -\max(0, x_\gamma)$. Clearly, $\mathbb{x} = \mathbb{x}^+ - \mathbb{x}^-$.

Definition 5.3.8. We define

$$\mathsf{Dom}(S^{\mathsf{supfin}}) := \{\mathbb{x}\colon \text{ both } \mathsf{FS}(\mathbb{x}^+) \text{ and } \mathsf{FS}(\mathbb{x}^-) \text{ have a supremum}\}$$

and $S^{\mathsf{supfin}}(\mathbb{x}) := \sup(\mathsf{FS}(\mathbb{x}^+)) - \sup(\mathsf{FS}(\mathbb{x}^-))$ if this exists.

Furthermore, observe that $\mathsf{FS}(\mathbb{x})$ can only have a supremum if $\mathsf{cof}(\mathsf{FS}(\mathbb{x})) = \mathsf{bn}(\mathbb{K})$ or if $\mathsf{FS}(\mathbb{x})$ has a maximal element (Proposition 2.3.5). Therefore, if $\mathbb{x}$ is a sequence of length $\lambda < \mathsf{bn}(\mathbb{K})$, then $\mathsf{FS}(\mathbb{x})$ can only have a supremum if $\mathsf{FS}(\mathbb{x})$ has a maximal element which implies that $\mathsf{supp}(\mathbb{x})$ is finite. The main result of this section, Theorem 5.3.12, proves that this is true in general, i.e. all summable sequences have finite support.

In particular, if $\mathsf{bn}(\mathbb{K}) > \omega$, then $\bar{1}_\omega$ is not S^{supfin}-summable, and therefore, S^{supfin} does not satisfy $\mathbf{Cons}_e$.

Lemma 5.3.9. For any $\mathbb{K}$, S^{supfin} satisfies **Ext**, **Ind**, **El**, and is positive.

Proof. Positivity and **Ext** are trivial. For **Ind**, let $\mathbb{x} \in \mathbb{K}^\kappa$, and $f : \kappa \to \kappa$ be a bijection, then note that $\mathsf{FB}(\mathbb{x}) = \mathsf{FB}((x_{f(\alpha)})_{\alpha\in\kappa})$. For **El**, note that $\mathsf{FB}(\mathsf{contract}(\mathbb{x})) = \mathsf{FB}(\mathbb{x})$. □

Proposition 5.3.10. If $\mathbb{x}$ and $\mathbb{y}$ are non-negative S^{supfin}-summable sequences of length κ and $a \in \mathbb{K}$ with $a \geq 0$, then so are $\mathbb{x} + a.\mathbb{y}$ and $\mathbb{x}^\frown\mathbb{y}$ and

$$S^{\mathsf{supfin}}(\mathbb{x} + a.\mathbb{y}) = S^{\mathsf{supfin}}(\mathbb{x}) + a.S^{\mathsf{supfin}}(\mathbb{y}) \text{ and}$$
$$S^{\mathsf{supfin}}(\mathbb{x}^\frown\mathbb{y}) = S^{\mathsf{supfin}}(\mathbb{x}) + S^{\mathsf{supfin}}(\mathbb{y}).$$

In other words, S^{supfin} satisfies **Lin** and **Conc** for non-negative sequences of length κ.

Proof. Since the sequences are non-negative, we do not need to split them into their positive and negative part. Let $x := \sup(\mathsf{FS}(\mathbb{x}))$ and $y := \sup(\mathsf{FS}(\mathbb{y}))$. We show that $\mathsf{FS}(\mathbb{x} + a.\mathbb{y})$ has supremum $x + a.y$. First of all, $x + a.y$ is an upper bound of $\mathsf{FS}(\mathbb{x} + a.\mathbb{y})$: if $I \in [\kappa]^{<\omega}$, then $\sum_{\gamma\in I}(x_\gamma + a.y_\gamma) = \sum_{\gamma\in I} x_\gamma + a.\sum_{\gamma\in I} y_\gamma$, so each element of $\mathsf{FS}(\mathbb{x} + a.\mathbb{y})$ is below $x + a.y$.

Let $\varepsilon > 0$. Find $b \in \mathsf{FS}(\mathbb{x})$ and $c \in \mathsf{FS}(\mathbb{y})$ such that $b + a.c \geq x + ay - \varepsilon$. Let I and J be finite sets such that $b = \sum_{\gamma \in I} x_\gamma$ and $c = \sum_{\delta \in J} y_\delta$. Then

$$\sum_{\gamma \in I \cup J} x_\gamma + a.y_\gamma \geq b + a.c \geq x + a.y - \varepsilon.$$

Thus $x + a.y$ is the least upper bound of $\mathsf{FS}(\mathbb{x} + a.\,\mathbb{y})$.

Using **El**, we get that $S^{\mathsf{supfin}}(\overline{0}_\mu \frown \mathbb{x}) = S^{\mathsf{supfin}}(\mathbb{x} \frown \overline{0}_\mu) = S^{\mathsf{supfin}}(\mathbb{x})$, so as in the proof of Proposition 5.1.6, we write $\mathbb{x} \frown \mathbb{y}$ as $(\overline{0}_\kappa \frown \mathbb{y}) + (\mathbb{x} \frown \overline{0}_\kappa)$ (i.e. the pointwise sum of sequences $(\overline{0}_\kappa \frown \mathbb{y})$ and $(\mathbb{x} \frown \overline{0}_\kappa)$) and obtain **Conc** from the linearity claim. ☐

By noting that Lemma 5.1.5 also holds when restricted to non-negative sequences, we immediately have that $\overline{1}_\alpha$ is not S^{supfin}-summable for any infinite α. To prove the general result, that no infinite positive sequence is S^{supfin}-summable, we use the following lemma:

Lemma 5.3.11. Let $\mathbb{x}$ be a sequence such that there is some α with $x_\alpha > 0$ and there are infinitely many β such that $x_\beta \geq x_\alpha$. Then $\mathbb{x}$ is not S^{supfin}-summable.

Proof. It is enough to show that $\mathbb{x}^+$ is not summable, so without loss of generality, we can assume that $\mathbb{x}$ is positive. By **Ind**, the order of elements does not matter, so again without loss of generality, assume that $\alpha = 0$, i.e. $x_0 > 0$ and there are infinitely other values at least as big.

Let $x'_\gamma := x_{1+\gamma}$, i.e. $\mathbb{x}'$ is the sequence $\mathbb{x}$ with the 0^{th} entry removed. Towards a contradiction, we assume that $\mathbb{x}$ is summable; by **Ext** and **Conc** for non-negative sequences (Lemma 5.3.9 & Proposition 5.3.10), we have that $S^{\mathsf{supfin}}(\mathbb{x}) = x_0 + S^{\mathsf{supfin}}(\mathbb{x}') > S^{\mathsf{supfin}}(\mathbb{x}')$. However, for each $\sum_{\gamma \in I} x_\gamma \in \mathsf{FS}(\mathbb{x})$, if $0 \in I$, we can find J with $J = \{\beta\} \cup I \backslash \{0\}$ and $\beta \notin I$ such that $\sum_{\gamma \in I} x_\gamma \leq \sum_{\gamma \in J} x_\gamma \in \mathsf{FS}(\mathbb{x}')$. Thus $\sup(\mathsf{FS}(\mathbb{x})) \leq \sup(\mathsf{FS}(\mathbb{x}'))$ and thus $S^{\mathsf{supfin}}(\mathbb{x}') < S^{\mathsf{supfin}}(\mathbb{x}')$. But this is a contradiction. ☐

Theorem 5.3.12. Let $\mathsf{bn}(\mathbb{K}) > \omega$ and $\mathbb{x}$ be S^{supfin}-summable. Then $\mathsf{supp}(\mathbb{x})$ is finite.

Proof. Suppose $\mathbb{x}$ is summable, so both $\mathbb{x}^+$ and $\mathbb{x}^-$ are summable. We show that $\mathbb{x}^+$ has finite support; the same proof shows that $-\mathbb{x}^-$ has finite support whence $\mathbb{x}^-$ has finite support, so the whole sequence $\mathbb{x}$ has finite support.

We create a sequence $\mathbb{y}$ that is a re-ordering of $\mathbb{x}^+$ with the property that for $\alpha < \beta$, $y_\alpha \geq y_\beta$. By **Ind**, $\mathbb{y}$ is summable if and only of $\mathbb{x}^+$ is summable. Also $|\mathsf{supp}(\mathbb{y})| = |\mathsf{supp}(\mathbb{x}^+)|$. Note that $y_\omega = 0$, because otherwise, $\mathbb{y}$ has infinitely many values at least as big as y_ω, and therefore contradicts Lemma 5.3.11. Thus $\mathsf{supp}(\mathbb{y})$ is an ordinal $\alpha \leq \omega$.

If $\mathsf{supp}(\mathbb{y}) = \omega$, then no finite sum of elements of $\mathbb{y}$ can be an upper bound of $\mathsf{FS}(\mathbb{y})$, so $\mathsf{cof}(\mathsf{FS}(\mathbb{y})) = \omega$. But that means that $\mathsf{FS}(\mathbb{y})$ has no supremum (since $\omega < \mathsf{bn}(\kappa)$), so $\mathbb{y}$ is not summable. Therefore $\mathsf{supp}(\mathbb{y})$ is a natural number whence $\mathsf{supp}(\mathbb{x}^+)$ is finite. ☐

This means that S^{supfin} trivialises by being equal to S^{finite} from Example 5.1.2. As a Pyrrhic victory, this, of course, implies that S^{supfin} satisfies all of the desiderata that finite sums satisfy, i.e. **Lin** and **Conc** for arbitrary sequences, **Comp**, and **Tr**. It does not satisfy $\mathbf{Ext}_{\mathbb{Q}}$.[3]

Simplest upper bounds

The reason for the fact that S^{supfin} trivialised was that in most cases, the supremum $\mathsf{FS}(\mathbb{x})$ does not exist. One idea is to replace the supremum operation with some other operation that is guaranteed to find an upper bound.

Let $\mathbb{K}$ be an initial subfield of No. Recall that if $\mathsf{FB}(\mathbb{x})$ is non-empty, Conway's simplicity theorem (Theorem 2.3.9) guarantees the existence of a simplest element. So we let $\mathsf{Dom}(S^{\mathsf{simple}}) = \{\mathbb{x}\colon \mathsf{FB}(\mathbb{x}) \text{ is non-empty}\}$ and let $S^{\mathsf{simple}}(\mathbb{x})$ the unique element of minimal length in $\mathsf{FB}(\mathbb{x})$ which exists by Theorem 2.3.9.

Clearly, S^{simple} satisfies **Ind** and **El**, by definition. Now, if $\mathsf{bn}(\mathbb{K}) > \omega$, $\overline{1}_\omega$ becomes summable, since the ordinal ω is the simplest surreal bigger than all finite partial sums, so $S^{\mathsf{simple}}(\overline{1}_\omega) = \omega$. But the same is true for any infinite sequence of 1s, so if $\lambda \geq \omega$, $S^{\mathsf{simple}}(\overline{1}_\lambda) = \omega$. Consequently, S^{simple} cannot satisfy **Conc**, $\mathbf{Cons}_e$ (for any ordinal embedding e), and **Lin**.

Proposition 5.3.13. The sum S^{simple} satisfies **Comp**.

Proof. Consider $\mathbb{x}$ and $\mathbb{y}$ such that $x_\alpha \leq y_\alpha$; then for each finite I, $\sum_{\alpha\in I} x_\alpha \leq \sum_{\alpha\in I} y_\alpha$, and so $\mathsf{FB}(\mathbb{x}) \supseteq \mathsf{FB}(\mathbb{y})$. Thus, if $\mathsf{FB}(\mathbb{y})$ is non-empty, then so is $\mathsf{FB}(\mathbb{x})$. Furthermore, if $m_{\mathbb{x}}$ and $m_{\mathbb{y}}$ are the unique elements of minimal length of $\mathsf{FB}(\mathbb{x})$ and $\mathsf{FB}(\mathbb{y})$, respectively, then either $m_{\mathbb{x}} = m_{\mathbb{y}}$, in which case $S^{\mathsf{simple}}(\mathbb{x}) = S^{\mathsf{simple}}(\mathbb{y})$; or they are different and $m_{\mathbb{x}}$ is strictly shorter than $m_{\mathbb{y}}$. In that case, it is not possible that $m_{\mathbb{y}} \leq m_{\mathbb{x}}$ (since otherwise $m_{\mathbb{x}} \in \mathsf{FB}(\mathbb{y})$, contradicting the minimality of $m_{\mathbb{y}}$), so $S^{\mathsf{simple}}(\mathbb{x}) = m_{\mathbb{x}} < m_{\mathbb{y}} = S^{\mathsf{simple}}(\mathbb{y})$. □

Even **Ext** fails badly if $\mathsf{bn}(\mathbb{K}) > \omega$: if x, y are surreal numbers such that $x < y$, and the length of y is smaller than the length of x, then the length one sequence of value x will not get x assigned as the sum. E.g. the simplest surreal in the class $\{x\colon x \geq \omega - 1\}$ is the ordinal ω. Therefore the length one sequence with single value $\omega - 1$ would be assigned the sum $\omega > \omega - 1$.

5.3.3 Hahn field sums

Recall the definition of a *Hahn field*, $\mathbb{H}(\mathbb{R}, \mathbb{G})$, where $\mathbb{G}$ is a divisible ordered group (Definition 2.3.16). If $f \in \mathbb{H}(\mathbb{R}, \mathbb{G})$, we think of f as a formal power series $\mathsf{S}_{i\in\mathsf{supp}(f)} f(i)X^i$. The natural transfinite sum on a Hahn field is pointwise (see [36, page 40], [92, §3.1], [93], and [184]).

Definition 5.3.14. Let $\mathbb{H} = \mathbb{H}(\mathbb{R}, \mathbb{G})$ be a Hahn field and $\mathbb{K} \subseteq \mathbb{H}$ and $\mathbb{h}$ a sequence in $\mathbb{K}$ of length μ. We say that $\mathbb{h} = (\mathsf{S}_{i\in\mathsf{supp}(f_\alpha)} f_\alpha(i)X^i)_{\alpha\in\mu}$ is S^{point}-summable (for "pointwise") if the following hold:

[3] An iterative version of this sum, taking $\sup(\mathsf{FS}(\mathbb{x}^+)) - \sup(\mathsf{FS}(\mathbb{x}^-))$ at limits steps only, is trivial for similar reasons at the first limit step. This contrasts with iterative infinite sums on ordinals, where standard ordinal addition and Lipparini's transfinite version of Hessenberg addition [133, Definitions 3.1 & 5.1] are non-trivial. Indeed, Lipparini's sum satisfies semiring versions of **El** [133, Lemma 3.3(4)], **Ext**, but fails **Conc** [133, page 517] and **Ind** [133, page 523].

1. the set $I := \bigcup_{\alpha\in\mu} \mathsf{supp}(f_\alpha)$ is well-ordered;
2. for each $i \in I$, $g(i) := \sum_{\alpha\in\mu} f_\alpha(i)$ converges in $\mathbb{R}$ (if $i \notin I$, let $g(i) := 0$);
3. the function $g \in \mathbb{K}$.

Then $S^{\mathsf{point}}(\mathbb{f}) := g$.

Intuitively, we arrange a sequence into an array of coefficients, and sum the $\mathbb{R}$-coefficients in each column $i \in \mathbb{G}$.[4]

Proposition 5.3.15. Suppose that $\mathbb{G}$ is an ordered group with a strictly increasing sequence $(i_\alpha)_{\alpha<\kappa}$. Then $\kappa \subseteq \mathsf{spect}(S^{\mathsf{point}})$ on $\mathbb{H}_\kappa(\mathbb{R}, \mathbb{G})$ and $\kappa + 1 \subseteq \mathsf{spect}(S^{\mathsf{point}})$ on $\mathbb{H}(\mathbb{R}, \mathbb{G})$.

Proof. Consider x_α with $\mathsf{supp}(x_\alpha) = \{i_\alpha\}$ and $x_\alpha(i_\alpha) = 1$. Then $\mathbb{x}$ is S^{point}-summable in $\mathbb{H}(\mathbb{R}, \mathbb{G})$ and all proper initial segments of $\mathbb{x}$ are S^{point}-summable in both $\mathbb{H}_\kappa(\mathbb{R}, \mathbb{G})$ and $\mathbb{H}(\mathbb{R}, \mathbb{G})$. □

If $|\kappa^{<\kappa}| = \kappa$, each of $\mathsf{No}_{<\kappa}$, and $\mathsf{No}_{\leq\kappa}$ are Hahn fields, and No is a “Hahn Field” (the class-sized analogue of a Hahn field) [36, Theorem 21]. Thus, S^{point} is defined on $\mathbb{R}_\kappa \subsetneq \mathsf{No}_{\leq\kappa}$. This sum satisfies most desiderata.

Proposition 5.3.16 (Folklore)**.** Let $\mathbb{K}$ be a subfield of the Hahn field, $\mathbb{H}(\mathbb{R}, \mathbb{G})$, such that for all $r \in \mathbb{R}$, $r.X^0 \in \mathbb{K}$. Then S^{point} satisfies **Ext**, **Lin**, **Conc**, **El**, and **Ext**$_\mathbb{Q}$ on $\mathbb{K}$.

Proof. All properties are directly inherited from the corresponding properties of the $\mathbb{R}$-sum used in the definition. □

Remark 5.3.17. If $\mathbb{K}$ is a subfield of a Hahn field, we have a small amount of **Ind**, using that ordinary $\mathbb{R}$-sum satisfies **Ind** for absolutely convergent sequences: let $(x_\alpha)_{\alpha\in\lambda}$ be S_{point}-summable. Suppose $x_\alpha = \mathsf{S}^{\mathbb{K}}_{I_\alpha} r^\alpha_i X^i$ is such that for all $i \in \bigcup_{\alpha\in\lambda} I_\alpha$, $(\sum_{\alpha\leq\beta} r^\alpha_i)_{\alpha\in\beta}$ is absolutely convergent in $\mathbb{R}$. Then S_{point} satisfies **Ind** for this sequence.

Proposition 5.3.18. On $\mathbb{R}_\kappa$, S^{point} does not satisfy **Ind**, **Comp**, **Cons**$_e$, or **Geom**.

Proof. Let $\mathbb{x} = (x_n)_{n\in\omega}$ be a sequence such that the series $\sum_{n\in\omega} x_n$ converges, but does not converge absolutely. So, by the Riemann sum theorem [180, Theorem 3.54], there is a permutation π such that $\sum_{n\in\omega} x_{\pi(n)}$ does not converge. Then define $f_n(0) := x_n$ and $f^\pi_n(0) := x_{\pi(n)}$. Clearly, $\mathbb{f}^\pi$ is a re-ordering of $\mathbb{f}$, so **Ind** implies that if one of these is summable, then so is the other. But $\mathbb{f}$ is summable and $\mathbb{f}^\pi$ is not.

By Theorem 5.2.3 and Proposition 5.3.16, **Cost**$_e$ fails. As $(\frac{1}{n})_{n\in\omega}$ is S^{point}-summable, so is $(\frac{\omega}{n})_{n\in\omega}$. But then $(\frac{\omega}{n})_{n\in\omega}$ pointwise bounds $\overline{1}_\omega$, which, by Theorem 5.2.3, is not S^{point}-summable. Hence, **Comp** fails. Observing that $r.X^0 \in \mathbb{R}_\kappa$ for all $r \in \mathbb{R}$, Propositions 5.2.2 and 5.3.16 imply that **Geom** fails. □

[4]Variations of this sum in the literature typically have an additional condition which prevents pathologies, such as the sum of a negative telescoping series, $(X^n - X^{n+1})_{n\in\omega}$, being positive. E.g. Conway requires that there is a descending sequence $(i_\beta)_{\beta\in\alpha}$ in $\mathbb{G}$ such that if $j \neq i_\beta$ then $r_{j,\alpha} \neq 0$ for only finitely many α ([36, page 40], alternatively, see [14, Definition 2.9]).

This reveals the main issue with S_{point}, that coefficients for different $g, h \in \mathbb{G}$ don't interact: there is no sequence, $(x_\alpha)_{\alpha\in\mu}$, where $x_\alpha = r_\alpha.X^g$ such that $S_{\mathsf{point}}((x_\alpha)_{\alpha\in\mu}) = \mathsf{S}^{\mathbb{K}}_{h\in J} r_h.X^h$ for $g \notin J$.

5.3.4 Point-wise sequential limits

The following definition of a sum on the surreal numbers was suggested in [134, 150]. Surreal numbers are sequences, so the sequence of partial sums can be considered as a sequence of sequences; at limits, we can ask whether the digits of this sequence stabilise pointwise. E.g., for each $n \in \mathbb{N}$, $\sum_n(\frac{1}{2^m}) = 1 - \frac{1}{2^{n+1}} = 1^\frown 0^\frown \overline{1}_n$. This sequence stabilises every digit, so a natural limit is $1^\frown 0^\frown \overline{1}_\omega$.

Let $\mathbb{K}$ be an initial subfield of No. If $\mathbb{x}$ is any sequence of surreals of limit length λ, we say that $\mathbb{x}$ *stabilises pointwise* if, for each α, there is some β such that for all $\gamma, \gamma' > \beta$, $x_\gamma(\alpha) = x_{\gamma'}(\alpha)$ (including the case "$x_\gamma(\alpha)$ is undefined"). In this case, we call the unique surreal, x, defined by $x(\alpha) = 0$ if $\mathbb{x}$ stabilises on 0 at digit α and $x(\alpha) = 1$ if $\mathbb{x}$ stabilises on 1 at digit α the *pointwise limit of* $\mathbb{x}$ and denote it by $\mathsf{pwlim}(\mathbb{x})$.

Let $\mathbb{x}$ be any sequence of elements of $\mathbb{K}$ of length μ. We inductively define the sequence $\mathbb{s} = (s_\alpha)_{\alpha<\mu}$ of partial sums of $\mathbb{x}$ by

$$\begin{aligned} s_0 &:= 0, \\ s_{\alpha+1} &:= s_\alpha + x_\alpha, \text{ and} \\ s_\lambda &:= \mathsf{pwlim}(\mathbb{s}\restriction\lambda) \text{ if } \lambda \text{ is a limit ordinal and } \mathbb{s}\restriction\lambda \text{ stabilises pointwise.} \end{aligned}$$

We say that $\mathbb{x} \in \mathsf{Dom}(S^{\mathsf{seq}})$ if, for all limit ordinals $\lambda \leq \mu$, s_λ is defined. Then $S^{\mathsf{seq}}(\mathbb{x}) = s_\mu$.

It is easy to see that S^{seq} satisfies **Ext** and **El**, but problems occur already at sequences of length ω, as observed by Mező [150, page 3]: if $\mathbb{x}$ is any divergent positive sequence of reals, then each element of $\mathbb{x}$ is a surreal of length $\leq\omega$ and the sequence of partial sums $\mathbb{s}$ eventually grows bigger than every n, i.e., for each $n < \omega$, the first n digits eventually stabilise on 1. Therefore, $s_\omega = \overline{1}_\omega$. Since this is independent of the choice of $\mathbb{x}$, **Lin** and **Conc** must fail: $S^{\mathsf{seq}}(\overline{1}_\omega + \overline{1}_\omega) = S^{\mathsf{seq}}(\overline{2}_\omega) = \omega \neq \omega+\omega = S^{\mathsf{seq}}(\overline{1}_\omega) + S^{\mathsf{seq}}(\overline{1}_\omega)$ and $S^{\mathsf{seq}}(\overline{1}_1{}^\frown\overline{1}_\omega) = \omega \neq \omega+1 = S^{\mathsf{seq}}(\overline{1}_1) + S^{\mathsf{seq}}(\overline{1}_\omega)$.

Proposition 5.3.19. In general, the sum S_{seq} violates **Comp**.

Proof. Let $\mathbb{x} := (x_n)_{n\in\omega}$ be defined like so:

$$x_n := \begin{cases} \overline{1}_{n+1}{}^\frown\overline{0}_\omega{}^\frown 0 & \text{if } n \text{ is even and} \\ \overline{1}_{n+1}{}^\frown\overline{0}_\omega{}^\frown 1 & \text{if } n \text{ is odd,} \end{cases}$$

$y_0 := x_0$, and $y_{n+1} := x_{n+1} - x_n$. Note that $0 \leq y_n \leq 2$. Since $\overline{2}_\omega$ is summable (by the above, its sum is ω), if **Comp** holds, then $\mathbb{y}$ should be summable. However, if $\mathbb{s}$ is the sequence of partial sums of $\mathbb{y}$, then by induction $s_n = x_n$. But the sequence $\mathbb{x}$ does not stabilise in digit ω, so s_ω is not defined, and thus $\mathbb{y}$ is not summable. □

Lipparini and Mező have also observed that **Ind** fails for S_{seq} [134, page 7].

5.3.5 Sums as integrals of step function

Let $\mathbb{K}$ be an rcf and $\varkappa \in \mathbb{K}^\lambda$ for some limit ordinal, λ. By Proposition 2.3.7, we can assume that $\mathsf{bn}(\mathbb{K}) \subseteq \mathbb{K}$ via an ordinal embedding. We define the step function:

$$\mathsf{step}_\varkappa(y) = \begin{cases} x_\alpha & \text{if } y \in (\alpha, \alpha+1), \\ 0 & \text{otherwise.} \end{cases}$$

If we could integrate this step function $\mathsf{step}_\varkappa$ to a function $F_\varkappa \colon [0,\lambda] \to \mathbb{K}$ with $F(0) = 0$, we could define the sum of $\varkappa$ to be $F_\varkappa(\lambda)$.[5]

We show that the easy option, using κ-continuous[6] functions whose derivatives are step functions, is trivial (Theorem 5.3.21, see Section 4.2.5 for more on differentiability). We then comment on an alternative (Remark 5.3.5).

Definition 5.3.20. We say that $\varkappa$ is $S_\int$-summable if there is a κ-continuous $F : \mathbb{K} \to \mathbb{K}$ which is differentiable at every $x \in \mathbb{K} \setminus \mathsf{bn}(\mathbb{K})$, such that $F' = \mathsf{step}_\varkappa$, where $F(0) = 0$. In which case, we define $S_\int(\varkappa) := F(\lambda + 1)$.

If $\mathbb{K}$ is an η_κ rcf, and $\varkappa \in \mathbb{K}^{\leq\kappa}$ has finite support, we can specify a piecewise linear function, F, on (finitely many) intervals with $\mathsf{bn}(\mathbb{K})$-endpoints, as required for $\varkappa$. This F is differentiable except at the finitely many endpoints of the intervals, and by Lemmata 4.1.11 and 4.2.4, F is κ-continuous. Hence, $S_\int$ satisfies **Ext**. But $S_\int$ is ultimately trivial, like S^{supfin}:

Theorem 5.3.21. Let $\mathbb{K}$ be an η_κ rcf and $\mathsf{bn}(\mathbb{K}) > \omega$. If $\varkappa \in \mathbb{K}^\lambda$ is $S_\int$-summable, then $\mathsf{supp}(\varkappa)$ is finite.

Proof. Without loss of generality, suppose $\varkappa \in (\mathbb{K}^{>0})^\omega$. By construction, F is constant on $C = [\neg\omega, \omega)$, suppose with value c. By construction, $(\sum_{m=1}^n x_m)_{n\in\omega}$ is cofinal in $F((-\infty, \neg\omega])$. By definition, F is increasing, so $\mathsf{cof}(F((-\infty, \neg\omega])) = \omega$. But by [75, Corollary 1.11(ii)], $(\sum_{m=1}^n x_m)_{n\in\omega}$ does not converge to c. So, there is some $a < c$ such that $a > \sum_{m=1}^n x_m$ for all $n \in \omega$. Hence F violates $\mathsf{IVT}(\mathbb{K})$, so is not κ-continuous, a contradiction. □

Hence, like S^{supfin}, $S_\int$ satisfies **Lin**, **Conc**, **Comp**, **El**, and **Ind**.

Remark 5.3.22. Several notions of No-integration have been developed (e.g. [62]). A notably inclusive notion is due to Costin, Ehrlich, and Friedman [38, 39]. They define an integral operator on a large class of functions with many desirable properties. Alas, the class of functions on which their operator is defined does not include step functions.

Costin conjectures that this can be partially overcome;[7] yet, Costin and Ehrlich conjecture that their operator cannot be extended to include all functions $\mathsf{step}_\varkappa$, even for $\varkappa \in \mathbb{R}^\omega$ for the following reason: such an operator would allow for an explicit construction of Banach limits (see [39, page 7 & proof of Theorem 12]); however, the existence of Banach limits is independent of ZF [39, Theorem 13(3)].[8]

[5] This is the conceptual reverse of the classical case, where we build integrals from sums.

[6] Mere continuity would allow (multiple) F where the series of partial sums of a positive sequence is non-increasing (a similar phenomenon is observed in [183, page 35]); κ-continuity blocks such 'jumping' F (by Theorem 4.2.3).

[7] Personal e-mail, dated 8th September 2021.

[8] Personal e-mail, dated 9th September 2021.

5.4 Sums, generalised polynomials, and Weierstraß' Approximation Theorem

We close this chapter by connecting the discussion of transfinite sums to real analysis as discussed in Chapter 4.

The *Weierstraß' Approximation Theorem* says that if $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is continuous, then there is a sequence of polynomials $(p_n)_{n\in\omega} \in (\mathbb{Q}[X])^\omega$ which converge uniformly to f [207]. The straightforward generalisations fail for non-Archimedean fields (by [44, CA51]), for a concrete example, recall $J_{\neg\omega} : [0, \omega] \to [0, 1]$ from Example 4.1.4. This $J_{\neg\omega}$ is continuous, but is not uniformly approximated by any $(p_\alpha)_{\alpha\in\kappa} \in (\mathbb{K}[X]([a, b]))^\kappa$. In fact, this fails even for the most restrictive class:[9]

Proposition 5.4.1. If $\mathbb{K}$ is an rcf with $\mathsf{bn}(\mathbb{K}) = \kappa$, then there is a κ-supercontinuous function that is not uniformly approximated by polynomials.

Proof. Recall Example 4.2.8, $\mathsf{upwardsstumble}(x)$. Fix $a, b \in \mathbb{K}$. There are $c, d, e, f \in \mathbb{K}$ so that $\mathsf{upwardsstumble}^{-1}(a, b) = (c, d)$ or $(c, d) \cup (e, f)$, so $\mathsf{upwardsstumble}$ is κ-continuous. Moreover, there are $g, h \in \mathbb{K}$ so that $\mathsf{upwardsstumble}([a, b]) = [g, h]$, so $\mathsf{upwardsstumble}$ is κ-supercontinuous. It also clearly has ω-many turning points. Suppose for a contradiction that there is a $p \in \mathbb{K}[X]([a, b])$ such that for all $x \in [0, \omega]$, $|\mathsf{upwardsstumble}(x) - p(x)| < \frac{1}{3}$. By the standard argument, as p is a polynomial, it is differentiable and piecewise-monotone between turning points. Clearly, p is not constant. As a polynomial has only finitely many turning points, let t_0 be the largest finite turning point of p, and let t_1 be the smallest infinite turning point, if it exists, otherwise let $t_1 = \omega$. Let $2n \in (t_0, t_1)$, so also $2n+\frac{3}{2} \in (t_0, t_1)$. Note that

$$\mathsf{upwardsstumble}(2n) = \mathsf{upwardsstumble}(2n + \frac{3}{2}) = 2n + 1.$$

Moreover, $2n + 1 \in (t_0, t_1)$, and $\mathsf{upwardsstumble}(2n + 1) = 2n$. By assumption $|p(x) - \mathsf{upwardsstumble}(x)| \leq \frac{1}{3}$. As p is piecewise monotone, it must have a turning point in $(2n, 2n + \frac{3}{2})$, contradicting that t_0 is the largest finite turning point. □

The inability of classical (finite exponent) polynomials to approximate functions results in a desire to define generalisations of polynomials using transfinite sums.

Let $\mathbb{K}$ be an ordered exponential field, $\mathbb{N} \subseteq A = \{a_\alpha \colon \alpha < \mu\} \subseteq \mathbb{K}$ a well-ordered set of *exponents*, and S a sum on $\mathbb{K}$. We say that $f\colon \mathbb{K} \to \mathbb{K}$ is an (A, S)*-polynomial* if there is an $I \subseteq \mu$ and a map $c\colon I \to \mathbb{K}$ such that for each $x \in \mathbb{K}$, the sequence $z^x_\alpha := c(\alpha).x^{a_\alpha}$ is S-summable and $S(\mathbb{z}^x) = f(x)$.

We write $\mathsf{Poly}(A, S)$ for the class of (A, S)-polynomials. Note that if $A = \mathbb{N}$ and $S = S^{\mathsf{finite}}$, then $\mathsf{Poly}(A, S) = \mathbb{K}[X]$. Assuming **Lin** and **Comp**, any $(\mathbb{K}^{\geq 0}, S)$-polynomial is continuous, using a simultaneous $\varepsilon\delta$-argument:

[9] In [74, Theorem 4.7.1], Galeotti showed that every κ-continuous function can be approximated at a point by polygonal functions.

Proposition 5.4.2. Let $\mathsf{bn}(\mathbb{K}) = \kappa$ and S satisfy **Lin** and **Comp**. Suppose $\beta < \kappa$ and $(c_\alpha)_{\alpha\in\beta}, (i_\alpha)_{\alpha\in\beta} \in (\mathbb{K}^{\geq 0})^\beta$ are such that $p(X) := S_{\alpha\in\beta}(c_\alpha X^{i_\alpha}) \in \mathsf{Poly}(\mathbb{K}^{\geq 0}, S)$. Then $p(x)$ is continuous.

Proof. As $\mathbb{K}$ is exponential, for all $b \in \mathbb{K}^{\geq 0}$, X^b maps intervals to intervals, hence is continuous. Fix some a and some $\varepsilon > 0$. We show that there is a $D > 0$ such that for all $x \in \mathbb{K}$ with $|x - a| < D$, $|p(a) - p(x)| \leq \varepsilon$.

As $\mathsf{bn}(\mathbb{K}) > \beta$, there is a lower bound $0 < C < c_\alpha$ for all $\alpha < \beta$. As $p(X) \in \mathsf{Poly}(\mathbb{K}^{\geq 0}, S)$, we know that $p(\frac{1}{C})$ is defined, i.e. $(c_\alpha \times \frac{1}{C^{i_\alpha}})_{\alpha\in\beta}$ is S-summable. As $(c_\alpha \times \frac{1}{C^{i_\alpha}})_{\alpha\in\beta}$ pointwise bounds $\overline{1}_\beta$, **Comp** implies $\overline{1}_\beta$ is S-summable. Suppose $S(\overline{1}_\beta) = M$.

Fix an $\alpha < \beta$. As $c_\alpha X^{i_\alpha}$ is a continuous function, there is $\delta_\alpha > 0$ such that for all x with $|x - a| < \delta_\alpha$, we have that $|c_\alpha \times x^{i_\alpha}| < \frac{\varepsilon}{M}$. These δ_α form a positive β-sequence, $(\delta_\alpha)_{\alpha\in\beta}$. As $\beta < \kappa = \mathsf{coi}(\mathbb{K}^{>0})$, there is a lower bound $D > 0$ such that for all $\alpha < \beta$, $D < \delta_\alpha$. Let $|x - a| < D$. Without loss of generality assume $x < a$. So, by **Lin** and **Comp** with $\overline{0}_\beta$, $0 = S(\overline{0}_\beta) \leq p(a) - p(x) = S_\beta(c_\alpha \times a^{i_\alpha}) - S_\beta(c_\alpha \times x^{i_\alpha})$. Then, by **Comp** with $(c_\alpha \times a^{i_\alpha})_{\alpha\in\beta}$, $(c_\alpha \times (a^{i_\alpha} - x^{i_\alpha}))_{\alpha\in\beta}$ is S-summable, and by **Lin**, $S_\beta(c_\alpha \times a^{i_\alpha}) - S_\beta(c_\alpha \times x^{i_\alpha}) = S_\beta(c_\alpha \times (a^{i_\alpha} - x^{i_\alpha}))$. By **Comp** again, and using the assumption that $|a - x| < D$, $p(a) - p(x) = S_\beta(c_\alpha(a^{i_\alpha} - x^{i_\alpha})) \leq S_\beta(\overline{\frac{\varepsilon}{M}}_\beta)$ which equals $\frac{\varepsilon}{M} \times S(\overline{1}_\beta)$ by **Lin**. Putting the pieces together, $|p(a) - p(x)| \leq \frac{\varepsilon}{M} \times M = \varepsilon$. □

Definition 5.4.3. Let $\mathbb{K}$ be an ordered exponential field, $\mathbb{N} \subseteq A = \{a_\alpha \colon \alpha < \mu\} \subseteq \mathbb{K}$ a well-ordered set of exponents, and S a sum on $\mathbb{K}$. Let $\mathcal{C}$ be any class of functions. We say that $\mathbb{K}$ satisfies the (A, S)-*Weierstraß' Approximation theorem for* $\mathcal{C}$, in symbols $\mathsf{WAT}(A, S, \mathcal{C})$, if, for any $f \in \mathcal{C}$, there is a sequence of (A, S)-polynomials which uniformly approximates f.[10]

It is natural to ask whether an appropriate choice of A and S would allow us to recover $\mathsf{WAT}(A, S, \mathcal{C})$ for some natural class $\mathcal{C}$, e.g. the κ-continuous or κ-supercontinuous functions.

Remark on alternative approximations

Other classes of functions can be uniformly approximated on $\mathbb{K}$. An example is that the uniformly continuous functions [180, Definition 4.18] can be uniformly approximated by λ-*step functions*, i.e. functions which are constant on each interval, I_α, where $(I_\alpha)_{\alpha\in\lambda}$ partitions $\mathbb{K}$. The Heine-Cantor theorem [180, Theorem 4.19] can be generalised straightforwardly to η_κ ordered fields where $\mathsf{wei}(\mathbb{K}) = \kappa$. Using this, we can show that if $f : [a, b] \to \mathbb{K}$ is uniformly continuous, and Q is order-dense in $\mathbb{K}$, then there is a sequence $(f_\alpha)_{\alpha\in\mathsf{bn}(\mathbb{K})}$ of Q-valued $\mathsf{ip}(\mathbb{K})$-step functions which converges uniformly to f. However, the class of Q-valued $\mathsf{ip}(\mathbb{K})$-step functions has size at least $\mathsf{wei}(\mathbb{K})^{\mathsf{ip}(\mathbb{K})}$, e.g. on $\mathbb{R}_\kappa$, $\mathsf{wei}(\mathbb{R}_\kappa)^{\mathsf{ip}(\mathbb{R}_\kappa)} = \kappa^\kappa$. To strengthen the analogy with the classical case, we might also ask whether there is a class, $\mathcal{C}$, where $\mathsf{WAT}(A, S, \mathcal{C})$ holds and $|\mathsf{Poly}(A, S)| < |\mathcal{C}|$.

[10] See [24] and [9, Theorem 2.23] for a different approach to generalising the Weierstraß' Approximation Theorem.

Chapter 6

Capping the Topology: Descriptive Set Theory on κ-Topologies

This chapter is based on joint work with Luca Motto Ros (Università degli Studi di Torino) and Claudio Agostini (Technische Universität Wien); all contributors contributed equally to this work.

In this chapter, we study generalised descriptive set theory on κ-topologies. The chapter is structured as follows. Section 6.1 concerns 'general κ-topology' for arbitrary spaces. In Section 6.2, we define and study the Borel$_\kappa$ sets, focusing on our main spaces: κ^κ, 2^κ, and certain ordered spaces. We show that if κ is inaccessible, then the Borel$_\kappa$ hierarchy has length at most 3 (Lemma 6.2.6). In Section 6.2.2, we show that the standard classical proofs of the non-collapse of the Borel$_\kappa$ sets of κ^κ fail, as there are no suitable complete or universal Borel$_\kappa$ sets. Instead, we characterise the Borel$_\kappa$ sets into an appropriate infinite difference hierarchy (Proposition 6.2.28). Section 6.2.3 gives corresponding results for linearly ordered spaces.

Section 6.3 studies κ-topological analyticity. The classically equivalent notions of analyticity form a strictly increasing hierarchy on κ^κ, from projections which add no complexity (Proposition 6.3.3) to the Borel$_\kappa$ images of κ^κ. We describe the bianalytic and strictly analytic sets for these notions, and compare them to the ordinary topology (summarised in Tables 6.2 and 6.3). Significantly, the generalised Suslin's theorem fails (Theorem 6.3.18). We prove analogous results for interval κ-topologies, and place the notions of analyticity on ordered spaces into a hierarchy which may differ from that of κ^κ (Proposition 6.3.49 and Corollary 6.3.51).

Section 6.4 describes how to construct many non-κ-homeomorphic κ-topologies for the same full topology (Proposition 6.4.3).

For this chapter, we again assume that κ is a regular uncountable cardinal. Recall that if κ is inaccessible, then $\kappa^{<\kappa} = \kappa$, and if $\kappa^{<\kappa} = \kappa$ then κ is regular (page 10). For this chapter, we call topologies *full topologies* to distinguish them from κ-topologies.

6.1 κ-Topological spaces

We define some basic κ-topological properties of κ-topologies on our main spaces, building on Section 2.2.1. Much of this generalises straightforwardly from full topologies. Other properties generalise the full topology case, but not straightforwardly (e.g. Proposition 6.1.5). Yet others are disanalogous to the full topology case (e.g. Remark 6.1.11 and Proposition 6.1.14).

We first need a little extra notation, which is used throughout this chapter. If Q is order dense in X, and we wish to distinguish between elements of Q and elements of $X \setminus Q$, we use q for elements of Q (and $a, b, ..., x, y, ...$ for elements of $X \setminus Q$ as usual). We write $B \sqcup C$ for $(\{0\} \times B) \cup (\{1\} \times C)$, which we treat as a disjoint union of (copies of) B and C (as $(\{0\} \times B) \cap (\{1\} \times C)$ is empty). If $B \subseteq X$, we call $f : X \to B$ a *retract of* B if $f(b) = b$ for all $b \in B$. If $A, A' \subseteq X^Y$ (or $A, A' \subseteq X^{<\kappa}$), we say that A' *refines* A if for every $s' \in A'$ there is a $s \in A$ such that $s' \subseteq s$. If $A \subseteq X^{<\alpha}$, let $\uparrow A := \bigcup_{s \in A} N_s$. We denote bounded κ-topology (on X^α) by τ_κ^{b}.

We say that $\mathcal{B} \subseteq \mathscr{P}(X)$ is *κ-additive* if for any $B_\alpha \in \mathcal{A}$ and any $\mu < \kappa$, $\bigcap_{\alpha \in \mu} B_\alpha \in \mathcal{B}$. Throughout, unless otherwise specified, we call a κ-topological space κ-additive if τ_κ is κ-additive (rather than if the full topology is κ-additive). Any subspace, X, of an α-additive κ-topological space Y is again α-additive, as if $\{O_\beta \cap X : \beta \in \mu\}$ is any family of κ-open subsets of X of size $\mu < \alpha$, then $\bigcap_{\beta \in \mu}(O_\beta \cap X) = (\bigcap_{\beta \in \mu} O_\beta) \cap X$ is κ-open in X by α-additivity of Y.

Our principal examples are κ-additive full topologies, for example any κ-Polish spaces. Even so, the κ-topology generated by a basis for a full topology need not be κ-additive. For example, the interval κ-topology on $\mathbb{R}_\kappa$ is not κ-additive (Proposition 6.1.13), but the full interval topology on $\mathbb{R}_\kappa$ is κ-additive.

We say that a κ-topology on X is *κ-zero-dimensional* if it has a basis, $\mathcal{B}$, such that every element of $\mathcal{B}$ is both κ-open and κ-closed.

Each set $\mathcal{B} \subseteq \mathscr{P}(X)$ naturally gives us a notion of $\mathcal{B}$-connectedness: $B \subseteq X$ is called *$\mathcal{B}$-connected* if there is no partition of B into sets $C, D \in \mathcal{B}$. We say that a κ-topological space is *κ-connected* if it is τ_κ-connected.

Theorem 6.1.1 ([74, Theorem 3.2.14])**.** If B is $\mathcal{B}$-connected, and $f : X \to X$ is $\mathcal{B}, \mathcal{B}$-continuous, then $f(B)$ is $\mathcal{B}$-connected.

Definition 6.1.2. Let (X, τ) be a topological space. A set $B \subseteq X$ is G_δ^κ if $X = \bigcap_{\alpha \in \kappa} O_\alpha$ for $O_\alpha \in \tau$. Likewise, B is F_σ^κ if $X \setminus B$ is G_δ^κ.

We define κ-homeomorphisms and product κ-topologies in the natural way: if $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, a function $f : X \to Y$ is a $\mathcal{B}_X, \mathcal{B}_Y$-*homeomorphism* if f is a bijection, f is $\mathcal{B}_X, \mathcal{B}_Y$-continuous and f^{-1} is $\mathcal{B}_Y, \mathcal{B}_X$-continuous. If $\mathcal{B}_X, \mathcal{B}_Y$ are κ-topologies, we say that f is a *κ-homeomorphism*. Moreover, if $(X, \tau_\kappa^X), (Y, \tau_\kappa^X)$ are κ-topological spaces generated by $\mathcal{B}_X, \mathcal{B}_Y$ respectively, the *product basis* is the set $\mathcal{B}_{X \times Y} := \{B \times C : B \in \mathcal{B}_X \wedge C \in \mathcal{B}_Y\}$, and the *product κ-topology* on $X \times Y$ is the set $\langle \mathcal{B}_{X \times Y} \rangle_\kappa$.

Proposition 6.1.3. If $(X, \tau_\kappa^X), (Y, \tau_\kappa^Y)$ are κ-topological spaces then the product κ-topology on $X \times Y$ can be characterised like so: $\pi_\kappa = \{\bigcup_{\alpha \in \lambda}(U_\alpha \times V_\alpha) : \lambda \in \kappa, U_\alpha \in \tau_\kappa^X, V_\alpha \in \tau_\kappa^Y\}$.

Proof. Clearly $\pi_\kappa \subseteq \{\bigcup_{\alpha\in\lambda} U_\alpha \times V_\alpha : \lambda < \kappa, U_\alpha \in \tau_\kappa^X, V_\alpha \in \tau_\kappa^Y\}$, as every basis element is κ-open. Conversely, suppose $P \in \pi_\kappa$. Then $P = \bigcup_{\alpha\in\lambda}(U_\alpha \times V_\alpha) = \bigcup_{\alpha\in\lambda}((\bigcup_{\gamma\in\lambda_\alpha} B_\gamma) \times (\bigcup_{\delta\in\mu_\alpha} B_\delta))$ where $\lambda_\alpha, \mu_\alpha < \kappa$, B_γ is basic open in X, and B_δ is basic open in Y. As κ is regular, $\max(\lim_{\alpha\in\lambda}, \lambda_\alpha \lim_{\alpha\in\lambda} \mu_\alpha) =: \nu < \kappa$. So, $P = \bigcup_{\gamma,\delta\in\nu}(B_\gamma \times B_\delta)$, as required. □

6.1.1 κ-Baire space and κ-Cantor space

In this section, we state and prove the basic κ-topological properties of κ-Baire space and κ-Cantor space. We show that, up to κ-homeomorphism, there are just two spaces of the form λ^κ: κ^κ and 2^κ.

Definition 6.1.4. Let $f : \kappa \to \kappa$. We define the following set:

$$D(f) := \{x \in \kappa^\kappa : x(\alpha) < f(\alpha) \text{ for every } \alpha \in \kappa\}.$$

We equip $D(f)$ with the subspace κ-topology inherited from κ^κ.

It is well-known that the full topology of the diagonal space on κ, $D(\text{id})$, is homeomorphic to 2^κ. This can be strengthened to a κ-homeomorphism:

Proposition 6.1.5. Let κ be inaccessible, and $f : \kappa \to \kappa$ be such that for all $\alpha < \kappa$, $f(\alpha) > 2$. Then $D(f)$ is κ-homeomorphic to 2^κ.

Proof. Let $T(f) = \{s \in \kappa^{<\kappa} : s(\alpha) < f(\alpha) \text{ for every } \alpha < \mathsf{len}(s)\}$, so $\mathcal{B}_T = \{N_s \cap D(f) : s \in T(f)\}$ is a basis for the topology of $D(f)$. The structure of the proof is as follows: we inductively prove a claim that there are club-many levels of $T(f)$ and $2^{<\kappa}$ which have the same cardinality. We use this to inductively construct an order and sequence-length preserving map, g, between a basis $\mathcal{A}$ for $D(f)$ and $\mathcal{B}$ for 2^κ. We then use g to define a function $\phi : D(f) \to 2^\kappa$. Finally, we prove ϕ is a κ-homeomorphism.

First, we prove that there is *some* bijection between levels of $T(f)$ and levels of $2^{<\kappa}$:

Claim 6.1.6. For every $s \in T(f)$ and $s' \in 2^{<\kappa}$, there is a club $C(s, s') \subseteq \{\alpha \in \kappa : |\mathsf{Lev}_\alpha(T(f)_s)| = |\mathsf{Lev}_\alpha((2^{<\kappa})_{s'})|\}$.

Proof. We prove this by induction (following [1, Claim 3.14.1], [37, Proposition 6.5]). Fix an ordinal $\alpha < \kappa$. We will find an $\alpha' > 0$ such that $|\mathsf{Lev}_{\alpha+\alpha'}(T(f)_s)| = |\mathsf{Lev}_{\alpha+\alpha'}((2^{<\kappa})_{s'})|$. We define a sequence $(\lambda_n)_{n\in\omega}$ such that $\lambda_n < \kappa$ so that $\sup_{n\in\omega} \lambda = \alpha'$.

Firstly, let $\lambda_0 = 0$. Suppose that for all $k \leq n$, λ_n has been defined. Then let $\lambda_{\leq n} := \sum_{k\leq n} \lambda_k$. Clearly $\lambda_{\leq n} < \kappa$. Let $\gamma < \kappa$ be such that:

1. $\max\{2^{|\alpha+\lambda_{\leq n}|}, |\mathsf{Lev}_{\alpha+\lambda_{\leq n}}(T(f)_s)|\} \leq \gamma$, and
2. for all $t \in \mathsf{Lev}_{\alpha+\lambda_{\leq n}}(T(f)_s)$, $\lambda_n \leq |\mathsf{Lev}_{\alpha+\lambda_{\leq n}+\gamma}(T(f)_t)|$.

The following shows that such a γ exists: for Property 1. note that $2^{|\alpha+\lambda_{\leq n}|} < \kappa$ (as κ is inaccessible), then note that there is a $n' > \lambda_{\leq n}$ such that $|\mathsf{Lev}_{\alpha+\lambda_{\leq n}}(T(f)_s)| \leq |\mathsf{Lev}_{\alpha+n'}(T(f)_s)|$, whilst Property 2. is trivial.

Let λ_{n+1} be the least such γ, and $\alpha' = \sup\{\lambda_n : n \in \omega\}$. By construction, $|\mathsf{Lev}_{\alpha+\alpha'}((2^{<\kappa})_{s'})| = |2^{\alpha+\alpha'}| = \prod_{n\in\omega} 2^{|\alpha+\lambda_n|} = \prod_{n\in\omega} |\alpha + \lambda_n|$. Moreover, for every $t \in \mathsf{Lev}_\alpha(T(f)_s)$, we have that:

$$\prod_{n\in\omega} |\alpha + \lambda_n| \leq |\mathsf{Lev}_{\alpha+\alpha'}(T(f)_t)| \leq |\mathsf{Lev}_{\alpha+\alpha'}(T(f)_s)| \leq \prod_{n\in\omega} |\alpha + \lambda_n|.$$

The first inequality follows from Property 2., while the last one follows from Property 1. Thus, $C = \{\alpha < \kappa : |\mathsf{Lev}_\alpha(T(f)_s)| = |\mathsf{Lev}_\alpha((2^{<\kappa})_{s'})|\}$ is unbounded in κ. We now prove that C contains a club. Let $C(s, s') := \{\alpha \in \kappa : |\mathsf{Lev}_\alpha(T(f)_s)| = |\mathsf{Lev}_\alpha((2^{<\kappa})_{s'})|$ and $| - \mathsf{len}(s') + \alpha| = |\alpha|\}$, and let β be a limit ordinal such that $J = C(s, s') \cap \beta$ is a cofinal in β. Let $(\alpha_\delta)_{\delta\in\gamma}$ be a cofinal sequence in J. Notice that $|\mathsf{Lev}_{\alpha_\delta}(T(f)_s)| = |\mathsf{Lev}_{\alpha_\delta}((2^{<\kappa})_{s'})| = 2^{|-\mathsf{len}(s')+\alpha_\delta|} = 2^{|\alpha_\delta|}$. As J is cofinal, the following holds:

$$|\mathsf{Lev}_\beta(T(f)_s)| \leq \prod_{\delta\in\gamma} |\mathsf{Lev}_{\alpha_\delta}(T(f))| \leq (\sup\{|\mathsf{Lev}_{\alpha_\delta}(T(f))| : \delta < \gamma\})^\gamma.$$

But we can easily bound this, as $\alpha_\delta, \gamma < \beta$:

$$|\mathsf{Lev}_\beta(T(f)_s)| \leq (2^{|\beta|})^{|\gamma|} \leq 2^{|\beta|} = |\mathsf{Lev}_\beta((2^{<\kappa})_{s'})|.$$

Then, we have

$$2^{|\beta|} = \prod_{\alpha\in\beta} 2 \leq |\mathsf{Lev}_\beta(T(f)_s)| \leq 2^{|\beta|} = |\mathsf{Lev}_\beta((2^{<\kappa})_{s'})|.$$

Hence $\beta \in C(s, s')$, and so $C(s, s')$ is a club. □

Next, proceeding recursively, we find a cofinal family of $I \subseteq \kappa$ and, for each $\alpha \in I$, we find families $\mathcal{A}_\alpha \subseteq T(f)$ and $\mathcal{B}_\alpha \subseteq 2^{<\kappa}$, and an order-preserving bijection $g_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha$ such that $\mathsf{len}(s) = \mathsf{len}(g_\alpha(s))$ for all $s \in \mathcal{A}_\alpha$ with the following properties for $\alpha < \beta \in I$:

1. $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \subseteq T(f)$ and $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta \subseteq 2^{<\kappa}$,
2. $g_\alpha \subseteq g_\beta$,

   from this, we define $\mathcal{A}_{<\gamma} := \bigcup_{\alpha\in\gamma} \mathcal{A}_\alpha$, $\mathcal{B}_{<\gamma} := \bigcup_{\alpha\in\gamma} \mathcal{B}_\alpha$, and $g_{<\gamma} := \bigcup_{\alpha\in\gamma} g_\alpha$; obviously $g_{<\gamma} : \mathcal{A}_{<\alpha} \to \mathcal{B}_{<\alpha}$; lastly
3. there is a $\gamma_0 > \beta$ such that $\mathcal{A}_\beta = \mathcal{A}_{<\beta} \cup \mathsf{Lev}_{\gamma_0}(T(f))$ and $\mathcal{B}_\beta = \mathcal{B}_{<\beta} \cup \mathsf{Lev}_{\gamma_0}(2^{<\kappa})$.

For the base case, let $g_0 : \{0\} \to \{0\}$ with $g(0) = 0$, i.e. send the root of $T(f)$ to the root of $2^{<\kappa}$. So, $\mathcal{A}_0 = \{0\}$ and $\mathcal{B}_0 = \{0\}$.

Suppose $0 < \beta < \kappa$ is such that $g_{<\beta} : \mathcal{A}_{<\beta} \to \mathcal{B}_{<\beta}$ has already been defined.

For every $s \in \delta(\mathcal{A}_{<\beta})$, the frontier of $\mathcal{A}_{<\beta}$, we can define $g(s) = \bigcup\{g(t) : t \in \mathcal{A}_{<\beta}$ with $t \subseteq s\}$. For each $s \in \delta(\mathcal{A}_{<\beta})$, we can use Claim 6.1.6 to find a club $C(s, g(s))$. As a $<\kappa$-intersection of clubs is a club, $\bigcap_{s\in A_{<\beta}} C(s, g(s))$ is also a club. Hence we can find a γ_0 such that for all $s \in \delta(\mathcal{A}_{<\beta})$, $|\mathsf{Lev}_{\gamma_0}(T(f)_s)| = |\mathsf{Lev}_{\gamma_0}((2^{<\kappa})_{g(s)})|$. Here we use that κ being inaccessible implies $|\delta(\mathcal{A}_{<\beta})| \leq 2^{\mu} < \kappa$ for some $\mu < \kappa$. Choose a bijection $h_s : \mathsf{Lev}_{\gamma_0}(T(f)_s) \to \mathsf{Lev}_{\gamma_0}((2^{<\kappa})_{g(s)})$.

For every tree, T, for every $A \subseteq T$, and for every $t \in T$, there is $s \in \delta(A)$ such that $t \in T_s$. Hence $\bigcup_{s\in\delta(A)} \mathsf{Lev}_\alpha(T_s) = \mathsf{Lev}_\alpha(T)$. Then let $\mathcal{A}_\beta := \mathcal{A}_{<\beta} \cup \mathsf{Lev}_{\gamma_0}(T(f))$ and $\mathcal{B}_\beta := \mathcal{B}_{<\beta} \cup \mathsf{Lev}_{\gamma_0}(2^{<\kappa})$.

Finally, let $g_\beta := g_{<\beta} \cup \bigcup_{s\in\mathcal{A}_{<\beta}} h_s$. Clearly, Conditions 1., 2., and 3. are satisfied for β. By construction, g_β is an order and sequence-length preserving bijection, as required.

We let $g := g_{<\kappa}, \mathcal{A} := \mathcal{A}_{<\kappa}$, and $\mathcal{B} := \mathcal{B}_{<\kappa}$. By construction, $g : \mathcal{A} \to \mathcal{B}$ is an order and sequence-length preserving bijection, as required.

Proceeding as in [97, Lemma 2.1], we use this g to define a κ-homeomorphism, $\phi : D(f) \to 2^\kappa$. Let $\phi(x) = \bigcup\{g(x{\restriction}\alpha) : \alpha < \kappa$ and $x{\restriction}\alpha \in \mathcal{A}\}$ (i.e., $\phi(x)$ is the unique element of $\bigcap\{N_{g(x\restriction\alpha)} : \alpha < \kappa$ and $x{\restriction}\alpha \in \mathcal{A}\}$). First, ϕ is well-defined, since Condition 3. ensures that for every $x \in D(f)$, there are cofinally many $\alpha < \kappa$ such that $x{\restriction}\alpha \in \mathcal{A}$, and the fact that g is length-preserving ensures that $\bigcup\{g(x{\restriction}\alpha) : \alpha < \kappa$ and $x{\restriction}\alpha \in \mathcal{A}\} \in 2^\kappa$. It remains to show that ϕ is a κ-homeomorphism.

Note that if $s \in \mathcal{A}$, then $\phi(N_s \cap D(f)) = N_{g(s)}$ is κ-open in 2^κ. Exactly similarly, if $t \in \mathcal{B}$, then $\phi^{-1}(N_t) = N_{g^{-1}(s)} \cap D(f)$ is κ-open.

So, let $s \in T(f) \setminus \mathcal{A}$: we want to show that $\phi(N_s \cap D(f))$ is κ-open in 2^κ. Given $s \in T(f)$, we can find an $\alpha > \mathsf{len}(s)$ such that $\mathsf{Lev}_\alpha(T(f)) \subseteq \mathcal{A}$. Then $N_s \cap D(f)$ is a union of some elements from $\mathsf{Lev}_\alpha(T(f))$, each of which is sent to an element of $\mathcal{B}$ of length α. However, $|\mathsf{Lev}_\alpha(T(f))| \leq 2^{|\alpha|} < \kappa$ since κ is inaccessible. Thus, $\phi(N_s \cap D(f))$ is the union of $<\kappa$-many elements of $\mathcal{B} \subseteq 2^{<\kappa}$ and so it is κ-open.

A similar argument works for all $t \in 2^{<\kappa} \setminus \mathcal{B}$. □

Remark 6.1.7. Note that in Proposition 6.1.5, it suffices for $f(\alpha) > 2$ cofinally often.

Corollary 6.1.8. If κ is inaccessible, and $\lambda < \kappa$, the κ-topological space λ^κ is κ-homeomorphic to 2^κ.

We now describe some of the key κ-topological properties of 2^κ and κ^κ. Significantly, 2^κ is κ-zero-dimensional, whilst κ^κ is not, though both are zero-dimensional with the full topology [112, page 35]. First, note that the κ-Cantor space 2^κ with the bounded topology is not κ-connected, as $2^\kappa = N_1 \sqcup N_0$.

Proposition 6.1.9. Let κ be inaccessible. Then for 2^κ and κ^κ, the corresponding τ^{b}_κ are κ-additive.

Proof. Let $\{A_\alpha : \alpha \in \lambda\}$ be a family of κ-open sets for some $\lambda < \kappa$, then for each $\alpha < \lambda$, there is a family of cones, $\{N^\alpha_{s_\beta} : \beta \in \mu_\alpha\}$, such that $A_\alpha = \bigcup_{\beta\in\mu_\alpha} N^\alpha_{s_\beta}$ and $\mu_\alpha < \kappa$. Then, for some map, ϕ:

$$O := \bigcap_{\alpha\in\lambda} \bigcup_{\beta\in\mu_\alpha} N^\alpha_{s_\beta} = \bigcup_{\phi\in\prod_{\alpha\in\lambda}\mu_\alpha} \Big(\bigcap_{\beta\in\lambda} N^\alpha_{s_{\phi(\alpha)}}\Big).$$

Since κ is inaccessible, we have that $|\prod_{\alpha\in\lambda}\mu_\alpha| \leq \mu^\lambda < \kappa$ for $\mu = \sup\{\mu_\alpha : \alpha < \lambda\}$.

Since every non-empty intersection of cones is still a cone, O is κ-open, as required. As 2^κ is a subspace of κ^κ, 2^κ is κ-additive. □

Proposition 6.1.10. The κ-Cantor space, 2^κ, is κ-zero-dimensional.

Proof. If $s \in 2^{<\kappa}$, then $2^\kappa \setminus N_s = \bigcup_{\alpha\in\mathsf{len}(s)} N_{s\restriction\alpha^\frown(1-s_\alpha)}$, i.e. the union of all cones immediately off the stem of N_s. As $\mathsf{len}(s) < \kappa$, $2^\kappa \setminus N_s$ is κ-open, so N_s is κ-clopen. □

This argument does not work for κ^κ, as every $t \subseteq s$ has κ-many neighbours, so the smallest antichain A such that $\bigcup_{s\in A} N_s = \kappa^\kappa \setminus N_t$ has size κ. In fact:

Remark 6.1.11. The κ-Baire space, κ^κ, is not κ-zero-dimensional: suppose, for a contradiction, that $\{B, C\}$ is a partition of κ^κ into two κ-open sets. Then, there are antichains $A_B, A_C \subseteq \kappa^{<\kappa}$ such that $\uparrow A_B = B$ and $\uparrow A_C = C$. As B and C are disjoint, $A_B \cup A_C$ is an antichain in $\kappa^{<\kappa}$. But, for any antichain $A \subseteq \kappa^{<\kappa}$, if $\uparrow A = \kappa^\kappa$, then $|A| = \kappa$. So, $|A_B \cup A_C| = \kappa$. Hence, at least one of A_B and A_C is size κ, without loss of generality we can suppose $|A_B| = \kappa$. But then $\uparrow A_B$ is not κ-open, a contradiction.

6.1.2 Linearly ordered spaces

Our other main class of example spaces are linearly ordered sets with their order κ-topologies, principally the interval κ-topologies on $\mathbb{R}_\kappa$. For most of this chapter, we only use the order properties of $\mathbb{R}_\kappa$. In Section 6.3.4, we use that $\mathbb{Q}_\kappa$ and $\mathbb{R}_\kappa$ are fields. Both $\tau_\kappa^{\mathbb{Q}_\kappa}$ and $\tau_\kappa^{\mathbb{R}_\kappa}$ generate the same full topology, namely the interval topology, but are distinct (Corollary 6.2.50). Equipped with the $\mathbb{Q}_\kappa$-interval κ-topology, $\mathbb{R}_\kappa$ is κ-connected [74, Corollary 3.2.11]. Unlike the bounded κ-topology (Corollary 6.2.30), for all $q \in Q$, $\{q\}$ is κ-closed in the Q-interval κ-topology, and so Borel$_\kappa$, as $\{q\} = X \setminus ((-\infty, q) \cup (q, \infty))$.

Other natural examples of κ-interval topologies include:

1. λ^κ with the λ^κ-interval κ-topology, where we order λ^κ lexicographically, i.e. $s^\frown\alpha^\frown x < s^\frown\beta^\frown y$ for any $\alpha < \beta$; and

2. the set $(\lambda^* \sqcup \kappa)^\kappa$ with the basis of lexicographic $(\lambda^* \sqcup \kappa)^\kappa$-intervals, $-\alpha < \beta$ for all $-\alpha \in \lambda^*, \beta \in \kappa$, where λ^* is κ with the reverse ordering.

The interval κ-topologies on ω^ω and κ^κ are not compatible with any field structure, as each has a minimal element, $\overline{0}_\omega$ and $\overline{0}_\kappa$ respectively (so there is no $\overline{0}_\omega - \overline{0}_\omega < \overline{0}_\omega$). So too for $(\lambda^* \sqcup \kappa)^\kappa$:

Example 6.1.12. Let λ be a regular infinite cardinal, and let $X = (\lambda^* \sqcup \kappa)^\kappa$. If G is a gap in X, then there is a $\mu < \kappa$, $\varkappa \in (\lambda^* \sqcup \kappa)^\mu$, $\alpha \in \kappa$, $\beta \in \lambda$, and $\gamma, \delta \in \lambda^* \sqcup \kappa$ such that $(\varkappa^\frown(\gamma)^\frown\overline{\alpha}_\kappa)_{\alpha\in\kappa}$ is cofinal with G and $(\varkappa^\frown(\delta)^\frown\overline{-\beta}_\kappa)_{\beta\in\lambda}$ is coinitial with

G.[1] Hence, any gap in X is a (λ,κ)-gap. Moreover, X has (λ,κ)-gaps, e.g. the one defined by $(0^\frown\overline{\alpha}_\kappa)_{\alpha\in\kappa}$ and $(1^\frown\overline{-\beta}_\kappa)_{\beta\in\lambda}$.

So, by Proposition 2.3.3, no ordered group (or field) structure is compatible with the interval κ-topology on X.

Proposition 6.1.13. 1. If $\kappa^{<\kappa}=\kappa$, then $\mathbb{R}_\kappa$ with the $\mathbb{R}_\kappa$- or $\mathbb{Q}_\kappa$-interval κ-topology is neither κ-zero-dimensional, nor ω_1-additive, and so not κ-additive.

2. The 2^κ-interval κ-topology on 2^κ is not ω_1-additive, and so not κ-additive.

3. The $(\kappa^*\sqcup\kappa)^\kappa$-interval κ-topology on $(\kappa^*\sqcup\kappa)^\kappa$ is not κ-additive.

Proof. 1. For non-κ-zero-dimensionality, note that the complement of a basic open $(-\infty,x)$ is $[x,\infty)$, as $\mathsf{bn}(\mathbb{K})=\kappa$, no sequence of length $<\kappa$ approaches x, so $[x,\infty)$ is not κ-open. For non-ω_1-additivity, note that $\bigcap_{n\in\omega}(0,\frac{1}{n})=(0,\frac{1}{\neg\omega})$ which is not κ-open (see Example 4.1.4).

2. Let $A:=\bigcap_{n\in\omega}(\overline{0}_\kappa,\overline{0}_n{}^\frown1^\frown\overline{0}_\kappa)=(\overline{0}_\kappa,\overline{0}_\omega{}^\frown\overline{1}_\kappa]$. Suppose that $A=\bigcup_{\alpha\in\lambda}(a_\alpha,b_\alpha)$ for some $a_\alpha,b_\alpha\in2^\kappa$ and $\lambda<\kappa$. There is some $\alpha<\kappa$ such that $\overline{0}_\omega{}^\frown\overline{1}_\kappa\in(a_\alpha,b_\alpha)$, thus $b_\alpha>\overline{0}_\omega{}^\frown\overline{1}_\kappa$, which means that $b_\alpha>\overline{0}_n{}^\frown\overline{1}_\kappa$ for some n, a contradiction.

3. As in Part 2.,

$$N_{\overline{0}_\omega}=\bigcap_{n\in\omega}(\overline{0}_n{}^\frown(-1)^\frown\overline{0}_\kappa,\overline{0}_n{}^\frown1^\frown0^\kappa)=\bigcup_{\alpha\in\kappa}(\overline{0}_\omega{}^\frown(-\alpha)^\frown\overline{0}_\kappa,\overline{0}_\omega{}^\frown\alpha^\frown0^\kappa).$$

Suppose that $N_{\overline{0}_\omega}=\bigcup_{\alpha\in\lambda}(a_\alpha,b_\alpha)$ for some $\lambda<\kappa$ and $a_\alpha,b_\alpha\in(\kappa^*\sqcup\kappa)^\kappa$, again assuming b_α are increasing. Then $N_{\overline{0}_\omega}$ has coinitiality $<\lambda$, as witnessed by $(b_\alpha)_{\alpha\in\lambda}$. As $(\overline{0}_\omega{}^\frown\overline{\beta}_\kappa)_{\beta\in\kappa}$ is cofinal, we can assume that $(b_\alpha)_{\alpha\in\lambda}$ is a subsequence of $(\overline{0}_\omega{}^\frown\overline{\beta}_\kappa)_{\beta\in\kappa}$. If $b_\alpha=\overline{0}_\omega{}^\frown\overline{\beta}_\kappa$, let $b'_\alpha:=\beta$. Then $(b'_\alpha)_{\alpha\in\lambda}$ is cofinal in κ. But this contradicts the regularity of κ. □

The witnesses to the failure of κ-additivity in Proposition 6.1.13 Part 1. is in fact κ-closed in $(0,\infty)$: $(0,\infty)\setminus(0,\frac{1}{\neg\omega})=\bigcup_{n\in\omega}(\frac{1}{n},\infty)$. Indeed, if X is a linear order, $Q\subseteq X$, and $A=\bigcap_{\alpha\in\lambda}(a_\alpha,b_\alpha)$ where $a_\alpha,b_\alpha\in Q\cup\{\pm\infty\}$, $(a_\alpha)_{\alpha\in\lambda}$ is strictly increasing, $(b_\alpha)_{\alpha\in\lambda}$ is strictly decreasing, and $\lambda<\kappa$, then A is κ-closed in the Q-interval κ-topology.

We can use κ-topologies to make finer distinctions of properties of (κ-)topological spaces, as full topologies may be generated from distinct κ-topologies. We first show the difference between the interval κ-topology and the bounded κ-topology on $(\kappa^*\sqcup\kappa)^\kappa$, and hence prove that the witness in Proposition 6.1.13 Part 3. is not κ-closed:

[1] Fix $(\mathbb{z}_\alpha)_{\alpha\in\mu},(\mathbb{w}_\beta)_{\beta\in\nu}$, where $\mathbb{z}_\alpha,\mathbb{w}_\beta\in(\lambda^*\sqcup\kappa)^\kappa$, such that $(\mathbb{z}_\alpha)_{\alpha\in\mu},(\mathbb{w}_\beta)_{\beta\in\nu}$ is cofinal with G, and $(\mathbb{w}_\beta)_{\beta\in\nu}$ is coinitial with G. By passing to a subsequence if necessary, we can assume that there is a least coordinate, α_0, and smallest level, ζ_0, such that for all $\alpha\in\mu$ with $\alpha\geq\alpha_0$, and for all $\zeta>\zeta_0$, we have that $\mathbb{z}_\alpha(\zeta)\in\kappa$, i.e. $\mathbb{z}_\alpha$ only takes values in κ. Likewise, for $(\mathbb{w}_\beta)_{\beta\in\nu}$. Let $\mathbb{x}$ be the greatest common subsequence of any tail element of $(\mathbb{z}_\alpha)_{\alpha\in\mu}$ and $(\mathbb{w}_\beta)_{\beta\in\nu}$. Either $\mathbb{z}_\alpha(\eta_1)+1=\mathbb{w}_\beta(\eta_1)$, or $\mathbb{z}_\alpha(\eta_1)$ is the copy of 0 in λ^*, and $\mathbb{w}_\beta(\eta_1)$ is the copy of zero in κ (otherwise G is not a gap). So, $\mathbb{z}_\alpha=\mathbb{x}^\frown(\gamma)^\frown\mathbb{z}'_\alpha$ and $\mathbb{w}_\beta=\mathbb{x}^\frown(\delta)^\frown\mathbb{w}'_\beta$ for suitable $\gamma,\delta,\mathbb{z}'_\alpha$, and $\mathbb{w}'_\beta$. The only such gap is the one defined by $(\mathbb{x}^\frown(\gamma)^\frown\overline{\alpha}_\kappa)_{\alpha\in\kappa}$ and $(\mathbb{x}^\frown(\delta)^\frown\overline{-\beta}_\kappa)_{\beta\in\lambda}$.

Proposition 6.1.14. Let $s \in (\kappa^* \sqcup \kappa)^{<\kappa}$ have successor length. Then $N_s = \{s^\frown x : x \in (\kappa^* \sqcup \kappa)^\kappa\}$ is neither κ-open nor κ-closed in the $(\kappa^* \sqcup \kappa)^\kappa$-interval κ-topology on $(\kappa^* \sqcup \kappa)^\kappa$.

Proof. That N_s is not κ-open is by cofinality, exactly as in Proposition 6.1.13 Part 3. (with $N_s = \bigcup_{\alpha\in\kappa}(s^\frown\overline{-\alpha}_\kappa, s^\frown\overline{\alpha}_\kappa)$). So, we prove that N_s is not κ-closed: by assumption $s = t^\frown\alpha$ for some $\alpha \in \kappa^* \sqcup \kappa$. Note that $(\kappa^* \sqcup \kappa)^\kappa \setminus N_s = (\bigcup_{\beta\in\kappa}(-\infty, t^\frown\overline{\beta}_\kappa)) \cup (\bigcup_{\beta\in\kappa}(t^\frown\overline{-\beta}_\kappa), \infty))$. Both of the sets $\bigcup_{\beta\in\kappa}(-\infty, t^\frown\overline{\beta}_\kappa)$ and $\bigcup_{\beta\in\kappa}((t^\frown\overline{-\beta}_\kappa), \infty)$ are not κ-open, by a cofinality argument.

Suppose for a contradiction that $(\kappa^* \sqcup \kappa)^\kappa \setminus N_s$ is κ-open. As the intersection of two κ-open sets is κ-open, $((\kappa^* \sqcup \kappa)^\kappa \setminus N_s) \cap (\infty, s^\frown\overline{0}_\kappa) = \bigcup_{\beta\in\kappa}(-\infty, t^\frown\overline{\beta}_\kappa)$ is κ-open, a contradiction. Hence $(\kappa^* \sqcup \kappa)^\kappa \setminus N_s$ is not κ-open. So, N_s is not κ-closed. □

6.2 The Borel$_\kappa$ hierarchy

In this section, we define the κ-topology analogue of the Borel hierarchy. We show that if κ is inaccessible, then this hierarchy collapses after only a few levels. We give a characterisation of the Borel$_\kappa$ sets of κ^κ and certain linearly ordered spaces. We also develop the descriptive set theory of these Borel$_\kappa$ sets, including a non-collapsing difference hierarchy, and show that these pointclasses have no universal or Lipschitz, κ-complete sets.

Definition 6.2.1 (Borel$_\kappa$ hierarchy)**.** Let $\mathcal{A} \subseteq \mathscr{P}(X)$, and $\alpha \in \mathsf{Ord}$. We define the following pointclasses:[2]

1. $\kappa\mathbf{\Sigma}^0_1(\mathcal{A}) := \mathcal{A}$,
2. $\kappa\mathbf{\Pi}^0_\alpha(\mathcal{A}) := \neg\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) = \{X \setminus O : O \in \kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A})\}$,
3. $\kappa\mathbf{\Sigma}^0_2(\mathcal{A}) := \{\bigcup_{\beta\in\lambda} A_\beta : A_\beta \in \kappa\mathbf{\Pi}^0_1(\mathcal{A}) \cup \kappa\mathbf{\Sigma}^0_1(\mathcal{A}) \wedge \lambda < \kappa\}$,
4. if $\alpha > 2$, $\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) = \{\bigcup_{\gamma\in\lambda} A_\gamma : A_\gamma \in \bigcup_{\beta\in\alpha} \kappa\mathbf{\Pi}^0_\beta(\mathcal{A}) \wedge \lambda < \kappa\}$,
5. $\kappa\mathbf{\Delta}^0_\alpha(\mathcal{A}) := \kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) \cap \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A})$, and
6. $\mathsf{Bor}_\kappa(\mathcal{A}) := \bigcup_{\alpha\in\kappa} \kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A})$. A set $S \in \mathsf{Bor}_\kappa(\mathcal{A})$ is called *Borel$_\kappa$* in $(X, \mathcal{A})$.

In the definition of $\mathsf{Bor}_\kappa(\mathcal{A})$, we stop after κ-many steps, because, if κ is regular, then $\mathsf{Bor}_\kappa(\mathcal{A})$ forms a κ-algebra. Indeed, if κ is regular, then for any family $\mathcal{A}$, $\mathsf{Bor}_\kappa(\mathcal{A})$ is the smallest κ-algebra containing $\mathcal{A}$. Hence, if (X, τ_κ) is a κ-topological space with a basis, $\mathcal{B}$, then $\kappa\mathbf{\Sigma}^0_2(\tau_\kappa) = \kappa\mathbf{\Sigma}^0_2(\mathcal{B})$ (and thus $\mathsf{Bor}_\kappa(\tau_\kappa) = \mathsf{Bor}_\kappa(\mathcal{B})$).

[2] This is not the κ-Borel hierarchy of, e.g. [74, Definition 3.5.11] or [135, §1], where κ indicates the spaces considered have weight κ, and is otherwise the standard Borel hierarchy. In our terminology, Galeotti's $\mathbf{B}(X)$ is $\mathsf{Bor}_{\kappa^+}(\tau_X)$.

When the family τ_κ is clear from the context, we write $\kappa\mathbf{\Sigma}^0_1$ for $\kappa\mathbf{\Sigma}^0_1(\tau_\kappa)$, etc. Unlike the usual Borel hierarchy, in Borel$_\kappa$ hierarchy, $\kappa\mathbf{\Sigma}^0_2$ set are $<\kappa$-unions of κ-opens *and* κ-closed (a mixed source). This ensures that the hierarchy is increasing: otherwise, we do not in general know whether $\kappa\mathbf{\Sigma}^0_1 \subseteq \kappa\mathbf{\Sigma}^0_2$.[3]

Proposition 6.2.2. For every family of sets $\mathcal{A} \subseteq \mathscr{P}(X)$ and for every $\alpha < \beta < \kappa$, we have $\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) \cup \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A}) \subseteq \kappa\mathbf{\Delta}^0_\beta(\mathcal{A})$.

Proof. If $\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) \cup \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A}) \subseteq \kappa\mathbf{\Sigma}^0_\beta(\mathcal{A})$, then $\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) \cup \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A}) \subseteq \kappa\mathbf{\Pi}^0_\beta(\mathcal{A})$, so it suffices to prove that $\kappa\mathbf{\Sigma}^0_\alpha(\mathcal{A}) \cup \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A}) \subseteq \kappa\mathbf{\Sigma}^0_\beta(\mathcal{A})$.

We proceed by induction on β. The case $\beta = 2$ holds by definition. Suppose $\beta > 2$ and assume the statement holds for every $\beta' < \beta$. Let $A_\gamma \subseteq X$ for some $\gamma < \lambda < \kappa$, and $\beta' < \beta$. If $A_\gamma \in \kappa\mathbf{\Pi}^0_{\beta'}(\mathcal{A})$, then by definition, $\bigcup_{\gamma\in\kappa} A_\gamma \in \kappa\mathbf{\Pi}^0_\beta(\mathcal{A})$. So, for each $\gamma < \lambda$, suppose $A_\gamma \in \kappa\mathbf{\Pi}^0_{\beta_\gamma}(\mathcal{A})$ for some $\beta_\gamma < \beta$.

If β is a limit, then for each $\beta' < \beta$, $\beta'+1 < \beta$, so by induction $A_\gamma \in \kappa\mathbf{\Sigma}^0_{\beta_\gamma}(\mathcal{A}) \subseteq \kappa\mathbf{\Pi}^0_{\beta_\gamma+1}(\mathcal{A})$. So, the result holds by the first case.

If $\beta = \beta' + 1$, then we can assume without loss of generality that $\beta_\gamma = \beta'$ for all $\gamma < \lambda$. So, $A_\gamma = \bigcup_{\delta\in\mu_\gamma} P^\gamma_\delta$ for some $P^\gamma_\delta \in \bigcup_{\alpha\in\beta'} \kappa\mathbf{\Pi}^0_\alpha(\mathcal{A})$. So, $\bigcup_{\gamma\in\lambda} A_\gamma = \bigcup_{\gamma\in\lambda}\bigcup_{\delta\in\mu_\gamma} P^\gamma_\delta$. As κ is regular, we can reindex the right-hand side as $\bigcup_{\gamma\in\mu} P_\gamma$ for some $\mu < \kappa$. So, $\bigcup_{\gamma\in\lambda} A_\gamma \in \kappa\mathbf{\Sigma}^0_\beta(\mathcal{A})$ as required. □

By the standard arguments, we note that for any basis, $\mathcal{B}$, $\kappa\mathbf{\Sigma}^0_\xi(\mathcal{B})$ and $\mathsf{Bor}_\kappa(\mathcal{B})$ is closed under the preimages of κ-continuous functions.

6.2.1 Collapse of the Borel$_\kappa$ hierarchy

The Borel$_\kappa$ hierarchy collapses assuming that either τ_κ has certain strongly topological properties, or κ is inaccessible. For the former, notice that in a κ-additive space, unions and intersections of $<\kappa$-many κ-clopen sets is κ-clopen. Thus, in particular, every set of a κ-additive κ-zero-dimensional κ-topology is κ-clopen. So, if τ_κ is a κ-additive κ-zero-dimensional κ-topology, then $\mathsf{Bor}_\kappa(\tau_\kappa) = \tau_\kappa$. This observation, and Propositions 6.1.9 and 6.1.10, yield the following:

Remark 6.2.3. If κ is inaccessible, then, on 2^κ, $\mathsf{Bor}_\kappa(\tau^{\mathrm{b}}_\kappa)$ is exactly the κ-clopen sets of 2^κ.

Hence, we mainly focus on κ^κ. For any κ-topology, when κ is inaccessible, $\mathsf{Bor}_\kappa = \kappa\mathbf{\Delta}^0_3$. We prove this next. First, we need an algebraic lemma. Let Γ be a family of sets. We define $\mathsf{U}_\kappa(\Gamma) := \{\bigcup_{\alpha\in\lambda} G_\alpha : G_\alpha \in \Gamma, \lambda < \kappa\}$, and $\mathsf{I}_\kappa(\Gamma) := \{\bigcap_{\alpha\in\lambda} G_\alpha : G_\alpha \in \Gamma, \lambda < \kappa\}$. When κ is inaccessible, we can 'swap' the order of $<\kappa$-unions and $<\kappa$-intersections:

Lemma 6.2.4. Suppose κ is inaccessible and $\Gamma \subseteq \mathscr{P}(X)$. Then:

[3] This is the most natural level to mix: mixing after $\kappa\mathbf{\Sigma}^0_2$ is similar to mixing at $\kappa\mathbf{\Sigma}^0_2$, but delayed to the first mixed level. Meanwhile, mixing earlier, i.e. $\kappa\mathbf{\Sigma}^{0'}_1 := \{\bigcup_{\alpha\in\lambda} A_\alpha : A_\alpha \in \mathcal{B} \cup \neg\mathcal{B} \wedge \lambda < \kappa\}$, has several unsatisfying consequences. For example $\kappa\mathbf{\Sigma}^{0'}_1$ has a basis of clopen sets, which is not, in general, a natural assumption.

1. $\mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma)) = \mathsf{U}_\kappa(\mathsf{I}_\kappa(\Gamma))$.
2. If Γ is closed under relative complements with X, then $\mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma))$ is a κ-algebra.

Proof. In general:

$$A := \bigcap_{\beta\in\lambda}\bigcup_{\alpha\in\mu} A_{\beta,\alpha} = \bigcup_{s\in\mu^\lambda}(\bigcap_{\beta\in\lambda} A_{\beta,s(\beta)}).$$

If κ is inaccessible and $\lambda, \mu < \kappa$, then $\mu^\lambda < \kappa$, and thus previous equation implies that $A \in \mathsf{U}_\kappa(\mathsf{I}_\kappa(\Gamma))$ for every $A \in \mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma))$. The other direction follows similarly.

If Γ is closed under complements, then $\mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma))$ is also closed under complements, thus is a κ-algebra, since given $\bigcap_{\beta\in\lambda}\bigcup_{\alpha\in\mu} A_{\beta,\alpha} \in \mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma))$ we have:

$$X \setminus \bigcap_{\beta\in\lambda}\bigcup_{\alpha\in\mu} A_{\beta,\alpha} = \bigcup_{\beta\in\lambda}\bigcap_{\alpha\in\mu}(X \setminus A_{\beta,\alpha}) \in \mathsf{U}_\kappa(\mathsf{I}_\kappa(\Gamma))$$

and $\mathsf{U}_\kappa(\mathsf{I}_\kappa(\Gamma)) = \mathsf{I}_\kappa(\mathsf{U}_\kappa(\Gamma))$ by the previous point. □

Corollary 6.2.5. If κ is inaccessible and $\mathcal{B}$ is κ-additive, then $\langle\mathcal{B}\rangle_\kappa$ is κ-additive.

The converse is false, e.g. $\{\bigcup_{\alpha\leq\omega} N_{s_\alpha} : s_\alpha \in \kappa^{<\kappa}\}$, the basis of at most countable unions of cones generates the κ-additive bounded κ-topology, but is not itself κ-additive.

Hence, inaccessibility alone entails a short Borel_κ hierarchy:

Lemma 6.2.6. For any $\mathcal{B} \subseteq \mathscr{P}(X)$, if κ is inaccessible, then $\kappa\mathbf{\Sigma}^0_3(\mathcal{B}) = \kappa\mathbf{\Delta}^0_3(\mathcal{B}) = \mathsf{Bor}_\kappa(\mathcal{B})$.

Proof. Clearly, $\Gamma = \kappa\mathbf{\Sigma}^0_1 \cup \kappa\mathbf{\Pi}^0_1$ is closed under complements. So, by Lemma 6.2.4 Part 2., $\mathsf{U}_\kappa(\mathsf{I}_\kappa(\kappa\mathbf{\Sigma}^0_1 \cup \kappa\mathbf{\Pi}^0_1))$ is a κ-algebra. Moreover, $\kappa\mathbf{\Pi}^0_3 = \mathsf{U}_\kappa(\mathsf{I}_\kappa(\kappa\mathbf{\Sigma}^0_1 \cup \kappa\mathbf{\Pi}^0_1))$, so $\kappa\mathbf{\Sigma}^0_3 = \kappa\mathbf{\Pi}^0_3 = \mathsf{Bor}_\kappa$. □

6.2.2 κ-Baire space

We can characterise the Borel_κ sets of κ^κ in ways that are more informative than their height in the Borel_κ hierarchy. We first characterise Borel_κ by their *conelike partitions*. This yields two further characterisations: one syntactic and one in terms of κ-connected components. We then stratify the Borel_κ sets into a non-collapsing difference hierarchy. The non-collapse of the ordinary Borel hierarchy can be proved using complete and universal sets, which must fail to generalise to the Borel_κ sets; we show that this fails due to the lack of suitable universal and complete sets.

Representations of Borel_κ sets

Here, we prove that Borel_κ sets of κ^κ can be represented by a certain partition into *conelike* sets in a unique way. We also prove that if κ is weakly compact, then the intersections of these families are non-empty (Lemma 6.2.16), which in fact characterises weak compactness for inaccessible cardinals (Remark 6.2.18). Later, we use this to show that certain κ-analytic sets are close to their full topological closure (Proposition 6.3.32).

We say that an antichain $A \subseteq \kappa^{<\kappa}$ (ordered with the subset relation) is *optimal* for a set $B \subseteq \kappa^\kappa$ if, for every antichain $A' \subseteq \kappa^{<\kappa}$ such that $\uparrow A' = B$, and every $a' \in A'$, there is an $a \in A$ where $a \subseteq a'$. We say that a set $C \subseteq \kappa^\kappa$ is *conelike in* N_s, for some $s \in \kappa^{<\kappa}$, if $C = N_s \setminus \bigcup_{\beta \in I} N_{s^\frown \beta}$ for some $I \subseteq \kappa$ where $|I| < \kappa$.

Remark 6.2.7. If $s \in \kappa^{<\kappa}$, then $\{X \subseteq \kappa^\kappa : X \text{ is conelike in } N_s\}$ is closed under arbitrary unions: if $(C_\alpha)_{\alpha\in\lambda}$ are conelike for N_s, then for each α, $C_\alpha = N_s \setminus \bigcup_{\beta\in I_\alpha} N_{s^\frown\beta}$. So, $\bigcup_{\alpha\in\lambda} C_\alpha = N_s \setminus \bigcup_{\beta \in \bigcap_{\alpha\in\lambda} I_\alpha} N_{s^\frown\beta}$. But $|\bigcap_{\alpha\in\lambda} I_\alpha| \leq |I_0| < \kappa$. So, $\bigcup_{\alpha\in\lambda} C_\alpha$ is conelike. A similar argument shows that $\{X \subseteq \kappa^\kappa : X \text{ is conelike in } N_s\}$ is closed under $<\kappa$-sized intersections

Conelikeness engenders a kind of two-valued measure on Borel$_\kappa$ sets:

Remark 6.2.8. For any $s \in \kappa^{<\kappa}$, let $\mu_s : \mathsf{Bor}_\kappa \to \{0,1\}$, where $\mu_s(B) = 0$ if $B \subseteq \bigcup_{\alpha\in I} N_{s^\frown\alpha}$ where $I \subseteq \kappa$ is such that $|I| < \kappa$, and $\mu_s(B) = 1$ if $\mu_s(N_s \setminus B) = 0$. We can directly prove that if $B \in \kappa\mathbf{\Sigma}^0_3$ then $\mu_s(B)$ is well-defined, and hence if κ is inaccessible, μ_s is well-defined on all of Bor_κ.

Sets that are conelike in distinct cones are either disjoint or comparable. Obviously, for every P_s and P_t conelike in N_s and N_t, respectively, $P_s \subseteq P_t$ implies $s \subseteq t$. Hence:

Remark 6.2.9. Let $s, t \in \kappa^{<\kappa}$ be distinct. Suppose that P_s and P_t are conelike in N_s and N_t, respectively, and that $P_s \cap P_t \neq \emptyset$. Then, $P_s \subseteq P_t$ implies that $t \subseteq s$ (and not $s \subseteq t$).

Lemma 6.2.10. *Let $(P_{s_\alpha})_{\alpha\in\gamma}$ be decreasing with respect to inclusion, and let each P_{s_α} be conelike in N_{s_α}. If $\bigcap_{\alpha\in\gamma} P_{s_\alpha} = \emptyset$, then $\gamma = \kappa$, and there is an $\alpha < \gamma$ and an $s \in \kappa^{<\kappa}$ such that $s_\beta = s$ for every $\beta > \alpha$.*

Proof. By Remark 6.2.9, if $(P_{s_\alpha})_{\alpha\in\gamma}$ is decreasing with respect to inclusion, then $(s_\alpha)_{\alpha\in\gamma}$ is increasing with respect to inclusion. So, suppose there is a cofinal $J \subseteq \gamma$ such that $(s_\alpha)_{\alpha\in J}$ is strictly increasing. Let $x' = \bigcup_{\alpha\in J} s_\alpha$, and let $x = x'$ if $\gamma = \kappa$, or $x = x'^\frown \overline{0}_\kappa$ otherwise. Then, we have $x \in \bigcap_{\alpha\in J} N_{s_\alpha} = \bigcap_{\alpha\in J} P_{s_\alpha}$, by Remark 6.2.9, so $\bigcap_{\alpha\in J} P_{s_\alpha} \neq \emptyset$.

Otherwise, there is no such cofinal sequence. So, there is a $\beta < \gamma$ and an $s \in \kappa^{<\kappa}$ such that $s_\beta = s$ for every $\beta > \alpha$, but $\gamma < \kappa$, then $\bigcap_{\alpha\in\gamma} P_{s_\alpha}$ is conelike (and thus non-empty) by Remark 6.2.7. □

Definition 6.2.11. Let $\mathcal{F} \subseteq \kappa^{<\kappa}$ be a family of sequences and let $\mathcal{P} = \{P_s : s \in \mathcal{F}\}$ be a family of Borel$_\kappa$ sets. We say that $\mathcal{P}$ is a *conelike partition* of B if $\mathcal{P}$ is a partition of B and each P_s is conelike in N_s.

We say that $\mathcal{F}$ and $\mathcal{P}$ are *optimal* for B if, in addition, $|\mathcal{F}| < \kappa$ and every conelike partition $\mathcal{P}' = \{P_s : s \in \mathcal{F}'\}$ of B refines $\mathcal{P}$ (i.e. for every $s' \in \mathcal{F}'$ there is $s \in \mathcal{F}$ such that $P_{s'} \subseteq P_s$).

Lemma 6.2.12. *Let κ be inaccessible. For every Borel$_\kappa$ set $B \subseteq \kappa^\kappa$, there is a unique family $\mathcal{F}(B) \subseteq \kappa^{<\kappa}$, and a unique conelike partition, $\mathcal{P}(B) = \{P_s : s \in \mathcal{F}(B)\}$, such that $\mathcal{F}(B)$ and $\mathcal{P}(B)$ are optimal for B.*

Proof. We prove that the family, $\mathcal{S}$, of Borel$_\kappa$ sets with an optimal conelike partition is a κ-algebra containing the κ-open sets.

First, we show that all κ-open sets are in $\mathcal{S}$. Let $B \in \kappa\mathbf{\Sigma}^0_1(\tau_\kappa)$. Let A be an optimal antichain in $\kappa^{<\kappa}$ such that $B = \uparrow A$, i.e. A is the set of minimal elements of $\{s \in \kappa^{<\kappa} : N_s \subseteq B\}$. Then, setting $\mathcal{F}(B) = A$ and $P_s = N_s$ suffices.

Next, we show that $\mathcal{S}$ is closed under complements. Suppose $C \in \mathcal{S}$, let $\mathcal{F}(C) \subseteq \kappa^{<\kappa}$ be an optimal family with a conelike partition, $\mathcal{P}(C)$. Without loss of generality, let $C \neq \emptyset$. We want to define $\mathcal{F}(B)$ and $\mathcal{P}(B)$ for $B = \kappa^\kappa \setminus C$.

Claim 6.2.13. For every $s \in \kappa^{<\kappa}$ there is a set, P_s, which is conelike in N_s such that either $P_s \subseteq C$ or $P_s \cap C = \emptyset$.

Proof. Let $s \in \kappa^\kappa$, and let $J = \{\beta < \kappa : N_{s^\frown\beta} \cap C \neq \emptyset\}$. If $|J| < \kappa$, then $(N_s \setminus \bigcup_{\beta\in J} N_{s^\frown\beta}) \cap C = \emptyset$, as required. So, assume $|J| = \kappa$.

For every $\beta \in J$, let $t(\beta) \in \mathcal{F}(C)$ be such that $N_{s^\frown\beta} \cap P_{t(\beta)} \neq \emptyset$. Notice that $t(\beta) \subseteq s^\frown\beta$, as $P_{t(\beta)} \subseteq N_{t(\beta)}$. As $|\mathcal{F}(A)| < \kappa$, there is a $t \in \mathcal{F}(C)$ and $J' \subseteq J$ of size κ such that $t(\beta) = t$ for all $\beta \in J'$. This implies $t \subseteq s$, as $t \subseteq s^\frown\beta$ for all $\beta \in J$. If $t = s$, then $s \in F(C)$ and, by definition, there is a conelike $P_s \subseteq C$, as required. Otherwise, $t \subsetneq s$, and there is an I_t such that $P_t = N_t \setminus \bigcup_{\beta\in I_t} N_{t^\frown\beta}$, thus either $N_s \subseteq N_{s\restriction(\mathsf{len}(t)+1)} \subseteq P_t$ or $N_s \cap N_{s\restriction(\mathsf{len}(t)+1)} \subseteq P_t = \emptyset$. Since $N_s \cap P_t \neq \emptyset$, we must have $N_s \subseteq N_{s\restriction(\mathsf{len}(t)+1)} \subseteq P_t$, and thus $P_s = N_s \subseteq C$. □

Using Remark 6.2.7, for each $s \in \kappa^{<\kappa}$, there is a unique maximal set

$$P_s = \bigcup\{P'_s : P'_s \text{ is conelike in } N_s \text{ and either } P'_s \subseteq C \text{ or } P'_s \cap C = \emptyset\},$$

with the property that P_s is conelike in N_s and either $P_s \subseteq C$ or $P_s \cap A = \emptyset$. Since $\mathcal{P}(C)$ is optimal, and these P_s are maximal, we have that $\mathcal{P}(C) = \{P_s : s \in \mathcal{F}(C)\}$.

Then, define:

$$\mathcal{F}(B) = \{s \in \kappa^{<\kappa} : P_s \subseteq B \text{ and } P_s \nsubseteq P_{s\restriction\beta} \text{ for every } \beta < \mathsf{len}(s)\}.$$

We claim that $\mathcal{F}(B)$ and $\mathcal{P}(B) = \{P_s : s \in \mathcal{F}(B)\}$ are optimal as desired. By construction, every other conelike partition P' of B refines $\mathcal{P}(B)$: indeed, if P is conelike in some N_s and $P \subseteq B$, then $P \subseteq P_s$ by construction, and by the definition of $\mathcal{F}(B)$, there is a $t \subseteq s$ such that $P_s \subseteq P_t$ with $t \in \mathcal{F}(B)$. So, we just need to check that $|\mathcal{F}(B)| < \kappa$.

Claim 6.2.14. We can partition $\mathcal{F}(B)$ into three sets, F_1, F_2, and F_3, such that for each $i \in 3$, $|F_i| < \kappa$.

Proof. Let $F_1 = \{s \in \mathcal{F}(B) : P_s = N_s \text{ and } s \text{ has successor length}\}$. For every $s^\frown\beta \in F_1$, $s \in \mathcal{F}(C)$: if not, then $P_s \cup N_{s^\frown\beta}$ is conelike in N_s containing $P_{s^\frown\beta}$, contradicting the definition of $\mathcal{F}(B)$ or of P_s. For every $s \in \mathcal{F}(C)$, let $I_s \subseteq \kappa$ be of size $<\kappa$, and such that $P_s = N_s \setminus \bigcup_{\alpha\in I_s} N_{s^\frown\alpha}$. Then, we must have $|F_1| \leq |\bigcup_{s\in\mathcal{F}(C)} I_s| < \kappa$, since κ is regular and $|\mathcal{F}(C)| < \kappa$.

Now let $F_2 = \{s \in \mathcal{F}(B) : P_s = N_s \text{ and } s \text{ has limit length}\}$. Let $\gamma = \sup\{\mathsf{len}(s) : s \in \mathcal{F}(C)\}$ and let $T = \{s \in \kappa^{<\kappa} : s \subseteq t \text{ for some } t \in \mathcal{F}(A)\}$. Then $|T| \leq |\mathcal{F}(C) \times \gamma| < \kappa$. We claim that for every $\alpha < \mathsf{len}(s)$ we have $s\restriction\alpha \in T$.

This way, each element $s \in F_2$ is uniquely assigned a subset of T, and thus, as κ is inaccessible, $|F_2| \leq |\mathscr{P}(T)| < \kappa$. By the definitions of $\mathcal{F}(B)$ and $P_{s\restriction\alpha}$, for every $\alpha < \mathsf{len}(s)$, we have $N_{s\restriction\alpha} \cap C \neq \emptyset$ and we can find $t \in \mathcal{F}(C)$ such that $N_{s\restriction\alpha} \cap P_t \neq \emptyset$. Using the same argument as above, t must be comparable with $s\restriction\alpha$, and furthermore we cannot have $t \subseteq s\restriction\alpha$, as otherwise $P_{s\restriction\alpha} \subseteq N_{s\restriction\alpha} \subseteq P_t$. Thus $s\restriction\alpha \subseteq t$, and so $s\restriction\alpha \in T$ by the definition of t.

Let $F_3 = \{s \in \mathcal{F}(B) : P_s \neq N_s\}$. Let $T = \{s \in \kappa^{<\kappa} : s \subseteq t \text{ for some } t \in \mathcal{F}(C)\}$ be defined as above: we claim $F_3 \subseteq T$. Let $s \in F_3$ and let β be such that $N_{s^\frown\beta} \cap P_s = \emptyset$, then there is $t \in \mathcal{F}(A)$ such that $P_t \cap N_{s^\frown\beta} \neq \emptyset$. Arguing as before, we must have $s \subseteq t$, and so $s \in T$. Therefore, we have $|F_3| \leq |T| < \kappa$. □

In order to prove that $\mathcal{S}$ is closed $<\kappa$-sized unions, let $\mu < \kappa$. For all $\gamma < \mu$, let $B_\gamma \in \mathcal{S}$ where we write P^γ_s for the $P_s \in \mathcal{P}(B_\gamma)$. Let $B = \bigcup_{\gamma\in\mu} B_\gamma$. We show that $B \in \mathcal{S}$. For any $s \in \bigcup_{\gamma\in\mu} \mathcal{F}(B_\gamma)$, let $P_s := \bigcup\{P^\gamma_s : \gamma \in \mu, s \in \mathcal{F}(B_\gamma)\}$. By construction, for all such s, P_s is Borel$_\kappa$, and either $P_s = N_s$, or P_s is conelike in N_s. We need to remove any P_s which is covered by a lower P_t. For every $x \in B$, let $s(x)$ be the minimal $s \in \bigcup_{\gamma\in\mu} \mathcal{F}(B_\mu)$ such that $x \in P_s$. Let $\mathcal{F}(B) := \{s(x) : x \in B\}$. By construction, $\mathcal{P}(B) = \{P_s : s \in \mathcal{F}(B)\}$ is a partition. As κ is regular, $|\mathcal{F}(B)| \leq \sum_{\gamma\in\mu} |\mathcal{F}(B_\gamma)| < \kappa$. So, proving that $\mathcal{P}(B)$ is optimal amounts to checking that every conelike partition of B refines $\mathcal{P}(B)$. Indeed, suppose $\mathcal{P}'$ is a conelike partition of B. Fix P'_s which is conelike in N_s and $P'_s \subseteq B$. Then, we must have that $P'_s \subseteq \bigcup\{P_t : t \in \mathcal{F}(B), P_t \cap P'_s \neq \emptyset\}$. By Remark 6.2.9, there are two cases:

1. there is an $r \subsetneq s$ with $r \in \mathcal{F}(B)$ such that $P'_s \subseteq P_r$, or
2. $P'_s \subseteq \bigcup\{P_t : s \subseteq t, t \in \mathcal{F}(B)\}$. Suppose for a contradiction that $s \subsetneq t$ for all $t \in \mathcal{F}(B)$ such that $s \subseteq t$. Then, since $|\mathcal{F}(B)| < \kappa$, we have that $P_t \subseteq \bigcup\{P_t : s \subseteq t\} \subseteq \bigcup\{N_t : s \subseteq t\}$ is a union of $<\kappa$-many cones, thus is not conelike in N_s, a contradiction. So, $s \in \mathcal{F}(B)$ and $P'_s \subseteq P_s$ by the definition of P_s.

So, P'_s is refined by some $P_t \in \mathcal{P}(B)$, as required. Hence, $\mathcal{S}$ contains the κ-open, and is closed under $<\kappa$-unions and complements. To conclude, recall that Bor_κ is the smallest κ-algebra containing the κ-open sets. □

Remark 6.2.15. If B, B' are Borel$_\kappa$ and $B \subseteq B'$ then $\mathcal{F}(B)$ refines $\mathcal{F}(B')$.

Lemma 6.2.16. Assume κ is weakly compact. Let $(B_\alpha)_{\alpha\in\kappa}$ be decreasing with respect to inclusion, where each B_α is non-empty and Borel$_\kappa$. If $\bigcap_{\alpha\in\kappa} B_\alpha = \emptyset$, then there is a $\gamma < \kappa$ such that $\bigcap_{\gamma<\alpha<\kappa} \mathcal{F}(B_\alpha) \neq \emptyset$.

Proof. For every B_α, let $\mathcal{P}(B_\alpha)$ be the unique conelike partition from Lemma 6.2.12. Note that if there are $\alpha < \beta$, $r \in \mathcal{F}(B_\alpha)$, and $t \in \mathcal{F}(B_\beta)$ such that $s \subseteq t$, then, for every γ satisfying $\alpha \leq \gamma \leq \beta$, there is a $s \in \mathcal{F}(B_\gamma)$ such that $r \subseteq s \subseteq t$. Indeed, since $\mathcal{P}(B_\beta)$ refines $\mathcal{P}(B_\gamma)$, and $\mathcal{P}(B_\gamma)$ refines $\mathcal{P}(B_\alpha)$, there is an $s \in \mathcal{F}(B_\gamma)$ such that $P^\beta_t \subseteq P^\gamma_s \subseteq P^\alpha_r$, and so we must have $r \subseteq s \subseteq t$ by Remark 6.2.9. This implies that if $s \in \mathcal{F}(B_\alpha)$ for cofinally many $\alpha < \kappa$, then $s \in \bigcap_{\alpha\in\kappa} \mathcal{F}(B_\alpha)$.

Now assume towards contradiction that for all $\gamma < \kappa$, $\bigcap_{\gamma<\alpha<\kappa} \mathcal{F}(B_\alpha) = \emptyset$. We show that there is a κ-Aronszajn tree, and thus κ is not weakly compact.

Let $T' = \bigcup_{\alpha\in\kappa} \mathcal{F}(B_\alpha)$. Then, for every $s \in T'$ there is $\alpha < \kappa$ such that $s \notin \mathcal{F}(B_\alpha)$, by the previous argument. So, by the pigeonhole principle, $|T'| = \kappa$. Let

$$F_\alpha = \{s \in \mathcal{F}(B_\alpha) : |\{t \in T' : s \subseteq t\}| = \kappa\}.$$

Claim 6.2.17. $F_\alpha \neq \emptyset$ for every $\alpha < \kappa$.

Proof. Indeed, $T'' = T' \setminus \bigcup_{\epsilon\in\alpha} \mathcal{F}(B_\epsilon)$ has size κ, and for every $t \in T''$, there is $s \in \mathcal{F}(B_\alpha)$ such that $s \subseteq t$, since $\mathcal{F}(B_\alpha)$ refines all $\mathcal{F}(B_\beta)$ for $\beta \geq \alpha$. Thus, by the pigeonhole principle, we can find $s \in \mathcal{F}(B_\beta)$ such that $s \subseteq t$ for κ-many $t \in T''$. □

Proceeding recursively, we define trees $T_\alpha \subseteq \bigcup_{\alpha\in\kappa} F_\alpha$ of height $\alpha+1$ such that all levels of T_α have size $<\kappa$. Let $T_0 = \{\emptyset\}$, and given α limit, let $T'_\alpha = \bigcup_{\gamma\in\alpha} T_\gamma$ and let $T_\alpha = T'_\alpha \cup \delta(T'_\alpha)$. It is clear that they satisfy all the required properties (since $|\delta(T'_\alpha)| < \kappa$ if κ is inaccessible).

Suppose that T_β is as required, and that $\alpha = \beta + 1$. Then, $L_\beta = \mathsf{Lev}_\beta(T_\beta)$ has size $<\kappa$. By the definition of T', the set $\{t \in T' : s \subseteq t\}$ has size κ for any $s \in L_\beta$. Also, for every $s \in L_\beta$, there is an $\alpha_s < \kappa$ such that $s \notin F_{\alpha'} \subseteq \mathcal{F}(B_{\alpha'})$ for every $\alpha' \geq \alpha_s$ (otherwise $s \in \bigcap_{\alpha_s<\alpha<\kappa} \mathcal{F}(B_\alpha)$ by the previous argument). Let $\gamma_\alpha = \sup\{\alpha_s : s \in L_\beta\}$, then $\gamma_\alpha < \kappa$ since $|L_\beta| < \kappa$. Let L_α be the set of minimal elements of $\bigcup_{s\in L_\beta}\{t \in F_{\gamma_\alpha} : s \subseteq t\}$, and let $T_\alpha = T_\beta \cup L_\alpha$. Then, $L_\alpha \cap T_\alpha = \emptyset$, thus T_α has height $\alpha + 1$, and $\mathsf{Lev}_\alpha(T_\alpha) = L_\alpha$ has size $<\kappa$ since $|F_{\gamma_\alpha}| \leq |\mathcal{F}(B_{\gamma_\alpha})| < \kappa$, as required.

We claim that $T = \bigcup_{\alpha\in\kappa} T_\alpha$ is a κ-Aronszajn tree. We just need to check that T has no branch of size κ, since all other properties are guaranteed by the construction. Assume for a contradiction that $b \subseteq T$ is a branch of size κ, and let $x = \bigcup b$. Then, $x \in \bigcap_{s\in b} N_s \subseteq \bigcap_{s\in b} P_s$, contradicting $\bigcap_{\alpha\in\kappa} B_\alpha = \emptyset$. □

Remark 6.2.18. The converse holds in the following sense. Suppose κ is inaccessible, and any decreasing sequence of Borel_κ sets, B_α, such that $\bigcap_{\alpha\in\kappa} B_\alpha = \emptyset$, is such that $\bigcap_{\alpha\in\kappa} \mathcal{F}(B_\alpha) \neq \emptyset$. Then κ is weakly compact.

Thus, this property characterises the weakly compact cardinals amongst the inaccessible ones.

This gives us a very informative representation of the Borel_κ sets. One particular use of this is in Theorem 6.3.32.

If κ is *not* inaccessible, then the Borel_κ hierarchy may be strictly longer:

Proposition 6.2.19. If κ is not inaccessible, there is a $\kappa\boldsymbol{\Sigma}^0_3$ set with no optimal conelike partition.

Proof. As κ is not inaccessible, let $\lambda < \kappa$ be such that $2^\lambda = \kappa$. Let $A = [T]$ be a closed set in the full topology of 2^λ such that $|2^\lambda \setminus A| = |A| = \kappa$. For any $\gamma < \lambda$, and $N_s \subseteq \kappa^\kappa$, let $B_\gamma := \bigcup_{s\in\mathsf{Lev}_\gamma(T)} N_s$. As T is a λ-tree, B_γ is κ-open in κ^κ. Then $\uparrow A := \bigcap_{\gamma\in\lambda} B_\gamma = \bigcup_{s\in A} N_s$ is $\kappa\boldsymbol{\Sigma}^0_3$. But $|A| = \kappa$, and it is clear that no conelike partition for $\uparrow A$ can have cardinality $<\kappa$. So, $\uparrow A$ has no optimal conelike partition. The same argument works for $\uparrow(2^\lambda \setminus A)$. □

Based on Lemma 6.2.12, we can characterise the Borel$_\kappa$-sets syntactically, as a union of certain 'V-like' sets (i.e. cones with κ-open sets removed). In fact, a Borel$_\kappa$ set $B = \bigcup_{\beta\in\mu} N_{s_\beta} \setminus O_\beta$ where N_{s_β} are cones, O_β are κ-open sets, and $\mu < \kappa$ (see Figures 6.1, 6.2, and 6.3).

Corollary 6.2.20. If κ is inaccessible, then $X \subseteq \kappa^\kappa$ is Borel$_\kappa$ if and only if it has one of the following forms:

1. $X = \bigcup_{s\in A} N_s$ where $A \subseteq \kappa^{<\kappa}$ is an antichain and $|A| < \kappa$, when X is κ-open;
2. $X = \kappa^\kappa \setminus \bigcup_{s\in A} N_s$ where $A \subseteq \kappa^{<\kappa}$ is an antichain and $|A| < \kappa$, when X is κ-closed;
3. $X = (\kappa^\kappa \setminus \bigcup_{s\in A} N_s) \cup \bigcup_{t\in A'} N_t$ where $A, A' \subseteq \kappa^{<\kappa}$ are antichains where $|A \cup A'| < \kappa$, and such that for every $t \in A'$, there is a $s \in A$ with $s \subsetneq t$, when X is $\kappa\mathbf{\Sigma}^0_2$;
4. $X = \bigcup_{i\in I}(N_{s_i} \setminus \bigcup_{t\in A_i} N_{s^\frown t})$ where $I, A_i \subseteq \kappa^{<\kappa}$ are antichains, $|I| < \kappa$, $|\bigcup_{i\in I} A_i| < \kappa$, and for all $i, j \in I$, $i \neq j$, $s_i \perp s_j$, when X is $\kappa\mathbf{\Pi}^0_2$; and
5. $X = \bigcup_{i\in I}(N_{s_i} \setminus \bigcup_{t\in A_i} N_{s^\frown t})$ where $I, A_i \subseteq \kappa^{<\kappa}$ are antichains and $|\bigcup_{i\in I} A_i| < \kappa$, when X is $\kappa\mathbf{\Sigma}^0_3$.

Proof. The proof is essentially a syntactic check, using Lemma 6.2.12. □

Remark 6.2.21. By Proposition 6.2.20 Part 4., $\kappa\mathbf{\Pi}^0_2 = \{U \setminus V : U, V \in \kappa\mathbf{\Sigma}^0_1\}$.

The κ-topology on κ^κ is 'more' κ-connected than the ordinary topology is connected, in the sense that κ-closed sets are κ-connected, as are the V-shaped sets (sets of the form $N_s \setminus N_{s^\frown s_\alpha}$):

Lemma 6.2.22. The following subsets of κ^κ are κ-connected:

1. N_s, for any $s \in \kappa^{<\kappa}$;
2. $N_s \setminus N_t$ for any $s, t \in \kappa^{<\kappa}$ with $t \supsetneq s$,
3. $N_s \setminus \bigcup_{\alpha\in\lambda} N_{s_\alpha}$ for any $s, s_\alpha \in \kappa^{<\kappa}$ with $s_\alpha \supsetneq s$ and $\lambda < \kappa$; and
4. $C \subseteq \kappa^\kappa$ which is κ-closed.

Proof. Part 1. is clear. We prove Part 2., as Part 3. is a simple elaboration. Suppose $N_s \setminus N_t$ is partitioned by disjoint κ-open sets A, B. Then $\{A \cup N_t, B\}$ is a κ-open partition of N_s, contradicting Part 1.

For Part 4., as in the classical case, κ^κ and $\emptyset$ are κ-connected. Otherwise, $C = \kappa^\kappa \setminus \bigcup_{\alpha\in\lambda} N_{s_\alpha}$ for some $s, s_\alpha \in \kappa^{<\kappa}$ with $s_\alpha \supsetneq s$ and $\lambda < \kappa$. Suppose for a contradiction that C is not κ-connected, so $C = U \sqcup V$ where U and V are κ-open. Without loss of generality assume $\lambda = 1$, i.e. $C = \kappa^\kappa \setminus N_s$ for some $s \in \kappa^{<\kappa}$. So, $C = \bigcup_{s\in J} N_j$ where $J \subseteq \kappa^{\mathsf{len}(s)} \setminus \{s\}$ is a maximal antichain. But the smallest such possibility is $|J| = \kappa$, so without loss of generality $U = \bigcup_{s\in J'} N_s$ and $V = \bigcup_{s\in J''} N_s$ for $J', J'' \subseteq J$. So, at least one of J', J'' has size κ, contradicting that U, V are κ-open. □

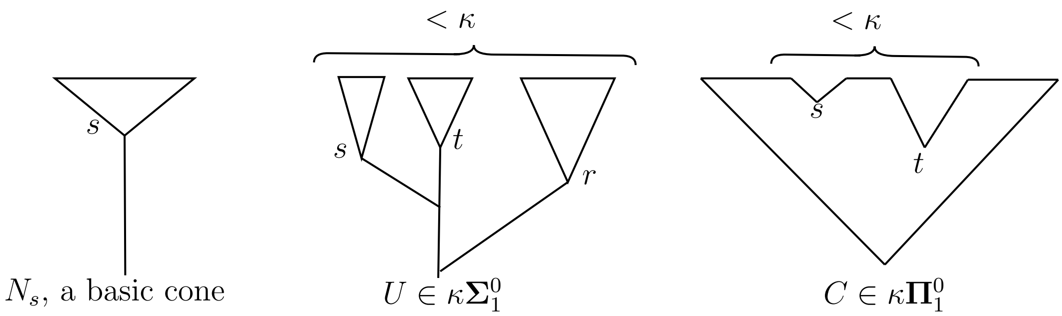

Figure 6.1: κ-open and κ-closed sets of κ^κ

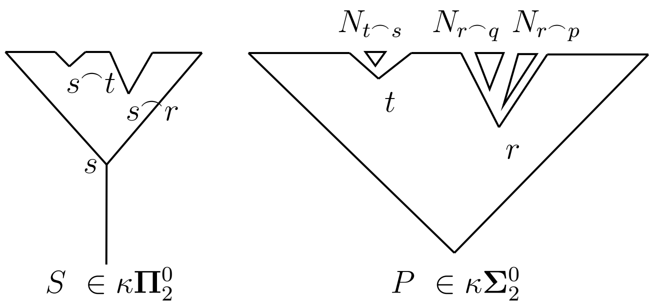

Figure 6.2: $\kappa\mathbf{\Sigma}^0_2$ and $\kappa\mathbf{\Pi}^0_2$ sets of κ^κ

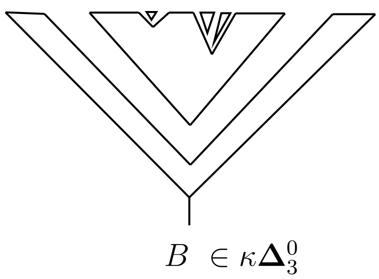

Figure 6.3: $\kappa\mathbf{\Delta}^0_3$ sets of κ^κ

The only κ-connected κ-open sets are the cones. So, e.g. every κ-connected κ-open $U = N_s \subseteq \kappa^\kappa$ is κ-homeomorphic to κ^κ itself. Hence, we can represent the Borel$_\kappa$ sets in terms of κ-connected sets:

Corollary 6.2.23. Suppose κ is inaccessible. If $B \in \mathsf{Bor}_\kappa(\tau^{\mathrm{b}}_\kappa)$, then B is a $<\kappa$-union of κ-connected Borel$_\kappa$ components.

Proof. By Lemma 6.2.6, every Borel$_\kappa$ set, B, is of the form $B = \bigcup_{\alpha\in\lambda} N_{s_\alpha} \setminus V_\alpha$ for some κ-open V_α and some $\lambda < \kappa$ (with possibly nested N_{s_α}). By Lemma 6.2.22, each of these components is κ-connected. □

We conclude that not all fully clopen sets are Borel$_\kappa$:

Proposition 6.2.24.
1. Every Borel$_\kappa$ subset of κ^κ is clopen in the full topology.
2. If $\kappa^{<\kappa} = \kappa$, then there are κ-many Borel$_\kappa$ sets. So, there are sets, $X \subseteq \kappa^\kappa$ which are clopen in the full topology which are not Borel$_\kappa$.

Proof. For Part 1., every cone N_s is clopen in the full topology, as usual. Trivially, the complement of a clopen is clopen. Moreover, the family of clopen sets is κ-additive. Finally, recall that Bor_κ is the smallest κ-algebra containing the κ-open sets.

For Part 2., for each $s \in \kappa^{<\kappa}$, N_s is a distinct Borel$_\kappa$ set. For the upper bound, we inductively prove that for each $\alpha < \kappa$, $|\kappa\mathbf{\Sigma}^0_\alpha| \leq \kappa$. First, note that $\kappa^{<\kappa} = \kappa$ implies that $|\kappa\mathbf{\Sigma}^0_1| = \kappa$. For the induction step, if this holds for all levels $\alpha < \beta$, then $|\kappa\mathbf{\Pi}^0_\alpha| \leq \kappa$ for all $\alpha < \beta$, so $|\kappa\mathbf{\Sigma}^0_\beta| \leq |\{U \subseteq \bigcup_{\alpha\in\beta} \kappa\mathbf{\Pi}^0_\alpha : |U| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$. So, there are at most $\kappa \times \kappa = \kappa$-many Borel$_\kappa$ sets. For the last part, note that there are 2^κ-many sets which are clopen in the full topology. □

We can construct a set which can only be represented as a union of κ-many disjoint cones, and the same is true of its complement, which is thus 'too complex' to be Borel$_\kappa$:

Example 6.2.25. Let κ be inaccessible. Then $\bigcup_{\alpha\in\kappa} N_{(2\alpha)}$ is clopen in the full topology, but, by Lemma 6.2.12, it is not Borel$_\kappa$.

Actually, $\bigcup_{\alpha\in\kappa} N_{(2\alpha)}$ is clopen in the full topology but not Borel$_\kappa$ even when κ is not inaccessible, by a separate argument.

The difference hierarchy on $\kappa\mathbf{\Sigma}^0_1$

The representation in Proposition 6.2.20 indicates a non-collapsing stratification of Bor_κ, namely the difference hierarchy on κ-open sets: Borel$_\kappa$ sets are small unions of 'V-shaped' sets ($N_s \setminus N_t$ with $s \subsetneq t$) and cones, where some of these V-shaped sets can fall strictly within other V-shaped sets ($N_s \setminus N_t \cup N_q \setminus N_r$ where $s \subsetneq t \subseteq q \subsetneq r$). The depth of the nesting of these V-shaped sets corresponds to a difference hierarchy on the κ-open sets (building on the classical Hausdorff difference hierarchy of [112, 22.E]):

Definition 6.2.26. Let $(C_\beta)_{\beta\in\alpha}$ be an increasing sequence of subsets of κ^κ. For all $\alpha<\kappa$, we define $\alpha\text{-}\mathsf{Diff}((C_\beta)_{\beta\in\alpha}) := \{x \in \bigcup_{\beta\in\alpha} C_\beta\colon$ the least $\beta<\alpha$ with $x\in C_\beta$ has parity opposite to that of $\alpha\}$.

Let $\Gamma \subseteq \mathscr{P}(\kappa^\kappa)$. We define $\alpha\text{-}\mathsf{Diff}(\Gamma) := \{\alpha\text{-}\mathsf{Diff}((C_\beta)_{\beta\in\alpha}) : (C_\beta)_{\beta\in\alpha}$ is an increasing sequence of $C_\beta\in\Gamma\}$. We let $\kappa\text{-}\mathsf{Diff}(\Gamma) := \bigcup_{\alpha\in\kappa}\alpha\text{-}\mathsf{Diff}(\Gamma)$.

Proposition 6.2.27. For all $\alpha<\beta<\kappa$, $\alpha\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1) \subsetneq \beta\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$.

Proof. Let $(s_\gamma)_{\gamma\in\beta}$ be a sequence of $s_\gamma\in\kappa^{<\kappa}$ such that $s_\gamma \subsetneq s'_\gamma$ for all $\gamma<\gamma'<\beta$. Then a simple check shows that $\beta\text{-}\mathsf{Diff}((N_{s_\gamma})_{\gamma\in\beta}) \in \beta\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)\setminus\alpha\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. □

We can exactly characterise the Borel$_\kappa$ sets as the κ-difference hierarchy on the κ-open set:

Proposition 6.2.28. For any $C\subseteq\kappa^\kappa$, $C\in\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$ if and only if C is Borel$_\kappa$.

Proof. First, we prove $\subseteq$: if $C\in\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$, then $C=\bigcup_{\beta\in\lambda}B_\beta$ for some $\lambda<\kappa$, where each B_β is the difference of two open sets, i.e. the difference of two sets of the sequence defining C. So, B_β is Borel$_\kappa$. Hence, C is Borel$_\kappa$.

In the other direction, note that $\kappa\mathbf{\Sigma}^0_1\subseteq\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$, that $\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$ is obviously closed under complements. So, it suffices to prove that $\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$ is closed under $<\kappa$-unions. Suppose that $(C_\beta)_{\beta\in\lambda}$ in $\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. As κ is regular, there is some $\mu<\kappa$ such that for all $\beta<\lambda$, $C_\beta\in\mu\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. So, by the standard argument (as in [112, Ex 22.26(i)]), $\bigcup_{\beta\in\lambda}C_\beta\in(\mu+1)\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. Hence, $\kappa\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$ is a κ-algebra containing the open sets. □

This is very unlike the classical case, where $\mathbf{\Delta}^0_\alpha\subsetneq\mathsf{Diff}(\mathbf{\Delta}^0_\alpha)\subseteq\mathbf{\Delta}^0_{\alpha+1}$ [89], [130, §37.III]. An immediate corollary of Proposition 6.2.20 is the following:

Corollary 6.2.29. We have that $\kappa\mathbf{\Pi}^0_2 = 2\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$.

Corollary 6.2.30. Singletons in κ^κ are not Borel$_\kappa$.

Proof. Fix $x\in\kappa^\kappa$. The result follows by induction on α, that $\{x\}\notin\alpha\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. Clearly, $\{x\}$ is not κ-open, nor is it the difference of two κ-open sets. Later steps only add components, so clearly do not include $\{x\}$. □

Complete and universal sets

A standard proof of the non-collapse of the classical Borel hierarchy constructs universal sets for each class, then diagonalises (e.g. [112, Theorem 22.4]). An alternative shows that there is a complete set for continuous Lipschitz function [6, Proposition 3.9]. By Lemma 6.2.6, neither argument generalises to the Borel$_\kappa$ hierarchy. Here we are more precise, we prove that there are no sets which are complete for Lipschitz κ-continuous functions; nor are there universal sets. We first show that there *are* κ-complete sets.

As usual, for $C,D\subseteq X$, we say $f:X\to X$ is a *reduction of C to D* if $C=f^{-1}(D)$.

Definition 6.2.31. Let (X,τ_κ) be a κ-topological space, and let $\Gamma\subseteq\mathscr{P}(X)$ be such that $X\in\Gamma$. A subset $C\subseteq X$ is called *κ-complete for* Γ, if every set in Γ can be reduced to C with a κ-continuous function.

Every κ-open set, U, is κ-complete for κ-open sets, by noting that checking $x \in U$ only uses $<\kappa$-many bits (as in the full topology), but also that checking $x \notin U$ uses only $<\kappa$-many bits. Formally:

Proposition 6.2.32. If $\kappa^{<\kappa} = \kappa$, then any non-empty κ-open set, $U \subsetneq \kappa^\kappa$, is κ-complete for $\kappa\mathbf{\Sigma}^0_1$.

Proof. Let $U \subsetneq \kappa^\kappa$ be non-empty and κ-open, and let V be a κ-open set. We construct a function $f : \kappa^\kappa \to \kappa^\kappa$ such that $f^{-1}(U) = V$. Let A, B be antichains such that $|B|, |A| < \kappa$, $\uparrow A = U$, and $\uparrow B = V$. Let $A_0 = \{a(0) : a \in A\}$. Let $\beta := \sup\{\mathsf{len}(b) : b \in B\}$ (as κ is regular, $\beta < \kappa$). Let $I := \{s \in \kappa^\beta : s \perp B\}$. If $|B| \geq |A|$, let $\chi : B \to A$ be surjection and let $\phi : I \to \kappa \setminus A_0$ be a bijection. We define f like so:

$$f(s^\frown y) = \begin{cases} \chi(s)^\frown y & \text{if } s \in B, \text{ and } y \in \kappa^\kappa, \\ \phi(s)^\frown y & \text{if } s \in I, \text{ and } y \in \kappa^\kappa. \end{cases}$$

Otherwise, $|B| < |A|$. Then let $\psi : |A| \to A$ and $\upsilon : \{t^\frown\alpha : t \in B, \alpha \notin |A|\} \cup I \to \kappa \setminus A_0$ be bijections. Then we define f like so:

$$f(x) = \begin{cases} \psi(\alpha)^\frown y & \text{if } x = t^\frown\alpha^\frown y \text{ for some } t \in B, \alpha \in |A|, \text{ and } y \in \kappa^\kappa, \\ \upsilon(\alpha)^\frown y & \text{if } x = t^\frown\alpha^\frown y \text{ for some } t \in B, \alpha \notin |A|, \text{ and } y \in \kappa^\kappa, \\ \upsilon(s)^\frown y & \text{if } x = s^\frown y \text{ for some } s \in I, y \in \kappa^\kappa. \end{cases}$$

Clearly, $f^{-1}(U) = V$. It remains to check that f is κ-continuous. First, suppose $|B| \geq |A|$. If $s = t^\frown r$ for some S such that $\phi^{-1}(s) \perp B$, then $f^{-1}(N_s) = N_{s'}$ for some $s' \in \kappa^{<\kappa}$. Otherwise, $f^{-1}(N_s) = \bigcup_{s' \in S} N_{s'}$ where $|S| \leq |B|$, so $f^{-1}(N_s)$ is κ-open. The case where $|B| < |A|$ is similar. □

This reduction of N_s to N_t need not be a retraction, nor even surjective. One consequence of Proposition 6.2.32, using the standard proof in e.g. [112, Proposition 22.15], is that the only κ-clopen sets are $\emptyset$ and κ^κ:

Proposition 6.2.33. If $\Gamma \subseteq \mathscr{P}(\kappa^\kappa)$ is closed under κ-continuous preimages, but not under complements, and every (non-empty) set in Γ is κ-complete, then $\Gamma \cap \neg\Gamma \subseteq \{\emptyset, \kappa^\kappa\}$.

Corollary 6.2.34. If $\kappa^{<\kappa} = \kappa$, then $\kappa\mathbf{\Delta}^0_1 = \{\emptyset, \kappa^\kappa\}$.

A parallel to Proposition 6.2.32 holds for $\kappa\mathbf{\Sigma}^0_2$, which yields a parallel triviality result for $\kappa\mathbf{\Delta}^0_2$:

Proposition 6.2.35. Suppose $\kappa^{<\kappa} = \kappa$. If $B \in \kappa\mathbf{\Sigma}^0_2 \setminus (\kappa\mathbf{\Pi}^0_1 \cup \kappa\mathbf{\Sigma}^0_1)$, then B is κ-complete for $\kappa\mathbf{\Sigma}^0_2$. So, any $B \in 2\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1) \setminus 1\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$ is κ-complete for $2\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$.

Proof. The proof is exactly similar to that of Proposition 6.2.32. One first proves that if $s \subsetneq t$, then $N_s \setminus N_t$ is κ-complete for $\kappa\mathbf{\Sigma}^0_2$. Then one proves that if B_1, B_2

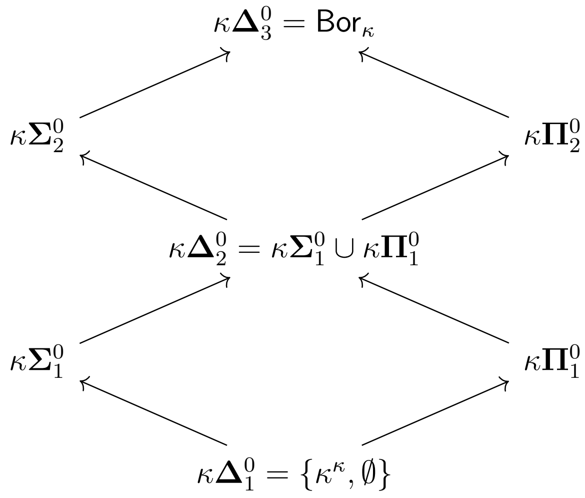

Figure 6.4: The Borel$_\kappa$ hierarchy on κ^κ

are non-empty and κ-open, and $B_1 \subsetneq B_2$, then $B_1 \setminus B_2$ is κ-complete for $\kappa\mathbf{\Sigma}^0_2$. The other cases are a simple elaboration.

For the second part, by Corollary 6.2.29, $\kappa\mathbf{\Pi}^0_2 = 2\text{-}\mathsf{Diff}(\kappa\mathbf{\Sigma}^0_1)$. The result then follows by duality. □

So, arguing as usual again (e.g. [112, Proposition 22.15]), we can show that if $\kappa^{<\kappa} = \kappa$, then $\kappa\mathbf{\Delta}^0_2 = \kappa\mathbf{\Sigma}^0_1 \cup \kappa\mathbf{\Pi}^0_1$. This gives us the representation of Borel$_\kappa$ hierarchy displayed in Figure 6.4.

The non-collapse of the ordinary Borel hierarchy in, e.g. [6, Proposition 3.9], relies on sets which are complete for *Lipschitz* functions, not just continuous functions. But we show that there are no sets which are complete for any Borel$_\kappa$ class, even for functions which are κ-continuous and Lipschitz.

First, recall that if $\phi : \kappa^{<\kappa} \to \kappa^{<\kappa}$, we define $f_\phi : \kappa^\kappa \to \kappa^\kappa$ as $f_\phi(x) = \bigcup_{\alpha\in\kappa} \phi(x{\restriction}\alpha)$.

Lemma 6.2.36 (Folklore, e.g. [6, Lemma 3.7])**.** A function $f : \kappa^\kappa \to \kappa^\kappa$ is continuous if and only if there is a function $\phi : \kappa^{<\kappa} \to \kappa^{<\kappa}$ with $f_\phi = f$ such that:

1. ϕ is monotone, and
2. $\mathsf{len}(\bigcup_{\alpha\in\kappa} \phi(x{\restriction}\alpha)) = \kappa$ for all $x \in \kappa^\kappa$ (i.e. ϕ is continuous).

As usual, a function $\phi : \kappa^{<\kappa} \to \kappa^{<\kappa}$ is called *Lipschitz* if it is monotone and for all $s \in \kappa^{<\kappa}$ we have that $\mathsf{len}(\phi(x)) = \mathsf{len}(x)$. A function $f : \kappa^\kappa \to \kappa^\kappa$ is called *Lipschitz* if $f = f_\phi$ for some Lipschitz ϕ. Both Lipschitz and κ-continuity imply continuity, so any proposed 'κ-topology version' of Lipschitz, L_κ, would imply being Lipschitz. So, if there are no sets which are complete for κ-continuous and Lipschitz functions, then there are no sets which are complete for L_κ functions, for any such notion, L_κ.

Definition 6.2.37 (following [6, page 23])**.** Let (X, τ_κ) be a κ-topological space, and let $\Gamma \subseteq \mathscr{P}(X)$ be such that $X \in \Gamma$. A subset $C \subseteq X$ is called *L, κ-hard for* Γ

if every set it Γ can be reduced to C by a fully Lipschitz, κ-continuous function. If, in addition, $C \in \Gamma$, then we say that C is *L, κ-complete for* Γ.

Proposition 6.2.38. If κ is inaccessible, then there is no Borel$_\kappa$ set which is L, κ-hard for $\kappa\mathbf{\Sigma}^0_1$. So, for each $\alpha \in \kappa$, there is no set which is L, κ-complete for $\kappa\mathbf{\Sigma}^0_\alpha$.

Proof. Let B be any set such that there is an $\alpha < \kappa$, and an antichain $A \subseteq \kappa^{<\alpha}$ such that $B = \uparrow A$. Notice that every κ-open set satisfies this property. By Lemma 6.2.12, if κ is inaccessible, every Borel$_\kappa$ set is also representable as $\uparrow A$ for such an antichain.

We prove that B is not L, κ-hard for κ-open sets. If B is empty, then it is clearly not L, κ-hard, so we are done. So, assume that there is an $x \in B$, and let $s = x\restriction\alpha$ and $t = x\restriction(\alpha + 1)$. Suppose that B is L, κ-hard for κ-open sets. Then, there is a Lipschitz, κ-continuous function, ϕ_t, such that $f^{-1}_{\phi_t}(B) = N_t$. Since $B = \uparrow A$ for $A \subseteq \kappa^{<\alpha}$ and ϕ is Lipschitz, we have that either $N_{\phi_t}(s) \subseteq B$ or $N_{\phi_t}(s) \cap B = \emptyset$: thus, either $N_s \subseteq f^{-1}_{\phi_t}(B)$ or $N_s \cap f^{-1}_{\phi_t}(B) = \emptyset$, but either way $f^{-1}_{\phi_t}(B) \neq N_t$, a contradiction. □

The situation is somewhat simpler for universal sets on the Borel$_\kappa$ hierarchy: if κ is inaccessible, then there are no universal sets for the Borel$_\kappa$ classes. As usual, if $C \subseteq X \times Y$ and $x \in X$, let $C_x = \{y \in Y : (x, y) \in X\}$. We consider the case where $X = Y = \kappa^\kappa$.

Proposition 6.2.39. If κ is inaccessible, then there is no set which is universal for $\kappa\mathbf{\Sigma}^0_1$.

Proof. Let $U \subseteq (\kappa^\kappa)^2$ be κ-open. Then $U = \bigcup_{\alpha\in\mu} N_{s_\alpha} \times N_{t_\alpha}$ for some $\mu < \kappa$. Note that we may have distinct $x, y \in \pi_1(U)$ such that $U_x = U_y$ (Figure 6.5). As $U_x = (\bigcup_{\alpha\in S} N_{s_\alpha} \times N_{t_\alpha})_x$, U_x can be uniquely assigned a set $S \subseteq \mu$ consisting of those α such that $x \in N_{t_\alpha}$. This characterises the possible U_x in terms of subsets of μ. Using this injection, $|\{U_x : x \in X\}| \leq 2^\mu$. As κ is a strong limit, $2^\mu < \kappa$. The bounded κ-topology has weight κ, so $|\kappa\mathbf{\Sigma}^0_1| \geq \kappa$. Hence, not every κ-open subset of X is a projection of U, i.e. U is not universal for $\kappa\mathbf{\Sigma}^0_1$. □

Proposition 6.2.40. If κ is inaccessible, then there is no set which is universal for $\kappa\mathbf{\Sigma}^0_2$.

Proof. Note that $|\kappa\mathbf{\Sigma}^0_2| = \kappa$. So, it suffices to prove that any $\kappa\mathbf{\Sigma}^0_2$ set has $<\kappa$-many distinct slices. By regularity, if $B \in \kappa\mathbf{\Sigma}^0_2$, then $B = C \cup U$ where C is κ-closed, and U is κ-open. As C is κ-closed in $(\kappa^\kappa)^2$, there is a $\lambda < \kappa$ and basic open rectangles, R_α, for each $\alpha < \lambda$, such that $C = (\kappa^\kappa)^2 \setminus \bigcup_{\alpha\in\lambda} R_\alpha$. In other words, C has λ-many 'holes' in it. As U is κ-open, there is a $\mu < \kappa$ and basic open rectangles R'_β for each $\beta < \mu$ such that $U = \bigcup_{\beta\in\mu} R'_\beta$. So, U has μ-many 'chunks'. For each $y \in Y$, $B_y = \kappa^\kappa \setminus (\bigcup_{\alpha\in\lambda}(R_\alpha)_y) \cup \bigcup_{\beta\in\mu}(R'_\mu)_y$. So, every slice B_y corresponds to a pair (S, P), where $S \subseteq \lambda$ defines the holes in C_y, and $P \subseteq \mu$ defines the chunks of U_y. Hence, $|\{B_y : y \in \kappa^\kappa\}| = |\mathscr{P}(\mu) \times \mathscr{P}(\lambda)|$. As κ is a strong limit, $\gamma < \kappa$ implies $|\mathscr{P}(\gamma)| < \kappa$, so $|\mathscr{P}(\mu) \times \mathscr{P}(\lambda)| < \kappa$. Hence, there are at most $<\kappa$-many distinct slices of any $\kappa\mathbf{\Sigma}^0_2$ set. □

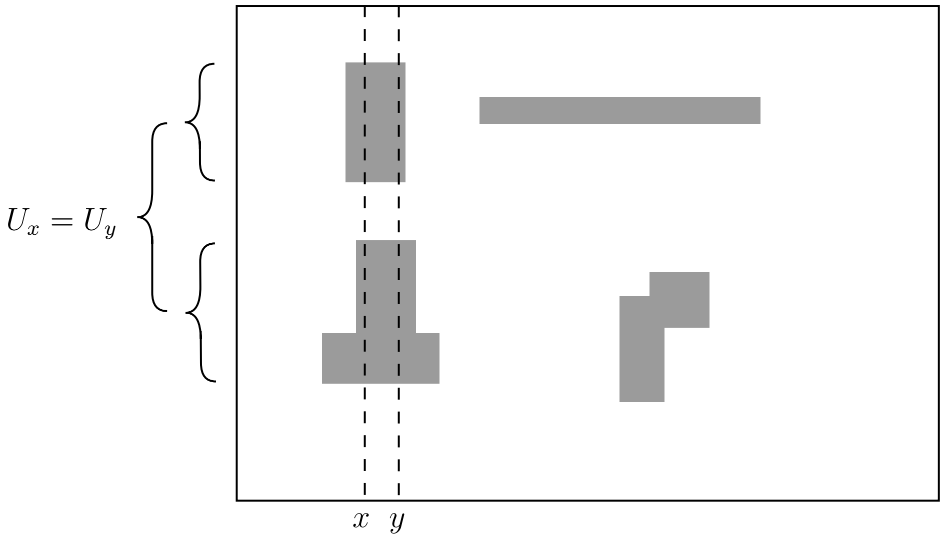

Figure 6.5: A κ-open set $U \subseteq (\kappa^\kappa)^2$ has few slices

Proposition 6.2.41. If κ is inaccessible, then there is no set which is universal for $\kappa\mathbf{\Sigma}^0_3$.

Proof. By Lemma 6.2.6, every Borel$_\kappa$ set is $\kappa\mathbf{\Sigma}^0_3$. So, $\kappa\mathbf{\Sigma}^0_3$ is closed under complements and κ-continuous preimages. Suppose $U \in \kappa^\kappa \times \kappa^\kappa$ were universal for $\kappa\mathbf{\Sigma}^0_3$ sets. Then we diagonalise: let $f : \kappa^\kappa \to \kappa^\kappa \times \kappa^\kappa$ be such that $f(x) = (x, x)$. Let $D = f^{-1}(U)$. As f is clearly κ-continuous, and as $\kappa\mathbf{\Sigma}^0_3$ is closed under κ-continuous preimages, we know that $D \in \kappa\mathbf{\Sigma}^0_3$. As $\kappa\mathbf{\Sigma}^0_3$ is closed under complements, $\kappa^\kappa \setminus D \in \kappa\mathbf{\Sigma}^0_3$. If U is universal for $\kappa\mathbf{\Sigma}^0_3$, then there is x_0 such that $\pi_{x_0}(U) = \kappa^\kappa \setminus D$, i.e. $(x, x) \notin U \iff (x_0, x) \in U$. So, $(x_0, x_0) \notin U \iff (x_0, x_0) \in U$, a contradiction. □

6.2.3 Linearly ordered spaces

The κ-Baire space is universal amongst κ-Polish spaces in the sense that each κ-Polish space is (fully) homeomorphic to a closed subset of κ^κ, however the situation is more complicated for κ-topologies (Proposition 6.2.44). This means we need a local theory of Borel$_\kappa$ sets on other κ-Polish spaces. We do so here on linearly ordered spaces. First, we define κ-Polishness.

Definition 6.2.42 ([1, Definition 2.3], [37, Definition 2.2])**.** Assume X has a basis, $\mathcal{B}$, where $|\mathcal{B}| = \kappa$.

1. We say that X is *strong κ-Choquet* (SC_κ) if player II has a winning strategy for the strong κ-Choquet game on the (full) topology $\langle\mathcal{B}\rangle_{\kappa^+}$.

2. We say that X is *strong fair κ-Choquet* ($f\mathsf{SC}_\kappa$) if player II has a winning strategy for the strong fair κ-Choquet game on the (full) topology $\langle\mathcal{B}\rangle_{\kappa^+}$.

By [1, Theorem 2.21], a κ-topological space, (X, τ_κ), is κ-Polish if $\langle\tau_\kappa\rangle_{\kappa^+}$ is regular, Hausdorff, κ-additive, and $f\mathsf{SC}_\kappa$.

Remark 6.2.43. The properties defined by replacing the full topology by the κ-topology in the definitions of SC_κ is equivalent to being SC_κ, and exactly similarly for $f\mathsf{SC}_\kappa$: we say that a set, A, is *κ-topology strong Choquet* if player II has a winning strategy in the strong κ-Choquet game on A where the open sets must be chosen from the basis. Note that A is SC_κ if and only if player II has a winning strategy in the A-game where both players must choose basic open sets [1, Remark 2.4], [37, Lemma 2.5]. Hence, SC_κ implies κ-topology strong Choquet. The converse is trivial. Likewise for $f\mathsf{SC}_\kappa$.

Proposition 6.2.44. Not every κ-Polish X is κ-homeomorphic to a Borel$_\kappa$ $B \subseteq \kappa^\kappa$.

Proof. Fix a space (X, τ) which is homeomorphic to a closed set in the full topology of κ^κ, but not to any clopen set in the full topology (for example $\{x \in 2^\kappa : x$ has finitely many 0s$\}$).

Suppose that X is κ-homeomorphic to a Borel$_\kappa$ set $B \subseteq \kappa^\kappa$. By Proposition 6.2.24 Part 1., B is clopen in the full topology. A κ-homeomorphism is a homeomorphism. Hence, X is homeomorphic to a clopen set, namely B, in the full topology, a contradiction. □

As in Corollary 6.2.23, we can characterise the Borel$_\kappa$ set of linearly ordered spaces as small unions of suitable components, in this case convex components. This yields a characterisation of the Borel$_\kappa$ sets in analogy to Lemma 6.2.6.

The idea is as follows: if $\mathsf{owei}(X) \leq \kappa$, then the Borel$_\kappa$ sets of the X-interval κ-topology are formed by glueing $<\kappa$-many convex Borel$_\kappa$ sets together. These convex sets fall into three types: the 'open-ended' convex sets are κ-open, the 'closed-ended' are κ-closed, and the 'half-open-ended' are $\kappa\mathbf{\Sigma}^0_2$. So, their unions are $\kappa\mathbf{\Sigma}^0_2$, hence $\mathsf{Bor}_\kappa = \kappa\mathbf{\Sigma}^0_2$. Formally: let $\mathsf{Conv}_\kappa(X) := \{B \subseteq X :$ there is a $\lambda < \kappa$ and pairwise disjoint convex Borel$_\kappa$ $C_\alpha \subseteq X$ such that $B = \bigcup_{\alpha\in\lambda} C_\alpha\}$.

Theorem 6.2.45. Let κ be inaccessible. Let X be a dense linear order such that $Q \subseteq X$ is order dense. Equip X with the Q-interval κ-topology. Then B is Borel$_\kappa$ if and only if $B \in \mathsf{Conv}_\kappa(X)$.

Proof. Clearly $\mathsf{Conv}_\kappa(X) \subseteq \mathsf{Bor}_\kappa(X)$. So, we prove the converse. We show that $\mathsf{Conv}_\kappa(X)$ is a κ-algebra containing the κ-open sets. First, if $B \in \kappa\mathbf{\Sigma}^0_1$, then B is $\bigcup_{\alpha\in\lambda} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for some $\lambda < \kappa$. By construction, each B_α is convex, so $\bigcup_{\alpha\in\lambda} B_\alpha$ is a suitable representation.

Closure under $<\kappa$-unions is immediate: let $B = \bigcup_{\alpha\in\lambda} B_\alpha$ such that $A_\alpha = \bigcup_{\beta\in\lambda_\alpha} B_{\alpha,\beta}$ where $B_{\alpha,\beta}$ are convex and Borel$_\kappa$, then $B = \bigcup_{\alpha\in\sup_{\alpha\in\lambda}\lambda_\alpha} B_\alpha$. By regularity, $\sup_{\alpha\in\lambda}\lambda_\alpha < \kappa$. Note that the components may overlap, but this still constitutes a $<\kappa$ union of convex Borel$_\kappa$ sets.

So, it suffices to show closure under complementation: let $B = \bigcup_{\alpha\in\lambda} C_\alpha$ for convex $C_\alpha \in \mathsf{Bor}_\kappa$ and $\lambda < \kappa$. We prove that $X \setminus B = \bigcup_{\alpha\in\lambda} C'_\alpha$ for some convex $C'_\alpha \in \mathsf{Bor}_\kappa$.

Claim 6.2.46. If $C_\alpha < C_\beta$ are convex and Borel$_\kappa$, then $C[\alpha, \beta] := \{x \in X : C_\alpha < x < C_\beta\}$ is Borel$_\kappa$.

Proof. By assumption, $C_\alpha \cup C_\beta \in \mathsf{Bor}_\kappa$, so $C[\alpha,\beta] \subseteq X \setminus (C_\alpha \cup C_\beta)$. Then let $a \in C_\alpha$ and $b \in C_\beta$. Then $X \setminus (C_\alpha \cup C_\beta \cup (-\infty, a) \cup (b, \infty)) = C[\alpha,\beta]$. So, $C[\alpha,\beta]$ is Borel$_\kappa$. □

Let $\{C'_\alpha : \alpha \in \zeta\}$ be a partition of $X \setminus B$ into maximal convex components. Both $(-\infty, \mathsf{bot}(B)]$ and $[\mathsf{top}(B), \infty)$ are also convex Borel$_\kappa$ (as $\lambda < \kappa$). So, fix a C'_β which is not an initial or final segment. There are four cases describing how the components of C_α may be arranged. In each case, we show that the convex set, C'_β, between the components is Borel$_\kappa$.

1. Suppose that there are $\alpha, \gamma < \lambda$ with $C_\alpha < C_\gamma$, such that $C_\alpha \cup C'_\beta \cup C_\gamma$ is convex, i.e. there is no δ such that $C_\alpha < C_\delta < C_\gamma$. Then $C'_\beta = C[\alpha,\gamma]$, so C'_β is Borel$_\kappa$ by Claim 6.2.46.

2. There is an α such that $C_\alpha < C'_\beta$ and $C_\alpha \cup C'_\beta$ is convex, but no γ such that $C'_\beta < C_\gamma$ and $C'_\beta \cup C_\gamma$ is convex. In which case, there is an infinite[4] (strictly) monotone sequence $(C_{\alpha_\gamma})_{\gamma\in\mu}$ with $\mu \leq \lambda$ such that $(C_{\alpha_\gamma})_{\gamma\in\mu}$ is $<_X$-decreasing, each $C'_\beta < C_{\alpha_\gamma}$, such that no C_{γ_0} is such that for all $\gamma < \mu$, $C_{\alpha_\gamma} < C_{\gamma_0} < C'_\beta$. Without loss of generality, assume $(C_{\alpha_\beta})_{\beta\in\mu}$ is increasing. By assumption, Q is order dense in X, so let $(q_\beta)_{\beta\in\mu}$ be a μ-length sequence approaching $A := \mathsf{bot}(\bigcup_{\beta\in\mu} C_{\alpha_\beta})$ (which is possibly a gap) with elements from Q. Then $\bigcup_{\beta\in\mu}(-\infty, q_\beta) = (\infty, A) \in \kappa\mathbf{\Sigma}^0_1$. So, $[A, \infty) \in \kappa\mathbf{\Pi}^0_1$. Let C_γ be the $<_X$-least such that $C_{\alpha_\beta} < C_\gamma$ for all $\beta \in \mu$, and let $r \in C_\gamma$. Then $\{x \in X : \forall\beta \in \mu(C_{\alpha_\beta} < x < C_\gamma)\} = [A, r) \setminus C_\gamma = C'_\beta$, so C'_β is convex Borel$_\kappa$ as required.

3. There is an γ such that $C'_\beta < C_\gamma$ and $C'_\beta \cup C_\gamma$ is convex, but no α such that $C_\alpha < C'_\beta$ and $C_\alpha \cup C'_\beta$ is convex. But this is exactly dual to Case 2. (i.e. symmetrically, $(C_{\alpha_\gamma})_{\gamma\in\mu}$ is $<_X$-decreasing, each $C_{\alpha_\gamma} > C'_\beta$, such that no C_{γ_0} is such that for all $\gamma < \mu$, $C_{\alpha_\gamma} > C_{\gamma_0} > C'_\beta$).

4. There is neither a γ such that $C'_\beta < C_\gamma$ and $C'_\beta \cup C_\gamma$ is convex, nor α such that $C_\alpha < C'_\beta$ and $C_\alpha \cup C'_\beta$ is convex. This is a combination of Cases 2. and 3. By assumption, $C'_\beta \neq \emptyset$. By Cases 2. and 3., $C'_\beta = [G, G']$ where G, G' are gaps, $\mathsf{cof}(G) < \kappa$, and $\mathsf{coi}(G') < \kappa$, hence C'_β is Borel$_\kappa$.

So, $X \setminus B = (-\infty, \mathsf{bot}(B)] \cup \bigcup_{\beta\in\zeta} C'_\beta \cup [\mathsf{top}(B), \infty)$ is a ζ-union of convex Borel$_\kappa$ sets. Hence, it remains to show that $\zeta < \kappa$. Every C'_β is either between exactly two C_α, or is defined by a (pair of) subsequence(s) of λ. Using this surjection, $\zeta \leq |2^\lambda|$. As κ is a strong limit, $\zeta < \kappa$. So, $X \setminus B \in \mathsf{Conv}_\kappa(X)$. □

If the order topology on X has large weight, then, by checking the endpoints of the Theorem 6.2.45 components, the only convex Borel$_\kappa$ sets are intervals themselves:

Corollary 6.2.47. Let κ be inaccessible. Let X be a dense linear order such that for all (α,β)-gaps in X, $\alpha, \beta \geq \kappa$. Then for any order dense Q, a convex set, $A \subseteq X$, is Borel$_\kappa$ in the Q-interval κ-topology if and only if A is an interval with $Q \cup \{\pm\infty\}$-endpoints.

[4]Finite subsets are obviously separable, as Q is order dense.

Meanwhile, if the weight of X is small then the Borel$_\kappa$ hierarchy collapses even more quickly than in general (as in Lemma 6.2.6).

Theorem 6.2.48. Let κ be inaccessible. Let X be a dense linear order. Let $Q \subseteq X$ be order dense in X. Suppose that, for all (λ, μ)-gaps in X, $\lambda, \mu \leq \kappa$. Let τ_κ^Q be the Q-interval κ-topology. Then $\mathsf{Bor}_\kappa(\tau_\kappa^Q) = \kappa\mathbf{\Sigma}_2^0(\tau_\kappa^Q)$, and so $\mathsf{Bor}_\kappa(\tau_\kappa^Q) = \kappa\mathbf{\Delta}_2^0(\tau_\kappa^Q)$.

Proof. By Theorem 6.2.45, every Borel$_\kappa$ set $B = \bigcup_{\alpha \in \lambda} C_\alpha$, for some convex, Borel$_\kappa$ C_α, and some $\lambda < \kappa$. As $\kappa\mathbf{\Sigma}_2^0$ is closed under $<\kappa$-unions, it suffices to show that all convex Borel$_\kappa$ sets are $\kappa\mathbf{\Sigma}_2^0$.

Recall that for any $q \in Q$, $\{q\}$ is a convex κ-closed set. Hence, if $C \subseteq X$ is a convex $\kappa\mathbf{\Sigma}_\alpha^0$ set such that $C < x$ and there is no $y \in X$ such that $C < y < q$, then $C \cup \{q\}$ is a convex $\kappa\mathbf{\Sigma}^0_{\max\{\alpha,2\}}$ set. So, it suffices to show that all convex Borel$_\kappa$ subsets with 'open' ends are $\kappa\mathbf{\Sigma}_2^0$, formally we prove the following: if $L, R \in \mathsf{Ded}(X) \cup \{\pm\infty\}$, and (L, R) is Borel$_\kappa$, then $(L, R) \cap X$ is $\kappa\mathbf{\Sigma}_2^0$.

If $L, R \in Q \cup \{\pm\infty\}$ then the result is trivial. So, we assume that $L \in \mathsf{Ded}(X) \setminus Q$ (R is similar), i.e. L is a (μ, λ)-gap, by assumption $\mu, \lambda \leq \kappa$. There are three options.

1. If $\mu < \kappa$, then there is a Q-sequence, $(q_\beta)_{\beta \in \mu}$, approaching L from above. Without loss of generality assume $q_\beta < R$ for all $\beta \in \mu$. Hence, $(L, R) = \bigcup_{\beta \in \mu}(q_\beta, R)$.

2. If $\lambda < \kappa$, suppose that $(q'_\beta)_{\beta \in \lambda}$ is a Q-sequence approximating L from below. Then $(L, R) = X \setminus (\bigcup_{\beta \in \lambda}(-\infty, q'_\beta) \cup [R, \infty))$. Note that $[R, \infty) \in \kappa\mathbf{\Pi}_1^0$, so $\bigcup_{\beta \in \lambda}(-\infty, q'_\beta) \cup [R, \infty) \in \kappa\mathbf{\Sigma}_2^0$, so $(L, R) \in \kappa\mathbf{\Sigma}_2^0$.

3. If $\mu = \lambda = \kappa$, we argue by induction that (L, R) is not Borel$_\kappa$. For the base case, suppose $(L, R) \in \kappa\mathbf{\Sigma}_1^0$. Then, as Q is order dense, $(L, R) = \bigcup_{\alpha \in \eta}(a_\alpha, b_\alpha)$ for some $a_\alpha, b_\alpha \in Q$, and some $\eta < \kappa$. As κ is regular, $\mathsf{cof}(\bigcup_{\alpha \in \eta}(a_\alpha, b_\alpha)) \leq \eta < \kappa = \mathsf{cof}((L, R))$, a contradiction.

   So, suppose the hypothesis holds for $\kappa\mathbf{\Sigma}_\alpha^0$. Suppose $\kappa\mathbf{\Pi}_\alpha^0$ contains some (L, R) where one of L, R is a (κ, κ)-gap. Without loss of generality, suppose R is the (κ, κ)-gap. By order density, let $q \in Q \cap (L, R)$. Then $(X \setminus (L, R)) \cap (q, \infty) = (R, \infty)$ is in $\kappa\mathbf{\Sigma}_\alpha^0$. But by assumption, R is a (κ, κ)-gap. This contradicts that $\kappa\mathbf{\Sigma}_\alpha^0$ has no convex sets with (κ, κ)-gaps as endpoints.

   Finally, the $\kappa\mathbf{\Sigma}_\alpha^0$ step is exactly like the base case, by taking an increasing $<\kappa$-union. $\square$

So, for example, the Borel$_\kappa$ sets of $\mathbb{R}_\kappa$ with the $\mathbb{Q}_\kappa$-interval κ-topology are exactly the $\kappa\mathbf{\Delta}_2^0$ sets. Even so, there may be convex sets which are not Borel$_\kappa$ with the interval κ-topology (strengthening the fact that $[\neg\omega, \infty)$ is not κ-open, see Example 4.1.4):

Example 6.2.49. Let κ be inaccessible. There is a (κ, κ)-gap, G, in $\mathbb{R}_\kappa$ (and hence in $\mathbb{Q}_\kappa$). So, (G, ∞) is convex but not Borel$_\kappa$ in the $\mathbb{Q}_\kappa$-interval κ-topology.

Moreover, the $\mathbb{Q}_\kappa$-interval κ-topology Borel_κ hierarchy doesn't even include the intervals with irrational endpoints (Corollary 6.2.50). This is a stark reminder of how different the two κ-topologies which generate the same topology can be.

Corollary 6.2.50. Let κ be inaccessible, and let $a \in \mathbb{R}_\kappa \setminus \mathbb{Q}_\kappa$. Then (a, ∞) is not Borel_κ for the $\mathbb{Q}_\kappa$-interval κ-topology on $\mathbb{R}_\kappa$.

Proof. Let $\tau_\kappa^{\mathbb{Q}_\kappa}$ be the $\mathbb{Q}_\kappa$-interval κ-topology on $\mathbb{R}_\kappa$. Let τ' be the $\mathbb{Q}_\kappa$-interval κ-topology on $\mathbb{Q}_\kappa$. We define a bijection $\phi : \tau' \to \tau_\kappa^{\mathbb{Q}_\kappa}$ like so: $\phi((p, q)) = (p, q)$. Extend ϕ to $\Phi : \mathsf{Bor}_\kappa(\tau') \to \mathsf{Bor}_\kappa(\tau_\kappa^{\mathbb{Q}_\kappa})$ in the natural way. Let $a \in \mathbb{R}_\kappa \setminus \mathbb{Q}_\kappa$. Note that $\Phi^{-1}(a, \infty)$ is a convex subset of $\mathbb{Q}_\kappa$ which starts with a gap. By Theorems 6.2.45 and 6.2.48, every convex Borel_κ subset which starts with a gap must start with a (λ, μ)-gap with at least one of $\mu, \lambda < \kappa$. As $\mathsf{bn}(\mathbb{R}_\kappa) = \kappa$, if $a \in \mathbb{R}_\kappa \setminus \mathbb{Q}_\kappa$, then $\text{cof}((-\infty, a)) = \text{coi}((a, \infty) = \kappa$. So, a is a (κ, κ)-gap in $\mathbb{Q}_\kappa$, so (a, ∞) is not Borel_κ. $\square$

Hence, $\{x\} \neq \bigcap_{\alpha \in \lambda} U_\alpha$ for any $\lambda < \kappa$ and κ-open U_α, so $\mathbb{R}_\kappa$ does not satisfy the κ-analogue of being a G_δ space (defined as in [56, 1.5H]). However, as we define $\kappa\mathbf{\Pi}_2^0$ as the $<\kappa$-unions of both $\kappa\mathbf{\Sigma}_1^0$ and $\kappa\mathbf{\Delta}_1^0$, $\{x\}$ is a $\kappa\mathbf{\Pi}_2^0$ set.

More generally, by 'filling in' the small gaps in $\mathbb{R}_\kappa$ (i.e. let $P := \{p \in \mathsf{Ded}(\mathbb{R}_\kappa) : \alpha \in \lambda$ and p is an (α, κ)-gap or (κ, α)-gap$\}$, and let $X := P \cup \mathbb{R}_\kappa$), we can construct an order which has no small gaps which is $f\mathsf{SC}_\kappa$, but need not be κ-Polish (as it is not necessarily κ-additive).

6.3 Analytic sets

Moving up the complexity hierarchy, we have the κ-analogues of analyticity on κ^κ and linearly ordered spaces. On ω^ω, there are many classically equivalent definitions of analyticity, including projections, continuous images, and Borel images of, variously, ω^ω, closed sets, and Borel sets. On κ^κ, with the full topology and Borel hierarchy, not all of these are equivalent [135, Theorem 1.5]. The situation is more complex still for the κ-analogues of analyticity. Definitions using projections are trivial: these are identical to the Borel_κ sets (Section 6.3.1). We show that the other candidates yield a substantive theory, and that the natural generalisations form a strict order. We begin by showing that projections do not in general increase complexity, vindicating Lebesgue's famous mistake [161, page 2]. We then state some general facts about the graphs of functions on κ^κ. Then, we define the notions of κ-analyticity, construct strict and bianalytic sets for these notions, and describe their relation to the full topology.

6.3.1 Projections of κ-closed and Borel_κ sets

Under standard conditions, the projections of κ-closed sets on a product $X \times Y$ is still κ-closed in X.

Proposition 6.3.1. If κ is inaccessible and τ_κ is a κ-additive κ-topology on X, then the Borel_κ sets on $X \times \kappa^\kappa$ are of the form $\bigcup_{\alpha \in \lambda}(R_\alpha \setminus \bigcup_{\beta \in \mu_\alpha} R_\beta^\alpha)$ where R_α, R_β^α are κ-open in $X \times \kappa^\kappa$ and $\lambda, \mu_\alpha < \kappa$.

Proof. By Lemma 6.2.6, every Borel$_\kappa$ subset of $X \times \kappa^\kappa$ is at most complexity $\kappa\mathbf{\Sigma}^0_3$, i.e. $\bigcup_{\alpha\in\nu}\bigcap_{\beta\in\nu_\alpha} S_{\alpha,\beta}$ where $S_{\alpha,\beta}$ are either κ-open or κ-closed in $X \times \kappa^\kappa$, and $\nu, \nu_\alpha < \lambda$. Fix a Borel$_\kappa$ B. Clearly if $S_{\alpha,\beta}$ are κ-closed, then $\bigcap_{\beta\in\nu_\alpha} S_{\alpha,\beta}$ is also κ-closed. Likewise, by κ-additivity, if $S_{\alpha,\beta}$ are κ-open, then $\bigcap_{\beta\in\nu_\alpha} C_{\alpha,\beta}$ is κ-open. So, $B = \bigcup_{\beta\in\nu'}(O_\beta \cap C_\beta)$ where O_β is κ-open, C_β is κ-closed, and $\nu' < \kappa$. In other words, $B = \bigcup_{\beta\in\nu'}(O_\beta \setminus (X\times\kappa^\kappa\setminus C_\beta))$. Finally, by Proposition 6.1.3, we can assume that O_β are rectangles R_β. □

So, the Borel$_\kappa$ subsets of $X \times \kappa^\kappa$ are of the form $\bigcup_{\alpha\in\lambda}(R_\alpha \setminus O_\alpha)$ where O_α is κ-open in $X \times \kappa^\kappa$. We now show that projections of κ-closed and Borel$_\kappa$ sets remain κ-closed and Borel$_\kappa$ respectively.

Proposition 6.3.2. Let κ be inaccessible and Y be κ-additive. Then, for any space X, and for any set $C \subseteq Y \times X$ which is κ-closed in $Y \times X$, $\pi_1(C)$ is κ-closed in Y.

Conversely, if D is κ-closed in Y, there is a C which is κ-closed in $Y \times X$ such that $\pi_1(C) = D$.

Proof. As κ is regular, each κ-closed set $C \subseteq Y \times X$ can be represented as $Y \times X$ with $\lambda < \kappa$-many κ-open rectangles, F_β, removed. Then $\pi(C) = Y \setminus \bigcup_{\alpha\in\mu} S_\alpha$, where $\mu < \kappa$ and $S_\alpha = \bigcap_{F_\beta\in R_\alpha}\pi(F_\beta)$, where R_α is some collection of removed rectangles such that the second coordinate covers X. There may be several such collections S_α, but crucially only $<\kappa$-many at most. Without loss of generality, we can assume that $F_\beta = U_\beta \times V_\beta$ where U_β is open in Y and V_β is open in Y. So, $\pi_1(F_\beta) = U_\beta$. So, $\pi(C) = Y \setminus \bigcup_{S\in\mathscr{P}(\lambda)}\bigcap_{\beta\in S}\pi(F_\beta) = Y \setminus \bigcup_{S\in\mathscr{P}(\lambda)}\bigcap_{\beta\in S} U_\beta$ for some Y_β κ-open in X. By assumption, Y is κ-additive, so $\bigcap_{\beta\in S} U_\beta$ is κ-open. But as κ is inaccessible, $|\mathscr{P}(\lambda)| < \kappa$, so $\bigcup_{S\in\mathscr{P}(\lambda)}\bigcap_{\beta\in S} U_\beta$ is κ-open. Hence, $\pi(C)$ is κ-closed.

Conversely, let D be κ-closed in Y. Then $\pi_1(C \times X) = C$. It is simple to check that $C \times X$ is κ-closed. □

We have a corresponding result for the projection of Borel$_\kappa$ sets, irrespective of the κ-additivity of Y:

Proposition 6.3.3.

1. Assume κ is inaccessible. Let $C \in X \times Y$ be Borel$_\kappa$. Then $\pi(C)$ is Borel$_\kappa$ in X.

2. Conversely, if B is Borel$_\kappa$ in X there is set $D \subseteq X \times Y$ which is Borel$_\kappa$ such that $\pi_1(D) = B$.

Proof. For Part 1., by Lemma 6.2.6, as κ is inaccessible, each Borel$_\kappa$ set in $X \times Y$ consists of μ-many (possibly nested) κ-open rectangles, $E \in \mathcal{E}$, with λ-many κ-open rectangles, F_β, removed, for some $\lambda, \mu < \kappa$. Fix such an E. This $E = U \times V$ for U, V κ-open in X. The contribution of E to $\pi(B)$ consists of $U \setminus \bigcup_{\alpha\in\lambda'} S_\alpha$, where $\lambda' < \kappa$ and $S_\alpha = \bigcap_{F_\beta\in R_i}\pi(F_\beta)$, where R_α is some collection of removed rectangles such that the second coordinate covers V of E. There may be several such collections S_α, but crucially only $<\kappa$-many at most. So, $\pi(E) = U \setminus \bigcup_{S\in\mathscr{P}(\lambda)}\bigcap_{\beta\in S}\pi(F_\beta) = U \setminus \bigcup_{S\in\mathscr{P}(\lambda)}\bigcap_{\beta\in S} I_\beta$ for some I_β κ-open in κ^κ.

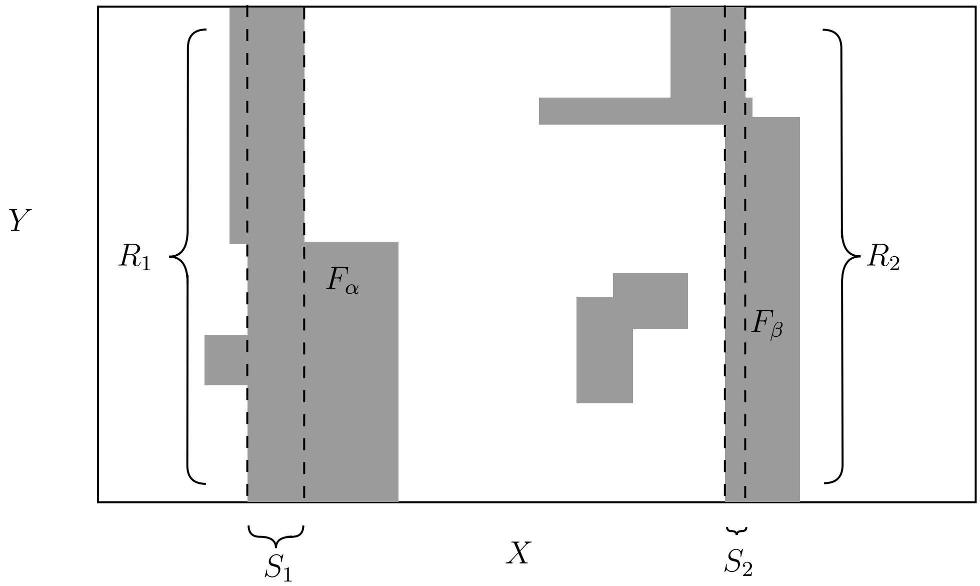

Figure 6.6: Projections of κ-closed sets

But as κ is inaccessible, $|\mathscr{P}(\lambda)| < \kappa$, so $\pi(E)$ is Borel$_\kappa$. Then $\pi(B) = \pi(\bigcup_{E \in \mathcal{E}} E)$, so $\pi(B)$ is a $\mu < \kappa$ union of Borel$_\kappa$ sets, hence is Borel$_\kappa$.

For Part 2., again check that $B \times Y$ is Borel$_\kappa$, and that $\pi_1(B \times Y) = B$. □

Corollary 6.3.4. If κ is inaccessible, either:

1. Y is κ-additive, in which case $C \subseteq X$ is κ-closed if and only if $\pi_1(D) = C$ for some κ-closed set D in $Y \times X$, or

2. Y is not κ-additive, in which case $\pi_1(D) = C$ for some D which is κ-closed in $Y \times X$ if and only if $C = X \setminus \bigcap_{\alpha \in \lambda} O_\alpha$ where O_α are κ-open and $\lambda < \kappa$.

Proof. The first case is precisely Proposition 6.3.2.

The second case resembles Proposition 6.3.2: $\pi(C) = X \setminus \bigcup_{\alpha \in \mu} \bigcap_{\beta \in \lambda} O_{\alpha,j}$ for $\mu, \lambda < \kappa$, so, by inaccessibility, $\pi(C) = X \setminus \bigcap_{\alpha' \in S} \bigcup_{\beta' \in T} O_{\alpha',\beta'}$ for $|S|, |T| < \kappa$ and some relabelling of the $O_{\alpha,\beta}$. So, $\pi(C) = X \setminus \bigcap_{\gamma \in \nu} O_\gamma$ for some κ-open O_γ and $\nu < \kappa$. □

So, for inaccessible κ, Lebesgue was right! 'Projection' as a function on Borel$_\kappa$ sets does not increase descriptive complexity (unlike the classical case, [161, page 2]). So, we focus on other notions of κ-analyticity, which do increase complexity (Definition 6.3.6).

6.3.2 Notions of analyticity

Section 6.3.1 trivialises the generalisations of analyticity based on projections. This leaves three remaining distinct notions, which can be ordered by inclusion. For this, we first define Borel$_\kappa$ functions:

Definition 6.3.5. Let τ_κ and σ_κ be κ-topologies on X and Y respectively. We say that $f : X \to Y$ is *Borel$_\kappa$* if for every $U \in \sigma_\kappa$ is such that $f^{-1}(U) \in \mathsf{Bor}_\kappa(\tau_\kappa)$.

Definition 6.3.6. Let X, Y be κ-Polish spaces with respective κ-topologies. A set $A \subseteq Y$ is called:

(a) *strongly X-analytic* if $A = f(X)$ where $f : X \to Y$ is κ-continuous,

(b) *intermediately X-analytic* if $A = f(B)$ where $B \subseteq X$ is Borel_κ and $f : B \to Y$ is κ-continuous (in the subspace κ-topology on B), and

(c) *weakly X-analytic* if $A = f(X)$ where $f : X \to Y$ is a Borel_κ function.

We write *weakly analytic* for weakly κ^κ-analytic, and so forth.

As in [135, page 3], we are also interested in injective κ-continuous and Borel_κ images. We say that A is *weakly injective-analytic* if it is witnessed by an injective κ-continuous function, likewise for intermediately injective-analytic and strongly injective-analytic.

Obviously, every strongly injective-analytic set is strongly analytic (and so forth), and every Borel_κ set is intermediately X-analytic (with the identity). As a concrete example, for all $s \in \kappa^{<\kappa}$, N_s is strongly analytic, as witnessed by $f(x) = s^\frown x$.

Proposition 6.3.7. Let X be a κ-Polish space with a κ-topology. Then, strong X-analyticity implies intermediate X-analyticity, which implies weak X-analyticity.

Proof. The implications from strongly X-analytic to intermediately X-analytic is trivial, so it suffices to prove intermediately X-analytic implies weakly X-analytic. Suppose h is κ-continuous, $B \subseteq X$ is Borel_κ, and $A = h(B) \neq \emptyset$. By assumption, there is a $y \in A$. Let g be defined like so:

$$g(x) = \begin{cases} x & \text{if } x \in B, \\ y & \text{if } x \in X \setminus B. \end{cases}$$

Let U be a basic open set in X, so $U \cup X \setminus B$ is Borel_κ. Note that $g^{-1}(U) = U \cap B$ or $g^{-1}(U) = U \cup X \setminus B$. So, g is a Borel_κ function. Then $h \circ g : X \to X$ is a Borel_κ function. Moreover, $(h \circ g)(X) = A$. □

Corollary 6.3.8. Let X be a κ-Polish space with a κ-topology. For a set $A \subseteq X$, the following are equivalent:

1. A is weakly X-analytic,

2. $A = f(C)$ where $f : X \to Y$ is a Borel_κ function (in the subspace κ-topology on C) and $C \subseteq Y$ is κ-closed, and

3. $A = f(B)$ where $f : B \to X$ is Borel_κ (in the subspace κ-topology on B) and where $B \subseteq Y$ is Borel_κ.

Proof. It suffices to prove that Case 3. implies Case 1. An exactly similar argument to the proof of Proposition 6.3.7 suffices. □

However, κ-continuous functions do not exhibit generalisations of all classical behaviour, in particular their graphs need not be Borel$_\kappa$. This prohibits the generalisation of the proof of the equivalence in Proposition 2.1.13.

Example 6.3.9. Let κ be inaccessible, and let $f((2\alpha)^\frown x) = f((2\alpha+1)^\frown x) = (2\alpha)^\frown x$. Then f is κ-continuous, but $\mathsf{Graph}(f)$ is not Borel$_\kappa$, as it cannot be represented as a union of $<\kappa$-many differences of κ-opens (use Corollary 6.2.25).

Example 6.3.10. Let κ be inaccessible. Let $D := \{(x,x) : x \in \kappa^\kappa\}$, i.e. $\mathsf{Graph}(\mathrm{id})$. Then D is not Borel$_\kappa$.

Example 6.3.11. Let κ be inaccessible. Let $c_x : \kappa^\kappa \to \kappa^\kappa$ be the constant x-function. Clearly, c_x is κ-continuous. Its graph, $\mathsf{Graph}(c_x) = \kappa^\kappa \times \{x\}$, is not Borel$_\kappa$: if it were, then by Proposition 6.3.3, $\{x\}$ would be Borel$_\kappa$, contradicting Corollary 6.2.30.

Example 6.3.12. Let κ be inaccessible. Fix a constant $c \in 2^\kappa$. We define $f : 2^\kappa \to 2^\kappa$ as

$$f(x) := \begin{cases} c & \text{if } x = 0^\frown y \text{ for some } y \in 2^\kappa \\ y & \text{if } x = 1^\frown y \text{ for some } y \in 2^\kappa. \end{cases}$$

Then $f(N_0) = \{c\}$, which is not open in $2^\kappa = f(2^\kappa)$. Hence, f is κ-continuous but not an open map.

So, if κ is inaccessible, then there are κ-continuous functions $f, g : \kappa^\kappa \to \kappa^\kappa$ such that $\mathsf{Graph}(f)$ is not Borel$_\kappa$ in $(\kappa^\kappa)^2$ and g is not an open map in the full topology. Hence, the Borel Graph Theorem is not κ-topologically generalisable. Indeed, later, we show that the notions of analyticity can be distinguished on κ^κ, and that at least two can be distinguished on $\mathbb{Q}_\kappa$ (Corollary 6.3.51).

6.3.3 κ-Baire space

We now focus on κ^κ and our three notions of analyticity (i.e. of κ^κ-analyticity). By Section 6.3.2, we know that classical equivalences of notions of analyticity may fail. In fact, on κ^κ, we can prove an equivalence of strong analyticity (Corollary 6.3.14), which can be compared to Proposition 6.3.49, which shows that the κ-continuous images of $\kappa\mathbf{\Sigma}^0_2(\mathbb{R}_\kappa)$ sets are exactly intermediately $\mathbb{R}_\kappa$-analytic sets.

Proposition 6.3.13. Every κ-closed set is the κ-continuous retract of κ^κ, and hence strongly analytic.

Proof. Let $C = \kappa^\kappa \setminus \bigcup_{\alpha\in\lambda} N_{s_\alpha}$, and let $N_s \subseteq C$. We define f like so:

$$f(x) = \begin{cases} s^\frown y & \text{if } x = s_\alpha{}^\frown y \text{ for some } \alpha \in \lambda, y \in \kappa^\kappa, \\ x & \text{if } x \in C. \end{cases}$$

Then $f(\kappa^\kappa) = C$, and f is clear a κ-continuous retraction. $\square$

Corollary 6.3.14. A set $A \subseteq \kappa^\kappa$ is strongly analytic if and only if $A = f(C)$ where $C \subseteq \kappa^\kappa$ is κ-closed and $f : C \to \kappa^\kappa$ is κ-continuous (in the subspace κ-topology on C).

Proof. Let C be κ-closed, $g : C \to \kappa^\kappa$ be κ-continuous (in the subspace κ-topology), and $A = g(C)$. Let $f : \kappa^\kappa \to C$ be a κ-continuous retract. Then $(f \circ g)(\kappa^\kappa) = A$ and $f \circ g$ is κ-continuous. □

Singletons are strongly analytic, using the constant map c_x, and hence weakly and intermediately analytic, but not Borel$_\kappa$ (by Corollary 6.2.30). In contrast, sets of intermediate cardinality are not strongly analytic:

Proposition 6.3.15. If $A \subseteq \kappa^\kappa$ is such that $1 < |A| < \kappa$, then A is not strongly analytic.

Proof. Suppose $f(\kappa^\kappa) = A$ for a κ-continuous f. All the points of $a \in A$ are isolated i.e. there are non-overlapping N_{s_a} such that $a \in N_{s_a}$, so the $f^{-1}(N_{s_a})$ form a κ-open partition of κ^κ of size $|A| < \kappa$, each of which is non-empty, so contains a cone. But this contradicts the fact that κ^κ is κ-connected. □

Proposition 6.3.16. If $A \subseteq \kappa^\kappa$ is such that $|A| < \kappa$, then A is intermediately analytic.

Proof. Let $A = (x_\alpha)_{\alpha\in\lambda}$ be enumeration with $\lambda < \kappa$, and choose a $y \notin A$. Define f like so:

$$f(x) = \begin{cases} x_\alpha & \text{if } x \in N_\alpha, \\ y & \text{otherwise.} \end{cases}$$

So, for any $s \in \kappa^{<\kappa}$, $f^{-1}(N_s)$ is a union of $<\kappa$-many κ-open sets (in particular $f^{-1}(N_s) = N_s \cup \bigcup_{\alpha\in S} N_\alpha$ where $S \subseteq \lambda$). So, f is κ-continuous. Then $B := \bigcup_{\alpha\in\lambda} N_\alpha$ is Borel$_\kappa$ (indeed, B is κ-open) and $f(B) = A$, as required. □

Corollary 6.3.17. The strongly analytic sets are a strict subset of the intermediately analytic sets.

However, there are some strongly analytic sets of size κ: let $C := \{\overline{\alpha}_\kappa : \alpha \in \kappa\}$, consisting of the constant sequences. Then C is strongly analytic (using $f : \alpha^\frown x \to \overline{\alpha}_\kappa$), but not Borel$_\kappa$. If C were Borel$_\kappa$, $C \cap N_0 = \{\overline{0}_\kappa\}$ is Borel$_\kappa$, contradicting Corollary 6.2.30.

Bianalytic, strict analytics, and Suslin's theorem

In this section, we show that the analogue of Suslin's theorem fails for κ-analytic and Borel$_\kappa$ sets, but that the weakly and intermediately analytic sets are closed under $<\kappa$-unions.

As in the classical situation, a set A is called *weakly bianalytic* if A and $\kappa^\kappa \setminus A$ are weakly analytic, and A is called *strictly weakly analytic* if it is weakly analytic and not weakly bianalytic. We likewise define the equivalent notions for strong and intermediate analyticity. In this section, we prove the following failure of Suslin's theorem:

Theorem 6.3.18. The following proper inclusions hold:

1. $\mathsf{Bor}_\kappa \subsetneq \{A \subseteq \kappa^\kappa : A \text{ is weakly bianalytic}\}$,
2. $\mathsf{Bor}_\kappa \subsetneq \{A \subseteq \kappa^\kappa : A \text{ is intermediately bianalytic}\}$, and
3. $\mathsf{Bor}_\kappa \not\subseteq \{A \subseteq \kappa^\kappa : A \text{ is strongly analytic}\}$ and $\mathsf{Bor}_\kappa \not\subseteq \{A \subseteq \kappa^\kappa : \kappa^\kappa \setminus A \text{ is strongly analytic}\}$.

For this, we prove the following lemma:

Lemma 6.3.19. Let $\kappa = \kappa^{<\kappa}$.

1. There are 2^κ-many κ-continuous κ-open functions.
2. There are 2^κ-many strongly bianalytic sets which are not Borel$_\kappa$.

Proof. First, for each $z \in 2^\kappa$, we define $f_z : \kappa^\kappa \to \kappa^\kappa$ like so:

$$f_z(x) = \begin{cases} x & \text{if } z(\alpha) = 1, y \in \kappa^\kappa, \text{and } x = (2\alpha)^\frown y, \\ (2\alpha)^\frown y & \text{if } z(\alpha) = 1, y \in \kappa^\kappa, \text{and } x = (2\alpha+1)^\frown y, \\ (2\alpha+1)^\frown y & \text{if } z(\alpha) = 0, y \in \kappa^\kappa, \text{and } x = (2\alpha)^\frown y, \\ x & \text{if } z(\alpha) = 0, y \in \kappa^\kappa, \text{and } x = (2\alpha+1)^\frown y. \end{cases}$$

Let $s \in \kappa^{<\kappa}$. Then $f_z(N_s) = N_t$ and $(f_z)^{-1}(N_s) = N_r \cup N_u$ for some $t, r, u \in \kappa^{<\kappa}$. Hence, f_z is κ-continuous and κ-open.

For Part 1., each $z \in 2^\kappa$ defines a distinct map, f_z, so there are at least 2^κ-many such maps. There are only 2^κ-many continuous functions, and every κ-continuous function is continuous, hence there are exactly 2^κ-many such maps.

For Part 2., there are only κ-many Borel$_\kappa$ sets, but $f_z(\kappa^\kappa)$ is distinct for each $z \in 2^\kappa$, so 2^κ-many of these sets are not Borel$_\kappa$. Moreover, for each $z \in 2^\kappa$ $f_z(\kappa^\kappa)$, there is a $z' \in 2^\kappa$ such that $f_z(\kappa^\kappa) = \kappa^\kappa \setminus f_{z'}(\kappa^\kappa)$. So, $f_z(\kappa^\kappa)$ is strongly bianalytic. □

This immediately implies that Suslin's theorem fails (i.e. Theorem 6.3.18 holds). For every $z, z' \in 2^\kappa$, $f_z(\kappa^\kappa)$ to $f_{z'}(\kappa^\kappa)$ are κ-homeomorphic: the map which swaps at every coordinate that differs between z, z' is a κ-homeomorphism. Using the argument from Corollary 6.2.25, we can identify some of the $f_z(\kappa^\kappa)$ which are not Borel$_\kappa$:

Proposition 6.3.20. Let κ be inaccessible, and let $z \in 2^\kappa$ be such that $|\{\alpha \in \kappa : z(\alpha) = 0\}| = |\{\alpha \in \kappa : z(\alpha) = 1\}|$. Then $f_z(\kappa^\kappa)$ is not Borel$_\kappa$.

Singletons are also κ-analytic and non-Borel$_\kappa$. For cardinality reasons, a singleton is neither weakly injective-analytic nor intermediately injective-analytic. So, unlike the classical case, the weakly injective-analytic sets are a proper subset of the weakly analytic sets. There are also non-Borel$_\kappa$ injective-analytic sets:

Proposition 6.3.21. If κ is inaccessible, then there is a non-Borel$_\kappa$ set A which is intermediately injective-analytic.

Proof. The set $A := \bigcup_{\alpha\in\kappa} N_{(2\alpha)}$ from Corollary 6.2.25 suffices: let $\phi : \kappa \to \{2\alpha : \alpha \in \kappa\}$ be a bijection. Let $f(\alpha^\frown x) = \phi(\alpha)^\frown x$. Then f is injective and κ-continuous. □

Despite the failure of Suslin's theorem, the bianalytic pointclasses exhibit some nice closure properties, in analogy with the Borel$_\kappa$ sets:

Proposition 6.3.22. If κ is inaccessible, then the intermediately analytic sets and the weakly analytic sets are closed under $<\kappa$-unions.

Proof. For the weak case, suppose $A_\alpha = g_\alpha(\kappa^\kappa)$ where g_α are Borel$_\kappa$ maps for some $\alpha < \lambda$ with $\lambda < \kappa$. Fix an $\alpha < \lambda$ and some $y_0 \in A_\alpha$. Let $g : \kappa^\kappa \to \kappa^\kappa$ be defined like so:

$$g(x) = \begin{cases} g_\alpha(y) & \text{if } x = \alpha^\frown y \text{ for some } \alpha < \lambda, y \in \kappa^\kappa, \\ y_0 & \text{otherwise.} \end{cases}$$

This g is Borel$_\kappa$, as $g^{-1}(N_{\beta^\frown s}) = g_\alpha^{-1}(N_s)$ (if $\beta < \lambda$), otherwise $g^{-1}(N_{\beta^\frown s}) = \kappa^\kappa \setminus \bigcup_{\alpha\in\lambda} g_\alpha^{-1}(\kappa^\kappa)$ (if $s(0) \geq \lambda$), both of which are Borel$_\kappa$ by the assumptions that g_α are all Borel$_\kappa$ and $\lambda < \kappa$. Clearly, $A = g(\kappa^\kappa)$.

For the intermediate case, let $A = \bigcup_{\alpha\in\lambda} A_\alpha$ where A_α are intermediately analytic for some $\lambda < \kappa$. So, for each $\alpha < \lambda$, $A_\alpha = f_\alpha(B_\alpha)$ for some κ-continuous map, f_α, and Borel$_\kappa$ set, B_α. Let $B := \bigcup_{\alpha\in\lambda} \alpha^\frown B_\alpha$. Clearly, B is Borel$_\kappa$. Let $f : \kappa^\kappa \to \kappa^\kappa$ be defined like so:

$$f(x) = f_\alpha(y) \text{ if } x = \alpha^\frown y \text{ for some } \alpha < \lambda, y \in \kappa^\kappa.$$

Obviously, $f(B) = \bigcup_{\alpha\in\lambda} B_\alpha$. Moreover, f is κ-continuous, as for any cone, N_s, $f^{-1}(N_s) = \bigcup_{\alpha\in I} f_\alpha^{-1}(N_s)$ for some $I \subseteq \lambda$, and $f_\alpha^{-1}(N_s)$ is κ-open. □

The strong analyticity analogue of Proposition 6.3.22 fails, by Proposition 6.3.15. A much more substantial pathology of strong analyticity is that it does not even contain all κ-open sets:

Proposition 6.3.23. A κ-open set, A, is strongly analytic if and only if $A = N_s$ for some $s \in \kappa^{<\kappa}$. Hence, not all κ-open sets are strongly analytic.

Proof. For each $s \in \kappa^{<\kappa}$, $f_s(x) = s^\frown x$ is a Borel$_\kappa$ function, and $f_s(\kappa^\kappa) = N_s$. Conversely, by Theorem 6.1.1, if A is κ-connected and f_s is Borel$_\kappa$ then $f_s(A)$ is κ-connected. By Lemma 6.2.22, κ^κ is κ-connected, so any strongly analytic set is κ-connected. But the only κ-connected κ-open sets are the cones (as in Lemma 6.2.22). For the final part, let $s, t \in \kappa^{<\kappa}$ be such that $s \perp t$. Then $N_s \sqcup N_t$ will do. □

The full topology and analyticity

So far, we have proved that the Borel$_\kappa$ sets are clopen in the full topology, and that the κ-analytics are strictly richer. In this section, we are more precise: we show that the intermediately analytic sets are of low complexity in the full topology, whilst the weakly analytic sets are of slightly higher complexity.

Proposition 6.3.24. Assume κ is weakly compact. If A is intermediately analytic, then A is closed in the full topology. Hence, intermediately bianalytic sets are clopen in the full topology.

Proof. Let $A = f(B)$ for a κ-continuous f, and a Borel$_\kappa$ B. Fix an $x \in \mathsf{cl}(A)$, it suffices to prove that $x \in A$. For all $\alpha < \kappa$, let $B_\alpha := f^{-1}(N_{x\restriction\alpha}) \cap B$. As $x \in \mathsf{cl}(A)$, for each α, $B_\alpha \neq \emptyset$. If $x \in A$, then we are done.

Otherwise, suppose $\bigcap_{\alpha\in\kappa} B_\alpha = \emptyset$. By Lemma 6.2.12, for each $\alpha < \kappa$, there is an optimal conelike partition, $\mathcal{P}(B_\alpha)$ and $\mathcal{F}(B_\alpha)$, for B_α. Let $P_s^{B_\alpha}$ be the conelike elements of $\mathcal{P}(B_\alpha)$ for $s \in \mathcal{F}(B_\alpha)$. Likewise, let $\mathcal{P}(B)$ and $\mathcal{F}(B)$ be the optimal conelike partition of B, and let P_s^B be the conelike elements of $\mathcal{P}(B)$ for $s \in \mathcal{F}(B)$.

Fix an $\alpha < \kappa$ and an $s \in \mathcal{F}(B_\alpha)$. As f is κ-continuous, $f^{-1}(N_{x\restriction\alpha})$ is a κ-open set, i.e. a $<\kappa$-union of cones. But B_α is the intersection of B with these cones. So, if $P_s^{B_\alpha} \subsetneq N_s$, then $s \in \mathcal{F}(B)$ and $P_s^{B_\alpha} = P_s^B$. Hence, there are two possible cases:

1. $s \notin \mathcal{F}(B)$ and $P_s^{B_\alpha} = N_s$, or
2. $s \in \mathcal{F}(B)$ and $P_s^{B_\alpha} = P_s^B$.

Next, we prove that $\bigcap_{\alpha\in\kappa} B_\alpha = \emptyset$. Suppose, for a contradiction, that $\bigcap_{\alpha\in\kappa} B_\alpha$ is not-empty. By Lemma 6.2.16, there is an $s \in \bigcap_{\gamma<\alpha<\kappa} \mathcal{F}(B_\alpha)$. So, there are two corresponding cases:

1. $s \in \mathcal{F}(B)$, in which case, for all $\alpha > \gamma$, $P_s^{B_\alpha} = P_s^B$. As $\beta < \gamma$ implies $B_\beta \subseteq B_\gamma$, we have that $P_s^B \subseteq \bigcap_{\alpha\in\kappa} B_\alpha$ and thus $\bigcap_{\alpha\in\kappa} B_\alpha$ is non-empty, a contradiction.
2. $s \notin \mathcal{F}(B)$, in which case, for all $\alpha > \gamma$, $P_s^{B_\alpha} = N_s$. Thus $N_s \subseteq \bigcap_{\alpha\in\kappa} B_\alpha$, again contradiction.

So, $\bigcap_{\alpha\in\kappa} B_\alpha$ is non-empty. So, there is $z \in \bigcap_{\alpha\in\kappa} B_\alpha$. By definition ($B_\alpha \subseteq f^{-1}(N_{x\restriction\alpha})$) is mapped to $f(z) = x$. So, $x \in A$. □

This implies that there are strictly strongly analytic sets, and strictly intermediately analytic sets:

Corollary 6.3.25. Suppose κ is weakly compact, and $x \in \kappa^\kappa$. Then:

1. $\kappa^\kappa \setminus \{x\}$ is not intermediately analytic, so $\{x\}$ is strictly intermediately analytic, , and
2. $\kappa^\kappa \setminus \{x\}$ is not strongly analytic, so $\{x\}$ is strictly strongly analytic.

Proof. It suffices to prove Part 1. This follows immediately from Proposition 6.3.24, as $\kappa^\kappa \setminus \{x\}$ is not closed in the full topology. □

For a concrete example, let the *doubling map* $d : \kappa^\kappa \to \kappa^\kappa$ be: $d(x) = y_x$ where $y_x(2\alpha) = y_x(2\alpha + 1) = x(\alpha)$, i.e. $d(x) = (x(0), x(0), x(1), x(1), ...)$: if A is intermediately analytic, $d(A)$ is intermediately analytic (as d is κ-continuous) but $d(A)$ is not intermediately coanalytic (as $N_s \not\subseteq d(A)$ for all $s \in \kappa^{<\kappa}$, then by Proposition 6.3.24).

Conversely, singletons are weakly bianalytic.

Proposition 6.3.26. Suppose κ is inaccessible. Then $\kappa^\kappa \setminus \{x\}$ is weakly analytic, so $\{x\}$ is weakly bianalytic. Moreover, $\kappa^\kappa \setminus \{x\}$ is weakly injective-analytic.

Proof. For simplicity, let $x = \overline{0}_\kappa$. Let for any $\beta > 0$ and any α, let $g : \alpha^\frown\beta^\frown y \mapsto \overline{0}_\alpha{}^\frown(1+\beta)^\frown y$. Then g is a Borel$_\kappa$ map, because $g^{-1}(N_s)$ is either a cone, or is a κ-closed set. Finally, $g(\kappa^\kappa) = \kappa^\kappa \setminus \{\overline{0}_\kappa\}$. For the last part, note that g is injective. □

It is natural to ask whether the class of weakly analytic sets is in fact closed under complementation.

Intermediate analyticity is further stymied in that it does not include all the sets which are closed in the full topology:

Proposition 6.3.27. If κ is inaccessible, then there is a set $A \subsetneq \kappa^\kappa$ which is closed in the full topology, weakly analytic, but not intermediately analytic.

Proof. Let $C = \{\overline{0}_\alpha{}^\frown 1^\frown \overline{0}_\kappa : \alpha \in \kappa\} \cup \{\overline{0}_\kappa\}$. The only limit point of C is $\overline{0}_\kappa$, so C is closed. This C is weakly analytic with the map $\alpha^\frown x \to \overline{0}_\alpha{}^\frown 1^\frown \overline{0}_\kappa$

We prove that C is not intermediately analytic. By Corollary 6.2.23, every Borel$_\kappa$ set is a $<\kappa$-union of κ-connected sets. So, by Theorem 6.1.1, every intermediately analytic set is a $<\kappa$-union of κ-connected sets.

For each, $x \in C \setminus \{\overline{0}_\kappa\}$, $\{x\}$ is κ-clopen in the subspace κ-topology on C. Suppose, for a contradiction, $C = \bigcup_{\alpha\in\lambda} C_\alpha$ for some $\lambda < \kappa$ and κ-connected C_α. Then, by the pigeonhole principle, there is a C_α of size at least 2. But if $x_\alpha \in C_\alpha \setminus \{\overline{0}_\kappa\}$ and $C_\alpha \neq \{x_\alpha\}$, then $\{x_\alpha\}$ is a proper clopen subset of C_α, a contradiction. □

It is not known to the author whether, assuming κ is inaccessible, there is a set $A \subsetneq \kappa^\kappa$ which is clopen in the full topology, but not intermediately analytic.

Meanwhile, the weakly analytic sets are strictly richer:

Proposition 6.3.28. There is a set $A \subsetneq \kappa^\kappa$ which is G^κ_δ but not open or closed in the full topology, such that A is weakly injective-analytic.

Proof. Note that $A := \{x \in \kappa^\kappa : \forall\alpha\exists\beta > \alpha(x(\beta) \neq 0)\}$ is a complete G^κ_δ set (so is neither open nor closed in the full topology). We define a function $f : \kappa^\kappa \to \kappa^\kappa$ which 'throws away' κ-many points:

$$f(x) = \overline{0}_{x(0)}{}^\frown(1+x(1))^\frown\overline{0}_{x(2)}{}^\frown(1+x(2))^\frown\ldots$$

In which case, $f(\kappa^\kappa) = A$. We show that f is Borel$_\kappa$. Let $D := \{s \in \kappa^{<\kappa} : \forall\alpha(s(\alpha) = 0 \to \exists\beta > \alpha \text{ where } s(\beta) \neq 0)\}$, and $\phi : \kappa^{<\kappa} \to D$ be the corresponding map to f. Clearly, ϕ is a bijection. If $s \in D$, then $f^{-1}(N_s) = N_{\phi^{-1}(s)}$. If $s \notin D$, then there is an $s' \in D$ such that $s = s'^\frown\overline{0}_\alpha$ for some $\alpha < \kappa$. So, the preimage of N_s is $\bigcup_{\beta\geq\alpha} N_{\phi^{-1}(s')^\frown\beta}$. Hence, $f^{-1}(N_s)$ is conelike in $N_{\phi^{-1}(s')}$, so is κ-closed. So, f is Borel$_\kappa$. □

This means we can properly separate the remaining notions of analyticity.

Corollary 6.3.29. Assume κ is weakly compact. Then not all weakly analytic sets are intermediately analytic, and not all intermediately analytic sets are strongly analytic.

Proof. The weak inclusions are by Proposition 6.3.7. Strictness follows from Corollary 6.3.17, and Propositions 6.3.28 & 6.3.24 respectively. □

Question 6.3.30. If $A \subseteq \kappa^\kappa$ is clopen in the full topology, is A weakly analytic (or equivalently weakly bianalytic)?

The weak analytics are not rich enough to include the closed sets of the full topology, but are almost rich enough, in that every weakly analytic A is only a few points away from a closed set. As usual, let $\mathsf{cl}(A)$ be the closure of A in the full topology.

Proposition 6.3.31. Let $\kappa^{<\kappa} = \kappa$. There are sets which are closed in the full topology which are not weakly analytic.

Proof. Every Borel$_\kappa$ function is continuous in the full topology, so let C be a closed set which is not the continuous image of κ^κ (as in [135, Theorem 1.5]). □

Theorem 6.3.32. Let κ be weakly compact. If A is weakly analytic then $|\mathsf{cl}(A) \setminus A| \leq \kappa$.

Proof. Let g be a Borel$_\kappa$ function such that $g(\kappa^\kappa) = A$. Fix an $x \in \mathsf{cl}(A) \setminus A$. As $x \in \mathsf{cl}(A)$, we know that $N_{x\restriction\alpha} \cap A \neq \emptyset$ for every $\alpha < \kappa$. For each $\alpha < \kappa$, we define $B^x_\alpha := g^{-1}(N_{x\restriction\alpha})$. As f is Borel$_\kappa$, the sequence $(B^x_\alpha)_{\alpha\in\kappa}$ is a chain of non-empty Borel$_\kappa$ sets which is decreasing with respect to inclusion (by Remark 6.2.15). As $x \notin A$, we know that $\bigcap_{\alpha\in\kappa} B^x_\alpha = \emptyset$. So, by Lemma 6.2.16, $\bigcap_{\alpha\in\kappa} \mathcal{F}(B^x_\alpha) \neq \emptyset$, so pick an $s_x \in \bigcap_{\alpha\in\kappa} \mathcal{F}(B^x_\alpha)$. We define $f : \mathsf{cl}(A) \setminus A \to \kappa^{<\kappa}$ by $f(x) = s_x$. We just need to prove that f is injective, and then the result follows from $|\kappa^{<\kappa}| = \kappa$.

Suppose $x, y \in \mathsf{cl}(A) \setminus A$ are such that $x \neq y$. Then there is an $\alpha < \kappa$ such that $N_{x\restriction\alpha} \cap N_{y\restriction\alpha} = \emptyset$. So, $B^x_\alpha \cap B^y_\alpha = g^{-1}(N_{x\restriction\alpha}) \cap g^{-1}(N_{y\restriction\alpha}) = \emptyset$. Note that $s = f(x) \in \mathcal{F}(B^x_\alpha)$, whilst $t = f(y) \in \mathcal{F}(B^y_\alpha)$. But now, if $s = t$ and P^x_s and P^y_t are the element associated to s in $\mathcal{P}(B^x_\alpha)$ and $\mathcal{P}(B^x_\alpha)$, respectively, then we would have $B^x_\alpha \cap B^y_\alpha \supset P^x_s \cap P^y_s$, but $P^x_s \cap P^y_s$ is non-empty since both sets are conelike in the same N_s, a contradiction (by Remark 6.2.7). Thus $s \neq t$ as required. □

Remark 6.3.33. Not all weakly analytic sets, A, are such that $|\mathsf{cl}(A) \setminus A| < \kappa$. For example, let $f : \kappa^\kappa \to \kappa^\kappa$ be such that for all $\alpha, \beta, \gamma < \kappa$ and all $x \in \kappa^\kappa$, we let $f(\alpha^\frown\beta^\frown\gamma^\frown x) = \alpha^\frown\overline{0}_\beta{}^\frown(\gamma+1)^\frown x$. Then f is Borel$_\kappa$ (as in Proposition 6.3.26). Then $f(\kappa^\kappa) = \kappa^\kappa \setminus \{\alpha^\frown\overline{0}_\kappa : \alpha \in \kappa\}$, whilst $\mathsf{cl}(f(\kappa^\kappa)) = \kappa^\kappa$, so $|\mathsf{cl}(f(\kappa^\kappa)) \setminus f(\kappa^\kappa)| = \kappa$.

As a consequence of Theorem 6.3.32, all weakly analytic sets are $f\mathsf{SC}_\kappa$:

Corollary 6.3.34. Let (X, τ_κ) be a κ-topological space. Every weakly analytic set of (X, τ_κ) is G^κ_δ in the full topology (X, τ), and thus an $f\mathsf{SC}_\kappa$-space.

Proof. Let A be weakly analytic. Then $A = (\mathsf{cl}(A) \setminus A) \cup \mathsf{cl}(A)$. By Theorem 6.3.32, $|\mathsf{cl}(A) \setminus A| \leq \kappa$. Singletons are closed in the full topology, so $\mathsf{cl}(A) \setminus A$ is F^κ_σ, so $X \setminus (\mathsf{cl}(A) \setminus A) = B$ is G^κ_δ. Note that $A = \mathsf{cl}(A) \cap B$ is the intersection of two G^κ_δ sets, hence A is G^κ_δ. The last part follows from [1, Proposition 2.10]. □

Whilst it is not known whether the fully clopen sets are weakly analytic, the superclosed sets are known to be so:

Definition 6.3.35 ([1, page 9])**.** A tree $T \subseteq \kappa^{<\kappa}$ is called *$<\kappa$-closed* if for every limit $\alpha < \kappa$, and every α-sequence in T, $(s_\beta)_{\beta\in\alpha}$, T has a level α node s above $(s_\beta)_{\beta\in\alpha}$. If, in addition, T is pruned, T is called *superclosed*. A set $C \subseteq \kappa^\kappa$ is *superclosed* if there is a superclosed T such that $[T] = C$.

Proposition 6.3.36. Let $C \subseteq \kappa^\kappa$ be a non-empty superclosed set. Then there is a Borel_κ retraction $f : \kappa^\kappa \to C$.

Proof. The proof proceeds as in [135, Proposition 1.3], whilst ensuring that the function obtained is Borel_κ. Let $C = [T]$ for some superclosed $T \subseteq \kappa^{<\kappa}$. Without loss of generality, we may assume that T is such that $N_s \cap C \neq \emptyset$ for all $s \in T$. We want to define a Borel_κ retraction $f : \kappa^\kappa \to C$.

For every $z \in C$, we set $f(z) = z$, as required for a retraction.

To define f on $\kappa^\kappa \setminus C$, let $A_s = \{s^\frown\beta : \beta < \kappa, s^\frown\beta \notin T\}$, and $B_s = \{s^\frown\beta : \beta < \kappa, s^\frown\beta \in T\}$ for every $s \in T$. We define functions $\phi_s : A_s \to B_s$ for every $s \in T$.

Let $U_s = \bigcup\{N_{s^\frown\beta} : \beta < \kappa, s^\frown\beta \notin T\} = \bigcup_{t\in A_s} N_t$. Notice that U_s is not Borel_κ if and only if $|A_s| = |B_s| = \kappa$. If U_s is Borel_κ, choose a γ such that $s^\frown\gamma \in T$ and define $\phi_s(s^\frown\beta) = s^\frown\gamma$ for all $\beta < \kappa$ with $s^\frown\beta \in A_s$. Otherwise, let ϕ_s be a bijection between A_s and B_s.

Let g be a choice function that associates a point $g(s) \in N_s \cap C$ to every $s \in T$. For every $z \in \kappa^\kappa \setminus C$, there is a unique $\alpha_z < \kappa$ such that $z{\restriction}\alpha_z \in \delta(T)$. Since T is superclosed, we know that α_z is a successor ordinal, so let β_z be such that $\alpha_z = \beta_z + 1$. Then, for every $z \in \kappa^\kappa \setminus C$, define $f(z) = g(\phi_{z\restriction\beta_z}(z{\restriction}(\beta_z + 1)))$.

It is clear that f is a retraction. We claim that f is also Borel_κ. Let $s \in T$. First, notice that for every $z \in N_s$, we have $f(z) \in N_s \cap C$, thus $N_s \subseteq f^{-1}(N_s)$. If, instead, $z \in \kappa^\kappa \setminus N_s$, we have that $f(z) \in N_s$ if and only if $z \notin C$ and $f(z) = g(s{\restriction}(\beta+1))$ for some $\beta < \mathsf{len}(s)$ such that $g(s{\restriction}(\beta+1)) \in N_s$ (one can take $\beta = \beta_z$, for concreteness). Let $B'_\alpha = f^{-1}(g(s{\restriction}(\alpha + 1))) \setminus N_s$, and let

$$B_\alpha = \bigcup\{N_{s\restriction\alpha^\frown\beta} : \beta < \kappa, s{\restriction}\alpha^\frown\beta \notin T \text{ and } \phi_{s\restriction\alpha}(s{\restriction}\alpha^\frown\beta) = s{\restriction}(\alpha + 1)\}.$$

Then, $B'_\alpha = \bigcup\{B_\beta : g(s{\restriction}(\alpha + 1)) = g(s{\restriction}(\beta + 1))\}$, and thus

$$f^{-1}(N_s) = N_s \cup \bigcup\{B_\alpha : \alpha < \mathsf{len}(s), g(s{\restriction}(\alpha + 1)) \in N_s\}.$$

If $U_{s\restriction\alpha}$ is Borel_κ, by the definition of $\phi_{s\restriction\alpha}$, we have that either $B_\alpha = U_{s\restriction\alpha}$ or $B_\alpha = \emptyset$, and either way B_α is Borel_κ.

Otherwise, $U_{s\restriction\alpha}$ is not Borel_κ. Then, by the definition of $\phi_{s\restriction\alpha}$, we have that $B_\alpha = N_{s\restriction\alpha^\frown\beta}$ for some β, and thus B_α is Borel_κ again. Thus, $f^{-1}(N_s)$ is the union of $<\kappa$-many Borel_κ sets, and so it is Borel_κ. Since the cones, N_s, form a basis for the κ-topology of κ^κ, this proves that f is Borel_κ. ☐

Corollary 6.3.37. Every superclosed subset of κ^κ is weakly analytic.

Indeed, every disjoint union of SC_κ sets is SC_κ, e.g. $N_s \cup N_t$ is SC_κ, so not all SC_κ are strongly analytic. So, by [1, Theorem 2.17], not all superclosed sets are strongly analytic.

We can summarise these properties of the analytic pointclasses in three tables. Table 6.1 summarises the closure properties from Proposition 6.3.22 and Corollary 6.3.25:

	closed under complements	closed under $<\kappa$-unions
strongly analytic	no	no
intermediately analytic	no	yes
weakly analytic	?	yes

Table 6.1: Closure of pointclasses

In Table 6.2, 'No' in cell (a, b) means that not all elements of column a are elements of row b, and so on.

	clopen	open	closed	superclosed
strongly analytic	No	No	No	No
intermediately analytic	?	?	No	?
weakly analytic	?	?	No	Yes

Table 6.2: Inclusions between pointclasses

In Table 6.3, 'Yes' in cell (a, b) means that there is an example of a set which is properly in column a (i.e. not of lower complexity) which is in row b.

	clopen	open	closed	F_σ	G_δ
strongly analytic	Yes	?	Yes	No	No
intermediately analytic	Yes	?	Yes	No	No
weakly analytic	Yes	Yes	Yes	?	Yes

Table 6.3: Proper witnesses in pointclasses

Generalising the ordinary case, we let Proj_κ be the closure of $\{\kappa^\kappa\}$ under the images of Borel$_\kappa$ functions, and complements. By Corollary 6.3.34, we know that weakly coanalytics are F^κ_σ. But it is not known if the Borel$_\kappa$ image of a (fully) Borel set is (fully) Borel. Hence, a natural question is:

Question 6.3.38. Let $\mathcal{B} \subseteq \mathscr{P}(\kappa^\kappa)$ be the smallest κ^+-algebra containing all the cones (i.e. the full Borel pointclass). Is $\mathsf{Proj}_\kappa \subseteq \mathcal{B}$?

6.3.4 Linearly ordered spaces

Next, we study the κ-analytic set on linearly ordered spaces. Recall that we call a set strongly analytic, etc., if it is strongly κ^κ-analytic. In this section, we focus on strong, intermediate, and weak X-analyticity, for a linearly ordered space, X. Projections are again dismissed as trivial (by Proposition 6.3.3). As in Section 6.2.3, the definition of X-analyticity on an arbitrary κ-Polish space, X, is complicated by the fact that κ^κ is not κ-universal for κ-Polish spaces. This means we must choose the domain space carefully. Clearly, if $g(X) = \kappa^\kappa$ for some κ-continuous g, then every strongly analytic set is strongly X-analytic, and so on. But (κ^κ-)analyticity can be pathological for some κ-Polish spaces:

Proposition 6.3.39. There is a κ-Polish space, X, such that X is strongly X-analytic, but not weakly analytic. Hence, for each such X:

1. weakly X-analytic and weakly analytic are distinct,
2. intermediately X-analytic and intermediately analytic are distinct, and
3. strongly X-analytic and strongly analytic are distinct.

Proof. Any X is automatically strongly X-analytic. So, it suffices to give an X which is not weakly analytic. Note that every Borel$_\kappa$ function is continuous, and there are sets $X \subseteq \kappa^\kappa$ which are not the continuous image of κ^κ [135, Theorem 1.5]. By [1, Theorem 2.21], such a set X is a κ-Polish space. □

For this reason, we focus on X-analytic sets on X, and particularly on interval κ-topologies. Note that Corollary 6.3.8 holds for such an X, but the analogue of Corollary 6.3.14 fails:

Remark 6.3.40. There is a κ-continuous image of a κ-closed set which is not strongly $\mathbb{R}_\kappa$-analytic: all strongly $\mathbb{R}_\kappa$-analytic sets are κ-connected, as $\mathbb{R}_\kappa$ is κ-connected and κ-continuity preserves κ-connectedness, but $\{0,1\} = \mathrm{id}(\{0,1\})$ is κ-closed and not κ-connected. So too for any ordered field, $\mathbb{K}$, equipped with the Q-interval κ-topology, where $\{0,1\} \subseteq Q \subseteq \mathbb{K}$.

As in Section 6.3.3, some strongly X-analytic sets are not Borel$_\kappa$. This we now prove. Throughout this section, we use both the *field* and the *order* properties of $\mathbb{Q}_\kappa$ and $\mathbb{R}_\kappa$. Recall the definition of κ-homogeneity from Definition 4.1.5.

Proposition 6.3.41. Let $\mathbb{K}$ be an κ-homogeneous ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$, which has a (κ,κ)-gap. Equip $\mathbb{K}$ with the $\mathbb{K}$-interval κ-topology. Then, if $C \subseteq \mathbb{K}$ is a convex set, such that C has cofinality and conitiality κ or 1 (including singletons), then C is strongly $\mathbb{K}$-analytic.

Proof. Let $a \in \mathbb{K}$. For $\{a\}$, use the constant map $f_a(x) = a$. Next we consider the improper intervals. For $[a,\infty)$, use the map $f(x) = X^2 + a$. For (a,∞), we follow the construction in the comment following Example 4.1.13. Pick a (κ,κ)-gap, G, in $\mathbb{K}$. By κ-homogeneity, let b be an order-reversing bijection from $(-\infty, G)$ to (a,∞), and let r be an order isomorphism from (G,∞) to (a,∞). Then let $f := b \cup r$. Clearly, $f(\mathbb{K}) = (a,\infty)$. Note that b is strictly decreasing, and r is strictly increasing. Hence, by Lemma 4.2.7, f is κ-continuous. So, (a,∞) is strongly $\mathbb{K}$-analytic. The improper interval $(-\infty,a)$ and $(-\infty,a]$ are similar, as are there convex sets $(-\infty, G)$ and $[H, \infty)$, where G is a (κ, λ)-gap and H is a (μ, κ)-gap for some $\mu, \lambda \leq \kappa$).

The remaining case is that of the intervals. Each case is similar, so we consider $[c, d)$ for some $c, d \in \mathbb{K}$. By the κ-homogeneity of $\mathbb{K}$, let $b : [0, \infty) \to [c, d)$ be an order isomorphism. Let $g : (-\infty, 0] \to \{c\}$ be the constant function with value c. Then $b \cup g$ is κ-continuous by the Glueing Lemma (Lemma 4.2.4). Finally, note that $(b \cup g)(\mathbb{K}) = [c, d)$. The bounded convex set case is similar, e.g. if G is a (κ, λ)-gap in $\mathbb{K}$, and $a \in \mathbb{K}$, then we use the same technique for $(a, G]$. □

Remark 6.3.42. In fact, using the same method, if $I, J \subseteq \mathbb{K}$ are intervals or improper intervals such that $|I| > 1$ (equivalently, $|I| = |\mathbb{K}|$), then there is a κ-continuous $f : I \to \mathbb{K}$ such that $f(I) = J$.

Corollary 6.3.43. Let κ be inaccessible. Let $\mathbb{Q}_\kappa$ have the $\mathbb{Q}_\kappa$-interval κ-topology. Then there are strongly $\mathbb{Q}_\kappa$-bianalytic sets which are not Borel$_\kappa$.

Proof. By Propositions 2.3.11 and 2.3.20, $\mathbb{Q}_\kappa$ has the requisite properties. By Theorem 6.2.48 and Example 6.2.49, if G is a (κ, κ)-gap in $\mathbb{Q}_\kappa$, then $(a, G]$ is not Borel$_\kappa$. By Proposition 6.3.41, $(a, G]$ is strongly $\mathbb{Q}_\kappa$-bianalytic. □

Remark 6.3.44. If $\mathbb{R}_\kappa$ is κ-homogeneous, then Proposition 6.3.41 and Corollary 6.3.43 (using Corollary 6.2.50 this time) also hold for $\mathbb{R}_\kappa$, with the $\mathbb{R}_\kappa$-interval κ-topologies.

The approach to Suslin's theorem in Corollary 6.3.43 and Remark 6.3.44 uses specific features of $\mathbb{Q}_\kappa$ and $\mathbb{R}_\kappa$, particularly that they have base number κ, and they have (κ, κ)-gaps (for Proposition 6.3.41). They show the failure of Suslin's theorem for *strong* X-bianalyticity (hence also intermediate and weak X-bianalyticity) for $\mathbb{Q}_\kappa$, and also for $\mathbb{R}_\kappa$, if $\mathbb{R}_\kappa$ is κ-homogeneous (see Question 4.1.10).

More generally, Suslin's theorem fails for *weakly* X-bianalytics, using a strategy somewhat like that of Lemma 6.3.19.

Proposition 6.3.45. Let κ be inaccessible. Let X be an order, and equip X with the X-interval κ-topology. Let X be densely ordered, κ-homogeneous, and $\mathsf{cof}(X) = \mathsf{owei}(X) = \kappa$. Then there is a set $A \subseteq X$ which is weakly X-analytic, but not Borel$_\kappa$.

Proof. Let $(p_\alpha)_{\alpha \in \kappa}$ be cofinal in X. For all $\alpha \in \kappa$, let $P_\alpha = (p_\alpha, p_{\alpha+1}]$. Note that $\bigcup_{\alpha \in \kappa} P_\alpha \subseteq X$, and for each limit $\alpha < \kappa$, the set $C_\alpha := \{x \in X : \forall \beta < \alpha (p'_\beta < x < p_\alpha\}$ may be nonempty. The collection $\{P_\alpha : \alpha \in \kappa\} \cup \{C_\alpha : \alpha < \kappa$ is a limit ordinal$\}$ is a partition of X.

As X is densely ordered, for each α, there is a $q_\alpha \in (p_\alpha, p_{\alpha+1})$, so let $P'_\alpha = (p_\alpha, q_\alpha]$, and $P''_\alpha = (q_\alpha, p_{\alpha+1}]$. As X is κ-homogeneous, for each α, there is an order-isomorphism $h_\alpha : P''_\alpha \to P'_\alpha$. Note that h_α is a κ-homeomorphism.

Now let $\alpha < \kappa$ be a limit such that $C_\alpha \neq \emptyset$. As $\mathsf{owei}(X) = \kappa$, we know that $\mathsf{coi}(C_\alpha) \leq \kappa$. If $\mathsf{coi}(C_\alpha) = \kappa$, we use the same method as in P''_α: X is κ-homogeneous, so there is an order isomorphism, $g_\alpha : C_\alpha \to P'_\alpha$, which is therefore a κ-homoeomorphism. Otherwise, $\mathsf{coi}(C_\alpha) < \kappa$. In which case, let $g_\alpha : C_\alpha \to \{p_\alpha\}$ be the constant map.

Then we define $f : X \to \bigcup_{\alpha\in\kappa} P'_\alpha$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in \bigcup_{\alpha\in\kappa} P'_\alpha \\ h_\alpha(x) & \text{if } x \in P''_\alpha, \\ g_\alpha(x) & \text{if } x \in C_\alpha. \end{cases}$$

Claim 6.3.46. The function f is Borel$_\kappa$.

Proof. Let $(a, b) \subseteq X$. Let $I \subseteq \kappa$ be such that for all $\alpha \in I$, $(C_\alpha \cup P'_\alpha) \cap (a, b) \neq \emptyset$. Note that I is a convex set in κ. As $(p_\alpha)_{\alpha\in\kappa}$ is cofinal, I is bounded above by some p_{α_0}. So, in particular, $|I| < \kappa$. Let β be the least in I and γ be the least $\gamma \in \kappa$ such that $\gamma > I$.

If $\gamma = \gamma_0 + 1$ for some $\gamma_0 < \kappa$, then $f^{-1}(a, b) = (a, b) \cup I''_a \cup I''_b \cup C_a \cup C_b$, where $I''_a \subseteq P''_\beta$ and $I''_b \subseteq P''_\gamma$ are open intervals, and $C_a \subseteq C_\beta$ and $C_b \subseteq C_\gamma$ are convex sets. If β is such that (C_β is non-empty and) g_β is not constant, then C_a is an interval; otherwise, if g_β is constant, then $C_a = C_\beta$, and as $\mathsf{coi}(C_\beta) < \kappa$ (and $\mathsf{top}(C_\beta) = p_\alpha$), then C_β is κ-open. The case of C_b is exactly similar.

Otherwise, γ is a limit. The case of C_a remains the same. For C_b, there is an extra step in the case where g_γ is constant: suppose that C_γ is non-empty. If g_γ is not constant, then C_a is an open interval. Otherwise, if g_γ is constant, then either:

1. $C_a = C_\beta$, in which case, as $\mathsf{coi}(C_\beta) < \kappa$ (and $\mathsf{top}(C_\beta) = p_\alpha$), we have that C_β is κ-open as in the successor case, or

2. $C_\alpha = \emptyset$, in which case note that $\mathsf{coi}(C_\alpha) = \kappa$, and by assumption we have that (a, b) contains a cofinal sequence of p_α with $\mathsf{bot}(C_\gamma)$. So, in fact, $b = p_\gamma$. Hence, by construction, $f^{-1}((a, p_\alpha)) = O \cup (a, \infty) \setminus \bigcup_{\beta\in\mathsf{coi}(C_\alpha)}(c_\beta, \infty)$, where $(c_\beta)_{\beta\in\mathsf{coi}(C_\alpha)}$ is coinitial in C_α, and $O < C_\gamma$ is a κ-open set. As $\mathsf{coi}(C_\alpha) < \kappa$, $f^{-1}((-\infty, p_\alpha))$ is the complement of a $<\kappa$-sized union, i.e. a κ-closed set. So, C_b is κ-closed.

Hence, $f^{-1}(a, b)$ is a union of finitely many open intervals and κ-closed sets, so is a Borel$_\kappa$ set. The (a, ∞) and $(-\infty, a)$ cases are similar. Hence f is Borel$_\kappa$. □

Finally note that $f(X) = \bigcup_{\alpha\in\kappa} P'_\alpha$, which consists of κ-many disjoint convex Borel$_\kappa$ subsets, so by Theorem 6.2.45, $f(X)$ is not Borel$_\kappa$. □

The same result holds if we instead assume that $\mathsf{coi}(X) = \kappa$, or indeed that there is an almost gap A in X such that $\mathsf{cof}(A) = \kappa$ or $\mathsf{coi}(A) = \kappa$.

Corollary 6.3.47. Let κ be inaccessible. Let X have the X-interval κ-topology, be densely ordered, κ-homogeneous, and $\mathsf{cof}(X) = \mathsf{owei}(X) = \kappa$. Then the Borel$_\kappa$ sets are strictly contained in the weakly X-bianalytic sets.

Proof. Every Borel$_\kappa$ set on X is weakly X-analytic. Let $f(X)$ be as in Proposition 6.3.45. This $f(X)$ is strongly X-analytic, hence weakly X-analytic. An exactly parallel proof to Proposition 6.3.45 shows that $X \setminus f(X)$ is not Borel$_\kappa$. □

Example 6.3.48. Let κ be inaccessible. On $\mathbb{Q}_\kappa$, the set $\bigcup_{\alpha\in\kappa}(2\alpha, 2\alpha+1]$ is weakly $\mathbb{Q}_\kappa$-analytic, but neither strongly $\mathbb{Q}_\kappa$-connected, nor Borel$_\kappa$ for the $\mathbb{Q}_\kappa$-interval κ-topology.

If X is as in Proposition 6.3.45 and has κ-many G_α, where each G_α is an $(\beta_\alpha, \gamma_\alpha)$-gap for some $\beta_\alpha, \gamma_\alpha < \kappa$, then the κ-open sets $(G_\alpha, G_{\alpha+1})$ partition X into κ-many pieces. By copying the method in Lemma 6.3.19, $\bigcup_{\alpha\in\kappa}(G_{2\alpha}, G_{2\alpha+1})$ is strongly X-bianalytic but not Borel$_\kappa$, so Suslin's theorem fails for strongly X-bianalytics.

On κ^κ, Corollary 6.3.14 holds. Likewise, if $\mathbb{K}$ is a κ-homogeneous ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$, which has a (κ, κ)-gap, then the intermediately X-analytic sets are exactly the κ-continuous images of $\kappa\mathbf{\Delta}^0_2$ sets (see also Remark 6.3.40). This gives us more of a description of the hierarchy of analyticity notions on κ^κ and $\mathbb{R}_\kappa$:

Corollary 6.3.49. Let κ be inaccessible. Let X be a dense linear order. Let $Q \subseteq X$ be order dense in X. Suppose that, for all (λ, μ)-gaps in X, $\lambda, \mu \leq \kappa$. Let X have the Q-interval κ-topology. A set $A \subseteq \mathbb{K}$ is intermediately $\mathbb{K}$-analytic if and only if $A = f(D)$ where $D \subseteq \mathbb{K}$ is $\kappa\mathbf{\Delta}^0_2$ and $f : D \to \mathbb{K}$ is κ-continuous (in the subspace κ-topology on D).

Proof. This is immediate from Theorem 6.2.48 □

But it is not known whether we can strengthen this to κ-closed sets (rather than $\kappa\mathbf{\Delta}^0_2$ sets).

Question 6.3.50. Suppose κ is inaccessible, $\mathbb{K}$ is a κ-homogeneous ordered field such that $\mathsf{bn}(\mathbb{K}) = \kappa$, which has a (κ, κ)-gap, and that $\mathbb{K}$ has the $\mathbb{K}$-interval κ-topology. Is a set $A \subseteq \mathbb{K}$ intermediately $\mathbb{K}$-analytic set $A \subseteq \mathbb{K}$ if and only if $A = f(C)$ where $C \subseteq \mathbb{K}$ is κ-closed and $f : C \to \mathbb{K}$ is κ-continuous (in the subspace κ-topology on C)?

This amounts to showing whether there is a κ-closed set, C, and a κ-continuous function, f, such that $f(C) = [0, \neg\omega)$. Using Remark 4.1.6, Example 4.1.13, the proof of Theorem 6.2.48, and the techniques in Proposition 6.3.41 and Remark 6.3.42, we can show the following: suppose κ is inaccessible, $\mathbb{K}$ is a κ-homogeneous ordered field such that $\mathsf{bn}(\mathbb{K}) =\kappa$, which has a (λ, λ)-gap for each infinite $\lambda \leq \kappa$, and $\mathbb{K}$ has the $\mathbb{K}$-interval κ-topology. Then, a set $A \subseteq \mathbb{K}$ is intermediately $\mathbb{K}$-analytic set $A \subseteq \mathbb{K}$ if and only if $A = f(C)$ where $C \subseteq \mathbb{K}$ is κ-closed and $f : C \to \mathbb{K}$ is κ-continuous (in the subspace κ-topology on C). However, this does not apply to $\mathbb{R}_\kappa$ or $\mathbb{Q}_\kappa$, as both are η_κ (so have no (ω, ω)-gaps).

Corollary 6.3.51.

1. Let κ be inaccessible. Let X be a dense linear order. Let $Q \subseteq X$ be order dense in X. Suppose that, for all (λ, μ)-gaps in X, $\lambda, \mu \leq \kappa$. Let X have the Q-interval κ-topology. Strong X-analyticity implies, but is not implied by, intermediate X-analyticity, and intermediate X-analyticity implies weak X-analyticity.

2. Let κ be weakly compact. Then strong analyticity implies, but is not implied by, intermediate analyticity, and intermediate analyticity implies, but is not implied by, weak analyticity.

Proof. The implications are exactly as in Proposition 6.3.7. The first non-implication is Remark 6.3.40. For the second pair of non-implications, we separate strong from intermediate analyticity using Corollary 6.3.17, and intermediate from weak analyticity using e.g. Proposition 6.3.27. □

It is natural to ask whether, on suitable order κ-topologies, weak and intermediate X-analyticity coincide:

Question 6.3.52. Let κ be inaccessible. Are weak and intermediate $\mathbb{R}_\kappa$-analyticity equivalent?

6.4 Distinct κ-topologies

Here, we show how to construct κ-topologies which are not κ-homeomorphic. Trivially, there are $2^{2^{\kappa^\kappa}}$ non-κ-homeomorphic κ-topologies on κ^κ (any non-fully homeomorphic full topologies will do, e.g. [132, Theorem 1.4])[5]. Also, if $|X| = \kappa^\kappa$, then there are at least two non-κ-homeomorphic κ-topologies on X (namely 2^κ and κ^κ), and if $\kappa^{<\kappa} = \kappa$, then there are at least three non-κ-homeomorphic κ-topologies on X (the extra is $\mathbb{R}_\kappa$). This follows from Table 6.4, and that κ-homeomorphisms preserve κ-additivity, κ-zero-dimensionality, and κ-connectedness. In this section, we prove the much stronger result that if κ is inaccessible, then there are 2^κ-many non-κ-homeomorphic κ-topologies on κ^κ whose full topology is fully homeomorphic to $(\kappa^\kappa, \langle\tau_b\rangle_\infty)$ (Corollary 6.4.4).

	κ-additive	κ-zero-dimensional	κ-connected
$2^\kappa, \tau^{\mathrm{b}}_\kappa$	yes	yes	no
$\kappa^\kappa, \tau^{\mathrm{b}}_\kappa$	yes	no	yes
$\mathbb{R}_\kappa, \tau^i_\kappa$	no	no	yes

Table 6.4: Non-κ-homeomorphic κ-topologies on a set of size κ^κ

Call a map a *κ-embedding* if it is a κ-homeomorphism on its image. Two κ-topologies (X, τ_κ) and (Y, σ_κ), are *mutually non-κ-embeddable* if there is neither a κ-embedding from X to Y nor a κ-embedding from Y to X.

By κ-additivity, the interval κ-topology on $\mathbb{R}_\kappa$ does not κ-embed into the bounded κ-topology on κ^κ. The converse fails too:

Remark 6.4.1. If $\kappa^{<\kappa} = \kappa$, then the bounded κ-topology on κ^κ does not κ-embed into the $\mathbb{Q}_\kappa$-interval topology on $\mathbb{R}_\kappa$: suppose $I \subseteq \mathbb{R}_\kappa$ is κ-homeomorphic to κ^κ, with homeomorphism $b : I \to \kappa^\kappa$. Then there are two cases. Either there is a $q \in \mathbb{Q}_\kappa$ such that $q \in \mathbb{Q}_\kappa \setminus I$ where there are $i, j \in I$ such that $i < q < j$, in which case $\{b((-\infty, q)), b((q, \infty))\}$ is a κ-clopen partition of κ^κ, contradicting that κ^κ is

[5] In fact, [132, Theorem 1.4] shows that there are $2^{2^{\kappa^\kappa}}$ distinct topologies. But as there are only 2^{κ^κ}-many bijections of κ^κ, each (full) homeoemorphism class has size at most 2^{κ^κ}, hence there are $2^{2^{\kappa^\kappa}}$ non-fully homeomorphic topologies (this argument is due to Stefan Geschke).

κ-connected. Or there is no such q, in which case let $q' \in I \cap \mathbb{Q}_\kappa$. Then $\{q'\}$ is κ-closed so $\{b(q')\}$ is κ-closed, contradicting Corollary 6.2.30.

Hence, the generalisation of Cantor's middle-third argument, by which κ^κ embeds into $\mathbb{Q}_\kappa$, yields a full embedding but not a κ-embedding.

Distinct κ-topologies may generate full topologies which are (fully) homeomorphic (in fact, identical), i.e. $\tau_\kappa \neq \sigma_\kappa$ such that $\langle \tau_\kappa \rangle_\infty = \langle \sigma_\kappa \rangle_\infty$. For example, the interval full topology on $\mathbb{R}_\kappa$ is generated by a κ^κ-sized basis, $\tau_\kappa^{\mathbb{R}_\kappa}$, and a κ-sized basis $\tau_\kappa^{\mathbb{Q}_\kappa}$. These even yield distinct Borel$_\kappa$ hierarchies (Corollary 6.2.50). We show that the standard κ-topologies $\mathbb{R}_\kappa$ and κ^κ generate homeomorphic (full) topologies, and moreover that we can build many pairwise distinct κ-topologies:

Theorem 6.4.2. Let $\kappa^{<\kappa} = \kappa$. Let $\tau_\kappa^{\mathbb{R}_\kappa}$ be the $\mathbb{R}_\kappa$-interval κ-topology on $\mathbb{R}_\kappa$, and let τ_κ^{b} be the bounded κ-topology on κ^κ. Then the full topologies $\langle \tau_\kappa^{\mathbb{R}_\kappa} \rangle_\infty$ and $\langle \tau_\kappa^{\mathrm{b}} \rangle_\infty$ are (fully) homeomorphic.

Proof. A space is (fully) homeomorphic to $(\kappa^\kappa, \langle \tau_\kappa^{\mathrm{b}} \rangle_\infty)$ if the full topology is κ-additive, SC_κ, and all κ-Lindelöf sets have empty interior [1, Theorem 3.9]. The full topology on $\mathbb{R}_\kappa$ is κ-additive: $\mathsf{bn}(\mathbb{R}_\kappa) = \kappa$ so $<\kappa$-intersections of intervals do not generate the intervals $[a, b]$ for $a, b \in \mathbb{R}_\kappa$. Next, no κ-Lindelöf $A \subseteq \mathbb{R}_\kappa$ includes any interval in $\mathbb{R}_\kappa$, as any interval can be divided into κ-many disjoint open sets using the dense set of (κ, κ)-gaps. So, it suffices to prove that $(\mathbb{R}_\kappa, \langle \tau_\kappa^{\mathbb{R}_\kappa} \rangle_\infty)$ is SC_κ. By Theorem 2.3.13, $\mathbb{R}_\kappa$ is η_κ, so is κ-spherically complete. So, by [1, Theorem 2.31], $\mathbb{R}_\kappa$ is SC_κ if and only if it is completely κ-metrisable. As $\mathbb{R}_\kappa$ metrises itself, $(\mathbb{R}_\kappa, \langle \tau_\kappa^{\mathbb{R}_\kappa} \rangle_\infty)$ is completely κ-metrisable, so is SC_κ. Hence, $(\mathbb{R}_\kappa, \langle \tau_\kappa^{\mathbb{R}_\kappa} \rangle_\infty)$ is homeomorphic to $(\kappa^\kappa, \langle \tau_\kappa \rangle_\infty)$. □

For interval κ-topologies, we can even guarantee that there are κ-many non-κ-bi-embeddable κ-topologies generating identical (not just κ-homeomorphic) full topologies:[6]

Proposition 6.4.3. Let κ be inaccessible. There are κ-many non-κ-biembeddable interval κ-topologies on $\mathbb{R}_\kappa$ generating the interval (full) topology.

Proof. For each $\lambda \leq \kappa$, let $\mathcal{B}_\lambda := \{\bigcap_{\beta \in \alpha} I_\beta : I_\beta \subseteq \mathbb{R}_\kappa$ is an interval and $\alpha < \lambda\}$. Note that the κ-topology $\langle \mathcal{B}_\lambda \rangle_\kappa =: \tau_\lambda$ is λ-additive. The (λ, κ)-gaps are dense in $\mathbb{R}_\kappa$ for every $\lambda \leq \kappa$, so each τ_λ is not λ^+-additive. Moreover, each τ_λ generates the ordinary interval topology as its full topology.[7]

We show that if $\lambda < \mu < \kappa$, then τ_λ does not κ-embed into τ_μ. Any subspace X of an α-additive κ-topological space Y is again α-additive. Thus if τ_λ embedded into τ_μ, then τ_λ is μ-additive. But this is a contradiction: there are (μ, κ)-gaps in $\mathbb{R}_\kappa$, hence τ_λ is not μ-additive.

Finally, as κ is inaccessible, there are κ-many cardinals below it, so the τ_λ witness the statement. □

[6] Interval κ-topologies essentially depend on the order, rather than the field structure of $\mathbb{R}_\kappa$. Appeals to field structure give a further way of distinguishing between interval κ-topologies on ordered fields (e.g. [3, Theorem 5.1]).

[7] Indeed, each τ_λ is $f\mathsf{SC}_\kappa$ (essentially by [37, Proposition 4.3(a)]).

While these κ-topologies are not κ-homeomorphic, they form a chain of κ-embeddable spaces: $\tau_\mu \hookrightarrow \tau_\lambda$ for all $\mu < \lambda$ (but the converse fails). Using the standard technique in e.g. [37, Proposition 6.5], we can then construct 2^κ-many distinct κ-topologies with the same full topology:

Corollary 6.4.4. Let κ be inaccessible. There are 2^κ-many non-κ-homeomorphic κ-topologies on κ^κ whose full topology is fully homeomorphic to $(\kappa^\kappa, \langle\tau_b\rangle_\infty)$.

Proof. By Proposition 6.4.3, there are κ-many non-κ-bi-embeddable κ-topologies on κ^κ. Enumerate them as τ_λ for $\lambda \in \kappa$. Let $(\kappa^\kappa)_\lambda$ be a (distinct) copy of κ^κ with the κ-topology τ_λ. For each non-empty $A \subseteq \kappa$, let X_A be the set $\bigsqcup_{\lambda\in A}(\kappa^\kappa)_\lambda$ together with the κ-topology τ_A generated by sets of the form $\bigsqcup_{\lambda\in\kappa} U_\lambda$ for U_λ κ-open (possibly empty or κ^κ) in $(\kappa^\kappa)_\lambda$. Then, (X_A, τ_A) contains each space $(\kappa^\kappa)_\lambda$ as a κ-clopen subspace. Since each $(\kappa^\kappa)_\lambda$ is κ-connected, then $\{(\kappa^\kappa)_\lambda : \lambda \in A\}$ are exactly the κ-connected components of X_A.

Suppose $h : X_A \to X_B$ is a κ-homeomorphism for some $A, B \subseteq \kappa$ such that $A \neq B$. Then, h maps each κ-connected component of $\bigsqcup_{\lambda\in A}(\kappa^\kappa)_\lambda$ to some κ-connected component of $\bigsqcup_{\lambda\in B}(\kappa^\kappa)_\lambda$. Since $A \neq B$, we can find $\lambda \in (A \setminus B \cup B \setminus A)$. But then, h gives a homeomorphism between $(\kappa^\kappa)_\lambda$ and some $(\kappa^\kappa)_\mu$ for $\mu \neq \lambda$, contradicting Proposition 6.4.3. □

Chapter 7

What are generalisations and what are they for?

> The essence of mathematics is that it consists of generalizations.
>
> *K. Gödel*, quoted in H. Wang [204, 9.2.14, page 300]

In this chapter, we provide a philosophical account of generalisation in mathematics. We attempt to answer the *what* and the *why* questions: firstly, what are generalisations? Secondly, why do mathematicians generalise at all?

We provide a philosophical explication of generalisation which answers the first question. To this end, we compare generalisation to other processes of change in mathematical practice. We then analyse *syntacticist* accounts of the nature of generalisations, which are inspired by the philosophical literature, and instead suggest an account based on the content of pieces of mathematics, which we call *semanticism*. Along the way, we provide a typology for generalisations that we have seen in Chapters 3 to 6, and from the mathematical literature more broadly.

In answering the second question, we assess whether certain traditional accounts of the motivation of mathematical change fit generalisations.

Our core conclusions are that 1) generalisation in mathematics is a *sui generis* process of mathematical change which cannot be reduced to other processes, 2) neither explantoriness nor simplicity is necessary for the success of a generalisation, and 3) a syntacticist account of the nature of generalisation is untenable, we must instead opt for a form of semanticism.

7.1 Introduction

When doing mathematics, there are times when mathematicians describe themselves as generalising[1] other pieces of mathematics. Moreover, this mathematical

[1]We use the word *generalisation* ambiguously to refer to the over-arching process of mathematical change, the classical case-generalised case pair, or the generalised case alone, as clarified by context.

practice tracks some underlying phenomenon. We see this in its use:

> "Measurability [of infinite cardinals] is a direct generalization of the existence of ultrafilters over ω" (Kanamori, [108, page 26])
>
> "[Boolean-valued models] are a generalization of ordinary models in the sense that the truth-values are not 0 and 1 ..., but are elements of a given complete Boolean algebra." (Jech, [104, page 55])
>
> "The concept of continuous mapping is a generalization of that of real-valued continuous function." (Nagata, [164, Ex II.12])

These uses, and those of mathematical practice more broadly, show clear and consistent patterns. We list further examples in Section 7.2.2. These connect generalisation to, amongst other things, expansion, corresponding properties, and more sophisticated structures with certain commonalities. The broader usage, as illustrated by our examples, gives us good reason to believe that generalisation is a genuine unified phenomenon in mathematics.

Mathematical practice is dynamic, featuring a number of *processes of change* (see [86, 122, 211]), of which mathematical generalisation is one. These processes are the ways by which mathematics as a practice develops, changes shape, expands (and contracts), is organised, restructured, clarified, and so forth. Whilst some processes are about restructuring 'old' mathematics (e.g. formalisation), others, including generalisation, describe the processes of generating 'new' mathematics.[2] We encounter these other processes of change later, as we distinguish generalisation from them (Section 7.3).

Why go to the trouble of giving an account of generalisations? Certainly, there is a broad benefit of philosophically clarifying more vague aspects of mathematical practice. To understand mathematical practice properly, we need an appropriate conceptual toolbox, and that conceptual toolbox must include some account of generalisation.

A more particular philosophical problem is the unity of generalisation. The natural pre-theoretic attempts to characterise mathematical generalisation do not capture its essence well. 'Increased generality' is insufficient: ZFC is more general than Pythagoras' theorem, but the former does not generalise the latter (cf. Sections 7.4 and 7.5.3). One appealing formalisation, to reduce generalisation to logical universal generalisation, is discussed and rebutted in Section 7.5.2. Meanwhile, Section 7.2.2 shows the diversity of generalisations, with various kinds (which we call *species*) of pieces of mathematics being appropriate for generalisation, and for generalisation acting differently for pieces of mathematics of the same species. These discordances stand in tension to the fact that generalisation seemingly forms

[2] A simple account of historical development suffices for our purposes (the real development need not be simple, see [186]). To avoid the well-rehearsed historiographical problems with a Whig conception of generalisation as 'progress', we describe these processes as *mathematical change*. Our focus is the structure, rather than the history, of known generalisation, though the latter guides the analysis. Likewise, we exclude cognitive generalisations (cf. Footnote 9), and those from mathematics education (cf. [55]).

a harmonious grouping of instances of mathematical change. What, then, underlies generalisations to unite them?

The structure of the chapter is as follows. Section 7.2 contains a diverse collection of examples, the *zoo*, which we use throughout, and an initial taxonomy of generalisations. We then make some initial observations on the nature of generalisation. Next, in Section 7.3, we argue that generalisation is a *sui generis* process of mathematical change, by showing that it cannot be reduced to or identified with domain expansions (as some generalisations expand domains whilst others do not), and abstraction (as generalisations may occur at a fixed level of abstraction). Section 7.4 describes the purposes and goals of generalisation. We see that no standard single goal motivates generalisation, instead the motivation is a more nuanced competing collection of goals. Finally, in Section 7.5, we broach the nature of generalisations. We argue that syntacticist accounts of the nature of generalisation are untenable. We propose an alternative non-syntactic schema for accounts, skeletal semanticism. Our proposed instantiation explains generalisation as pieces of content hanging together in an appropriately similar way, where the level of generality simultaneously increases.

7.2 The zoo

In this section, we describe the inhabitants of our zoo of generalisations. We classify these generalisations into various non-metaphysical types, called species (Section 7.2.1). Using these species, we identify a necessary condition on accounts of the nature of generalisation, the Adequacy Condition. We also classify accounts of the nature of generalisation based on how they treat the species. In Section 7.2.2, we meet the residents of the zoo. These are the example generalisations we use throughout. Lastly, in Section 7.2.3, we describe some initial features of generalisation which the examples demonstrate.

7.2.1 The species, and first dimensions

Generalisation is a process of change of real mathematics, and in discussing it, it is useful to have a classification of the types of pieces of real mathematics. With their purposes elsewhere, philosophers of mathematics typically classify mathematics metaphysically, into something like objects, syntactic strings/expressions, and concepts. This seems unhelpfully narrow for the analysis of mathematical methodology and practice. We describe a more refined, naturalistic classification of pieces of mathematics, which we call the species. The species are *not* intended as metaphysical categories, but something more like functional types, a form of scaffolding of mathematics. We make no claim about whether these map onto *ontological* types. And so, the species:

1. objects;[3]

2. concepts;

3. object-types (possibly accessed using syntactic definitions);

4. operations, relations, and functions;

5. axioms;

6. theorems;

7. proofs (including syntactic formal proofs);

8. structures (including syntactic theories); and

9. areas, fields, and sub-disciplines.

The list is not exhaustive (questions, conjectures, diagrammes, and proof techniques/procedures may be further species), but is sufficient for our purposes.

We can classify accounts of the nature of generalisations by whether they are *species-by-species*, i.e. explain generalisation for each species individually (e.g. 'this is how object-generalisation works, this is how axiom-generalisation works', and so on), or *unified*.

Concurrently, many different species are suitable for generalisation (see Section 7.2.2).[4] We treat each species as equally essential for candidate accounts of generalisation to explain, which yields the following:

> The Adequacy Condition: an account of the nature of generalisations in mathematics must explicate the nature of generalisations of each species which can be generalised.

7.2.2 Meet the residents

Now the residents. Each of the following varieties of generalisation exposes some features which any account of mathematical generalisation must explicate. We outline these features in Section 7.2.3.

Numbers and functions

A classic source of generalisations are those of *number*. First, we have the generalisations of finite arithmetic to 1) infinite cardinal arithmetic, 2) infinite ordinal arithmetic, and 3) numerosities respectively (see [141]).

[3] "Objects" isn't meant in a metaphysically strong way: all reference to actually existing mathematical objects should be treated as us looking through 'Plato-tinted spectacles' [22, page 11].

[4] Which species? We think any. This 'generalisation maximalism' dovetails with the account of mathematics as the study of all (suitable) structures, but it's not really wedded to such a position. In short, given a tool, mathematicians tend to use it on anything they can.

Secondly, a classic sequence of generalisations is from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ to $\mathbb{R}$, and finally to $\mathbb{C}$. These, the Cantorian generalisation of finite arithmetic, and the generalisation from arithmetical to arbitrary functions (Section 7.2.2), are the most discussed examples in the literature (e.g. [33, page 86], [81, page 96], [122, page 210]).

Another number generalisation is from $\mathbb{Z}$ to the object-type (or concept) *ring* (see Section 7.2.3).

A final numerical example is the generalisation from $\mathbb{R}$ to $\mathbb{R}_\kappa$ (detailed in Section 2.2.3). For each suitable cardinal, κ, the ordered field $\mathbb{R}_\kappa$ satisfies generalisations of many of the properties of $\mathbb{R}$. These generate a host of generalisations of theorems (e.g. Chapter 4). The generalisation of a topology (on $\mathbb{R}$) to a κ-topology (on $\mathbb{R}_\kappa$) and its resultant generalisation of continuity, κ-continuity, is important to us (see Chapters 4 and 6). The form of several of the resultant theorem-generalisations follow a recognisable pattern (and perhaps are predictable, more on this in Section 7.5.2), e.g. IVT (detailed in Section 4.2.2):

IVT1. Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ passes through all intermediate points.

IVT2. Every κ-continuous function from $\mathbb{R}_\kappa$ to $\mathbb{R}_\kappa$ passes through all intermediate points.

Meanwhile, other theorem generalisations have a less recognisable, unfamiliar pattern, which diverges in form, and may even be *unexpected* before the process of generalisation begins. Such a case is BWT (Section 4.3):

BWT1. Every bounded $\mathbb{R}$ sequence has a convergent subsequence.

BWT2. Every 'nicely bounded'[5] $\mathbb{R}_\kappa$ sequence has a convergent subsequence if and only if κ has the tree property.

In all generalisations, the classical case must be recoverable from the generalised case (see the Golden Rule, Section 7.2.3). This is satisfied for BWT2.: $\aleph_0$ has the tree property, so we recover the classical case using a little additional information about the relevant cardinal in the right-hand side of the generalised case.

A further broad class consists of the generalisations of functions. Some generalisations of functions are basically similar to those of number, like when the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is expanded to the set of integer functions. A notable example is from Riemann integration to Lebesgue integration. Here, the generalisation expands the domain of application on a function, but within a fixed source of entities, i.e. the set of Riemann-integrable real functions on a bounded interval is included in the set of Lebesgue-integrable real functions on a bounded interval, but all of these are from an original source, namely the real functions (see Section 7.5.2).

A very different kind of generalisation of function is that from the algebraic (or algebraic and trigonometric, and so on) functions to arbitrary functions, i.e. in ahistorical terms, from the constructive/model-theoretic to the set-theoretic. We return to this in Section 7.2.3.

[5]Specially, bounded interval-witnessed κ-sequences (Section 4.3).

Two theorem generalisations

Our next two examples are generalisations of theorems. We mentioned the iterative object-type generalisation from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$, and so on. Another example of iteration, for the species *theorem*, is the sequence of generalisations of BFPT (cf. Section 4.2.3). For our purposes, the classical and the most general case are:

BFPT1. If $C \subseteq \mathbb{R}$ is a non-empty closed bounded interval, and $f : C \to C$ is $\mathbb{R}$-continuous, then there is some $x \in C$ such that $f(x) = x$.

BFPT2. If V is a Banach space, $C \subseteq V$ is non-empty, closed, and convex, and $f : C \to C$ is V-continuous with a compact image, then there is some $x \in C$ such that $f(x) = x$.

Intermediate generalisations extend the result from $\mathbb{R}$ to $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$ [84, Chapter 0]. *Recastings* of this result are discussed in Section 7.5.2.

We also have what we might call *assumption-dropping* generalisations, where it is merely observed in a proof that a certain property is not necessary, e.g. generalisations of proofs for particular metric spaces to arbitrary metric spaces, like the following (essentially due to Fréchet [199]):

MS1. For any $n \in \mathbb{N}$, every Cauchy sequence in $\mathbb{R}^n$ has a limit.

MS2. For any complete metric space, X, every Cauchy sequence in X has a limit.

This is a theorem generalisation, which goes via a proof generalisation: in the proof, it is observed that only the metric properties of $\mathbb{R}^n$ are necessary, and hence the same proof holds for any metric space, hence the name 'assumption-dropping'.

Background assumptions and axiom elimination

Further generalisations turn on changing relevant (possibly implicit, possibly meta-theoretic) assumptions and axioms. These can be split into two kinds, those where we weaken object-theory, and those where we weaken the background mathematical theory.

The generalisation from Euclidean to Bolyai's absolute geometry is an example of the first kind.[6] The generalisation permits spaces of arbitrary curvature, so includes Euclidean geometries and (properly) non-Euclidean geometries (see [41, §8.3, §9.1], [81, page 104]). Here, the eliminated axiom (the Parallel Postulate) is in the target object-theory.

We see this more dramatically in mathematical logic, where the fine-grained control of theories is central. For any choice principle, P, where ZFP is strictly weaker than ZFC (i.e. a *fragment*, see Section 3.1), it is natural to say that ZFP generalises ZFC.

[6] A very strong proposal of this kind is that *constructive mathematics generalises classical mathematics*. This seems to be Moschovakis' position for (some substantial fragment of) descriptive set theory [160]. We set such logical concerns to one side, and assume a classical background logic throughout.

For an example turning on weakening the background mathematical theory, take real analysis as our object-theory. This might ordinarily be carried out using ZFC as the background theory. This can be generalised to real analysis with ZFP in the background (see [90, page 72], [116, Chapter 1]). Similarly, given a fixed definition, we can say that the object-kind consisting of the objects which satisfy this definition under ZFP generalises the object-kind consisting of the objects satisfying that same definition under ZFC, so e.g. the real functions of ZFP generalise the real functions of ZFC.

Trivialities, Frankenstein, and gruesome disjunks

Our final major class of examples are the edge cases: it is not immediately clear whether these are generalisations, and establishing this is one of our concerns. The three classes are 1) the 'underly general' trivialities, 2) the 'overly general' trivialities, and 3) the deliberately constructed 'Frankenstein' disjunctions.

The 'underly general' trivialities are when the classical case *is* the candidate generalised case. Conversely, the 'overly general' trivialities go very far. One might say, in keeping with some broad mathematical goal of maximality, that generalisations ought to go 'as far as possible'. This leads quickly to worries of *over-generalisation*. For example, one might ask whether the theory consisting of just the axiom Extensionality generalises ZFC, or whether any theorem can be generalised to the Verum, $\top$. We use these as test cases for accounts of generalisation, but note that the inclusion of candidate 'over-generalisations' as generalisations is not obviously fatal for an account.

The final puzzles are the Frankenstein examples. Frankensteins are most easily described for theorems: we move from theorem "φ" to theorem "$\varphi \vee \psi$", for some mathematical ψ. So, Cantor's theorem, "For any set X, $|X| < |\mathscr{P}(X)|$" is converted into "For any set X, $|X| < |\mathscr{P}(X)|$ or $2 + 2 = 4$". We call this junk disjunct a *disjunk*. Disjunks are reminiscent of gruesome properties in metaphysics (see [182]). Frankensteins can also be constructed for other species, e.g. using junk axioms in a theory or junk lines in a proof. Intuitively, it seems these should not count as generalisations. We discuss these in Sections 7.3.1, 7.4, and 7.5.3.

Odds and ends

Examples are not yet standardised in the scattered literature. For reasons of space, we cannot develop all the examples used therein to which we allude, but we include a list here:

1. Abelian groups are generalised to groups (see Section 7.3.2).

2. Continuity at a point in $\mathbb{R}$ is generalised to continuity over the whole of $\mathbb{R}$ (see Section 7.3.2 and [146, page 19]).

3. Metric continuity is generalised to topological continuity (see Sections 7.3, 7.5.2, and [202, page 15]).

7.2.3 Initial features

These examples easily yield some initial principal features of generalisation, which we describe here. We first state the relationality of generalisation. We show that generalisation-relation is non-injective, relative, and possibly anti-reflexive and non-transitive, in the senses we define. We also show that certain generalisations can be *recast* as different species, whilst generalisations may be *species independent*. We categorise generalisation as either intra-species or inter-species. In the subsequent sections, we investigate the more fundamental nature of generalisation. Any account of the nature of generalisation must dovetail with these initial features.

Relationality and the Golden Rule

Generalisation is relational, relating the classical and the generalised case. A necessary condition is:

Golden Rule. If a piece of mathematics G is a generalisation of a piece of mathematics C, then the classical case C is suitably recoverable from the generalised case G.

The simpler requirement, that the classical case must be an example of the generalisation, is too strong. This is because a piece of mathematics may be generalised in various ways: in logical universal generalisation, the classical case is recovered by replacing a variable with a fixed parameter; in cases where a certain property becomes constitutive of a new notion [202, page 46], e.g. from $\mathbb{Z}$ to *ring*, the classical case instantiates the new notion. A notable case is the recovery of BWT1. from BWT2., where we need a little additional information to recover the classical theorem (Section 7.2.2).

If the term 'suitable methods of recovery' is sufficiently generous, then the Golden Rule is not sufficient: Pythagoras' theorem can be recovered from ZFC, but the former does not generalise the latter. Indeed, from any theorem, we can (classically) recover any other theorem using one's preferred background theory of mathematics! So, only certain recovery methods suffice, including instantiation, fixing variable parameters and notions, adding axioms to theories, adding properties to definitions or object-types, and the fulfilment of certain antecedents in (bi)conditionals.

Order theory of the generalisation-relation

Next, we extract information about the *generalisation-relation*, i.e. the relation between the classical case and the generalised case.

Finite arithmetic can be generalised to cardinal, ordinal, and numerosity arithmetic. These generalisations do not form a single unified theory of arithmetic. Hence, one piece of mathematics might have several mutually incompatible generalisations, so the generalisation-relation is not injective (though incomparable generalisations might have a common further generalisation, as suggested in [83]).

The sequence of generalisations from $\mathbb{N}$ to $\mathbb{Z}$ to $\mathbb{Q}$ to $\mathbb{R}$ to $\mathbb{C}$ shows that generalisations are *relative*: a generalised piece of mathematics can, itself, be generalised. Hence also the terms 'the classical case' and 'the generalised case' are relative, e.g. $\mathbb{Z}$ plays both rôles.

Meanwhile, the generalisations from arithmetic to arbitrary functions show that some generalisations *are* (absolutely) maximal. It is natural to say that the most general notion of a function is just anything which takes an input to an output: anything more general stops being a function. The point is that *no* hypothesis is made, except for functionality (see [31, pages 390-393]).[7] Hence, the arbitrary functions are the most general notion we could reasonably use, i.e. they cannot be reasonably generalised. This means that not every piece of mathematics stands in the (forwards) generalisation-relation to some other piece of mathematics.

Some cases of iterated generalisation seem transitive. For example, it seems reasonable to say that $\mathbb{Q}$ generalises $\mathbb{N}$ (via $\mathbb{Z}$). But not all such two-step generalisations seem transitive. It is doubtful that the object-type *ring* is rightly called a generalisation of $\mathbb{N}$ (via $\mathbb{Z}$). This suggests the generalisation-relation is not (always) transitive. We leave open whether generalisation of a fixed sort is transitive.

Finally amongst the order-theoretic properties, we can ask if the generalisation-relation is reflexive, i.e. does a piece of mathematics generalise itself? If so, then 'underly general' trivialities count as generalisations. One can easily imagine this being a consequence of a formal theory of generalisations. In Section 7.5.2 we describe and critique a simple account which classifies pieces of mathematics as generalisations based on syntax alone, a basic version being: if '$F(c)$' is a theorem then a theorem '$\forall x(x \in D \rightarrow F(x))$', for some domain D, is a generalisation. In which case one generalisation would be '$\forall x(x \in \{c\} \rightarrow F(x))$'; but this is semantically equivalent to '$F(c)$'. So, pieces of mathematics would generalise themselves. Alternatively, one might have an intuition that a generalisation is, by definition, non-trivial (e.g. Villeneuve's account, see Section 7.5.2). Either option seems palatable: one could capture the latter view from the former perspective as "non-identity generalisations", then recapture the latter view by considering "generalisations or the identity".

Intra-Species, inter-Species, and function

We can characterise generalisations as being either intra-species or inter-species. Most, but not all, of our examples are naturally viewed as *intra-species*, for example the object-to-object generalisation from $\mathbb{N}$ to $\mathbb{Z}$. The generalisation from $\mathbb{Z}$ to a ring is instead *inter-species*: $\mathbb{Z}$ is an object, whilst *ring* is an object-type[8] (or concept).

One complication here is the possibility of *generalising a species*. We see this play out in the generalisation of *function* (Section 7.2.2). On one level, the gener-

[7] For example, real functions can be generalised to real distributions [33, page 148], [192, page 4], but real distributions are still a kind of arbitrary function, so the notion of an arbitrary function is not generalised.

[8] *Pace* structuralist construals of $\mathbb{Z}$ as an object-type; but presumably the structuralist has some gloss of the everyday distinction between objects and types of objects in mathematics.

alisation from arbitrary to abstract functions is similar to the previous examples; for particular domains, e.g. $\mathbb{R}$, the generalisation consists of a domain expansion $\{f : \mathbb{R} \to \mathbb{R} : f \text{ is algebraic}\}$ to the set $\{f : \mathbb{R} \to \mathbb{R}\}$. But it is more naturally phrased as a concept generalisation, especially within its historical context (see [31, 157, 186]). Rather than a domain expansion, the concept *function* itself is generalised, or perhaps replaced. One must tread cautiously here. However, for our purposes, this can be largely coded as concept or object-type generalisations.

A first pass at revisionism

Next, we show that pieces of mathematics can be *recast*, i.e. one piece of mathematics can be captured in multiple ways, notably with different species. Recasting might be articulated in terms of the preservation of mathematical content (see Section 7.5.3), but for now, we rely on an intuitive understanding of the term. We also note that recasting does not mean that generalisations of one species always come hand-in-hand with generalisations of other species. Section 7.5.2 analyses recasting in more detail.

The example generalisations typically have an obvious representation, which in turn has an obvious species. But we can often recast this representation. A very simple recasting is from "$\varphi \wedge \psi$" to "$\psi \wedge \varphi$". But recasting transcends rearranging clauses. Here are some possible phrasings of BFPT:

BFPT1. If $C \subseteq \mathbb{R}$ is a non-empty closed bounded interval, and $f : C \to C$ is $\mathbb{R}$-continuous, then there is some $x \in C$ such that $f(x) = x$; or

BFPT1a. $\mathsf{BFPT}(\mathbb{R})$ (i.e. $\mathbb{R}$ has a certain property of fields); or

BFPT1b. $\mathbb{R} \in BFPT$ (i.e. $\mathbb{R}$ is in the class of Brouwer's-fixed-point spaces).

These different statements are suitably mathematically equivalent, so we say that they recast the same piece of mathematics. Propositional recasting may be a kind of propositional equivalence (as in passive sentence-active sentence propositional equivalence, [65, §3]). Non-sentential recasting follows the same lines. We also appear to have inter-species recasting of generalisations, e.g. cardinal arithmetic generalises finite arithmetic (a theory generalisation), cardinal numbers generalise natural numbers (an object generalisation), and facts about cardinal numbers generalise facts about finite numbers (theorem generalisations). These seemingly capture the same generalisation.

Recasting can be used to analyse accounts of the nature of generalisation. Suppose we can recast A as B using recasting R, and B is an example of a generalisation. Is A then a generalisation? Does it depend on the type of recasting, R? For now, we allow a little flexibility, but favour accounts of generalisation which explain generalisations more naturalistically, i.e. those which don't resort to substantial recasting.

Finally, we observe that recasting does not require that generalisations of one species always correspond to generalisations in the other species. Instead, some generalisations happen *independently* of whether there is a corresponding generalisation in some other species. We see this in descriptive set theory, where some theorems have different proofs in ZF and ZFC (see [161, §7.F]). Here, the theorem is not generalised, only the proof is. So, we cannot recast every generalisation into any other species. The legitimate extent of recasting and revision is a major topic in Section 7.5.

7.3 What generalisations could not be: A lone figure

We can classify mathematical change into several patterns. Some patterns have a more tractable characterisation than generalisation, a few may even be formalisable. For example, (set-theoretic) reduction is the process of providing a set-theoretic Ersatz for each piece of mathematics which replicate all relevant properties. Reduction is clearly distinct from generalisation. For others, the demarcation is less clear; might generalisations be a variety of domain expansions? We reject this, and argue that generalisation is a *sui generis process of mathematical change*, which cannot be reduced to other processes. Some pieces of mathematics may instantiate two processes, e.g. a generalisation which is an abstraction. However, these are not generalisations *in virtue* of being abstractions, rather they exhibit properties which make them properly called generalisations and properly called abstractions. To justify this position, we argue that generalisation is either not co-extensional or not cointentional with two close processes of change of mathematics: domain expansion and abstraction.

Besides reduction, many processes are obviously conceptually distinct from generalisation, e.g. concept formation and axiomatisations [122, page 194], [211, page 163], so can be safely ignored. Specialisation acts in the opposite direction to generalisation, so is distinct [170, page 104]). Nor can mathematical generalisation be simply the mathematical instantiation of natural scientific generalisation.[9] Refinement is only slightly more difficult. A very rough description of theorem generalisation is:[10]

1. some generalisations say the same about more than the classical case; and
2. other generalisations say less about more than the classical case.

Meanwhile, we can roughly characterise a refinement as saying more about the same. Any reasonable precisification of these glosses suffices to distinguish the two processes. So, they are not coextensive.

[9] Gorskii unifies mathematical generalisation with scientific generalisation [81, 82]. We think Gorskii's unification fails: types $A2$ and $A3$ are essentially scientific, types $A4$ and $A5$ are essentially mathematical, whilst $A5$ is possibly instantiated by both, as a form of domain expansion. Gorskii's own mathematical instantiation of $A5$, optical to electromagnetic theory, does not work as this is really a *cognitive realisation*, rather than a metaphysical domain expansion. But we can find scientific examples of real domain expansion, e.g. special to general relativity (we can also find mathematical examples of cognitive realisations in mathematics). Either way, domain expansion cannot explain generalisation by Section 7.3.1.

[10] This description is not taken to be necessary, but may be sufficient.

Distinguishing refinement from generalisation is a test case for how we distinguish generalisation from other processes. Next, we show that domain expansion and abstraction do not cover all kinds of generalisation. So, we deny *Colyvan's Dichotomy* (loosely based on [33, page 99]), which states that generalisation is exactly the combination of domain expansion and abstraction:

To generalise is to:
1.) extend a system, or
2.) abstract a similarity between systems.

Colyvan's Dichotomy

We conclude with a revision of Colyvan's Dichotomy (Section 7.3.3).

7.3.1 Domain expansion

We begin with domain expansion (see [23, 142] for details). Some instances of generalisations are apparently also instances of domain expansions [33, page 99], [201, page 410]. The generalisation from the spaces $\mathbb{R}^n$ to the complete metric space is one example: generalising from MS1. to MS2. can be interpreted as a domain expansion, from the domain of $\mathbb{R}^n$ spaces to the domain of complete metric spaces (more on this on page 166).

So, in an uninteresting way, some generalisations are *also* domain expansions. But should we think that generalisation is simply some form of domain expansion? Some writers seem to use the terms interchangeably (e.g. Bellomo's reading of Peacock [12, page 377]). We disagree. We sketch an argument in the literature which would appear to separate the two processes straightforwardly, but show that it does not succeed. Instead, we provide further examples which do distinguish the processes.

Domain expansion does not include generalisation

Villeneuve proposes to separate domain expansion from generalisation with generalised concepts which are (exactly) coextensional with the classical concept they generalise [202, page 10]. His principal example is the generalisation from metric continuity to topological continuity [202, page 15]. These coincide on $\mathbb{R}$. So, the argument goes, it is a generalisation without a domain expansion. However, this misdescribes the situation, as the focus on $\mathbb{R}$ artificially restricts the notion of topological continuity. In reality, the *intended* domain of topological continuity properly extends the domain of metric continuity, to include functions on (non-metrisable) topological spaces. Restricting to the domain of metric functions is like saying that 'amongst non-negative integers, the theory of the integers is co-extensive with that of the natural numbers'. This coextensivity is only due to an artificial restriction on the domain.

However, there are generalisations which are not domain expansion. We focus on abstraction and those which lack an intended domain.[11]

[11] Sometimes domains are *replaced*. When generalising from $\mathbb{R}$ to $\mathbb{R}_\kappa$, rather than expanding the domain of $\mathbb{R}$, we build $\mathbb{R}_\kappa$ from a parametrised generalisation of $\mathbb{Q}$, $\mathbb{Q}_\kappa$ (Section 2.3.5). (A

Some generalisations are abstractive, for example from $\mathbb{Z}$ to the object-type (or concept) *ring*. Whilst the domain of $\mathbb{Z}$ is the integers, *ring* does not have a domain in the same sense (the 'underlying set' will not do: object-kinds/concepts do not have domains in the same way). One might suggest the class of rings as the domain of *ring*, but this does not yield a domain expansion: individual integers are not rings! Instead, $\mathbb{Z}$ is an *element* of the domain of *ring*.

Other generalisations may not be equipped with a domain at all, perhaps including the generalisation from ZFC to ZFP of page 163. Viewing this as a domain expansion of *the class of models* of ZFC does not help, as it stretches our restrictions on revisionism (the natural description is a generalisation of *axioms*), and this class is sensitive to the meta-theory, apparently unlike the generalisation itself.

Generalisation does not include domain expansion

Conversely, some domain expansions are not generalisations. We argue for the conceptual difference between domain expansion and generalisation, and hence find a non-generalisation domain expansion.[12]

Let P and Q be pieces of mathematics, with contextually associated domains D and E respectively. Here are two forms of account of domain expansion:

1. The simple view: Q is a domain expansion of P if and only if $D \subsetneq E$.
2. The refined view: Q is a domain expansion of P if and only if $D \subsetneq E$ and E is of the right kind, as determined by D.

With the simple view, work goes into establishing why certain domain expansions are fruitful [12, page 369]. A typical explanation is that certain domain expansions are valued as they "round off a domain and simplify its theory by adjoining elements" (Manders [142, page 554], see also [91, page 161], translated in [213, page 216]) but according to the simple view, this 'rounding off' is not necessary for being a domain expansion: domain expansion is *simply* to expand the domain, nothing more. Meanwhile, the refined view distinguishes domain expansion from plain domain inclusion, and may take the 'rounding off' to be constitutive

Distinguishing generalisation from domain expansion under the simple view is easy: we use the bruteness of domain expansion, compared with some aspect of preservation of content (or substance, material, see Section 7.5.3) in generalisation. So, we create a Frankenstein's theorem, using the fundamental theory of arithmetic.[13] Let $D := \mathbb{N} \cup \{-1\}$.

similar argument can be given if viewing $\mathbb{R}$ and $\mathbb{R}_\kappa$ axiomatically.) This gives $\mathbb{R}_\kappa$ a distinctly different flavour to domain expansion, making $\mathbb{R}_\kappa$ the κ-analogue of $\mathbb{R}$ (see Section 7.3.3).

[12] A radically revised predicate, like Q_{triv} (page 183), artificially turns any generalisation into a domain expansion. But this is clearly unnatural and unintended.

[13] Alternatively, extend a continuous $f : (0,1) \to \mathbb{R}$ which is right-continuous at 1 to $\hat{f} : (0,1] \to \mathbb{R}$ by setting $\hat{f}(1) = \lim_{x \to 1} f(x)$. Whilst $\mathsf{Dom}(f) \subsetneq \mathsf{Dom}(\hat{f})$, f and $\hat{f}$ apparently have the same content (e.g. they are inter-definable), with no increase in the 'level of generality' (see Section 7.5.3).

1. Every $n \in \mathbb{N}$ has a prime factorisation.

2. Every $n \in D$ either has a prime factorisation or is -1.

This is a domain expansion under the simple view: $\mathbb{N} \subsetneq D$. However, this does not seem to be a generalisation, as the disjunk is irrelevant to the classical theorem, nor does this generalise the nature of the natural numbers.

This may not suffice given the refined view of domain expansion; as one could argue that domain expansion also has a naturality constraint which this Frankenstein's theorem violates. Even then, we may have some examples of domain expansions which are precisifications/numerical refinements of theorems, which seem not to be generalisations (e.g. they do not increase the level of generality, see 7.5.3). One possibility concerns the proofs of fragments of Goldbach's conjecture. A *Goldbach fragment* is any result which specifies a domain $D \subseteq \mathbb{N} \setminus \{0, 1, 2\}$, and states that for all $n \in D$, n is the sum of two primes. Some Goldbach fragments are simple theorems of number theory, e.g. when $D = \{2 + p : p \text{ is prime}\}$. Others are numerical checks (at the time of writing, the lower bound is $D = \{n \in \mathbb{N} : n \leq 4 \times 10^{18}\}$, [169]). Improved Goldbach fragments are domain expansions (they expand the domain, and maintain naturality), but do not seem to be generalisations, for they are not more *general*, they are just (in some sense) more *complete*.

We conclude that we can distinguish domain expansion from generalisation in terms of extension and nature, and neither includes the other.

7.3.2 Abstraction

Can we also distinguish generalisation from abstraction? Some suggest that abstraction always involves generalisation [146, page 19], whilst others presuppose some unstated division between the two (e.g. [85, page 329], [138, page 39], possibly also [28]). We tow a middle path: generalisations are not a type of abstraction, but some generalisations are abstractions. We distinguish these processes via differences in nature, by arguing that there are generalisations which are at fixed level of abstraction which cannot be abstractions.

Like generalisation, abstraction is difficult to get a good handle on. The Aristotelian characterisation of abstraction is subtractive: we exclude some of the properties of a fixed class of mathematical individuals, and thereby access a wider class. But this is not a full articulation. Abstraction principles [60, 140] give some further insight, but cannot be the full story either (see [13, page 82] on the diversity of abstraction).

We first articulate a rough notion of a *level of abstraction* (as in [61, 149]), and in turn an even rougher outline of *abstractness*. The latter is deliberately imprecise, to fit a more full development.

Mathematicians talk about pieces of mathematics being more or less abstract. This can be cashed out as a hierarchy of levels of abstractions. We hold that abstractions must, by definition, increase the level of abstraction.[14] We then ar-

[14] This is notably different to abstraction in natural science, which (typically) involves redescriptions of the *same* phenomena at different levels of theory [124, page 51], [167, page 209], whilst mathematical abstraction links *different* pieces of mathematics.

gue that, for any plausible cashing out of these levels of abstraction, there are generalisations which are at the same level of abstraction.

First, a clarification: abstractness is closely tied to the metaphysical abstract-concrete debate for objects. Multiple levels of *metaphysical* abstractness would be dismissed by the Fregean, who condones only two levels of abstraction, abstract and non-abstract. But this cannot be the full story: there is some sense in which pieces of mathematics are more or less abstract, independent of whether they are metaphysically abstract or concrete. For example, Marquis holds that *group* and *Abelian group* are at the same level of abstraction ([147, page 14], see [13, pages 101-104] or [203, page 5] for further examples). We have a *new problem of abstractness*:

(NPoA). How do we account for the intra-mathematical scale of abstractness which is applied to pieces of mathematics?

The division between generalisation and abstraction can be defended without a precise account of the levels of abstraction, i.e. without a precise response to the new problem of abstractness. For this, we look at generalisations which are candidates of having a constant level of abstraction, however these levels are precisified.

We give a first-pass articulation of abstractness in mathematics, with just enough detail for our purposes (e.g. to distinguish it from *familiarity*). This should account for the intuitive epistemological dimension of mathematical abstractness (see [146, page 2]), and the structural/ontological complexity dimension. We want to avoid simplistic logical notions of abstractness, such as 'adding quantifiers' (possibly Villeneuve's position, see [202, page 112]). These simplistic notions seem too distinguishing, e.g. given a background set theory where we often code repeated variables together (e.g. $\forall x \forall y F(x, y) \equiv \forall z F(\pi_1(z), \pi_2(z))$, see [161, §1C.2]). We give the following gloss:

> 'mathematical abstractness' is a measure of the complexity (or the construction) of pieces of mathematics and the ease with which we can refer to such pieces.

Complexity could then be cashed out through set-theoretic methods including size, membership-chain length [149, page 39], or V or L-rank. This only applies to objects, which is sufficient for our arguments, but could be naturally extended to apply to other species. We would need to ensure that all abstractions do increase abstraction as just characterised, but this seems possible.

With this rough gloss, we can identify generalisations with a fixed level of abstraction. Consider the classical case of 'continuity at $x \in \mathbb{R}$', and its generalisation is 'for every $x \in \mathbb{R}$, continuity at x' [146, page 19]. This additional universal quantifier doesn't add any complexity to the construction, nor does it add any complexity in reference, so the level of abstraction is fixed.

Conversely, might all abstractions be generalisations? Marquis tentatively suggests so [146, page 19]. One line of thought is to suggest that, unlike generalisation, there is an *organisational* variety (or possibly rôle) of abstraction [13, page 109]. But we leave this open.

7.3.3 Colyvan's Dichotomy redux

Colyvan's Dichotomy fails as it overgenerates (e.g. including all domain expansions). We conclude that mathematical generalisation is a *sui generis* process of mathematical change. Here, we revise Colyvan's Dichotomy to include two further flavours of generalisation, parametrised analogues, and weakened background theory generalisation.

A possible new flavour is parametrised analogue generalisation (for example Footnote 11, see [123, page 296] or [170] for more on analogy), e.g. the generalisations from $\mathbb{R}$ to $\mathbb{R}_\kappa$. One could recast this as an abstraction to a class-function, $\mathbb{R}_{(-)}$. But this revision seems unnatural, as the practice supports talk of (a particular) $\mathbb{R}_\kappa$ as a fixed field which is the (κ^{th}) generalisation of $\mathbb{R}$. However, as $\mathbb{R}$ embeds into $\mathbb{R}_\kappa$ (Proposition 2.3.25), this generalisation might be viewed as a particularly regular form of domain expansion, so too for other parameter analogues like it.[15]

The other additional flavour is generalisation by weakening background assumption (somewhat like [146, page 16]). We might use the generalisation from ZFC- to ZFP-real analysis to show this flavour is distinct. It is obviously not a straightforward domain expansion; each time, the intended domain is the reals. It also is not an abstraction, as the level of abstraction remains constant: the theories have exactly corresponding objects, properties, and connections. Finally, no parameter is overtly changed, and a hidden parameter recapturing seems revisionist.[16] This gives our new classification (Table 7.1), where we do not commit to *all* examples of each generalisation method necessarily being generalisations.

Input-Output Species	Generalisation Method
intra-species	domain expansions
inter-species	abstraction
	weakened background theories
	parametrised analogues?

Table 7.1: Colyvan's Dichotomy Redux

[15] For example, any $\eta_{2^{\aleph_0}}$-ordered rcf is a domain expansion of the *ordered field* $\mathbb{R}$ (Remark 2.3.26), to be a domain expansion of the *order* $\mathbb{R}$, it suffices to be an η_1-ordered rcf (Proposition 2.3.24). This suggests that domain expansion cannot be reduced to the superset relation, i.e. lends weight to the refined view of domain expansion.

[16] In other words, where the background theory is the parameter (e.g., model-theoretically, $\mathbb{R}^{\mathsf{ZFC}}$).

7.4 Why bother?

> [T]o think that mathematics pursues generality for the sake of generality is to misunderstand the sound truth that a natural generalization simplifies by reducing the number of assumptions and by thus letting us understand certain aspects of the whole.
>
> *H. Weyl*, [209, page 454]

Why do mathematicians generalise? We set aside possible sociological or psychological reasons, such as enjoyment or pressure to publish, and instead focus on the philosophical and methodological motivation. There is some background noise from the overarching methodological purposes of mathematics, e.g. to push results further, or to open the door to further results. Refinement pushes results further, but need not generalise. In this section, we provide an account of the *particular* purposes (or measures of success) for generalising.

We outline three proposed comprehensive motivations for generalising, generality, explanation, and simplicity. We argue that each fails to be comprehensive. Instead, we picture a range of competing goals of generalisation, which must be balanced, where no goal is in itself necessary and sufficient. We suggest that the proposal for generalisation as being explanation-oriented reveals that generalisation is unification-oriented. Finally, we mention the identification of new structures as contributing goal of generalising.

Generality

It might be thought that mathematicians generalise for a single fundamental reason (with possible supervening secondary goals). Both explanation and simplicity have been suggested as comprehensive analyses of the reason mathematicians generalise [122, page 208], [123, page 297], [209, page 454]. Generality itself is a natural third candidate. Here, we argue that each of these suggested factors fails, that at best each contributes, but must be balanced against other motivations.

First, we show that generalisation is not for the purpose of generality alone. We begin by fleshing out the notion of *generality*.

We can approach generality as a motivation by first reviewing the literature on generality (see [30, 47]): generality may involve having many applications [29, page 61], or connecting large numbers of theorems [29, page 82]. Poincaré takes a high density of some fact holding to yield generality, and suggests that expressions with several quantifiers also suffices [30, page 16], [31, page 406], [177, page 200]. Similarly, generality is associated with high dimensionality [31, page 407], and with extending procedures [85]. These provide a range of *ways of attaining* generality (which may conflict [19, 123]), rather than identifying generality *itself*. Despite this complication, these flesh out the intuition of what generality ought to mean.

With some grip on generality itself, can we say that mathematicians generalise solely in order to increase generality? Certain histories of mathematics have suggested so [28], [165, §1]. For example, the generality of 19th century mathematics is considered successful, compared to the redundancy of the many special-case lemmas in ancient Greek geometry [165, page 33].

However, we claim that self-motivation is implausible. Firstly, motivating generalisation by generality just pushes the problem one level up: why do we value generality then? Moreover, generality alone seems insufficient. Following Weyl [209, page 454], generalisation is not 'for the sake of generality': "mathematicians ... have pointed out that generalization is not an end in itself; what is to be found is rather the right generalization or the interesting one" (Breger, [20, page 221]). This 'right degree' of generality prohibits generality alone from being a motivation: if it were so, then generalising *past* the right degree of generality would still be considered successful, which it is not (e.g. the overly general trivialities, Section 7.2).

But we do not want generality to be the enemy of generalisation. Rather, Weyl's comments hint at diverse goals for generalisation, each of which contributes to motivating generalisation.

Explanation and, from the ashes, unification

> [G]eneralization [is] constantly pursued ... as the means to reach really satisfactory explanations which account for scattered individual results.
>
> *S. Feferman*, [58, page 3]

Some mathematical goals are organisational: we want to structure the diverse and wide-ranging mass of, e.g. mathematical entities and facts. *Explanation* has been suggested as a comprehensive organisational motivation for generalisation. Generalisation, being relation, helps add connections amongst pieces of mathematics. These connections may help explain various classical cases, or may unify a number of classical cases together. Kitcher sees this as the core motivation to generalise: generalisations are successful when they explain ([122, page 208], corroborated in [71], [81, page 204]), and they explain when they unify [121]. We argue that a common-or-garden understanding of explanation is not sufficient for a comprehensive motivation for generalisation, but suggest that unification is plausibly a contributing motivation, to be balanced amongst others.

Kitcher's notion of explanatoriness is quite specific, turning on the *mere existence of a common explanatory pattern* for multiple pieces of mathematics (see [121, §6]). But this is primarily a notion of unification, rather than about explanations proper. Instead, we focus on an ordinary, explanation-focused notion of explanitoriness. We don't rely on a fixed notion of explanatoriness, but a rough gloss is 'it encourages understanding in the subject'.

The problem is that successful generalisations need not explain the classical case, nor the classical cases explain the generalised case.[17] We see both of these in the following example. A key property for the generalised real analysis of $\mathbb{R}_\kappa$-sequences is *interval witnessing* (Section 4.3). This helps describe the underlying structure of convergence in $\mathbb{R}_\kappa$. However, interval witnessing doesn't help explain any behaviour in $\mathbb{R}$: *every* $\mathbb{R}$-sequence is interval witnessed. It is simply invisible in $\mathbb{R}$. Interval witnessing holds in classical analysis, but is entirely unexplanatory there. It is only explanatory in the generalised setting, where it shows when an $\mathbb{R}_\kappa$-sequence behaves like any $\mathbb{R}$-sequences do (so might still be unifying). This generalisation is more difficult to understand, and introduces structure which is unnecessary for understanding the classical case. It even curtails explanation of the classical structure, where more illuminating tailored explanations are available. It simply muddies the waters.[18]

Conversely, the classical case doesn't explain the generalised case: the invisible structure of interval witnessing means that the tailored explanation of the classical convergence does not explain the behaviour in the generalised case. The classical explanation is too blunt, it says that $\mathbb{R}_\kappa$-sequences do not exhibit the exact behaviour of $\mathbb{R}$-sequences, without explaining the similarities to when an $\mathbb{R}_\kappa$-sequence is interval witnessing.

So, the success here does not derive from explaining the classical case or explaining the generalised case. But this does point towards a more plausible contributing goal: unification. Despite their failure to explain, unifying generalisations can be very successful. More fruitful unifications unify behaviour so as to 'extract a mathematical core', rather than simply corralling disparate pieces of mathematics. While interval witnessing does not *explain* the behaviour of $\mathbb{R}$-sequences, it does unify the behaviour of $\mathbb{R}$-sequences with (some) $\mathbb{R}_\kappa$-sequences.[19]

Still, unification does not seem to be a comprehensive explanation. Unification seems to work well in e.g. natural science, where one might aim to generalise theories of the four fundamental forces to a comprehensive unified theory. But this pattern does not fit all generalisations. When generalising from Euclidean to absolute geometry, we do not corral together some old pieces of mathematics, rather we locate or extract a form for new pieces of mathematics (i.e. isolate a new structure, see page 176).

[17] Identifying unification and explanatoriness has also been questioned in the context of natural science, see [159, page 4].
[18] Infinite cardinal arithmetic is similar: it does not obviously explain classical finite arithmetic. Numerosities are even worse! Their definition requires an additional, superficially irrelevant, structure, like an arbitrary fixed ultrafilter (for more details on this, see [140, page 142]). But such an ultrafilter plays no rôle in explanations of finite cardinal arithmetic, and its existence is even independent of ZFC [212, page 28], which far outstrips the everyday set-theoretical strength required for explanations of finite cardinal arithmetic.
[19] The strength of unification over explanatoriness may turn on agent-independence (for details, see page 187): unification depends on the behaviour of mathematical objects, whilst explanatoriness depends on *agents* accessing the mathematics.

Simplicity

> The advantages of *generalization* in geometry [are] to simplify theories and shed an intuitive light on them.
>
> *M. Chasles*, [27, pages 209-210], translated in [151, page 142], emphasis original

Explanation and unification are only partial accounts of the motivation to generalise. An intuitively plausible goal of generalisation is to simplify. It has been suggested that simplification is *necessary* for the success of certain processes of mathematical change. This is explicit by the late 18$^{\text{th}}$ century [165, §1.2.3.], [186, page 319]. More recently, simplicity has been suggested as the marker for the success for domain expansion [91, 142]. It has also been claimed for generalisations [123, page 297], [209, page 454]. We dispute this: structures can become much less simple when we generalise, whilst still being successful. As with explanatoriness, we argue that the *necessity* of simplification is implausible, but it is plausibly a contributing goal to be balanced against others.

We continue with the generalisation from $\mathbb{R}$ to $\mathbb{R}_\kappa$. The theory and objects themselves are just more complex: there are more objects, with more bizarre properties, fewer regularities, and the objects and theory are less graspable (see Chapter 4). The constructions of $\mathbb{R}_\kappa$ are more involved, and there are fewer equivalent constructions ($\mathbb{R}_\kappa$ can be constructed as the Cauchy, but *not* the Dedekind, completion of $\mathbb{Q}_\kappa$). Classical facts about generalised continuity no longer hold universally, with numerous pathologies due to order-theoretic gaps in the field. This goes beyond unfamiliarity with $\mathbb{R}_\kappa$. However simplicity is cashed out, the $\mathbb{R}_\kappa$ fields are less simple than $\mathbb{R}$. Nor can we say, following Weyl ([209, page 454], cf. [71]), that generalisations simplify by reducing the number of assumptions: we move from using ZFC for analysis on $\mathbb{R}$ to ZFC with large cardinal assumptions for $\mathbb{R}_\kappa$ (see Section 4.3 and Chapter 6). However, it is obvious that these generalisations are at least somewhat successful. So, generalisations need not be simplifying to be successful.

Isolating new structures

We conclude with another (non-comprehensive) contributing motivation. A contemporary account of mathematics is as the analysis of structures, but this gives no guidance on which structures are significant or interesting. Generalisation provides a way to identify significant new structures, by grounding them in classical cases: if the classical case is significant, generalisation may preserve (some of) this significance. This is unlike unification and simplicity, which concern *established* mathematics. Meanwhile, some generalisations *generate new structures*. Consider the generalisation of $\mathbb{N}$ to $\mathbb{Z}$. One could describe this generalisation as a unification of objects (e.g. unifying $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$) or a unification of theorems about $\mathbb{N}$ (unifying various special-case theorems which implicitly use negative integer, e.g.

"if $n < m$, then $m - n + k = k - n + m$"). But a natural description of the $\mathbb{N}$-to-$\mathbb{Z}$ generalisation is as a process which isolates a new object, and then 'start afresh' with our analysis of $\mathbb{Z}$. So too in the $\mathbb{R}$-to-$\mathbb{R}_\kappa$ generalisation. Rather than unifying disparate pieces of pre-existing mathematics, it seems that some generalisations involve the recognition of a new object or theory based on the classical cases.

7.5 Nature: The thing itself

> [G]eneralizing is much more than just picking constants and replacing them with variables.
>
> *J. Tappenden*, [197, page 264]

Let's take stock. We've seen various generalisations of several species (Section 7.2.2). Section 7.2.3 gave us some grip on the process of generalisation. We've argued that generalisation is a *sui generis* process of mathematical change, with non-trivial interactions with other processes (Section 7.3). Lastly, we saw that generalisation has its own distinctive, competing goals (Section 7.4).

In the remainder of this chapter, we propose an account of what generalisations are. We have already classified accounts of generalisation as being species-by-species or unified (Section 7.2.1). We briefly sketch two further dimensions, formality and resource use (Section 7.5.1). We use these to describe two approaches to explaining the nature of generalisation, syntacticism and semanticism. In Section 7.5.2, we analyse the *syntacticist programme*, a family of accounts which propose to explicate generalisation in terms of relations of form, with a principal focus on syntactic patterns of pieces of mathematics in their everyday capturings in natural or formal language. This has appealing simplicity and predictability/algorithmicality, and would place generalisations amongst the nicely formalisable process of mathematical change. We argue that even the most refined version of syntacticism is unsustainable, due ultimately to syntactic variance or a risk of recoursing to a substantial use of semantic resources. Instead, in Section 7.5.3, we propose a semanticist account which avoids the problems of syntacticism, at the cost of less predictability and simplicity. We outline a schema of semanticist accounts, and propose a version based on *content* (broadly construed). In Section 7.5.4, we submit semanticism to ordeals corresponding to those faced by syntacticism, and show that semanticism ultimately fares better.

7.5.1 Dimensions redux: Resources and formality

First, we classify accounts of the nature of processes of mathematical change based on their use of resources, and on their formality. We begin with resources. Pieces of mathematics exhibit only two fundamental features which could be used to ground an account of the nature of generalisation. These are (1) their everyday form in natural or mathematical language, and (2) meaning/nature. We call the former the *syntactic* features, and the later the *semantic* features. We elaborate our particular conceptions of each of these features as we go on. If not drawing uniformly on these resources, accounts are patchy and case-by-case, and so struggle to explain how generalisation is a unitary phenomenon.

We can also discern a *scale of formality* in accounts of processes of mathematical change. Some processes are amenable to formal accounts, e.g. the simple view of domain expansion, or Buzaglo's account of concept expansion from [23]. They give a formal framework that precisely describes the relevant process of change, ideally a mathematical or logical one. Formal accounts may employ substantial technical machinery (e.g. Buzaglo's uses model-theoretic embeddings). Meanwhile, informal accounts explicate a process without a formal framework, e.g. Wagner's account of abstraction as the "incomplete, underdetermined, intermittent and open-ended translation between systems of presentation" [203, page 3].

Unification:	unified	species-by-species
Resource Use:	semantic	syntactic
Formality:	formal	informal

Table 7.2: Dimensions of Accounts of the Nature of Generalisation

7.5.2 Syntacticism

Full formality is the gold standard in accounts of processes of mathematical change. The strongest formalisations allow us to *construct* an instance of mathematical change, but this is implausible for generalisation, where outputs may be *unpredictable* from the inputs (more on this later). Abandoning constructibility, a weaker account is that the classical and generalised case exhibit a fixed syntactic pattern, typically a similarity pattern. So, checking that a candidate is a genuine generalisation becomes formal and routine, principally consisting of checking a pattern of syntactic form of the ordinary capturing of a piece of mathematics, in either a natural or formal language. We call this *syntacticism*. We focus on syntacticism as it is a particularly simple representative of the syntax-focused approaches to generalisation and is implicit in the literature.[20]

Syntacticism is the easy road to generalisation. We don't need to muck about with the semantic content of mathematics, or do too much hard work in classifying types of generalisation: it is a simple, possibly unified, even formalisable account. However, we contend, it cannot be sustained. Our strategy has four steps. We first outline a basic *Ramsey syntacticism*, which fails the Adequacy Condition. We refine this to a more encompassing *translation syntacticism*. We then raise a problem concerning the choice of defining properties, and detail two responses (*Anything Goes* and *verificational translation syntacticism*), both of which are unsatisfactory. Finally, we describe a broader methodological problem, which faces syntacticism of any stripe (*syntactic variance*). It is not clear whether any version can overcome this challenge. Hence, in Section 7.5.3, we pursue an alternative semanticist account.

[20] Other possible syntax-focused approaches to generalisation include those based on the deductive power of sentences and theories (see Footnote 29).

Ramsey syntacticism

We start by refining *logical* generalisation to a basic syntacticist account, Ramsey syntacticism. Whilst not explicit in the literature, this basic account is the common core of Villeneuve's syntacticism below, and of a stock of simple, logical conceptions of mathematical generalisation. We show that it fails the Adequacy Condition.

The most basic version of syntacticism says that (theorem) generalisation is just the replacement of a fixed parameter with a variable in some piece of mathematics. This reduces mathematical generalisation to logical universal generalisation. Clearly, we also require truth (see [81, page 89]), and the possibility to restrict the new variable to some domain, D. Formally:

7.5.2.I. A piece of mathematics, $\forall x \in D(F(x))$, is a generalisation if and only if it is true and $F(c)$ is a classical piece of mathematics with $c \in D$, for a suitable domain, D.

In such cases, the pattern of syntactic form which 7.5.2.I. explicates is entirely clear. According to 7.5.2.I., given a classical piece of mathematics, checking a candidate generalisation has only three steps. The first concerns syntax only, where we check that the classical and generalised cases are of the form $F(c)$ and $\forall x \in D(F(x))$ respectively. The second and the last steps are non-syntactic, but do not depend on any connection between the classical and generalised case: we check that the generalised case is true, and check that $c \in D$.

However, many generalisations do not take this form required for 7.5.2.I., for example the BFPT generalisations (from $\mathbb{R}$ to $\mathbb{R}^2$, and so on). We could recast the $\mathbb{R}^2$-generalisation with the domain $D = \{\mathbb{R}, \mathbb{R}^2\}$, but the general solution is to refine 7.5.2.I. slightly:

RAMSEY. A piece of mathematics, $\forall x \in D(F(x))$, is a generalisation if and only if it is true and $F(c)$ or $\forall x \in D'(F(x))$ is a classical piece of mathematics with either $c \in D$ or $D' \subsetneq D$ respectively, for a suitable domain, D.

Let's call this *Ramsey syntacticism* (after [173]). Versions of it have been alluded to several times (e.g. [171, page 108] and, with more suspicion, [137, page 138] & [146, page 19]). Like 7.5.2.I., Ramsey syntacticism has specific and isolated minor uses of semantic resources. Crucially, it does not require a semantic connection between the classical and generalised case: RAMSEY. has non-syntactic conditions, but these conditions do not turn on a non-syntactic relation between the classical and generalised pieces of mathematics; rather, they concern the connection between associated objects, domains, and a check of a truth condition.

Ramsey syntacticism is formal, syntactically-focused, and appealingly simple and procedural: it provides the candidate generalisations, then the mathematician's job is to check that $c \in D$ (or $D \subseteq D'$) and the truth of the candidate generalised case. That's all there is to generalisation! Granted, there must be some care about the domain, D,[21] but Ramsey syntacticism makes generalisation considerably more algorithmic than might initially be thought.

[21]It would seem that only certain domains are suitable for 7.5.2.I. and RAMSEY.. A natural precisification is that D must be suitably determined by, e.g., the nature of c. This already sneaks in some substantial use of semantic resources.

It also has certain logicist merits, as it reduces a piece of mathematical methodology to (a mild elaboration of) a piece of logical methodology. This reduction is to be expected, if mathematics reduces to logic. More pragmatically, be we logicist or not, we need an account of logical methodology anyway, and Ramsey syntacticism gives us two for the price of one.

An involved example shows that Ramsey syntacticism undergenerates theorem generalisations.[22] But the real deathknell is its failure of the Adequacy Condition: it is mute on non-theorem generalisations (as observed in [146, page 19]). So, we next outline a more adequate version of syntacticism. Then we describe the critical problem for all versions of syntacticism: the Scylla and Charybdis of overgeneration and substantial appeals to semantics.

Translation syntacticism

> General objects and general methods appear to be, in [Chasles'] eyes, two sides of the same coin
>
> *K. Chemla*, [29, page 58]

Ramsey syntacticism contains a seed of hope. Many non-propositional generalisations can be recaptured using recastings which are suitable for explication via RAMSEY. Easy cases are proofs, theories, and definitions. An example of such an account is Villeneuve's, which ultimately derives from parameter replacement:

> "[logical universal generalisation] only applies to statements and, because our considerations are mathematical notions, we will develop the extended logical generalization process which will be characterized by a fixed-variable relation between the initial notion and the new notion." [202, page iv].

[22] Generalisations can reveal subtleties hidden in the classical case e.g. EVT (see Section 4.2.4).

EVT1. Every continuous function from a closed bounded interval $I \subseteq \mathbb{R}$ to $\mathbb{R}$ attains its extrema.

EVT2. Every continuous function from a closed bounded interval $I \subseteq \mathbb{R}_\kappa$ to $\mathbb{R}_\kappa$ attains its extrema.

From EVT1., Ramsey syntacticism predicts EVT2. But EVT2. is false. A cunning recasting of EVT1. provides a fix: we also vary the notion of continuity: let τ'_{α_0} be the ordinary topology, τ'_κ be the κ-topology, $\aleph_0$-schmontinuity be continuity, and κ-schmontinuity be κ-supercontinuity:

EVT3. Every $\aleph_0$-schmontinuous function on a $\tau'(\aleph_0)$-closed bounded interval $I \subseteq \mathbb{R}(\aleph_0)$ attains its extrema.

EVT4. Every κ-schmontinuous function on a $\tau'(\kappa)$-closed bounded interval $I \subseteq \mathbb{R}(\kappa)$ attains its extrema.

But this fix is wildly *ad hoc*: we must replace continuity with a function which haphazardly assigns notions of continuity based on the cardinal. This is gruesome to the extreme! It simply has no grounding in generalised real analysis (see Chapter 4 for details).

Taken literally, this fails the Adequacy Condition, as it is intentionally an account of notion (i.e. concepts) generalisation. But he gestures towards the wider project of capturing generalisations of various species through other species:

> "definitions of notions, theorems, examples, proofs, etc. ... will be formalised by formulas ... [which] will not be not so different, from the point of view of the study of generalisation processes, to the formalisation of natural language." [202, page 57].[23]

Let's take a (non-sentential!) generalisation to see this in action. Consider the following object-type in the variable r, which we capture using an open formula, and its generalisation:

OBJT1. $r \in \mathbb{R}$

OBJT2. $r \in \mathbb{R}_\kappa$

For generosity, let $\mathbb{R}_\kappa$ be a parametrised version of $\mathbb{R}$, i.e. we have a class function $\mathbb{R}(-)$, where $\mathbb{R}(\kappa) := \mathbb{R}_\kappa$ (cf. Section 7.3.3). Let $\mathsf{Def}(x, k)$ mean that x is defined to be of kind k, and Card be the class of cardinals. Then we recapture OBJT1. with a sentential definition:

DEFR3. $\forall r(\mathsf{Def}(r, (\aleph_0\text{-})\text{real number}) \leftrightarrow r \in \mathbb{R}(\aleph_0))$

A RAMSEY. generalisation, essentially universal generalisation of an open formula, yields the new definition:

DEFR4. $\forall \kappa \in \mathsf{Card} \forall r(\mathsf{Def}(r, \kappa\text{-real number}) \leftrightarrow r \in \mathbb{R}(\kappa))$

The generalisation here is spot on. This approach could apply to several species, so is not immediately stumped by the Adequacy Condition. It is best understood *translationally*: we translate pieces of mathematics into a fixed species, so that generalisation always happens *via* this species.[24] For example, the generalisation from Euclidean to absolute geometry could be captured with object-types, from *a Euclidean space* to *an absolute space*. This object-type is then translated into a (definitional) sentence, and the dance continues. Such translation is not very radical: it's ubiquitous in mathematics anyway (see Section 7.2.3). This approach could go via any 'fundamentally' generalisation-appropriate species, but the obvious candidate is the theorem. Non-sentential pieces of mathematics are translated into sentential pieces, generalised by RAMSEY., and then untranslated. Let's call this *translation syntacticism*.

[23] All translations of Villeneuve are the author's.

[24] An alternative *pluralist syntacticism* takes generalisation to consist of multiple processes. This is less unified, and may rely on translation anyway, so we focus on translational approaches. (It may also have commutativity problems: suppose $R_{S,S'}(-)$ recaptures pieces of mathematics from species S to S', and that G_S is the generalisation-method for species S; if P is of species S, why does $R_{S',S}(G_{S'}(R_{S,S'}(P))) = G_S(P)$?)

Subduing translation syntacticism: Diversity and defining properties

In the next two sections, we show that translation syntacticism has an unenviable dilemma, either being disunited, or the need to appeal substantially to semantic resources. The underlying tension is between the need for a unified account, and the syntactic diversity of generalisations. We state this as two demands:

Demand 1. (Unity) Generalisations must be explained by one/few permissible syntactic patterns (for generalisation is a unitary phenomenon).

Demand 2. (Diversity) Ill-fitting pieces of mathematics must be captured/recast into a form which fits these patterns, using without resorting to substantial uses of semantic resources.

Translation syntacticism fairs well on Demand 1., by building an account of generalisation out of syntactic patterns for a single species. They must then specify *what kind of capturings/recastings are permissible*, i.e. how much flexibility is required to satisfy Demand 2. We argue that this is difficult to give a sustainable specification.

To gain traction, we consider object generalisation. Here, the syntacticist must describe the object with a syntactic expression. The choice of a defining property is non-trivial (see [197, page 268]). The syntacticist ideal would be a syntactically-determined choice of the 'correct' property, but it is quite clear that there is not even a naturalistic method of choice, let alone a syntactic method. *Unpredictable analogues* are notably difficult; for example, to make DEFR3. suitable for generalisation, we had to recast it using a predicate $\mathbb{R}(-)$ (not the object $\mathbb{R}_\kappa$). This is not predicted from the initial object-type string in OBJT1., it was post-rationalised. In general, it need not be clear what the right formulation of the classical case should be so that we can apply RAMSEY. (Chapter 4 and Footnote 22 contain further examples). We simply cannot always predict the right syntactic form in advance. So, the question becomes: how can the translation syntacticist choose the 'correct' properties *at all*?

Defining properties via Anything Goes or verificational translation syntacticism

In this section, we consider two possible answers. One is to allow any property (Anything Goes), the other is to verify retrospectively (verificational translation syntacticism). We argue that the former fatally over-generates, whilst is doomed to succeed: it *must* be true, but is uninformative about the nature of generalisations. In the next section, we describe the challenge for the syntacticist to find a non-verificational solution to the property-choice problem, without employing semantic resources.

Suppose that the syntacticist says 'Anything Goes!': generalisation can be based on *any* property, i.e. any application of RAMSEY. to any mathematical statement constitutes a generalisation. We claim that this over-generates. Sometimes, we seem to have a unique correct generalisation (perhaps the infinite cardinals, see [140, page 147], cf. real definitions [197, 198], and unique concept expansion [23, chapter 7], [204, 7.3.3]). We also need to block contingent or accidental properties (as in [17, page 483], [29], [81, page 82]), on pains of triviality. Here's an example: let $D_1(x)$ and $D_2(x)$ define objects O_1 and O_2 respectively. We introduce a new definition, $P_{\text{triv}}(-)(-)$, like so:

1. $P_{\text{triv}}(1)(x) \leftrightarrow D_1(x)$, and
2. $(\forall i \in \{1,2\} P_{\text{triv}}(x)) \leftrightarrow D_2(x)$.

According to Ramsey syntacticism, $\forall i \in \{1,2\} P_{\text{triv}}(i)(x)$ generalises $P_{\text{triv}}(1)(x)$. So, by Anything Goes, any defined object generalises any other. This is clearly unintended. In generalising, we have only certain properties in mind,[25] blocking Anything Goes.

Instead, there must be a non-trivial choice of property. One option is to give up on *predicting* suitable properties, and instead verify retrospectively, saying "Yes, we did not know beforehand what generalisation-appropriate form of OBJT1. is, but in hindsight, there is a recapturing of OBJT1. in so that OBJT2. conforms to RAMSEY." We have encountered several mildly verificational revisions (e.g. the class function $\mathbb{R}(-)$ in DEFR3., or Footnote 22). Villeneuve hints at it too, e.g. a generalisation is conservative if "it is *possible* to explain the translation from N to N' using a using a fixed-variable" ([202, page 78], emphasis added). Taking the 'possibility' seriously means *any* redescription, even an unpredictable one, is a candidate;[26] if descriptions D and D' corefer, assuming a background theory of mathematics, then we can replace D with D' for the purposes of checking a known generalisation. We call this verificational translation syntacticism.[27]

Let's see verificational translation syntacticism in action. Villeneuve claims that it is impossible to recover sequential continuity from topological continuity (*pace* topological nets, [56, §1.6]). But a suitable choice of definition makes this recovery easy: we simply declare that sequential continuity means 'topological continuity with the metric topology'. Then this is a logical universal generalisation, so Villeneuve's problem evaporates.

However, this flexibility spells the undoing of verificational translation syntacticism: triviality. The setup resembles P_{triv}. Let C be a theorem, and G be its generalisation (elaborations to other species are also possible). We define a new predicate $Q_{\text{triv}}(-)$ so that $Q_{\text{triv}}(1) \leftrightarrow C$ and $(\forall x \in \{1,2\} Q_{\text{triv}}(x)) \leftrightarrow G$. Hence we can phrase C and G as, respectively:

TRIV1. $Q_{\text{triv}}(1)$

TRIV2. $\forall x \in \{1,2\} Q_{\text{triv}}(x)$

[25] It also automatically includes 'over-generalisations', e.g. BFPT to the Verum, $\top$, and 'under-generalisations', e.g. rigid expressions like 'being $\mathbb{R}$'.

[26] Villeneuve does not pursue this, instead claiming that some generalisation cannot be captured as logical universal generalisations. But his examples are only non-logical in their *natural* forms. Villeneuve allows some redescription away from the natural form (e.g. adding parameters [202, page 91]). So, even for Villeneuve, rediscription is a question of extent, not kind.

[27] There are natural versions of this for other species. Extra care is required for theorems: a redescribed theorem ought to say the same thing (perhaps formalised as inter-replacability). This suggests an overriding restriction on replacement, to preserve mathematical content(/etc., Section 7.5.3). This smells like using semantic resources already.

Applying RAMSEY. to TRIV1. generates TRIV2. So, any candidate generalisation can be put in the form for verificational translation syntacticism, making the account true. But the selfsame examples show that verificational translation syntacticism is uninformative about the nature of generalisation: generalisation remains unexplicated.

Syntactic variance

Translation syntacticism struggled with Demand 2. (Diversity), in the choice of defining properties for objects. We now give a methodological problem with syntacticism of any stripe, by pushing on Demand 1. (Unity), arguing that generalisations exhibit sufficiently strong syntactic variance to undermine *any* version of syntacticism. We outline a last-resort syntacticism, to specify a non-trivial canonical form, and present a significant challenge for this account.

There are both inter-generalisation and (our focus) intra-generalisation cases of syntactic variance. In the former, for a fixed species, the syntactic form of the classical case-generalisation pair can vary wildly between different generalisations:

PAVA1. An algebra, τ, is topology if it is closed under $|\tau|$-sized unions and finite intersections.

PAVA2. An algebra, τ, is κ-topology if it is closed under κ-sized unions and finite intersections.

DOEX1. The integers, $\mathbb{Z}$, are formed by closing $\mathbb{N}$ under negation.

DOEX2. The rationals, $\mathbb{Q}$, are formed by closing $\mathbb{Z}$ under division.

Each pair exhibits a different similarity pattern. The first pair is ideal for RAMSEY., with a constant that can be varied. The second pair requires substantial work: DOEX2. replaces both a constant ($\mathbb{N}$ to $\mathbb{Z}$) and a predicate ($-$-closure to $\div$-closure). Even if a RAMSEY.-appropriate revision is possible, it is more natural to say that this is simply a different pattern of similarity.

So, each species may have numerous possible axes of similarity between the generalised and classical case. To ensure unity between these axes, the syntacticist could appeal to a *meta-similarity*: whilst the two pairs do not share a pattern, the two patterns themselves somehow instantiate some further, more fundamental notion of similarity.

But meta-similarity is undone by intra-generalisation variance. Here, *within* a generalisation, the classical and generalised cases have radically different syntactic (natural) forms. Unpredictable analogies are particularly troubling. For example, the syntax of BWT1. and of BWT2. are radically different. Any syntactically similar analogue of BWT1. entirely misses the right-hand side of the generalisation!

Intra-generalisation syntactic variance is even worse for theories. Compare ZFP and ZFC. For some P, we can tell a clear syntactic story for the generalisation, e.g. AC_ω (see Section 3.1). But other P have wildly divergent syntax to AC, and are *not known* to be ZF-equivalent to some P′ with a suitable syntactic relation to AC.[28] This comes out fiercely when P is *about sets* (and nothing else) e.g.:

P_{31}. The union of a countable set of countable sets is countable [95, Form 31].

As P_{31} is syntactically unrelated to AC, the syntax of axiomatisations provides no evidence to think that $\mathsf{ZF}+\mathsf{P}_{31}$ generalises ZFC, and so the syntacticist must conclude that it does not. Yet $\mathsf{ZF}+\mathsf{P}_{31}$ generalises ZFC (for P_{31} is a proper fragment of AC)!

To avoid the substantial use of semantic resources, the syntacticist must give an account in terms of a kind of unsystematic family resemblance of many syntactic patterns, where sometimes the pattern is *not* syntactic similarity, which includes the ZFC to $\mathsf{ZF}+\mathsf{P}_{31}$ generalisation. What powers this family resemblance? One could be a syntacticist about generalisations themselves, with a semantic account of the family resemblance, e.g. use (i.e. the patterns are related for such-and-such a semantic reason). But this is semanticism *par excellence*. Semantics would then explicate the nature of generalisation, with the syntax being a non-explanatory artefact. Instead, syntacticism, properly so-called, should give an account of this family resemblance which turns on patterns of syntactic similarity, for which there is no clear candidate.[29]

As a final roll of the dice, the syntacticist could revert to an account of syntactic similarity, with an additional claim that there is a non-arbitrary *canonical* recasting of any piece of mathematics into a form which is appropriate for syntactism. Some canonicity may be possible (e.g. the canonical derivation for proofs, see [64, §3], [208]). But, in general, it is unclear what this canonical form should be. Recasting AC and P_{31} so that they are syntactically similar (via [95, Form 1C]) apparently uses resources substantially outstripping the syntax of axiomatisation (as in Footnote 29), for example sneaking in a substantial use of semantic resources by using model-theoretic properties. Similarly, enriching syntax *itself* enough to have non-arbitrary canonical translations 'built in' just imports semantics into the syntax (i.e. semantic pollution, cf. [174]). Specifying the canonical form is a significant challenge for the syntacticist.

Refinements of Ramsey syntacticism either cannot be sustained, or face the challenge of syntactically specifying a canonical form. And so the bell tolls for syntacticism. Syntax is clearly a *useful indicator*, which is practically helpful for

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[28] Sometimes P and P′ are ZF-equivalent, e.g:

$\mathsf{P}_{8\mathsf{E}}$. In a metric space, every sequentially continuous real-valued function is continuous.

As $\mathsf{P}_{8\mathsf{E}}$ is ZF-equivalent to AC_ω, thanks to substantial mathematical work (see [95]), there is a (pseudo-)syntactic story about why $\mathsf{ZF}+\mathsf{P}_{8\mathsf{E}}$ generalises ZFC, via $\mathsf{ZF}+\mathsf{AC}_\omega$ (only 'pseudo' as it is via model-theoretic equivalence, an apparently semantic notion). But this story is back-to-front! We knew that $\mathsf{ZF}+\mathsf{P}_{8\mathsf{E}}$ generalised ZFC *before* showing that $\mathsf{P}_{8\mathsf{E}}$ was ZF-equivalent to AC_ω: being a generalisation only depends on $\mathsf{P}_{8\mathsf{E}}$ being a weak choice principle.

[29] The syntacticist might point to the proof theory: everything which can be derived from $\mathsf{ZF}+\mathsf{P}_{31}$ can be derived from ZFC, but not vice-versa. But this certainly goes beyond syntacticism, which concern the similarities of the syntactic forms of the axiomatisations, rather the substantially theoretically richer comparison of deductive power or deductive closures (see Footnote 20).

recognising and suggesting generalisations. But to provide an account of their nature, we pursue a different approach: semantics.

7.5.3 Semanticism

Before one can generalise..., there must be a mathematical substance

H. Weyl, quoted in [210, page 171]

The resource dimension of accounts of generalisation maps out two systematic paths: accounts based on on a syntactic connection between the generalised and classical cases, and accounts which based on semantics connections. We have discussed syntactic approaches based on the similarity of syntactic form. Such accounts faced significant challenges in avoiding substantial uses of semantic resources. An alternative is to base an account on semantic resources themselves. 'Semantic resources' here is necessarily broad, for example BFPT and its generalisations share some clear similarity in meaning, whilst $\mathbb{R}$ and $\mathbb{R}_\kappa$ are similar in nature. We describe how semantic resources are best understood, focusing on the *semantic value* of pieces of mathematics. Based on this, we suggest a schema, *skeletal semanticism*, for accounts of generalisations:

SkelSem. A piece of mathematics, G, is a generalisation of a piece of mathematics, C, if and only if the semantic value of C and G are suitably related.

SkelSem. itself is obviously simple-minded, and 'suitably related' is too permissive. We do not recommend this as an account; instead, it must be precisified into more plausible candidates by providing appropriate interpretations of semantic value and a suitable relation.

We suggest that some notions of semantic value are not optimal to flesh this out. Of the remaining, we focus on *content* as a possible candidate semantic value. We give some restrictions on the suitability of a notion of content for our purposes, which excludes certain off-the-shelf notions of content, but do not commit to any particular notion of content. We then suggest that the suitable relation of content should increase the level of generality whilst also preserving content (possibly in the sense of *pieces hanging together* similarly). We defend the resulting informal account against revenge problems inherited from the syntacticist (Section 7.5.4), and provide some practical examples of how semanticism explicates generalisations.

A proposal: Content

Semanticism depends on a suitable relationship between the semantic values of the classical and generalised cases. So, we must clarify the notion of semantic value, and the suitable relationship. Here we focus on the semantic value. This presents a difficulty. We intend for semantic value to apply to all species of pieces of mathematics. We can think of semantic value in the ordinary way for, say, theorems (picking out, e.g., what a theorem says). For other species, semantic value must be different. For objects, this broad sense of semantic value is intended to pick out something like the *nature* of the object.

Some species present easy insights into how a notion of semantic value should work. The theorems "$\varphi \wedge \psi$" and "$\psi \wedge \varphi$" should have the same value, which should be built from the values of the theorems "φ" and "ψ" in a suitable way (for more on this, see [15, §4.6]). The semantic value of some species (e.g. theorems, theories, and proofs) is plausibly covered by a single broad notion of semantic value, perhaps an elaboration of propositional content. The challenge is how to use propositional meaning to account for the semantic value of objects (see Section 7.5.4).

Several notions of semantic value, including *material*, *substance*, and *content*, all seem to have the requisite breadth.[30] We tentatively suggest basing our account on content, as it is a little more tractable; next, we place constraints on accounts of that content suit our purposes.

Two constraints: Hyperintensionality and agent-dependency

We propose to flesh out the skeleton of semanticism with content. We do not intend to give an account of mathematical content (some options can be found in [94, page 7]). Instead, we give two restrictions for a notion of content, which excludes certain off-the-shelf accounts: suitable hyperintensionality, and a preference for agent-independence.

We intend content to be a broad semantic feature of pieces of mathematics from various species, including theorems and objects. Theorem content should be close to propositional content, and object content should be close to nature. We focus on formal accounts of theorem-content, with the hope that these might form the heart of an account of cross-species content.

The content of a theorem should capture 'what the theorem says about how the mathematical world is', and so pieces of mathematics with the same content should be inter-replaceable in a mathematical argument. So, our notion of content must be hyperintensional (for details, see [120, §2]). However, content cannot be *too* hyperintensional. In Sections 7.2.2 and 7.2.3 we had:

BFPT1. If $C \subseteq \mathbb{R}$ is a non-empty closed bounded interval, and $f : C \to C$ is $\mathbb{R}$-continuous, then there is some $x \in C$ such that $f(x) = x$; and

BFPT1a. $\mathsf{BFPT}(\mathbb{R})$.

We take it that these have the same content. However, they have different components, one talks of functions on subspaces of a topological space, and properties of that function, the other concerns a property of topological spaces only. 'Very structured' versions of structured content come perilously close to syntax (see

[30] We can exclude some other notions for our purposes. *Topic* and *subject matter* are insufficiently hyperintensional (e.g. differing theorems of real analysis have the same topic). Other notions are spurned for being too tied to particular species, e.g. *meaning* (tied to sentences), *nature* and, perhaps, *essence* (tied to objects). In [198, pages 351-352], (ontological) objects *and* concepts have essences; if objects and concepts exhaust ontology, then this essence is a broad semantic notion, all we need is to each reduce species to these two categories.

[119]), so would distinguish BFPT1. from BFPT1a. on the basis of their different components. This is too distinguishing. Instead, an appropriate notion of content must identify e.g. passive and active constructions (*à la* Frege [65, §3]), different subject-verb-object orderings (cf. [119, page 1360]), and the exemplar BFPT1.-to-BFPT1a.-like recapturings. Standard notions of structured propositional content (see [120, §3.1]) may suffer here. But they contain the kernel of a notion, as the 'pieces' of a proposition seemingly do help constitute the content. We return to this later in this section with the image of the pieces hanging together.

We also suggest that content should be agent-independent. Agent-dependent accounts of content [10, page 89], [43, page 13] naturally lead to an agent-dependent version of semanticism, i.e. whether a candidate is really a generalisation is agent-dependent. Agent-independent notions of content are preferable for semanticism, as they block problematic *coded* content (see Section 7.5.4),[31] by substantiating the intuition that coding and translating artificially interpret some unintended content from the intended content of some mathematics (as in the purely arithmetical content of [101]). The explication is agent-dependency: the *agent* artificially 'sees' the unintended coded content in the intended content, using the coding. Blocking coding blocks this problem.

Flesh on the skeleton

In this section, we develop a candidate precisification of SkelSem. The account is informal, unified, and semantic. Our characterisations of the nature of generalisation turn on preservation or suitable relationships of content using a scale of generality, and a picture of pieces hanging together.

Two things happen in any generalisation: (1) something is preserved, and (2) something is changed. More refined versions of syntacticism propose to explain both parts at once: something is preserved (the syntactic pattern), whilst something is changed in that the generalisation is (somehow) strictly stronger. The semanticist must also specify which changes and preservations are adequate.

The semanticist must track whether something has changed *in the appropriate way*. Changing "The sum of the degrees in a triangle is 180°." to "La somme des degrés des angles d'un triangle est 180°." won't do. These two sentences have the same *level of generality* (cf. levels of abstraction, Section 7.3.2). It's fairly clear that any naturalistic cashing out of levels of generality suffices: some statements are just more general than others.

Generalisations must increase the level of generality. But increasing the level of generality need not *automatically* yield a generalisation, as nothing needs to be preserved. For example, the object 'the right-angle triangle with sides of length $3, 4, 5$' is of a lower generality than ZFC. But these do not constitute a generalisation: there is nothing linking the two cases. Instead, we require a *connection in the content* between the cases for them to constitute a generalisation. The hard

[31] There does not seem to be a corresponding problem for the syntacticist: we typically take syntax to be worn on its sleeve (excluding extended/hidden notions of syntax, e.g. Villeneuve's hidden variables [202, page 91]). But this need not be so for content; some examples of hidden/unintended content can be seen in [43, page 14], [101], [181, page 32], and Section 7.5.4).

problem is: what is the required relationship between the content of the candidate generalised and classical case?[32]

The suitable relationship must be not too loose (to avoid the triangle-ZFC example), and not too strict (otherwise the classical and generalised cases would have *exactly the same content*). The problem cases from Section 7.2.2 help us gain traction. The Frankenstein examples seem to violate the required suitable relationship. But we can't just ban case distinction *altogether* (à la [118, page 12]), as disjunction can be mathematically natural [196, 197]. A semanticist explanation is that Frankensteins lack a relation of content. This could be understood as a claim about common patterns (and common pieces): if "$\varphi \vee \psi$" is to generalise "φ", then the contents of "φ" and "ψ" must exhibit some common semantic pattern.[33]

We claim that this common pattern can be explained with the image of pieces hanging together.Precisifying SkelSem. we get:

ConSem. A piece of mathematics, G, is a generalisation of a piece of mathematics, C, if and only if G is of a higher level of generality than G, and T and S, the content of C and G respectively, must hang together suitably similarly.

This picture is somewhat like structured propositional content, in that certain pieces (concepts, objects, etc.) are associated (or even constitutive) of the content. We call a number of pieces hanging together in a certain way a *fabric* of pieces. The picture we wish to develop is one where different fabrics exhibit similar (or the same) *pattern*. For some traction consider the ordinary mental visualisation of a mathematical structure. The picture of pieces hanging together, fabric, and so in, is exemplified in the ordinary way in such a mathematical structure: the objects and relations of a structure are pieces, as are substructures, and so on. In this case, the patterns that the pieces exhibit in a structure can be described through theorems, but the notion of a pattern is intended to be more inclusive (and less formal). For example, we can also think of pieces hanging in a certain way in an object-type, with a *gap* in the fabric for an object to fill. For the purposes of giving a semantic account of generalisation, there are various ways that some fabrics of pieces hanging together could be suitably similar. These include the following.

1. The fabric of a theorem is clearly sufficiently close to the fabric of a definition obtained from that theorem by removing one piece (i.e. definition by abstraction, Section 7.2.3).

2. A *zoomed in* subpattern is sufficiently close to a *zoomed out* pattern of which it is a subpattern.

[32] Gorskii suggests relevance logic ([81, pages 84 & 172], see [4, §3] for an introduction to relevance logic), but essential idea is that the generalised case must be relevant to the classical case, for a logical notion of relevance. But this will not do, as its semantics are typically Kripkean (i.e. using possible worlds [143, §1-3]), so is not hyperintensional enough to satisfy the conditions of page 187 (see [15] for more on the hyperintensional deficiencies of Kripke semantics, particularly [15, §2.1]).

[33] *Relevance* between the classical and generalised case doesn't suffice for an explanation here (cf. Footnote 32): however we flesh out the notion of relevance, we expect that "$\varphi \vee \psi$" is always relevant to "φ".

3. Fabrics with different pieces but the same pattern of pieces (à la homeomorphism) are sufficiently close.

Notably, the first way is inter-species. As an example, take the generalisation of EVT from $\mathbb{R}$ to $\mathbb{R}_\kappa$. It is very natural to say that the pieces hang together in the same way here, but some pieces are replaced by analogous pieces (e.g. continuity with κ-supercontinuity). Note that this is not simply a similarity of syntactic form, it is rather a similarity between the respective relationships between objects, domains, concepts, and so forth. This need not be the semantic 'version' of generalisations by variable replacement: the same image seems to work for abstraction. For example, the way various pieces hang together in $\mathbb{Z}$ is also the way the pieces hang together in *ring*, except before where $\mathbb{Z}$ occupies a position in this image of pieces hanging together, now that position is unoccupied to represent a definition of a new object-type. Meanwhile, in a Frankenstein, the pieces need not hang together at all similarly, violating the suitability relation.

This version of semanticism is clearly informal. But it does not prohibit formalisation. Compare this to logical entailment, which one might characterise thusly: "A conclusion c *follows from* premises P if and only if any case in which each premise in P is true is also a case in which c is true" (Beall and Restall, [11, **(V)**, page 2]). This permits a rich discussion about what cases are: classical models, states in Kripke model, situations for relevance logic, and so forth. By substituting formal components, ConSem. might be formalised. For example, a (non-causative) truthmaker articulation of content might be suitable (see [11]), perhaps cashed out in terms of some kind of mathematical events semantics.

Keeping score, the semanticist accounts for the change in a generalisation as an increased level of generality, and the preservation with a suitable relationship between the respective contents of the classical and generalised case. We suggest the image of *pieces hanging together* as a way of grasping the suitable relationship, but an open question is whether this can be more fully articulated or even formalised.

7.5.4 Three attempts at revenge against semanticism

Semanticism is only as good as its defence against the problems of syntacticism; so here we make sure that semanticism can mount a robust defence against these problems. We first block a disunity problem with using semantic resources. Then we block more particular problems with our proposed semanticism.

First revenge: Obligate pluralism

Syntacticism relied on translation. These approaches had difficulty explaining the unity generalisation across species. Likewise, the unification in semanticism relies on a *broad* semantic resource. One might complain that the unarticulated notion of broad content hides an untraversable difference between object content and theorem content. Perhaps articulating more fully would reveal that semanticism must base itself on at least two notions, propositional content and nature. This disjunctiveness would have downstream effects which could inhibit the explanation of the unity of generalisation.

Besides standing our ground (reassert the underlying similarity between the nature of objects and the propositional content), we could grasp the thistle by accepting this disjunctivism, and then argue that the 'two disjuncts' version of content is far more united than, hence preferable to, syntacticism. Where the syntacticist requires at least one different explication for each species, the semanticism makes do with two, namely (something like) nature and (something like) propositional content. The disjunction into propositional content and nature can also be diluted by a kind of correspondence: we can capture the generalisation from Euclidean to absolute geometry as either theory-generalisation or object-generalisation, so too for other theories. With such a correspondence between object-generalisation (based on nature) and theory-generalised (based on propositional content), one might say that even though there is an *ontological* difference in semantic resources, this difference does not correspond to a *disunited* notion of generalisation, as we can recapture the semantic content in each direction.

So, it seems we can draw on semantic resources in a way that is unified, either with a very minor disunity (two semantic notions in correspondence) or no disunity (broad semantics).

Second revenge: Translation and coding

Syntacticism needed some amount of translation between ways of capturing a piece of mathematics to get going. Semanticism might also need an amount of translation in the background machinery. If broad content is based on propositional content (possibly with nature for objects), then some translation is used to explicate the content of the other species, e.g. the content of proofs might supervene on the content of the sentences within them. This concern can be sharpened by apparently content-equivalent theories with different generalisation. Here, we outline and block this problem.

Different mathematical expressions can have the same content. If C and D have the same content, and G generalises C, the semanticist is compelled to commit to G being a generalisation of D. However, one might claim that some examples violate this. For example, theories may "simulate each other's intended content" (Martinot, [148, page 16]). But sometimes a generalisation of C seems not to generalise D. Suppose that equivalence of theorem content amounts to intertranslatability ([100, page 16], see [70] for further alternative notions of equivalence). For example, there is a certain version of finite ZF, known as $\mathsf{ZF{-}inf}^*$ (ZF with the infinity axiom replaced by its negation, plus the statement "every set is contained in a transitive set", see [175, pages 499-502]). Just as with ZFP, this $\mathsf{ZF{-}inf}^*$ generalises ZFC, but $\mathsf{ZF{-}inf}^*$ is also intertranslatable with PA [175, page 502]. However, PA does not seem to generalise ZFC: they are about different things, one is a theory of *sets*, the other of *numbers*.

The semanticist rejoinder is obvious, simply reject that intertranslatability implies the equivalence of content: PA clearly doesn't have equivalent content with *any* set theory, as it is not about sets. This 'translation' is really an *encoding* (we code PA into $\mathsf{ZF{-}inf}^*$). Content need not be preserved via coding. One could justify this using an *agent-independent* notion of content (see Section 7.5.3), by blocking the 'seeing' of unintended content via coding.

Third revenge: Variance

One might think that *suitability* generates a further revenge problem, based on variance. A major reason for dismissing syntacticism was that a suitable syntactic relationship must be highly disjunctive, consisting of multiple syntactic (sub)relations with no natural (syntactic) commonalities. This was deemed unjustifiable. The sceptic might have the same reservations for semanticism: isn't the suitable semantic relationship of content too disjunctive?

Certainly, *syntactic* variance won't cause a problem, as content has considerable syntactic independence (e.g. passive and active expressions may have the same content). But what of variance in the relationship of content? Unlike syntactic similarity, there are many possible instantiations of the pieces hanging together similarly. There are subpatterns, corresponding patterns with some pieces swapped, corresponding patterns but one has a hole (definition by abstraction), and so on. These must be unified.

There seems no obvious reason to think that a suitable semantic relationship of content *must* be disjunctive. These instantiations have close affinities, but we do not have a definitive argument for their unity. We suggest that the image of the pieces hanging together is enough to give purchase on a notion of a suitable relationship which is, at worst, not radically varied. Perhaps this picture represents a small number of different fundamental ways the pieces can hang together, but that would represent far more unity than syntacticism.

7.5.5 Putting semanticism to work

Here, we give semanticist explanations of both uncontroversial and more puzzling generalisations. One generalisation was from a special proof of MS1. for $\mathbb{R}^n$, to the general proof of MS2. for any complete metric space (Section 7.2.2). The relationship between the respective contents here is clearly suitable: the pieces hang together in *exactly* the same way (the proof only used some structural properties, so the pieces obviously hang together exactly similarly). The generalised case clearly is of a higher level of generality: it concerns a more general notion (*complete metric spaces* rather than $\mathbb{R}^n$).

For other generalisations, like the one from $\mathbb{N}$ to $\mathbb{Z}$, the higher level of generality is clear, but the connection of content needs some work. The higher generality follows from $\mathbb{Z}$ being more inclusive in the sense of domain expansion, and also structurally more inclusive (as it is closed under subtraction). For the suitable relationship of content, we can find the pattern of $\mathbb{N}$-pieces within the pattern of $\mathbb{Z}$-pieces, i.e. we can cover some pieces and their connections which constitute the content of $\mathbb{Z}$, and the remaining pieces hang together in exactly the way that the pieces hang together in $\mathbb{N}$. This could be formalised via substructures.

Next, we use semanticism to explicate some more puzzling candidate generalisations. The higher level of generality requirement explains some problem cases, for example it blocks the 'underly general' trivialities (Section 7.2.2). It is plausible that the level of generality is constant across Frankensteins as well. Theory generalisations were also puzzling. The generalisation from ZFC to ZFP, so difficult for syntacticism (Section 7.5.2), is explained easily in terms of content: this is a generalisation of *theories of sets*. For suitable P, both are clearly theories of sets (with varying conceptions on what counts as 'all sets', [99]). Hence their respective contents are sufficiently related. Meanwhile, the higher level of generality essentially turns on the fact that $\mathsf{ZFP} \nvDash \mathsf{AC}$.

We can also explain the problematic 'over-generalising' candidate generalisations (Section 7.2.2): they are excluded, as they do not preserve content. One case was ZFC to the theory of Extensionality. Whilst ZFP is a generalisation of ZFC, as they both are reasonably considered theories of *sets*, Extensionality (alone) is not a theory of sets (it over-generates, applying to classes, sets, pluralities, and so on, [99, page 11]). This change of *subject matter* is enough to guarantee that the respective contents are not suitably related. The story from a non-trivial theorem (e.g. BFPT) to $\top$ is similar, whilst BFPT has some content (about the fixed points of certain functions), $\top$ has no content at all. Indeed, a standard account of such logical tautologies is that they are *topic neutral* (see [139, §4]), so also must be content neutral.

7.6 Conclusion

Generalisations are ubiquitous in mathematics. Chapters 3 to 6 and Section 7.2 highlighted their diversity, yet there is a strong underlying sense of unity amongst them. Despite their mathematical prevalence, there is as yet much space for explicit, dedicated philosophical explications of generalisation. This chapter explores that space. We have grappled with certain micro-level philosophical phenomena surrounding mathematical generalisation. But taking a step back, our macro-level intention has been to show that generalisations are a genuine phenomenon and independent object of study in the philosophy of mathematics. Accounting for the nature of and motivation for generalisation has led us into rich philosophical territory, some quite traditional, whilst others less so. We've encountered ontology, content, divisions of mathematical practice, coding, translation, theoretical virtues, and methodological motivation. With this, we have attempted to show the philosophical weight which studying generalisation has, and that the analysis of generalisation can be self-sustaining and non-parasitical. We traced a framework for analysing generalisations, and proposed our own account of generalisation. This gives us more tools in our conceptual toolbox, and helps open up some little trod areas of mathematical practice for philosophical analysis, principally processes of mathematical change.

Our framework for understanding accounts of the nature of generalisations categorises them in various dimensions. For this, we introduced a non-metaphysical typology of species for pieces of mathematics, to better reflect mathematical practice (Section 7.2.1). With this, we can better scaffold a slew of debates in the philosophy of mathematical practice, and use this scaffolding to answer analogous questions around other processes of change. A key classification was between unified and non-unified accounts. We also isolated dimensions of formality and resource-use (Section 7.5.1). These dimensions help classify and assess further proposals for accounts of the nature of generalisation, and can be transferred to other processes of mathematical change. Our framework yields a classification of the pieces of mathematics themselves, such as input-output species classification of the residents of the zoo (Section 7.2.2). These residents also demonstrated order-theoretic properties of the generalisation-relation (Section 7.2.3), which can guide future proposals for (the semantics of) a logic of generalisation. We had three main ambitions. Let's see how we faced up.

Our first sizeable undertaking was to establish whether generalisation could be reduced to other processes of change in mathematics. In Section 7.3, we distinguished (mathematical) generalisation from the closest processes of change: domain expansion and abstraction. En route, we gestured at accounts of each process, and stressed the significance of levels of abstraction (Section 7.3.2). In these separations, we achieved our first major conclusion, that generalisation is a *sui generis* process of mathematical change, and so the philosophy of generalisation is not directly parasitical on such debates.

Our second undertaking was to give an account of the particular motivations of generalisation (Section 7.4). We countered candidate comprehensive analyses of the motivation to generalise, such as explanation and generality. This gave us our second main conclusion, that generalisation is motivated by a range of competing goals, like unification, simplicity, and isolating new structures. This opens up further bridges, this time between theoretical virtues and processes of mathematical change.

Our last, most significant, undertaking was to develop an account of the nature of generalisation. We uncovered what we take to be the fundamental divide on the nature of generalisation, the opposition between focusing on syntactic resources and on semantic resources. We inspected the champions of these families of accounts, syntacticism and semanticism, to see if they reflected the picture of generalisation we painted in the earlier parts. We saw that even the refined versions of syntacticism were unsustainable, and that there are underlying methodological problems with relying on the similarity of syntactic form *at all*, either being disunited or ultimately resorting to semantic resources (Section 7.5.2). Our counter-proposal was semanticism. We sketched a skeletal schema, then proposed to flesh this out on the basis of suitable preservations of mathematical content, with increased mathematical generality (Section 7.5.3). Our version of semanticism face up to the problems of unity, translation, and variance; and ultimately decided in favour of this version.

A major theme, playing a key rôle in unveiling the problems with syntacticism, was mathematical content (cf. Section 7.2.3). We took mathematical content to be broad and more inclusive than traditional notions of (say) propositional content, to better account for the meaning and about-ness of pieces of mathematics. This opens a bridge to a discussion about specifically mathematical semantics, and throws down a semantic gauntlet to philosophers to further articular a (possibly formal) notion of content, and of a suitable relationship of content. Those who rise to the challenge will lead us yet further into the sunlit uplands of semanticism.

It is the first mild day of March:
Each minute sweeter than before
The redbreast sings from the tall larch
That stands besides our door.

There is a blessing in the air,
Which seems a sense of joy to yield
To the bare trees, and mountains bare,
And grass in the green field.

My sister! ('tis a wish of mine)
Now that our morning meal is done,
Make haste, your morning tasks resign;
Come for and feel the sun.

Edward will come with you; – and pray,
Put on with speed your woodland dress;
And bring no book; for this one day
We'll give to idleness.

extract from *To My Sister* (1798), Wordsworth, the sainted forebear.

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Samenvatting

Dit proefschrift beslaat twee onderzoeksgebieden. Het eerste, de beschrijvende verzameligenleer, is wiskundig van aard. Het andere, veralgemenisering in de wiskunde, valt onder de filosofie. Beschrijvende verzamelingenleer is de studie naar het gedrag van de definieerbare deelverzamelingen van een gegeven structuur, bijvoorbeeld van de reële getallen. In de hoofdstukken van wiskundige aard, presenteren we wiskundige stellingen die de beschrijvende verzamelingenleer koppelen aan de veralgemeniseerde beschrijvende verzamelingenleer. Aan de hand hiervan geven we een filosofische beschrijving van de redenen voor, en de aard van, veralgemenisering in de wiskunde.

In Hoofdstuk 3 stratificeren we formele theorieën van verzamelingenleer op basis van deze beschrijvende complexiteit. Het aftelbare keuzeaxioma voor reële getallen is een van de meest elementaire verzwakkingen van het keuzeaxioma. Het is van belang voor veel deelgebieden van de wiskunde. Beschrijvende keuzeprincipes zijn varianten van deze verzwakking, die de beschrijvende complexiteit van verzamelingen in beschouwing nemen. Wij geven een techniek om de onafhankelijkheid van beschrijvende keuzeprincipes te bewijzen, gebaseerd op de techniek van Jensen forcing. Deze resultaten veralgemeniseren een stelling van Kanovei.

Hoofdstuk 4 geeft de essentie van een gegeneraliseerde reële analyse. Dat wil zeggen, een reële analyse op generalisaties van de reële getallen naar hogere oneindigheden. Dit bouwt voort op het werk van Galeotti en zijn coauteurs. We veralgemenen klassieke stellingen van de reële analyse tot bepaalde verzamelingen van functies, versterken de definitie van continuïteit, en weerleggen andere klassieke stellingen. We tonen ook aan dat een bepaalde eigenschap van kardinaalgetallen, de boomeigenschap, equivalent is aan de extremumstelling voor een bepaalde verzameling functies die de continue functies veralgemenen.

De vraag van Hoofdstuk 5 is of een robuuste notie van oneindige sommen kan worden ontwikkeld voor generalisaties van de reële getallen naar hogere oneindigheden. We geven enkele onverenigbaarheidsresultaten die suggereren van niet. We analyseren verschillende kandidaat-noties van oneindige sommen, zowel afkomstig uit de literatuur als origineel, en laten zien aan welke van de verlangde eigenschappen van een juist notie van som ze niet voldoen.

In Hoofdstuk 6 bestuderen we de beschrijvende verzamelingenleer die voortvloeit uit een veralgemenisering van de topologie, κ-topologie, die in de vorige twee hoofdstukken wordt gebruikt. We ontwikkelen deze op (geordende) veralgemeniseringen van de reële getallen, en op de veralgemeniseerde Baire-ruimte. We laten zien dat de theorie heel anders is dan die van de standaard (volledige) topologie. Voorbeelden van verschillen zijn: de ineenstorting van de Borel-hiërarchie, een gebrek aan universele of volledige verzamelingen, de 'grote fout' van Lebesgue (projecties verhogen de complexiteit niet), een strikte hiërarchie van noties van analyticiteit, en het falen van de stelling van Suslin.

Ten slotte geven we in Hoofdstuk 7 een filosofische uiteenzetting van de aard van veralgemenisering in de wiskunde, en beschrijven we de methodologische redenen waarom wiskundigen veralgemeniseren. Daarbij onderscheiden we veralgemenisering van andere veranderingsprocessen in de wiskunde, zoals abstractie en domeinuitbreiding. We geven een aantrekkelijke syntactische beschrijving van veralgemenisering, maar laten zien dat die tekortschiet. Uiteindelijk stellen we een semantische beschrijving van veralgemenisering voor, waarbij twee delen van de wiskunde samen een veralgemenisering vormen als ze een bepaalde inhoudelijke relatie hebben, en een verhoogd niveau van algemeenheid.

Abstract

This dissertation has two major threads, one is mathematical, namely descriptive set theory, the other is philosophical, namely generalisation in mathematics. Descriptive set theory is the study of the behaviour of definable subsets of a given structure such as the real numbers. In the core mathematical chapters, we provide mathematical results connecting descriptive set theory and generalised descriptive set theory. Using these, we give a philosophical account of the motivations for, and the nature of, generalisation in mathematics.

In Chapter 3, we stratify set theories based on this descriptive complexity. The axiom of countable choice for reals is one of the most basic fragments of the axiom of choice needed in many parts of mathematics. Descriptive choice principles are a further stratification of this fragment by the descriptive complexity of the sets. We provide a separation technique for descriptive choice principles based on Jensen forcing. Our results generalise a theorem by Kanovei.

Chapter 4 gives the essentials of a generalised real analysis, that is a real analysis on generalisations of the real numbers to higher infinities. This builds on work by Galeotti and his coauthors. We generalise classical theorems of real analysis to certain sets of functions, strengthening continuity, and disprove other classical theorems. We also show that a certain cardinal property, the tree property, is equivalent to the Extreme Value Theorem for a set of functions which generalise the continuous functions.

The question of Chapter 5 is whether a robust notion of infinite sums can be developed on generalisations of the real numbers to higher infinities. We state some incompatibility results, which suggest not. We analyse several candidate notions of infinite sum, both from the literature and more novel, and show which of the expected properties of a notion of sum they fail.

In Chapter 6, we study the descriptive set theory arising from a generalisation of topology, κ-topology, which is used in the previous two chapters. We show that the theory is quite different from that of the standard (full) topology. Differences include a collapsing Borel hierarchy, a lack of universal or complete sets, Lebesgue's 'great mistake' holds (projections do not increase complexity), a strict hierarchy of notions of analyticity, and a failure of Suslin's theorem.

Lastly, in Chapter 7, we give a philosophical account of the nature of generalisation in mathematics, and describe the methodological reasons that mathematicians generalise. In so doing, we distinguish generalisation from other processes of change in mathematics, such as abstraction and domain expansion. We suggest a semantic account of generalisation, where two pieces of mathematics constitute a generalisation if they have a certain relation of content, along with an increased level of generality.

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