LARGE CARDINALS AND DEFINABLE WELL-ORDERS, WITHOUT THE GCH

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Abstract. We show that there is a class-sized partial order $\mathbb{P}$ with the property that forcing with $\mathbb{P}$ preserves ZFC, supercompact cardinals, inaccessible cardinals and the value of 2^κ for every inaccessible cardinal κ and, if κ is an inaccessible cardinal and A is an arbitrary subset of ${}^\kappa\kappa$, then there is a $\mathbb{P}$-generic extension of the ground model V in which A is definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V[G]}}, \in\rangle$ by a Σ_1-formula with parameters.

We use this result to construct a class-sized partial order with the above preservation properties that forces the existence of well-orders of $\mathrm{H}(\kappa^+)$ definable in the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ for every inaccessible cardinal κ. Assuming the GCH, David Asperó and Sy-David Friedman showed in [AF09] and [AF12] that there is a class-sized partial order preserving ZFC and various large cardinals and forcing the existence of a well-order of the universe whose restriction to $\mathrm{H}(\kappa^+)$ is definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V[G]}}, \in\rangle$ by a parameter-free formula for every uncountable regular cardinal κ. Our second result can be interpreted as a boldface version of this result in the absence of the GCH.

1. Introduction

Given an uncountable regular cardinal κ, we call the set ${}^\kappa\kappa$ consisting of all functions $f : \kappa \longrightarrow \kappa$ the *generalized Baire Space for* κ. The study of the *descriptive set theory* of these spaces, i.e., of their definable subsets and the structural properties of these subsets, was initiated by Alan Mekler and Jouko Väänänen in [MV93] and deep links to model theory and logic were established (see, for example, [Vää95], [TV99], [Vää11] and [FHK]). A discussion of some of these results is contained in Chapter IV of [FHK]. In this paper, we study the definable subsets of this space when κ is a large cardinal, especially a supercompact cardinal.

Remember that an uncountable cardinal κ is *γ-supercompact* with $\gamma \geq \kappa$ if there is an elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $\gamma < j(\kappa)$ and ${}^\gamma M \subseteq M$. This is equivalent to the existence of a normal ultrafilter on the set $\mathcal{P}_\kappa(\gamma)$ of all subsets of γ of cardinality less than κ (see [Kan03, Theorem 22.7]). Given such an ultrafilter U, we let M_U denote the transitive collapse of the corresponding ultrapower $\mathrm{Ult}_U(\mathrm{V})$ and $j_U : \mathrm{V} \longrightarrow M_U$ denote the corresponding elementary embedding. Finally, we call a cardinal κ *supercompact* if κ is γ-supercompact for all $\gamma \geq \kappa$.

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Let κ be a supercompact cardinal and A be an arbitrary subset of ${}^{\kappa}\kappa$. We want to construct an outer model W of the ground model V such that κ is still supercompact in W, $(2^{\kappa})^{\mathrm{V}} = (2^{\kappa})^{\mathrm{W}}$ and A is definable in the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{W}}, \in\rangle$. By extending coding methods developed in [Lüc12], this aim is achieved in the following theorem.

Theorem 1.1. *There is a* ZFC*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

(i) *Let* κ *be a cardinal with the property that there is no singular limit of inaccessible cardinals* ν *with* $\nu^+ < \kappa \leq 2^{\nu}$*. Then forcing with* $\mathbb{P}$ *does not collapse* κ *and, if* κ *is regular, then* $\mathbb{P}$ *preserves the regularity of* κ*.*
(ii) $\mathbb{P}$ *preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*
(iii) *If* α *is an inaccessible cardinal and* G *is* $\mathbb{P}$ *generic over* V*, then* $(2^{\alpha})^{\mathrm{V}} = (2^{\alpha})^{\mathrm{V[G]}}$.
(iv) *If* κ *is an inaccessible cardinal and* A *is a subset of* ${}^{\kappa}\kappa$*, then there is a condition* p *in* $\mathbb{P}$ *with the property that* A *is definable in* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V[G]}}, \in\rangle$ *by a* Σ_1*-formula with parameters whenever* G *is* $\mathbb{P}$*-generic over* V *with* $p \in \mathrm{G}$.

In addition, if the class of inaccessible cardinals is bounded in On*, then* $\mathbb{P}$ *is forcing equivalent to a set-sized forcing.*

In particular, if the *Singular Cardinal Hypothesis* holds in the ground model, then forcing with $\mathbb{P}$ preserves cofinalities and cardinalities.

The proof of this result will actually show that certain degrees of supercompactness are preserved. Let κ be γ-supercompact such that γ is a cardinal with $\gamma = \gamma^{<\kappa}$, $2^{\gamma} = \gamma^+$ and $2^{\nu} \leq \gamma$, where ν is the supremum of all inaccessible cardinals smaller or equal to γ. Then κ will still be γ-supercompact after forcing with $\mathbb{P}$. Given a supercompact κ, we will use a classical result due to Robert Solovay to show that there is a proper class of cardinals γ that satisfy the above properties with respect to κ.

We want to use the above coding result to produce ZFC-models with definable well-orders of $\mathrm{H}(\kappa^+)$ for every supercompact cardinal κ. We give a brief overview of related existing results. A detailed discussion of this topic can be found in the first part of [Fri10]. In [FH11], Peter Holy and the first author constructed a class forcing that adds such definable well-orders of low quantifier complexity and preserves various large cardinals.

Theorem 1.2 ([Fri10, Theorem 9])**.** *There is a class forcing which forces the* GCH*, preserves all supercompact cardinals (as well as a proper class of* n*-huge cardinals for each* $n < \omega$*) and adds a well-order of* $\mathrm{H}(\kappa^+)$ *that is definable in* $\langle \mathrm{H}(\kappa^+), \in\rangle$ *by a* Σ_1*-formula with parameters for every uncountable regular cardinal* κ*.*

If the GCH holds in the ground model, then results due to David Asperó and the first author show that it is possible to produce *lightface* definable well-orders of $\mathrm{H}(\kappa^+)$ for every uncountable regular cardinal κ.

Theorem 1.3 ([AF09, Theorem 1.1] and [AF12, Theorem 1.1])**.** *Assume the* GCH*. There is a formula* $\varphi(x, y)$ *without parameters and a definable class-sized partial order* $\mathbb{P}$ *preserving* ZFC*, the* GCH *and cofinalities that satisfy the following statements.*

(i) $\mathbb{P}$ *forces that there is a well-order* $\leq$ *of the universe such that*

$$\{\langle a, b\rangle \in \mathrm{H}(\kappa^+)^2 \mid \langle \mathrm{H}(\kappa^+), \in\rangle \models \varphi(a, b)\}$$

is the restriction $\leq\restriction \mathrm{H}(\kappa^+)$ *and is a well-order of* $\mathrm{H}(\kappa^+)$ *whenever* κ *is a regular uncountable cardinal.*

(ii) *For all regular cardinals* $\kappa \leq \lambda$, *if* κ *is a* λ*-supercompact cardinal in* V, *then* κ *remains* λ*-supercompact after forcing with* $\mathbb{P}$.

The second result of this paper shows that it is possible to add definable well-orders of $\mathrm{H}(\kappa^+)$ for every inaccessible cardinal κ without assuming the GCH with a class forcing that preserves supercompact cardinals and failures of the GCH at inaccessible cardinals.

Theorem 1.4. *There is a* ZFC*-preserving class forcing* $\mathbb{P}$ *definable without parameters that satisfies the following statements.*

(i) *Let* κ *be a cardinal with the property that there is no singular limit of inaccessible cardinals* ν *with* $\nu^+ < \kappa \leq 2^\nu$. *Then forcing with* $\mathbb{P}$ *does not collapse* κ *and, if* κ *is regular, then* $\mathbb{P}$ *preserves the regularity of* κ.

(ii) $\mathbb{P}$ *preserves the inaccessibility of inaccessible cardinals and the supercompactness of supercompact cardinals.*

(iii) *If* α *is an inaccessible cardinal and* G *is* $\mathbb{P}$ *generic over* V, *then* $(2^\alpha)^{\mathrm{V}} = (2^\alpha)^{\mathrm{V}[\mathrm{G}]}$ *and there is a well-order of* $\mathrm{H}(\alpha^+)^{\mathrm{V}[\mathrm{G}]}$ *that is definable in the structure* $\langle \mathrm{H}(\alpha^+)^{\mathrm{V}[\mathrm{G}]}, \in\rangle$ *by a formula with parameters.*

In fact, the partial order $\mathbb{P}$ constructed in the proof of this result satisfies the statements listed in Theorem 1.1.

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2. Generic tree coding

The goal of this section is to construct a partial order that forces an arbitrary subset A of ${}^\kappa\kappa$ to be definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. This construction will be a variation of the *generic tree coding* developed in [Lüc12]. In this section, we present a detailed discussion of the properties of this forcing verified in [Lüc12], because most of these results will be needed in later proofs. In order to define this partial order, we give a brief review of our notation.

Given an ordinal λ and a set X, we let ${}^{<\lambda}X$ denote the set of all functions f with $\mathrm{dom}(f) \in \lambda$ and $\mathrm{ran}(f) \subseteq X$. If κ is a cardinal, then we let $\kappa^{<\lambda}$ denote the cardinality of ${}^{<\lambda}\kappa$. We call a set $T \subseteq ({}^{<\lambda}X)^n$ a *subtree of* $({}^{<\lambda}X)^n$ if the following statements hold.

(i) For all $\langle s_0, \ldots, s_{n-1}\rangle \in T$, $\mathrm{lh}(s_0) = \cdots = \mathrm{lh}(s_{n-1})$.

(ii) If $\langle s_0, \ldots, s_{n-1}\rangle \in T$ and $\alpha < \mathrm{lh}(s_0)$, then $\langle s_0 \restriction \alpha, \ldots, s_{n-1} \restriction \alpha\rangle \in T$.

Given $t = \langle t_0, \ldots, t_{n-1}\rangle \in T$, we define $\mathrm{lh}(t) = \mathrm{lh}(t_0)$ and call the ordinal $\mathrm{ht}(T) = \mathrm{lub}\{\mathrm{lh}(t) \mid t \in T\}$ the *height of* T. A tuple of functions $\langle x_0, \ldots, x_{n-1}\rangle \in \left({}^{\mathrm{ht}(T)}X\right)^n$ is called a *cofinal branch through* T if $\langle x_0 \restriction \alpha, \ldots, x_{n-1} \restriction \alpha\rangle \in T$ for all $\alpha < \mathrm{ht}(T)$. We let $[T]$ denote the set of all cofinal branches through T. If T is a subtree of $({}^{<\lambda}X)^{n+1}$ for some $\lambda \in \mathrm{On}$, then we define

$$p[T] = \{\langle x_0, \ldots, x_{n-1}\rangle \in \left({}^{\mathrm{ht}(T)}X\right)^n \mid (\exists x_n)\ \langle x_0, \ldots, x_n\rangle \in [T]\}.$$

Definition 2.1. Let κ be an infinite cardinal. A subset A of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$*-subset* if there is a subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with $A = p[T]$.

Given an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$, it is a well-known fact that a subset of ${}^{\kappa}\kappa$ is a $\mathbf{\Sigma}^1_1$-subset if and only if it is definable in the structure $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. A proof of this folklore result can be found in [Lüc12, Section 2].

We sketch the idea behind the definition of our forcing notion. Fix an uncountable regular cardinal κ with $\kappa = \kappa^{<\kappa}$ and an enumeration $\langle s_\alpha \mid \alpha < \kappa\rangle$ of all elements in ${}^{<\kappa}\kappa$. We say that $x \in {}^{\kappa}\kappa$ is *coded by* $z \in {}^{\kappa}2$ *and* $\gamma < \kappa$ if

$$s_\beta \subseteq x \iff z(\prec\gamma, \beta\succ) = 1$$

holds for all $\beta < \kappa$, where $\prec\cdot,\cdot\succ$ denotes the Gödel-pairing function. Given a subset A of ${}^{\kappa}\kappa$, our forcing will add a subtree T_G of ${}^{<\kappa}2$ with the property that, in the generic extension, A is equal to the set of all x that are coded by some $z \in [T_G]$ and $\gamma < \kappa$. This definition of A provides a tree T in the generic extension that satisfies $A = p[T]$.

Definition 2.2. Given a limit ordinal λ, we call a pair $\langle A, s\rangle$ a λ-*coding basis* if the following statements hold.

(i) A is a non-empty subset of ${}^{\lambda}\lambda$ and $s : \lambda \longrightarrow {}^{<\lambda}\lambda$.
(ii) $\mathrm{ran}(s)$ contains $\{x \restriction \alpha \mid x \in A,\ \alpha < \lambda\}$ and all constant functions in ${}^{<\lambda}\lambda$.
(iii) For all $\alpha < \lambda$, $\mathrm{lh}(s(\alpha)) \leq \alpha$ and $\{\beta < \lambda \mid s(\alpha) = s(\beta)\}$ is unbounded in λ.

For the rest of this section, we fix a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$. Given a κ-coding basis $\langle A, s\rangle$, we define a partial order $\mathbb{P}_s(A)$. The domain of $\mathbb{P}_s(A)$ consists of all triples $p = \langle T_p, f_p, h_p\rangle$ with the following properties.

(i) T_p is a subtree of ${}^{<\kappa}2$ that satisfies the following statements.
 (a) T_p has cardinality less than κ.
 (b) If $t \in T_p$ with $\mathrm{lh}(t) + 1 < \mathrm{ht}(T_p)$, then t has two immediate successors in T_p.
(ii) $f_p : A \xrightarrow{part} [T_p]$ is a partial function such that $\mathrm{dom}(f_p)$ is a non-empty set of cardinality less than κ.
(iii) $h_p : A \xrightarrow{part} \kappa$ is a partial function with the following properties.
 (a) $\mathrm{dom}(h_p) = \mathrm{dom}(f_p)$.
 (b) For all $x \in \mathrm{dom}(h_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$ with $\alpha = \prec h_p(x), \beta\succ$, we have
$$s(\beta) \subseteq x \iff f_p(x)(\alpha) = 1.$$

We define $p \leq_{\mathbb{P}_s(A)} q$ to hold if the following statements are satisfied.

(a) T_p is either equal to T_q or an end-extension of T_q.
(b) If $x \in \mathrm{dom}(f_q)$, then $x \in \mathrm{dom}(f_p)$ and $f_q(x)$ is an initial segment of $f_p(x)$.
(c) $h_q = h_p \restriction \mathrm{dom}(h_q)$.

Lemma 2.3. *$\mathbb{P}_s(A)$ is $<\kappa$-closed, satisfies the κ^+-chain condition and has cardinality at most 2^κ.*

Proof. If $\lambda \in \mathrm{Lim} \cap \kappa$ and $\langle p_\mu \mid \mu < \lambda\rangle$ is a strictly $\leq_{\mathbb{P}_s(A)}$-descending sequence in $\mathbb{P}_s(A)$, then we define $T = \bigcup_{\mu<\lambda} T_{p_\mu}$, $h = \bigcup_{\mu<\lambda} h_\mu$ and

$$f(x) = \bigcup\{f_{p_\mu}(x) \mid \mu < \lambda,\ x \in \mathrm{dom}(f_{p_\mu})\}$$

for all $x \in \mathrm{dom}(h)$. It is easy to see that $p = \langle T, f, h\rangle \in \mathbb{P}_s(A)$ and $p \leq_{\mathbb{P}_s(A)} p_\mu$ holds for all $\mu < \lambda$.

Next, assume that $\langle p_\mu \mid \mu < \kappa^+\rangle$ enumerates an antichain in $\mathbb{P}_s(A)$. By our assumptions, we can assume $T_{p_\mu} = T_{p_\rho}$ for all $\mu, \rho < \kappa^+$. A Δ-system argument shows that we may assume the existence of an $r \subseteq A$ with $r = \mathrm{dom}(f_{p_\mu}) \cap \mathrm{dom}(f_{p_\rho})$, $f_{p_\mu} \restriction r = f_{p_\rho} \restriction r$ and $h_{p_\mu} \restriction r = h_{p_\rho} \restriction r$ for all $\mu < \rho < \kappa^+$. But this shows that $\langle T_{p_0}, f_{p_0} \cup f_{p_1}, h_{p_0} \cup h_{p_1}\rangle$ is a common extension of p_0 and p_1, a contradiction.

Finally, the assumption $\kappa = \kappa^{<\kappa}$ implies that there are only κ-many such subtrees and 2^κ-many such partial functions of cardinality less than κ. □

The next lemma will allow us to show that various subsets of $\mathbb{P}_s(A)$ are dense.

Lemma 2.4. *Fix a condition p in $\mathbb{P}_s(A)$ and a sequence $\langle c_x \in {}^\kappa 2 \mid x \in \mathrm{dom}(f_p)\rangle$. There exists a $\leq_{\mathbb{P}_s(A)}$-descending sequence $\langle p_\mu \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) \leq \mu < \kappa\rangle$ such that $p = p_{\mathrm{ht}(T_p)}$ and the following statements hold for all $\mathrm{ht}(T_p) \leq \mu < \kappa$.*

(i) $\mathrm{dom}(f_{p_\mu}) = \mathrm{dom}(f_p)$ *and* $\mathrm{ht}(T_{p_\mu}) = \mu$.

(ii) *If $x \in \mathrm{dom}(f_p)$ and $\mu \neq \prec h_p(x), \beta \succ$ for all $\beta < \kappa$, then*

$$f_{p_{\mu+1}}(x)(\mu) = c_x(\mu).$$

(iii) *If $\mu \in \mathrm{Lim}$, then* $\mathrm{ran}(f_{p_\mu}) = T_{p_{\mu+1}} \cap {}^\mu 2$.

Proof. We construct the sequences inductively. If $\mu \in \mathrm{Lim}$, then we define $T_{p_\mu} = \bigcup\{T_{p_{\bar\mu}} \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}$. Given $x \in \mathrm{dom}(f_p)$, we define

$$f_{p_\mu}(x) = \bigcup\{f_{p_{\bar\mu}}(x) \mid \mathrm{ht}(T_p) \leq \bar\mu < \mu\}.$$

If $\mu = \bar\mu + 1$ with $\bar\mu \notin \mathrm{Lim}$, then $T_{p_{\bar\mu}}$ has a maximal level and there is only one suitable tree T_{p_μ} of height μ end-extending it. In particular, $f_{p_{\bar\mu}}(x) \in T_{p_\mu}$ for all $x \in \mathrm{dom}(f_p)$. For all $x \in \mathrm{dom}(f_p)$, we define $f_{p_\mu}(x)$ to be the unique element t of ${}^\mu 2$ with $f_{p_{\bar\mu}}(x) \subseteq t$ and

$$t(\bar\mu) = \begin{cases} 1, & \text{if } \bar\mu = \prec h_p(x), \beta \succ \text{ and } s(\beta) \subseteq x, \\ 0, & \text{if } \bar\mu = \prec h_p(x), \beta \succ \text{ and } s(\beta) \nsubseteq x, \\ c_x(\bar\mu), & \text{otherwise.} \end{cases}$$

Finally, assume that $\mu = \bar\mu + 1$ with $\bar\mu \in \mathrm{Lim}$. Then we set $T_{p_\mu} = T_{p_{\bar\mu}} \cup \mathrm{ran}(f_{p_{\bar\mu}})$. Since $[T_{p_\mu}] = \{t \in {}^\mu 2 \mid t \restriction \bar\mu \in \mathrm{ran}(f_{p_{\bar\mu}})\}$, we can define f_{p_μ} as in the first successor case. □

Corollary 2.5. *The following sets are dense subsets of $\mathbb{P}_s(A)$.*

(i) $C_\mu = \{p \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) > \mu\}$ *for all* $\mu < \kappa$.

(ii) $D_x = \{p \in \mathbb{P}_s(A) \mid x \in \mathrm{dom}(f_p)\}$ *for all* $x \in A$.

(iii) $E_{x,y} = \{p \in \mathbb{P}_s(A) \mid x, y \in \mathrm{dom}(f_p), f_p(x) \neq f_p(y)\}$ *for all* $x, y \in A$.

(iv) $F_z = \{p \in \mathbb{P}_s(A) \mid \mathrm{ht}(T_p) = \mu + 1,\ z \restriction \mu \notin T_p\}$ *for all* $z \in {}^\kappa 2$.

Proof. (i) This statement follows directly from Lemma 2.4.

(ii) Given $p \in \mathbb{P}_s(A)$ with $x \notin \mathrm{dom}(f_p)$ and $b \in [T_p]$, we define

$$q = \langle T_p, f_p \cup \{\langle x, b\rangle\}, h_p \cup \{\langle x, \mathrm{ht}(T_p)\rangle\}\rangle.$$

Then $q \in D_x$ and $q \leq_{\mathbb{P}(A)} p$.

(iii) Given $p \in \mathbb{P}_s(A)$, we can apply the above result to find $q \leq_{\mathbb{P}_s(A)} p$ with $x, y \in \mathrm{dom}(f_q)$. There is $\mathrm{ht}(T_q) \leq \mu < \kappa$ with $\prec h_q(x), \beta_0 \succ \neq \mu \neq \prec h_q(y), \beta_1 \succ$ for all $\beta_0, \beta_1 < \kappa$ and we can use Lemma 2.4 to find $q^* \leq_{\mathbb{P}_s(A)} q$ with $\mathrm{ht}(T_{q^*}) = \mu + 1$ and $f_{q^*}(x)(\mu) \neq f_{q^*}(y)(\mu)$.

(iv) Fix $p \in \mathbb{P}_s(A)$ and $\mathrm{ht}(T_p) \leq \mu < \kappa$ with $\mu \neq \prec h_p(x), \beta \succ$ for all $x \in \mathrm{dom}(f_p)$ and $\beta < \kappa$. Using Lemma 2.4, we can find $q \leq_{\mathbb{P}_s(A)} p$ with $\mathrm{ht}(T_q) = \mu + 1$, $\mathrm{dom}(f_q) = \mathrm{dom}(f_p)$ and $f_q(x)(\mu) = 1 - z(\mu)$ for all $x \in \mathrm{dom}(f_p)$. In particular, $z \restriction (\mu+1) \notin \mathrm{ran}(f_q)$. Another application of the above lemma gives us conditions $s \leq_{\mathbb{P}_s(A)} r \leq_{\mathbb{P}_s(A)} q$ with $\mathrm{ht}(T_s) = \mathrm{ht}(T_r) + 1 = \mathrm{ht}(T_q) + \omega + 1$, $\mathrm{dom}(f_s) = \mathrm{dom}(f_p)$ and $T_s \cap {}^{\mathrm{ht}(T_r)}2 = \mathrm{ran}(f_r)$. Since $z \restriction \mathrm{ht}(T_r) \neq f_r(x)$ for all $x \in \mathrm{dom}(f_p)$, we have $z \restriction \mathrm{ht}(T_r) \notin T_s$. □

Corollary 2.6. *Let G be $\mathbb{P}_s(A)$-generic over* V*. The following statements hold true in* V$[G]$.

(i) $T_G = \bigcup_{p \in G} T_p$ *is a subtree of* ${}^{<\kappa}2$ *of height* κ *with* $[T_G] \cap \mathrm{V} = \emptyset$.

(ii) *If we define* $F_G(x) = \bigcup\{f_p(x) \mid p \in G, x \in \mathrm{dom}(f_p)\}$ *for all* $x \in A$*, then* $F_G : A \longrightarrow [T_G]$ *is an injection.*

(iii) *Let* $H_G = \bigcup_{p \in G} h_g$. *Then* $H_G : A \longrightarrow \kappa$ *and*

$$s(\beta) \subseteq x \iff F_G(x)(\prec H_G(x), \beta \succ) = 1 \tag{1}$$

for all $x \in A$ *and* $\beta < \kappa$. □

Lemma 2.7. *If G is $\mathbb{P}_s(A)$-generic over* V*, then* $\mathrm{ran}(F_G) = [T_G]^{\mathrm{V}[G]}$.

Proof. Let $\dot{T} \in \mathrm{V}^{\mathbb{P}_s(A)}$ be the canonical name for T_G and $\dot{F} \in \mathrm{V}^{\mathbb{P}_s(A)}$ be the canonical name for F_G.

Assume, toward a contradiction, that there is an $x \in [T_G]^{\mathrm{V}[G]} \setminus \mathrm{ran}(F_G)$ and let $\tau \in \mathrm{V}^{\mathbb{P}_s(A)}$ be a name for x. By the above corollary, $x \notin \mathrm{V}$ and there is a $p \in G$ with

$$p \Vdash \text{“}\tau \in [\dot{T}] \wedge \tau \notin \check{\mathrm{V}} \wedge \tau \notin \mathrm{ran}(\dot{F})\text{”}.$$

For each $r \leq_{\mathbb{P}_s(A)} p$, we define a partial function $t_r : \kappa \xrightarrow{part} 2$ in V by setting

$$t_r = \bigcup\{t \in {}^{<\kappa}2 \mid r \Vdash \text{“}\check{t} \subseteq \tau\text{”}\}.$$

We have $t_r \in {}^{<\kappa}2$, because $r \Vdash$ “$\tau \notin \check{V}$”. Note that $r_1 \leq_{\mathbb{P}_s(A)} r_0 \leq_{\mathbb{P}_s(A)} p$ implies $t_{r_0} \subseteq t_{r_1}$. Since $\mathbb{1}_{\mathbb{P}_s(A)} \Vdash$ “$\tau \restriction \check{\alpha} \in \check{\mathrm{V}}$” holds for all $\alpha < \kappa$, the set $\{r \leq_{\mathbb{P}(A)} p \mid \alpha \subseteq \mathrm{dom}(t_r)\}$ is dense below p for all $\alpha < \kappa$.

Moreover, if $p' \leq_{\mathbb{P}_s(A)} p$, then we can find a $p'' \leq_{\mathbb{P}_s(A)} p'$ with the property that for every $x \in \mathrm{dom}(f_{p'})$ there is an $\alpha < \mathrm{ht}(T_{p''}) \cap \mathrm{dom}(t_{p''})$ with $f_{p''}(x)(\alpha) \neq t_{p''}(\alpha)$, because $\mathbb{P}_s(A)$ is $<\kappa$-closed.

Given $p_0 \leq_{\mathbb{P}_s(A)} p$, the above remarks allow us to construct a strictly $\leq_{\mathbb{P}_s(A)}$-descending sequence $\langle p_n \in \mathbb{P}_s(A) \mid n < \omega\rangle$ with the following properties.

(i) For all $n < \omega$, $\mathrm{ht}(T_{p_n}) \subseteq \mathrm{dom}(t_{p_{n+1}})$ and $\mathrm{dom}(t_{p_n}) \subsetneq \mathrm{ht}(T_{p_{n+1}})$.

(ii) For all $n < \omega$ and $x \in \mathrm{dom}(f_{p_n})$, there is an $\alpha \in \mathrm{dom}(t_{p_{n+1}})$ with

$$f_{p_{n+1}}(x)(\alpha) \neq t_{p_{n+1}}(\alpha).$$

The proof of Lemma 2.3 shows that there exists a greatest lower bound $p_\omega \in \mathbb{P}_s(A)$ of the sequence $\langle p_n \mid n < \omega\rangle$. This means $T_{p_\omega} = \bigcup_{n<\omega} T_{p_n}$ and $\mathrm{dom}(f_{p_\omega}) = \bigcup_{n<\omega} \mathrm{dom}(f_{p_n})$. Let $t = t_{p_\omega} \restriction \mathrm{ht}(T_{p_\omega})$. Since $p_\omega \Vdash$ “$\check{t} \subseteq \tau \wedge \tau \in [\dot{T}]$”, we have $p_\omega \Vdash$ “$\check{t} \in \dot{T}$”.

By our construction, we have $\mu = \mathrm{ht}(T_{p_\omega}) \in \mathrm{Lim}$ and $t \notin \mathrm{ran}(f_{p_\omega})$. We can apply Lemma 2.4 to find a condition $p^* \leq_{\mathbb{P}_s(A)} p_\omega$ with $\mathrm{ht}(T_{p^*}) = \mu + 1$ and $t \notin T_{p^*}$. This obviously implies $p^* \Vdash$ “$\check{t} \notin \dot{T}$”, a contradiction. □

Lemma 2.8. *Let G be $\mathbb{P}_s(A)$-generic over* V*. The following statements are equivalent for $y \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$.*

(i) *$y \in A$.*

(ii) *There is $z \in [T_G]^{\mathrm{V}[G]}$ and $\gamma < \kappa$ such that*

$$s(\beta) \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \tag{2}$$

holds for all $\beta < \kappa$.

Proof. If $y \in A$, then Corollary 2.6 shows that $F_G(y) \in [T_G]$ and $H_G(y) < \kappa$ witnesses that the second statement holds true.

Pick $y \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$, $z \in [T_G]^{\mathrm{V}[G]}$ and $\gamma < \kappa$ such that (2) holds. By Lemma 2.7, we have $z = F_G(x)$ for some $x \in A$. Pick $p \in G$ with $x \in \mathrm{dom}(f_p)$. Assume, toward a contradiction, that $\gamma \neq h_p(x) = H_G(x)$. By Lemma 2.4 and our assumptions on s, this implies that the set

$$D_t = \{q \leq_{\mathbb{P}_s(A)} p \mid \mathrm{ht}(T_q) = \mu + 1, \ \mu = \prec\gamma, \beta\succ, \ f_q(x)(\mu) = 0, \ s(\beta) = t\}$$

is dense below p for all $t \in \mathrm{ran}(s)$ and there is a $q \in G \cap D_{y\restriction 1}$ with $q \leq_{\mathbb{P}_s(A)} p$. Then there is a $\beta < \kappa$ with $\mathrm{ht}(T_q) = \prec\gamma, \beta\succ + 1$, $z(\prec\gamma, \beta\succ) = 0$ and $s(\beta) = y \restriction 1 \subseteq y$, contradicting (2). This shows $\gamma = H_G(x)$ and we can conclude that

$$s(\beta) \subseteq y \iff z(\prec\gamma, \beta\succ) = 1 \iff F_G(x)(\prec H_G(x), \beta\succ) = 1 \iff s(\beta) \subseteq x$$

holds for all $\beta < \kappa$. Since every initial segment of x is of the form $s(\beta)$ for some $\beta < \kappa$, we can conclude $y = x \in A$. □

Theorem 2.9 ([Lüc12, Theorem 1.5])**.** *If G is $\mathbb{P}_s(A)$-generic over* V*, then A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$ in* $\mathrm{V}[G]$.

Proof. In $\mathrm{V}[G]$, define T to be the set that consists of pairs $\langle t, u\rangle$ such that $t \in {}^{<\kappa}\kappa$, $u \in {}^{<\kappa}\kappa$ and there is $\gamma < \kappa$ and $v \in T_G$ with $\mathrm{lh}(t) = \mathrm{lh}(u) = \mathrm{lh}(v)$, $u(\alpha) = \prec\gamma, v(\alpha)\succ$ for all $\alpha < \mathrm{lh}(t)$ and

$$s(\beta) \subseteq t \iff v(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \mathrm{lh}(t)$ with $\prec\gamma, \beta\succ < \mathrm{lh}(t)$. It is easy to check that T is a tree.

If $\langle x, y\rangle \in [T]^{\mathrm{V}[G]}$, then there is $z \in [T_G]^{\mathrm{V}[G]}$ and $\gamma < \kappa$ with $y(\beta) = \prec\gamma, z(\beta)\succ$ and

$$s(\beta) \subseteq x \iff z(\prec\gamma, \beta\succ) = 1$$

for all $\beta < \kappa$. By Lemma 2.8, this implies $x \in A$.

Conversely, if $x \in A$ and $y \in ({}^\kappa\kappa)^{\mathrm{V}[G]}$ with $y(\alpha) = \prec H_G(x), F_G(x)(\alpha)\succ$, then $\langle x, y\rangle \in [T]$ by our assumptions on s and Lemma 2.8. □

We close this section by proving a structural property of our coding forcing that will be needed in the proof of supercompactness preservation.

Lemma 2.10. *Assume $P \subseteq \mathbb{P}_s(A)$ satisfies the following properties.*

(i) *$\eta = \mathrm{lub}\{\mathrm{ht}(T_p) \mid p \in P\} \in \mathrm{Lim} \cap \kappa$.*

(ii) *$D = \bigcup\{\mathrm{dom}(f_p) \mid p \in P\}$ has cardinality less than κ.*

(iii) *If $p_0, p_1 \in P$, then there is $q \in P$ with $q \leq_{\mathbb{P}_s(A)} p_0, p_1$.*

Then there is a unique condition $p_P \in \mathbb{P}_s(A)$ with $\mathrm{ht}(T_{p_P}) = \eta$, $\mathrm{dom}(f_{p_P}) = D$ and $p_P \leq_{\mathbb{P}_s(A)} p$ for all $p \in P$.

Proof. Set $T = \bigcup\{T_p \mid p \in P\}$. Then T is a tree of height η and an end-extension of T_p for all $p \in P$. If we define

$$F : D \longrightarrow [T];\ x \longmapsto \bigcup\{f_p(x) \mid p \in P,\ x \in \mathrm{dom}(f_p)\} \in [T],$$

then this is a well-defined function. Moreover, for all $x \in D$ there is a unique $H(x) < \kappa$ with $h_p(x) = H(x)$ for all $p \in P$ with $x \in \mathrm{dom}(f_p)$ and we can define $H : D \longrightarrow \kappa$ in this way.

If $x \in D$ and $\alpha, \beta < \eta$ with $\alpha = \prec H(x), \beta \succ$, then there is $p \in P$ with $x \in \mathrm{dom}(f_p)$ and $\alpha, \beta < \mathrm{ht}(T_p)$. We can conclude

$$s(\beta) \subseteq x \iff f_p(x)(\alpha) = 1 \iff F(x)(\alpha) = 1.$$

This shows that $p_P = \langle T, F, H\rangle$ is a condition in $\mathbb{P}$ with $p_P \leq p$ for all $p \in P$.

Let $q \in \mathbb{P}_s(A)$ be a condition with $\mathrm{ht}(T_q) = \eta$, $\mathrm{dom}(f_q) = D$ and $q \leq_{\mathbb{P}_S(A)} p$ for all $p \in P$. Since $\eta \in \mathrm{Lim}$, for every $t \in T_q$ there is a $p \in P$ with $\mathrm{lh}(t) < \mathrm{ht}(T_p)$ and therefore $t \in T_p$. This shows $T_q = \bigcup\{T_p \mid p \in P\} = T$. In the same way, we can show $f_q(x) = \bigcup\{f_p(x) \mid p \in P,\ x \in \mathrm{dom}(f_p)\} = F(x)$ and $h_q(x) = H(x)$ for all $x \in D$. This means $q = p_P$. □

3. Coding well-orders

In this section, we show how to apply the results of the last section to construct a definable well-order of $\mathrm{H}(\kappa^+)$ in a $\mathbb{P}_s(A)$-generic extension of the ground model. Throughout this section κ is an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$.

Given functions $x, y \in {}^\kappa\kappa$, let $\prec x, y\succ$ denote the unique function $z \in {}^\kappa\kappa$ such that

$$z(\prec\alpha, \beta\succ) = \begin{cases} x(\beta), & \text{if } \alpha = 0,\\ y(\beta), & \text{if } \alpha = 1,\\ 0, & \text{otherwise}\end{cases}$$

holds for all $\alpha, \beta < \kappa$. We say that a κ-coding basis $\langle A, s\rangle$ *codes a well-order of* ${}^\kappa\kappa$ if there is a well-order $\leq$ of ${}^\kappa\kappa$ such that $A = \{\prec x, y\succ \mid x, y \in {}^\kappa\kappa,\ x \leq y\}$.

Theorem 3.1. *If $\langle A, s\rangle$ is a κ-coding basis that codes a well-order of ${}^\kappa\kappa$ and G is $\mathbb{P}_s(A)$-generic over* V*, then there is a well-order of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *that is definable in* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ *by a formula with parameters.*

Proof. We work in $\mathrm{V}[G]$. Let $\leq$ denote the well-order of $({}^\kappa\kappa)^{\mathrm{V}}$ coded by A. By Theorem 2.9, both $\leq$ and $({}^\kappa\kappa)^{\mathrm{V}}$ are definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$.

Define R to be the set of all pairs $\langle a, x\rangle$ in $\mathrm{H}(\kappa^+) \times {}^\kappa 2$ such that there is a bijection $b : \kappa \longrightarrow \mathrm{tc}(\{a\} \cup \kappa)$ with the following properties.

(i) For all $\alpha, \beta < \kappa$, $x(\prec 0, \prec\alpha, \beta\succ\succ) = 1$ if and only if $b(\alpha) \in b(\beta)$.
(ii) For all $\alpha < \kappa$, $x(\prec 1, \alpha\succ) = 1$ if and only if $b(\alpha) \in a$.

This relation is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$. If $\langle a_0, x\rangle, \langle a_1, x\rangle \in R$, then it is easy to see that $a_0 = a_1$ holds. Moreover, if $\langle a, x\rangle \in R$ and $x \in \mathrm{V}$, then a is an element of $\mathrm{H}(\kappa^+)^{\mathrm{V}}$. This shows that $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$.

Since $\mathbb{P}_s(A)^{\mathrm{V}}$ is ${<}\kappa$-closed and A is definable in the above structure, we have $\mathbb{P}_s(A)^{\mathrm{V}} = \mathbb{P}_s(A) \subseteq \mathrm{H}(\kappa^+)$ and this partial order is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$. Given $y \in A$ and $\gamma < \kappa$, the proof of Lemma 2.8 shows that $H_G(y) = \gamma$ holds if and only if there is a $z \in [T_G]$ such that (2) holds for all $\beta < \kappa$. We can conclude that the function H_G is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$. In combination with (1), this implies that the function F_G is also definable in this structure. The filter G consists of all conditions p in $\mathbb{P}_s(A)$ such that T_G is an end-extension of T_p and, if $x \in \mathrm{dom}(f_p)$, then $f_p(x) = F_G(x) \restriction \mathrm{ht}(T_p)$ and $h_p(x) = H_G(x)$. Since all of these parameters are either elements of $\mathrm{H}(\kappa^+)$ or definable in this structure, we can conclude that G is definable in $\langle \mathrm{H}(\kappa^+), \in \rangle$.

Let N denote the set of all functions $n : \kappa \times \kappa \longrightarrow \mathbb{P}_s(A)$ in V with the property that $A^n_\alpha = \{n(\alpha, \beta) \mid \beta < \kappa\}$ is an antichain in $\mathbb{P}_s(A)$ for all $\alpha < \kappa$. We define E to be the set consisting of all pairs $\langle y, n \rangle \in {}^\kappa 2 \times N$ such that

$$y(\alpha) = 1 \iff A^n_\alpha \cap G \neq \emptyset$$

holds for all $\alpha < \kappa$. By identifying functions in N with nice names for subsets of κ, it is easy to see that the domain of E is ${}^\kappa 2$. Both relations N and R are all definable in the above structure.

Define $r : \mathrm{H}(\kappa^+) \longrightarrow ({}^\kappa 2)^{\mathrm{V}}$ to be the function that sends $a \in \mathrm{H}(\kappa^+)$ to the $\leq$-least $x \in ({}^\kappa 2)^{\mathrm{V}}$ such that $R(a, y)$, $E(y, n)$ and $R(n, x)$ for some $y \in {}^\kappa 2$ and $n \in N$. This function is definable in $\langle \mathrm{H}(\kappa^+), \in \rangle$ and yields a definable well-order of $\mathrm{H}(\kappa^+)$. □

Next, we introduce partial orders $\mathbb{C}_\alpha$ that *randomly* well-order ${}^\alpha\alpha$ if α is a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$. This coding is *random* in the sense that the generic filter chooses the well-order of ${}^\alpha\alpha$ that is coded using a partial order of the form $\mathbb{P}_s(A)$.

If α is not a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$, then we define $\mathbb{C}_\alpha$ to be the trivial partial order. Otherwise, we define the domain of $\mathbb{C}_\alpha$ to consist of conditions $\langle A, s, p \rangle$ such that either $A = s = p = \emptyset$ or $\langle A, s \rangle$ is an α-coding basis that codes a well-ordering of ${}^\alpha\alpha$ and $p \in \mathbb{P}_s(A)$. We set $\langle A, s, p \rangle \leq_{\mathbb{C}_\alpha} \langle B, t, q \rangle$ if either $B = \emptyset$ or $A = B \neq \emptyset$, $s = t$ and $p \leq_{\mathbb{P}_s(A)} q$.

Proposition 3.2. *Let α be a regular uncountable cardinal with $\alpha = \alpha^{<\alpha}$.*

(i) *$\mathbb{C}_\alpha$ is $<\alpha$-closed.*

(ii) *A filter G is $\mathbb{C}_\alpha$-generic over V if and only if there is an α-coding basis $\langle A, s \rangle$ coding a well-order of ${}^\alpha\alpha$ in V and H $\mathbb{P}_s(A)$-generic over V with*

$$G = \{\langle \emptyset, \emptyset, \emptyset \rangle\} \cup \{\langle A, s, p \rangle \in \mathbb{C}_\alpha \mid p \in H\}. \tag{3}$$

In particular, $\mathrm{V}[G] = \mathrm{V}[H]$ holds in the above situation, forcing with $\mathbb{C}_\alpha$ preserves cofinalities, cardinalities and 2^α and every set of ordinals of cardinality at most α in a $\mathbb{C}_\alpha$-generic extension of the ground model V is covered by a set that is an element of V and has cardinality α in V.

(iii) *If G is $\mathbb{C}_\alpha$-generic over V, then there is a well-order of $\mathrm{H}(\alpha^+)$ that is definable in $\langle \mathrm{H}(\alpha^+), \in \rangle$ by a formula with parameters.* □

Note that $\mathbb{C}_\alpha$ is uniformly definable in parameter α.

4. Iterated coding forcing

In this section, we use the coding forcing developed above in an iterated forcing construction. Our account of iterated forcing follows [Bau83] and [Cum10] and we will repeatedly use results proved there.

By the results of the last section, there is a unique forcing iteration

$$\langle \langle \vec{\mathbb{C}}_{<\alpha} \mid \alpha \in \mathrm{On} \rangle, \langle \dot{\mathbb{C}}_\alpha \mid \alpha \in \mathrm{On} \rangle \rangle$$

with *Easton support* (see [Cum10, Definition 7.5]) satisfying the following properties.

(i) If $\beta < \alpha$ and α is inaccessible, then $\vec{\mathbb{C}}_{<\beta}, \dot{\mathbb{C}}_\beta \in \mathrm{V}_\alpha$.
(ii) If α is not an inaccessible cardinal, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha$ is trivial".
(iii) If α is an inaccessible cardinal, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha = \mathbb{C}_{\check{\alpha}}$".

For all $\nu \leq \mu$, we let $\dot{\mathbb{C}}_{[\nu,\mu)}$ denote the canonical $\vec{\mathbb{C}}_{<\nu}$-name with

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\nu}} \Vdash \text{"}\dot{\mathbb{C}}_{[\nu,\mu)} \text{ is a partial order with domain } \{\vec{p} \restriction [\check{\nu}, \check{\mu}) \mid \vec{p} \in \check{\vec{\mathbb{C}}}_{<\mu}\}\text{"}$$

such that there is a dense embedding $e_{[\nu,\mu)} : \vec{\mathbb{C}}_{<\mu} \longrightarrow \vec{\mathbb{C}}_{<\nu} * \dot{\mathbb{C}}_{[\nu,\mu)}$ with $e_{[\nu,\mu)}(\vec{p}) = \langle \vec{p} \restriction \nu, \dot{q} \rangle$ and $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu}} \Vdash$ "$\dot{q} = \check{\vec{p}} \restriction [\check{\nu}, \check{\mu})$" (see [Bau83, Section 5]).

Proposition 4.1. *Let $\alpha < \mu$ and μ be a regular cardinal. Assume that there are no inaccessible cardinals in (α, μ) and $\vec{\mathbb{C}}_{<\alpha+1}$ has the property that every set of ordinals of cardinality less than μ in a $\vec{\mathbb{C}}_{<\alpha+1}$-generic extension of the ground model is covered by a set of cardinality less than μ in the ground model. Then*

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha+1}} \Vdash \text{"}\dot{\mathbb{C}}_{[\alpha+1,\nu)} \textit{ is } {<}\check{\mu}\textit{-closed"}$$

for all $\nu > \alpha$.

Proof. For all $\alpha < \beta < \mu$, we have $\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash$ "$\dot{\mathbb{C}}_\beta$ is trivial" by the definition of $\vec{\mathbb{C}}_{<\nu}$ and our assumptions on μ. This shows that $\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash$ "$\dot{\mathbb{C}}_\beta$ is $<\check{\mu}$-closed" holds for all $\beta > \alpha$. Moreover, $\vec{\mathbb{C}}_{<\beta}$ is an inverse limit for every limit ordinal $\beta > \alpha$ with $\mathrm{cof}(\beta) < \mu$. We can apply [Cum10, Proposition 7.12] to deduce the statement of the claim. □

Proposition 4.2. *If α is an inaccessible cardinal, then $\vec{\mathbb{C}}_{<\alpha}$ preserves the inaccessibility of α.*

Proof. Let G be $\vec{\mathbb{C}}_{<\alpha}$-generic over V. Fix $\beta < \alpha$ and let $G_{\beta+1}$ denote the corresponding filter in $\vec{\mathbb{C}}_{<\beta+1}$. If $\mu = (|\vec{\mathbb{C}}_{<\beta}|^+ + |\beta|)^+$, then there are no inaccessible cardinals in (β, μ) and $\dot{\mathbb{C}}^{\mathrm{V}[G_{\beta+1}]}_{[\beta+1,\alpha)}$ is $<\beta^+$-closed by Proposition 4.1. This shows $({}^\beta\alpha)^{\mathrm{V}[G]} \subseteq \mathrm{V}[G_{\beta+1}]$. Since $\vec{\mathbb{C}}_{<\beta+1} \in \mathrm{V}_\alpha$ and α is inaccessible in $\mathrm{V}[G_{\beta+1}]$, the statement of the claim follows directly. □

Proposition 4.3. *$\vec{\mathbb{C}}_{<\nu}$ preserves the inaccessibility of all inaccessible cardinals.*

Proof. By Proposition 4.2 and our assumptions, $\vec{\mathbb{C}}_{<\nu}$ preserves the cofinality, cardinality and inaccessibility of all inaccessible cardinals greater or equal to ν.

Let $\alpha < \nu$ be an inaccessible cardinal. By Proposition 4.2, $\vec{\mathbb{C}}_{<\alpha}$ preserves the inaccessibility of α and $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha}} \Vdash$ "$\dot{\mathbb{C}}_\alpha$ is not trivial". Proposition 3.2 shows that $\vec{\mathbb{C}}_{<\alpha+1}$ preserves the inaccessibility of α. If $\mu = (|\vec{\mathbb{C}}_{<\alpha+1}|^+ + \alpha)^+$, then there are no inaccessible cardinals in (α, μ) and $\mathbb{1}_{\vec{\mathbb{C}}_{<\alpha+1}} \Vdash$ "$\dot{\mathbb{C}}_{[\alpha+1,\nu)}$ is $<\check{\mu}$-closed". In particular, $\vec{\mathbb{C}}_{<\nu}$ preserves the inaccessibility of α. □

Lemma 4.4. *Let $\alpha < \nu$ and α be an inaccessible cardinal. Assume G is $\vec{\mathbb{C}}_{<\nu}$-generic over* V*, $\bar{G}$ is the corresponding filter in $\vec{\mathbb{C}}_{<\alpha}$ and G_α is the corresponding filter in $\dot{\mathbb{C}}^{\bar{G}}_\alpha$. Then $(2^\alpha)^{\mathrm{V}[G]} = (2^\alpha)^{\mathrm{V}}$, $\dot{\mathbb{C}}^{\bar{G}}_\alpha = \mathbb{C}^{\mathrm{V}[\bar{G}]}_\alpha$ is not the trivial partial order and, if $\langle A, s\rangle$ is an α-coding basis coding a well-order of ${}^\alpha\alpha$ in $\mathrm{V}[\bar{G}]$ with $\langle A, s, \mathbb{1}_{\mathbb{P}_s(A)}\rangle \in G_\alpha$, then A is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[G]$ and there is a well-order of $\mathrm{H}(\alpha^+)^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}(\alpha^+)^{\mathrm{V}[G]}, \in\rangle$ by a formula with parameters.*

Proof. It follows directly from the definition of the forcing iteration that the partial order $\vec{\mathbb{C}}_{<\alpha}$ has cardinality at most α. This implies $(2^\alpha)^{\mathrm{V}[\bar{G}]} = (2^\alpha)^{\mathrm{V}}$ and we can apply Lemma 2.3 to conclude $(2^\alpha)^{\mathrm{V}[\bar{G}][G_\alpha]} = (2^\alpha)^{\mathrm{V}}$. By Proposition 4.2, α is an inaccessible cardinal in $\mathrm{V}[\bar{G}]$ and there is an α-coding basis $\langle A, s\rangle$ in $\mathrm{V}[\bar{G}]$ such that $\langle A, s, \mathbb{1}_{\mathbb{P}_s(A)}\rangle \in G_\alpha$. Theorem 2.9 shows that A is a $\boldsymbol{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[\bar{G}][G_\alpha]$ and there is a well-order of $\mathrm{H}(\alpha^+)^{\mathrm{V}[\bar{G}][G_\alpha]}$ definable in $\langle \mathrm{H}(\alpha^+)^{\mathrm{V}[\bar{G}][G_\alpha]}, \in\rangle$ by a formula with parameters by Theorem 3.1. As above, it is easy to show that $\dot{\mathbb{C}}^{\bar{G}*G_\alpha}_{[\alpha+1,\nu)}$ adds no new α-sequences of ordinals. We can conclude $(2^\alpha)^{\mathrm{V}[G]} = (2^\alpha)^{\mathrm{V}}$, $({}^\alpha\alpha)^{\mathrm{V}[G]} = ({}^\alpha\alpha)^{\mathrm{V}[\bar{G}*G_\alpha]}$ and $\mathrm{H}(\alpha^+)^{\mathrm{V}[G]} = \mathrm{H}(\alpha^+)^{\mathrm{V}[\bar{G}][G_\alpha]}$. □

Proposition 4.5. *Let κ be an infinite cardinal with the property that $\kappa \notin (\nu^+, 2^\nu]$ holds whenever ν is a singular limit of inaccessible cardinals. Given $\mu > \kappa$, $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality of κ and, if κ is regular, then $\vec{\mathbb{C}}_{<\mu}$ preserves the regularity of κ.*

Proof. By Proposition 4.3, we may assume that κ is not inaccessible. Let

$$\nu = \sup\{\alpha < \kappa \mid \alpha \text{ is an inaccessible cardinal}\}.$$

If $\nu = 0$ or ν is inaccessible, then $\nu < \kappa$, $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the κ-chain condition and $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ "$\dot{\mathbb{C}}_{[\nu+1,\mu)}$ is $<\check{\kappa}^+$-closed" holds by Proposition 4.1.

If ν is singular and $\kappa = \nu$, then κ is a limit of inaccessible cardinals and $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality of κ by Proposition 4.3.

Let ν be singular and $\kappa = \nu^+$. Assume, toward a contradiction, that κ has cardinality less or equal to ν in some $\vec{\mathbb{C}}_{<\mu}$-generic extension $\mathrm{V}[G]$ of the ground model. Then there is an inaccessible cardinal α with $\mathrm{cof}(\kappa)^{\mathrm{V}[G]} < \alpha < \nu$. If $\bar{G}$ is the filter in $\vec{\mathbb{C}}_{<\alpha+1}$ induced by G, then $\mathrm{cof}(\kappa)^{\mathrm{V}[\bar{G}]} < \kappa$, because $\dot{\mathbb{C}}^{\bar{G}}_{[\alpha+1,\mu)}$ is $<\alpha$-closed by Proposition 4.1. But $\vec{\mathbb{C}}_{<\alpha+1}$ satisfies the κ-chain condition, a contradiction. This shows that $\vec{\mathbb{C}}_{<\mu}$ preserves the cardinality and cofinality of ν^+.

If ν is singular and $\kappa > 2^\nu$, then $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the κ-chain condition and $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ "$\dot{\mathbb{C}}_{[\nu+1,\mu)}$ is $<\check{\kappa}^+$-closed" holds by Proposition 4.1. □

5. Preserving supercompactness

This section is devoted to the proof of the following theorem.

Theorem 5.1. *Let γ be a cardinal with $2^\gamma = \gamma^+$ and $2^\nu \leq \gamma$, where*

$$\nu = \sup\{\alpha \leq \gamma \mid \alpha \textit{ is an inaccessible cardinal}\}.$$

If κ is γ-supercompact with $\gamma = \gamma^{<\kappa}$, then

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash \text{"}\check{\kappa} \textit{ is } \check{\gamma}\textit{-supercompact"}$$

holds for all $\lambda > \nu$.

Proof. By our assumptions, $\mathrm{cof}(\gamma) \geq \kappa$ and $\nu \in [\kappa, \gamma)$ is a strong limit cardinal.

Let U be a normal ultrafilter on $\mathcal{P}_\kappa(\gamma)$. We will prove a number of claims that will allow us to show that κ is γ-supercompact in every $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of the ground model. Given $\alpha \leq \beta \in \mathrm{On}$, we define $\vec{\mathbb{Q}}_{<\alpha} = \vec{\mathbb{C}}^{M_U}_{<\alpha}$, $\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{C}}^{M_U}_\alpha$ and $\dot{\mathbb{Q}}_{[\alpha,\beta)} = \dot{\mathbb{C}}^{M_U}_{[\alpha,\beta)}$.

Since ν is either an inaccessible cardinal or a limit of inaccessible cardinals, we have $\vec{\mathbb{C}}_{<\alpha} \in \mathrm{V}_\nu \subseteq M_U$ for all $\alpha < \nu$ and this shows $\vec{\mathbb{C}}_{<\nu} \in M_U$, because ${}^\gamma M_U \subseteq M_U$ holds. The definition of $\vec{\mathbb{C}}_{<\alpha}$ is absolute between V and M_U for every $\alpha \leq \nu$. Hence elementarity implies $\vec{\mathbb{C}}_{<\nu} = \vec{\mathbb{Q}}_{<\nu}$. In particular, if $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over V, then $\bar{G}$ is $\vec{\mathbb{Q}}_{<\nu}$-generic over M_U.

Claim 1. *If $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over* V*, then $({}^\gamma M_U[\bar{G}])^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$.*

Proof of the claim. Let $x \in \mathrm{V}[\bar{G}]$ with $x \subseteq \gamma$. We can find a $\vec{\mathbb{C}}_{<\nu}$-nice name $\tau = \bigcup_{\alpha<\gamma} \{\check{\alpha}\} \times A_\alpha$ with $x = \tau^{\bar{G}}$. By the above remarks, we have $\vec{\mathbb{C}}_{<\nu} \subseteq {}^\nu \mathrm{V}_\nu$ and every A_α has cardinality at most $2^\nu \leq \gamma$. This shows that every A_α is an element of M_U and we also get $\langle A_\alpha \mid \alpha < \gamma \rangle \in M_U$. Hence $\tau \in M_U$ and $x = \tau^{\bar{G}} \in M_U[\bar{G}]$. We can conclude $({}^\gamma 2)^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$.

Let $X \in \mathrm{V}[\bar{G}]$ with $X \subseteq \mathrm{On}$ and $|X|^{\mathrm{V}[\bar{G}]} \leq \gamma$. Since $\vec{\mathbb{C}}_{<\nu}$ satisfies the γ^+-chain condition in V, there is an $X_0 \in \mathrm{V}$ with $X \subseteq X_0$ and $|X_0|^{\mathrm{V}} \leq \gamma$. By our assumptions, $X_0 \in M_U$ and $|X_0|^{M_U} \leq \gamma$. Let $\langle \eta_\alpha \mid \alpha < \gamma \rangle$ be an enumeration of X_0 in M_U and $x = \{\alpha < \gamma \mid \eta_\alpha \in X\} \in \mathrm{V}[\bar{G}]$. By the above argument, $x \in M_U[\bar{G}]$ and this shows $X \in M_U[\bar{G}]$.

The argument shows $({}^\gamma \mathrm{On})^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$ and this implies the statement of the claim, because $M_U[\bar{G}]$ is a transitive ZFC-model with $\mathrm{On} \subseteq M_U[\bar{G}] \subseteq \mathrm{V}[\bar{G}]$. □

Claim 2. *If $\bar{G}$ is $\vec{\mathbb{C}}_{<\nu}$-generic over* V*, then $\dot{\mathbb{C}}_\nu^{\bar{G}} = \dot{\mathbb{Q}}_\nu^{\bar{G}}$.*

Proof of the claim. If ν is not an inaccessible cardinal in V, then ν is not inaccessible in M_U and both partial orders are trivial.

Now, assume that ν is inaccessible in V and M_U. By Lemma 4.4, $(2^\nu)^{\mathrm{V}[\bar{G}]} = (2^\nu)^{\mathrm{V}} \leq \gamma$ and Claim 1 implies $\mathcal{P}({}^\nu\nu)^{\mathrm{V}[\bar{G}]} = \mathcal{P}({}^\nu\nu)^{M_U[\bar{G}]}$. This allows us to conclude $\dot{\mathbb{C}}_\nu^{\bar{G}} = \mathbb{C}_\nu^{\mathrm{V}[\bar{G}]} = \mathbb{C}_\nu^{M_U[\bar{G}]} = \dot{\mathbb{Q}}_\nu^{\bar{G}}$. □

In particular, if G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over V, then G is $\vec{\mathbb{Q}}_{<\nu+1}$-generic over M_U.

Claim 3. *If G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over* V*, then $({}^\gamma M_U[G])^{\mathrm{V}[G]} \subseteq M_U[G]$.*

Proof of the claim. Let $\bar{G}$ be the filter in $\vec{\mathbb{C}}_{<\nu}$ corresponding to G and G_ν be the filter in $\dot{\mathbb{C}}_\nu^{\bar{G}}$ corresponding to G. By Proposition 3.2 and the above claims, there is a partial order $\mathbb{P}$ in $M_U[\bar{G}]$ and $H \in M_U[G]$ such that $\mathbb{P}$ satisfies the ν^+-chain condition in $\mathrm{V}[\bar{G}]$, H is $\mathbb{P}$-generic over $\mathrm{V}[\bar{G}]$ and H induces G_ν as in (3). Every antichain in $\mathbb{P}$ in $\mathrm{V}[\bar{G}]$ has cardinality at most γ in $\mathrm{V}[\bar{G}]$ and $({}^\gamma M_U[\bar{G}])^{\mathrm{V}[\bar{G}]} \subseteq M_U[\bar{G}]$, so we can repeat the proof of Claim 1 and deduce the statement of the claim. □

The proofs of the above claims show that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of V is covered by a set of cardinality γ in V. By our assumptions, this implies that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{Q}}_{<\nu+1}$-generic extension of M_U is covered by a set of cardinality γ in M_U. In particular, forcing with $\vec{\mathbb{Q}}_{<\nu+1}$ preserves $(\gamma^+)^{M_U} = (\gamma^+)^{\mathrm{V}}$.

Claim 4. *If G is $\vec{\mathbb{C}}_{<\nu+1}$-generic over* V*, then $\dot{\mathbb{Q}}_{[\nu+1,\mu)}^G$ is $<\gamma^+$-closed in $M_U[G]$ for all $\mu > \nu$ and the power set of $\dot{\mathbb{Q}}_{[\nu+1,j_U(\nu))}^G$ in $M_U[G]$ has cardinality at most γ^+ in* $\mathrm{V}[G]$.

Proof of the claim. In M_U, the interval (ν,γ^+) contains no inaccessible cardinals, because ${}^{\gamma}M_U \subseteq M_U$ holds and no ordinal in this interval is inaccessible in V. By the above remark and an application of Proposition 4.1 in M_U, we can conclude that $\dot{\mathbb{Q}}^G_{[\nu+1,\mu)}$ is $<\gamma^+$-closed in $M_U[G]$ for all $\mu > \nu$.

By the definition of the partial order $\dot{\mathbb{C}}_{[\alpha,\beta)}$ and elementarity, the cardinality of $\dot{\mathbb{Q}}^G_{[\nu+1,j_U(\nu))}$ in $M_U[G]$ is less than or equal to the cardinality of $\vec{\mathbb{Q}}_{<j_U(\nu)}$ in M_U. The above computations and elementarity show that the cardinality of $\vec{\mathbb{Q}}_{<j_U(\nu)}$ in M_U is at most $j_U(2^\nu)$ and this ordinal is smaller or equal to $j_U(\gamma)$. Since forcing with $\mathbb{C}_{<\nu+1}$ over M_U preserves the value of $2^{j_U(\gamma)}$, we can conclude that the power set of $\dot{\mathbb{Q}}^G_{[\nu+1,j_U(\nu))}$ has cardinality at most $j_U(2^\gamma)$ in $M_U[G]$. If $\alpha < j_U(2^\gamma)$, then α is represented in M_U by a function $f : \mathcal{P}_\kappa(\gamma) \longrightarrow 2^\gamma$ contained in V. By our assumptions, $\mathcal{P}_\kappa(\gamma)$ has cardinality γ in V and there are at most 2^γ-many such functions in V. Since $2^\gamma = \gamma^+$ holds in V and $(\gamma^+)^{\mathrm{V}[G]} = (\gamma^+)^{\mathrm{V}}$, this shows that $j_U(2^\gamma)$ has cardinality at most γ^+ in $\mathrm{V}[G]$. □

Since $\vec{\mathbb{C}}_{<\nu} \in M_U$ has cardinality at most γ in V, we have $j_U \restriction \vec{\mathbb{C}}_{<\nu} \in M_U$ and there is a sequence

$$\langle \dot{G}_\alpha \in (\mathrm{V}^{\vec{\mathbb{Q}}_{<\nu}})^{M_U} \mid j_U(\kappa) \leq \alpha < j_U(\nu)\rangle$$

of names in M_U with the property that $\dot{G}^{\bar{G}}_\alpha = \{j_U(\vec{p}) \restriction \alpha \mid \vec{p} \in \bar{G}\}$ for all $\alpha \in [j_U(\kappa), j_U(\nu))$ whenever $\bar{G}$ is $\vec{\mathbb{Q}}_{<\nu}$-generic over M_U.

Claim 5. *Let $\alpha \in [j_U(\kappa), j_U(\nu))$ be an inaccessible cardinal in M_U, H be $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ be the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H. If $\dot{G}^{\bar{G}}_\alpha \subseteq H$ and $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}^{M_U[H]}_\alpha}$ for some $\vec{p} \in \bar{G}$, then the following statements hold.*

(i) *There is a unique α-coding basis $\langle A_\alpha, s_\alpha\rangle$ coding a well-order of ${}^\alpha\alpha$ in $M_U[H]$ such that for all $\vec{p} \in \bar{G}$ with $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}^{M_U[H]}_\alpha}$ there is a $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$ with $j_U(\vec{p})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$.*

(ii) *The set*

$$P_\alpha = \{q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]} \mid (\exists \vec{p} \in \bar{G})\ j_U(\vec{p})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle\}$$

satisfies the statements (i)-(iii) *of Lemma 2.10 in $M_U[H]$.*

Proof of the claim. If $\vec{p} \in \bar{G}$ and $\beta < \nu$, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\beta}} \Vdash$ "$\vec{p}(\beta) \in \dot{\mathbb{C}}_\beta$". By elementarity, we have $\mathbb{1}_{\vec{\mathbb{Q}}_{<\alpha}} \Vdash$ "$j_U(\vec{p})(\alpha) \in \dot{\mathbb{Q}}_\alpha$" and, by Proposition 4.2, this implies

$$Q_\alpha = \{j_U(\vec{p})(\alpha)^H \mid \vec{p} \in \bar{G}\} \subseteq \dot{\mathbb{Q}}^H_\alpha = \mathbb{C}^{M_U[H]}_\alpha.$$

Given $\vec{p}_0, \vec{p}_1 \in \bar{G}$, there is a $\vec{p} \in \bar{G}$ with $\vec{p} \leq_{\vec{\mathbb{C}}_{<\nu}} \vec{p}_0, \vec{p}_1$ and hence $\vec{p} \restriction \beta \Vdash$ "$\vec{p}(\beta) \leq_{\dot{\mathbb{P}}_\beta} \vec{p}_0(\beta), \vec{p}_1(\beta)$" for all $\beta < \nu$. Since $j_U(\vec{p}) \restriction \alpha \in \dot{G}^{\bar{G}}_\alpha \subseteq H$, this argument shows that the elements of Q_α are pairwise compatible.

Pick $\vec{p}_* \in \bar{G}$ with $j_U(\vec{p}_*)(\alpha)^H \neq \mathbb{1}_{\mathbb{C}^{M_U[H]}_\alpha}$ and define $\langle A_\alpha, s_\alpha\rangle \in M_U[H]$ to be the unique α-coding basis coding a well-order of ${}^\alpha\alpha$ with $j_U(\vec{p}_*)(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$ for some condition $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$. By the above computations, every element of Q_α is either of the form $\mathbb{1}_{\mathbb{C}^{M_U[H]}_\alpha}$ or $\langle A_\alpha, s_\alpha, q\rangle$ for some $q \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$.

Since $\bar{G}$ has cardinality at most γ in $M_U[H]$, $\gamma < j_U(\kappa) \leq \alpha$ and α is regular in $M_U[H]$, we know that $\eta = \mathrm{lub}\{\mathrm{ht}(T_q) \mid q \in P_\alpha\} < \alpha$ and $\bigcup\{\mathrm{dom}(f_q) \mid q \in P_\alpha\}$ has cardinality less than α in $M_U[H]$.

We show that $\eta \in \mathrm{Lim} \cap \alpha$. Let $\vec{p} \in \bar{G}$ and $p \in \mathbb{P}_{s_\alpha}(A_\alpha)^{M_U[H]}$ with $\langle A_\alpha, s_\alpha, p\rangle = j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\mathbb{C}_\alpha^{M_U[H]}}$. Let D be the set consisting of all conditions $\vec{q} \in \vec{\mathbb{C}}_{<\nu}$ with $\vec{q} \leq_{\vec{\mathbb{C}}_{<\nu}} \vec{p}$ and

$$\vec{q} \restriction \beta \Vdash \text{“}(\forall A, s, p) \ \big[\big(\dot{\mathbb{C}}_\beta = \mathbb{C}_{\check{\beta}} \wedge \vec{p}(\beta) = \langle A, s, p\rangle \neq \mathbb{1}_{\mathbb{C}_{\check{\beta}}}\big) \longrightarrow (\exists \bar{p})[\vec{q}(\beta) = \langle A, s, \bar{p}\rangle \wedge \mathrm{ht}(T_p) < \mathrm{ht}(T_{\bar{p}})]\big]\text{”}$$

for all $\beta < \nu$. An easy inductive construction using Lemma 2.4 shows that D is dense below $\vec{p}$ in V. If $\vec{q} \in D \cap \bar{G}$ with $j_U(\vec{q})(\alpha)^H = \langle A_\alpha, s_\alpha, q\rangle$, then $\mathrm{ht}(T_q) > \mathrm{ht}(T_p)$ holds in $M_U[H]$ by elementarity. This shows that η is a limit ordinal.

Finally, the conditions in P_α are pairwise compatible, because the conditions in Q_α are pairwise compatible and the first part of the claim shows that every condition in P_α belongs to a condition in Q_α. □

In M_U, we define a sequence $\vec{q}_* = \langle \dot{q}_\alpha \in (\mathrm{V}^{\vec{\mathbb{Q}}_{<\alpha}})^{M_U} \mid \alpha < j_U(\nu)\rangle$ such that the following statements hold in M_U for all $\alpha < j_U(\nu)$.

(i) If $\alpha < j_U(\kappa)$ or α is not an inaccessible cardinal, then $\mathbb{1}_{\vec{\mathbb{Q}}_{<\alpha}} \Vdash$ “$\dot{q}_\alpha = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\alpha}$”.

(ii) If α is an inaccessible cardinal in $[j_U(\kappa), j_U(\nu))$, then $\dot{q}_\alpha$ is a canonical $\vec{\mathbb{Q}}_{<\alpha}$-name τ such that the following statements hold whenever H is $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ is the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H.
   (a) If $\dot{G}_\alpha^{\bar{G}} \subseteq H$ and $j_U(\vec{p})(\alpha)^H \neq \mathbb{1}_{\dot{\mathbb{Q}}_\alpha^H}$ for some $\vec{p} \in \bar{G}$, then $\tau^H = \langle A_\alpha, s_\alpha, p_{P_\alpha}\rangle$, where A_α, s_α and P_α are defined as in Claim 5 and p_{P_α} is defined as in Lemma 2.10.
   (b) Otherwise, $\tau^H = \mathbb{1}_{\dot{\mathbb{Q}}_\alpha^H}$.

Claim 6. $\vec{q}_* \in \vec{\mathbb{Q}}_{<j_U(\nu)}$.

Proof of the claim. Let $\alpha \in [j_U(\kappa), j_U(\nu))$ be a regular cardinal in M_U. For all $\vec{p} \in \vec{\mathbb{C}}_{<\nu}$ there is an $\bar{\alpha}_{\vec{p}} < \alpha$ with $j_U(\vec{p})(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\bar{\alpha}_{\vec{p}} \leq \beta < \alpha$. Since j_U"$\vec{\mathbb{C}}_{<\nu}$ is an element of M_U and has cardinality less than α in M_U, we can find an $\bar{\alpha} \in (j_U(\kappa), \alpha)$ with $j_U(\vec{p})(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\vec{p} \in \vec{\mathbb{C}}_{<\nu}$ and $\bar{\alpha} \leq \beta < \alpha$. If $\beta \in (\bar{\alpha}, \alpha)$ is an inaccessible cardinal, H is $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U and $\bar{G}$ is the filter in $\vec{\mathbb{Q}}_{<\nu}$ induced by H, then $j_U(\vec{p})(\beta)^H = \mathbb{1}_{\dot{\mathbb{Q}}_\beta^H}$ for all $\vec{p} \in \bar{G}$ and $\dot{q}_\beta^H = p_{P_\beta} = \mathbb{1}_{\dot{\mathbb{Q}}_\beta^H}$ by the uniqueness of p_{P_β}. By the definition of $\dot{q}_\beta$, this shows $\dot{q}_\beta = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$. Therefore $\vec{q}_*$ is a sequence with Easton support. □

Claim 7. *If H is $\vec{\mathbb{Q}}_{<j_U(\nu)}$-generic over M_U with $\vec{q}_* \in H$ and $\bar{G}$ is the corresponding filter in $\vec{\mathbb{Q}}_{<\nu}$, then j_U"$\bar{G} \subseteq H$.*

Proof of the claim. Let $\alpha \in [\nu, j_U(\nu))$ and F be $\vec{\mathbb{Q}}_{<\alpha}$-generic over M_U with $\vec{q}_* \restriction \alpha \in F$. Assume that F induces $\bar{G}$ in $\vec{\mathbb{Q}}_{<\nu}$ and

$$\vec{q}_* \restriction [\nu, \alpha) \leq_{\dot{\mathbb{Q}}^{\bar{G}}_{[\nu,\alpha)}} j_U(\vec{p}) \restriction [\nu, \alpha) \tag{4}$$

holds for all $\vec{p} \in \bar{G}$. Pick $\vec{p} \in \bar{G}$. There is a $\bar{\kappa} < \kappa$ such that $\vec{p}(\beta) = \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}$ for all $\beta \in [\bar{\kappa}, \kappa)$ and

$$j_U(\vec{p})(\beta) = \begin{cases} \vec{p}(\beta), & \text{if } \beta < \bar{\kappa}, \\ \dot{\mathbb{1}}_{\dot{\mathbb{Q}}_\beta}, & \text{if } \bar{\kappa} \leq \beta < \nu. \end{cases}$$

by the definition of $\vec{\mathbb{C}}_{<\nu}$. In particular, $\vec{p} \leq_{\vec{\mathbb{Q}}_{<\nu}} j_U(\vec{p}) \restriction \nu$. By our assumption, there is a $\vec{p}_* \in \bar{G}$ with $\vec{p}_* \leq_{\vec{\mathbb{Q}}_{<\nu}} \vec{p}$ and

$$\vec{p}_* * (\vec{q}_* \restriction [\nu, \alpha)) \leq_{\vec{\mathbb{Q}}_{<\nu} * \dot{\mathbb{Q}}_{[\nu,\alpha)}} i_{[\nu,\alpha)}(j_U(\vec{p}) \restriction \alpha).$$

This implies $j_U(\vec{p}) \restriction \alpha \in F$ and hence $\dot{G}_\alpha^{\bar{G}} \subseteq F$.

Next, we show that (4) holds in $M_U[G]$ for all $\vec{p} \in \bar{G}$ and $\alpha \in [\nu, j_U(\nu)]$ by induction. The case "$\alpha = \nu$" is trivial and the case "$\alpha \in \mathrm{Lim}$" follows directly from the induction hypothesis.

Assume $\alpha = \bar{\alpha}+1$ with $\bar{\alpha} \geq \nu$. We may assume that $\bar{\alpha}$ is an inaccessible cardinal in M_U. It suffices to show that

$$\vec{q}_*(\bar{\alpha})^F \leq_{\dot{\mathbb{Q}}_{\bar{\alpha}}^F} j_U(\vec{p})(\bar{\alpha})^F$$

holds in $M_U[F]$ whenever $\vec{p} \in \bar{G}$ and F is $\vec{\mathbb{Q}}_{<\bar{\alpha}}$-generic over M_U such that $\vec{q}_* \restriction \bar{\alpha} \in F$ and F induces $\bar{G}$ in $\vec{\mathbb{Q}}_{<\nu}$. We may assume that there is a $\vec{p} \in \bar{G}$ with $j_U(\vec{p})(\bar{\alpha})^F \neq \mathbb{1}_{\dot{\mathbb{Q}}_{\bar{\alpha}}^F}$. By the induction hypothesis and the above computations, we directly get $\dot{G}_{\bar{\alpha}}^{\bar{G}} \subseteq F$. The definition of $\vec{q}_*(\bar{\alpha})$ and Claim 5 imply

$$\vec{q}_*(\bar{\alpha})^F = \langle A_\alpha, s_\alpha, p_{P_{\bar{\alpha}}} \rangle \leq_{\dot{\mathbb{Q}}_{\bar{\alpha}}^F} j_U(\vec{p})(\bar{\alpha})^F$$

for all $\vec{p} \in \bar{G}$.

This induction shows that (4) holds if $\alpha = j_U(\nu)$ and $\vec{p} \in G$. This allows us to repeat the above computation and conclude j_U"$G \subseteq H$. □

Claim 8. $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ *"$\check{\kappa}$ is $\check{\gamma}$-supercompact".*

Proof of the claim. Let G be $\vec{\mathbb{C}}_{<\nu+1}$-generic over V, $\bar{G}$ be the corresponding filter in $\vec{\mathbb{C}}_{<\nu}$ and G_ν be the corresponding filter in $\dot{\mathbb{C}}_\nu^{\bar{G}}$. Claim 4 combined with Claim 3 shows that there is an $\bar{H} \in \mathrm{V}[G]$ such that $\vec{q}_* \in \bar{H}$, $\bar{H}$ is $\vec{\mathbb{Q}}_{<j_U(\nu)}$-generic over M_U and $\bar{H}$ induces G in $\vec{\mathbb{Q}}_{<\nu+1}$. By Claim 7, we have j_U"$\bar{G} \subseteq \bar{H}$ and we can apply [Cum10, Proposition 9.1] to define an elementary embedding $j : \mathrm{V}[\bar{G}] \longrightarrow M_U[\bar{H}]$ extending j_U in $\mathrm{V}[G]$ by setting $j(\tau^{\bar{G}}) = j_U(\tau)^{\bar{H}}$ for all $\tau \in \mathrm{V}^{\vec{\mathbb{C}}_{<\nu}}$.

We show that there is an $H_* \in \mathrm{V}[G]$ such that H_* is $\dot{\mathbb{Q}}_{j_U(\nu)}^{\bar{H}}$-generic over M_U and j"$G_\nu \subseteq H_*$. We may assume that ν is an inaccessible cardinal. This implies $(2^\nu)^{\mathrm{V}[\bar{G}]} = (2^\nu)^{\mathrm{V}} \leq \gamma$. By Proposition 3.2, there is a ν-coding basis $\langle A, s \rangle \in \mathrm{V}[\bar{G}]$ coding a well-order of ${}^\nu\nu$ and a filter $F_\nu \in \mathrm{V}[G]$ such that F_ν is $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$-generic over $\mathrm{V}[\bar{G}]$ and F_ν induces G_ν as in (3).

By Claim 3, we have $({}^\gamma\mathrm{On})^{\mathrm{V}[G]} \subseteq M_U[G] \subseteq M_U[\bar{H}] \subseteq \mathrm{V}[G]$ and this implies that $({}^\gamma M_U[\bar{H}])^{\mathrm{V}[G]} \subseteq M_U[\bar{H}]$ holds. In particular, both $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ and $j \restriction \mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ are elements of $M_U[\bar{H}]$, because $\mathbb{P}_s(A)^{\mathrm{V}[\bar{G}]}$ has cardinality at most γ in $\mathrm{V}[\bar{G}]$. If $j(\langle A, s \rangle) = \langle \bar{A}, \bar{s} \rangle$ and $P = j$"F_ν, then $\langle \bar{A}, \bar{s} \rangle$ is a $j_U(\nu)$-coding basis that codes a well-order of ${}^{j_U(\nu)}j_U(\nu)$ in $M_U[\bar{H}]$, $P \subseteq \mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ and $P \in M_U[\bar{H}]$, because F_ν is an element of $M_U[\bar{H}]$. As in the proof of Claim 5, the set P satisfies the statements (i)-(iii) of Lemma 2.10 in $M_U[\bar{H}]$ and we can find a condition $p_P \in \mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ as in the statement of the Lemma.

In $M_U[\bar{H}]$, $\mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$ is $<\gamma^+$-closed and has cardinality at most $j_U(\gamma)$. By the proof of Claim 4, $j_U(\gamma)$ has cardinality at most γ^+ in $\mathrm{V}[G]$ and there is an $F_* \in \mathrm{V}[G]$ such that $p_P \in F_*$ and F_* is $\mathbb{P}_{\bar{s}}(\bar{A})^{M_U[\bar{H}]}$-generic over $M_U[\bar{H}]$. If $H_* \in \mathrm{V}[G]$ is the filter in $\mathbb{C}_{j_U(\nu)}^{M_U[\bar{H}]}$ corresponding to F_*, then H_* is $\dot{\mathbb{Q}}_{j_U(\nu)}^{\bar{H}}$-generic over $M_U[\bar{H}]$ and our construction ensures $j"G_\nu \subseteq H_*$. Another application of [Cum10, Proposition 9.1] allows us to define an elementary embedding $j_* : \mathrm{V}[G] \longrightarrow M_U[\bar{H}][H_*]$ in $\mathrm{V}[G]$ that extends j. Since $({}^\gamma\mathrm{On})^{\mathrm{V}[G]} \subseteq M_U[\bar{H}][H_*] \subseteq \mathrm{V}[G]$, this argument shows that κ is γ-supercompact in $\mathrm{V}[G]$. □

Claim 9. *If $\lambda > \nu$, then $\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash$ "$\check{\kappa}$ is $\check{\gamma}$-supercompact".*

Proof of the claim. Let H be $\vec{\mathbb{C}}_{<\lambda}$-generic over V and G be the corresponding filter in $\vec{\mathbb{C}}_{<\nu+1}$. There are no inaccessible cardinals in (ν, γ^+) and the above computations show that $\vec{\mathbb{C}}_{<\nu+1}$ has the property that every set of ordinals of cardinality at most γ in a $\vec{\mathbb{C}}_{<\nu+1}$-generic extension of the ground model is covered by a set of cardinality γ in V. By Proposition 4.1, $\dot{\mathbb{C}}^G_{[\nu+1,\lambda)}$ is $<\gamma^+$-closed in $\mathrm{V}[G]$.

By Claim 8, there is a normal filter U^* on $\mathcal{P}_\kappa(\gamma)$ in $\mathrm{V}[G]$ and U^* is also a normal filter on $\mathcal{P}_\kappa(\gamma)$ in $\mathrm{V}[H]$, because $\mathrm{V}[H]$ is a $\dot{\mathbb{C}}^G_{[\nu+1,\lambda)}$-generic extension of $\mathrm{V}[G]$ and $<\gamma^+$-closed forcing preserve normal filters on $\mathcal{P}_\kappa(\gamma)$. □

This completes the proof of Theorem 5.1. □

The following result due to Robert Solovay shows that, given a supercompact cardinal κ, there is a proper class of cardinals γ satisfying the assumptions of Theorem 5.1 with respect to κ. Remember that an uncountable cardinal is *strongly compact* if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter on S. Every supercompact cardinal is strongly compact (see [Kan03, Corollary 22.18]).

Theorem 5.2 ([Sol74, Theorem 1])**.** *If κ is a strongly compact cardinal and γ is a singular strong limit cardinal greater than κ, then $2^\gamma = \gamma^+$.*

Let κ be a cardinal and $\gamma_0 \geq \kappa$. There is a singular strong limit cardinal $\gamma > \gamma_0$ such that $\mathrm{cof}(\gamma) \geq \kappa$ and there are no inaccessible cardinals in $(\gamma_0, \gamma]$. If κ is supercompact, then $2^\gamma = \gamma^+$ by Theorem 5.2 and γ satisfies the assumptions of Theorem 5.1. This proves the following statement.

Corollary 5.3. *If κ is supercompact and $\gamma \in \mathrm{On}$, then there is a $\nu \in \mathrm{On}$ with*

$$\mathbb{1}_{\vec{\mathbb{C}}_{<\lambda}} \Vdash \text{"}\check{\kappa} \text{ is } \check{\gamma}\text{-supercompact"}$$

for all $\lambda > \nu$. □

6. Proofs of the main results

Given $\alpha \leq \beta \in \mathrm{On}$, let $\epsilon_{\alpha,\beta} : \vec{\mathbb{C}}_{<\alpha} \longrightarrow \vec{\mathbb{C}}_{<\beta}$ denote the canonical embedding of partial orders. Let D be the class of all $\vec{p}$ such that there is a $\beta \in \mathrm{On}$ with $\vec{p} \in \vec{\mathbb{C}}_{<\beta}$ and $\vec{p} \neq \epsilon_{\alpha,\beta}(\vec{q})$ for all $\alpha < \beta$ and $\vec{q} \in \vec{\mathbb{C}}_{<\alpha}$. Define $\mathbb{P}$ to be the class forcing with domain D ordered by $\vec{p} \leq_{\mathbb{P}} \vec{q}$ if there are $\alpha, \beta, \gamma \in \mathrm{On}$ with $\alpha, \beta \leq \gamma$, $\vec{p} \in \vec{\mathbb{C}}_{<\alpha}$, $\vec{q} \in \vec{\mathbb{C}}_{<\beta}$ and $\epsilon_{\alpha,\gamma}(\vec{p}) \leq_{\vec{\mathbb{C}}_{<\gamma}} \epsilon_{\beta,\gamma}(\vec{q})$. This means that $\mathbb{P}$ is a direct limit of the directed system $\langle\langle\vec{\mathbb{C}}_{<\alpha} \mid \alpha \in \mathrm{On}\rangle, \langle\epsilon_{\alpha,\beta} \mid \alpha \leq \beta \in \mathrm{On}\rangle\rangle$. Since $\vec{\mathbb{C}}_{<\alpha}$ is uniformly definable in parameter α, $\mathbb{P}$ is definable without parameters.

Proof of Theorem 1.4. First, assume that the inaccessible cardinals are bounded in On and define

$$\nu = \sup\{\alpha \in \mathrm{On} \mid \alpha \text{ is an inaccessible cardinal}\}.$$

We have $\mathbb{1}_{\vec{\mathbb{C}}_{<\nu+1}} \Vdash$ "$\dot{\mathbb{C}}_{[\nu+1,\lambda)}$ is trivial" for all $\lambda > \nu$ and this shows that $\mathbb{P}$ is forcing equivalent to $\vec{\mathbb{C}}_{<\nu+1}$. Since ν is definable without parameters and each $\vec{\mathbb{C}}_\alpha$ is definable in parameter α, the partial order $\vec{\mathbb{C}}_{<\nu+1}$ is definable without parameters. Proposition 4.3, Lemma 4.4 and Corollary 5.3 show that $\vec{\mathbb{C}}_{<\nu+1}$ satisfies the statements listed in Theorem 1.4 under this assumption.

Now, assume that there are unboundedly many inaccessible cardinals in On. Let G be $\mathbb{P}$-generic over V.

For each $\beta \in$ On, define $G_\beta = \{\epsilon_{\alpha,\beta}(\vec{p}) \mid \alpha \leq \beta,\ \vec{p} \in \mathrm{G} \cap \vec{\mathbb{C}}_{<\alpha}\}$. Then G_β is $\vec{\mathbb{C}}_{<\beta}$-generic over V, V[G] is the union of all V$[G_\beta]$ and G_α is the filter induced by G_β in $\vec{\mathbb{C}}_{<\alpha}$ whenever $\alpha \leq \beta \in$ On.

Claim 1. *If α is an inaccessible cardinal in* V *and $x \in$* V[G] *is a subset of α, then $x \in$* V$[G_{\alpha+1}]$.

Proof of the claim. There is a $\beta > \alpha$ with $x \in$ V$[G_\beta]$. Since $\vec{\mathbb{C}}_{<\alpha+1}$ satisfies the α^+-chain condition in V, we can apply Proposition 4.1 to show that $\dot{\mathbb{C}}_{[\alpha+1,\beta)}$ is $<\alpha^+$-closed in V$[G_{\alpha+1}]$ and this implies $x \in$ V$[G_{\alpha+1}]$. □

Claim 2. *Let x be an element of* V[G]*. There is an inaccessible cardinal α such that $y \in$* V$[G_{\alpha+1}]$ *for all $y \in$* V[G] *with $y \subseteq x$. In particular,* V[G] *satisfies the* Power Set Axiom*.*

Proof of the claim. By our assumption, we can find an inaccessible cardinal α in V such that $x \in$ V$[G_{\alpha+1}]$ and $|x|^{\mathrm{V}[G_{\alpha+1}]} \leq \alpha$. Let $i : x \longrightarrow \alpha$ be an injection in V$[G_{\alpha+1}]$. If $y \in$ V[G] is a subset of x, then there is $\beta > \alpha$ with $y \in$ V$[G_\beta]$. By Claim 1, we have $i"y \in$ V$[G_{\alpha+1}]$ and therefore $y \in$ V$[G_{\alpha+1}]$. This argument shows that $\mathcal{P}(x)^{\mathrm{V}[G_{\alpha+1}]}$ is the power set of x in V[G]. □

Claim 3. V[G] *is a model of* ZFC*.*

Proof of the claim. Let $\vec{p}$ be a condition in $\mathbb{P}$, $A \in$ V and $\langle D_a \mid a \in A\rangle$ be a V-definable sequence of dense subclasses of $\mathbb{P}$. There is $\alpha \in$ On with $\vec{p} \in \vec{\mathbb{C}}_{<\alpha}$. Given $a \in A$, define $d_a = \{\vec{q} \upharpoonright \alpha \mid (\exists \beta \geq \alpha)\ \vec{q} \in D_a \cap \vec{\mathbb{C}}_{<\beta}\} \in$ V. Then $\langle d_a \mid a \in A\rangle \in$ V and each d_a is predense in $\mathbb{P}$. This shows that $\mathbb{P}$ is *pretame* with respect to V (see [Fri00, page 33]). By [Fri00, Lemma 2.19], this implies that V[G] is a model of ZFC$^-$. □

Claim 4. *Let κ be a cardinal in* V *with the property that there is no singular limit of inaccessible cardinals ν with $\nu^+ < \kappa \leq 2^\nu$ in* V*. Then κ is a cardinal in* V[G] *and, if κ is regular in* V*, then κ is regular in* V[G]*.*

Proof of the claim. By Proposition 4.5, κ is a cardinal in V$[G_\mu]$ for every $\mu \in$ On and, if κ is regular in V, then κ is regular in every V$[G_\mu]$. In combination with the above remarks, this directly implies the statement of the claim. □

Claim 5. *If κ is a supercompact cardinal in* V*, then κ is supercompact in* V[G]*.*

Proof of the claim. Given $\gamma \in$ On, Corollary 5.3 shows that there is a $\nu \in$ On such that κ is γ-supercompact in V$[G_\beta]$ for all $\beta > \nu$. By Claim 2, there is an inaccessible cardinal α such that $\mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V[G]}} = \mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\alpha]}$ and therefore $\mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\alpha]} = \mathcal{P}(\mathcal{P}_\kappa(\gamma))^{\mathrm{V}[G_\beta]}$ for all $\beta > \nu$. We can conclude that κ is γ-supercompact in V[G]. □

Claim 6. *If α is an inaccessible cardinal in* V*, then α is an inaccessible cardinal in* $\mathrm{V[G]}$ *and* $(2^\alpha)^{\mathrm{V[G]}} = (2^\alpha)^{\mathrm{V}}$*.*

Proof of the claim. By Proposition 4.3, α is an inaccessible cardinal in $\mathrm{V}[G_{\alpha+1}]$ and Lemma 4.4 shows that $(2^\alpha)^{\mathrm{V}[G_{\alpha+1}]} = (2^\alpha)^{\mathrm{V}}$ holds. The statement of the claim follows directly from Claim 1. □

Claim 7. *Let α be an inaccessible cardinal in* V*. There is a well-order of* $\mathrm{H}(\alpha^+)^{\mathrm{V[G]}}$ *that is definable in* $\langle \mathrm{H}(\alpha^+)^{\mathrm{V[G]}}, \in\rangle$ *by a formula with parameters.*

Proof of the claim. By Claim 2, there is a $\nu > \alpha$ with $\mathrm{H}(\alpha^+)^{\mathrm{V[G]}} = \mathrm{H}(\alpha^+)^{\mathrm{V}[G_\nu]}$. The statements of the Claim follows directly from Lemma 4.4. □

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.1. Let α be an inaccessible cardinal and A be a subset of ${}^\alpha\alpha$. There is a $\vec{\mathbb{C}}_{<\alpha}$-name $\dot{p}$ with the property that, whenever G is $\vec{\mathbb{C}}_{<\alpha}$-generic over V, then there is an α-coding basis $\langle \bar{A}, \bar{s}\rangle$ coding a well-order of ${}^\alpha\alpha$ in $\mathrm{V}[G]$ that satisfies the following statements in $\mathrm{V}[G]$.

(i) $\dot{p}^G = \langle \bar{A}, \bar{s}, \mathbb{1}_{\mathbb{P}_{\bar{s}}(\bar{A})^{\mathrm{V}[G]}}\rangle \in \dot{\mathbb{C}}^G_\alpha$.
(ii) There is a well-order $\leq$ of ${}^\alpha\alpha$ such that $\leq$ witnesses that the pair $\langle \bar{A}, \bar{s}\rangle$ codes a well-order of ${}^\alpha\alpha$ and the subset A forms an initial segment of ${}^\alpha\alpha$ of order-type $|A|$ with respect to this well-order.

Pick $\vec{p} \in \vec{\mathbb{C}}_{<\alpha+1}$ with $\vec{p}(\alpha) = \dot{p}$. Then p is a condition in $\mathbb{P}$.

Let G be $\mathbb{P}$-generic over V with $p \in \mathrm{G}$. For each $\beta \in \mathrm{On}$, define G_β as in the proof of Theorem 1.4 and let $\dot{p}^{G_\alpha} = \langle \bar{A}, \bar{s}, \mathbb{1}_{\mathbb{P}_{\bar{s}}(\bar{A})^{\mathrm{V}[G_\alpha]}}\rangle \in \mathrm{V}[G_\alpha]$. By Claim 2 in the above proof, there is a $\nu > \alpha$ with $\mathrm{H}(\alpha^+)^{\mathrm{V[G]}} = \mathrm{H}(\alpha^+)^{\mathrm{V}[G_\nu]}$. Lemma 4.4 implies that $\bar{A}$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\alpha\alpha$ in $\mathrm{V}[G_\nu]$ and therefore also in $\mathrm{V[G]}$. Let $\leq$ denote the well-order of $({}^\alpha\alpha)^{\mathrm{V}[G_\alpha]}$ produced by the above construction. Then $\leq$ is definable in $\langle \mathrm{H}(\alpha^+)^{\mathrm{V[G]}}, \in\rangle$ and A is either equal to the domain of $\leq$ or to the set of all $\leq$-predecessors of an element of this domain. This shows that A is definable in $\langle \mathrm{H}(\alpha^+)^{\mathrm{V[G]}}, \in\rangle$ by a Σ_1-formula with parameters. □

7. Open problems

We close this paper with some open problems related to the above results.

If the *Singular Cardinal Hypothesis* holds, then forcing with the class-sized partial order constructed in Theorem 1.4 does not collapse cardinals. It is not obvious if the converse of this implication also holds.

Question 7.1. *Is it consistent that the partial order constructed in the proof of Theorem 1.4 collapses cardinals?*

Given a κ-coding basis $\langle A, s\rangle$, an easy argument shows that forcing with $\mathbb{P}_s(A)$ adds a Cohen-subset of κ. Therefore, a positive answer to the above question would follow from the existence of certain *scales* (see [Jec03, Definition 24.6]). The proof of [Hon10, Observation 4.3] contains the idea behind this approach.

As mentioned in the abstract, Theorem 1.4 can be viewed as a *boldface* version of Theorem 1.3 in the absence of the GCH. We may therefore ask whether a *lightface* version of Theorem 1.4 is possible.

Question 7.2. *Let κ be a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and $2^\kappa > \kappa^+$. Is there a cardinal preserving partial order $\mathbb{P}$ with the property that, whenever $\mathrm{V}[G]$ is a $\mathbb{P}$-generic extension of the ground model, then there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a formula* without *parameters?*

In [FH12], Radek Honzik and the first author use a κ^{++}-strong cardinal to produce a model with a measurable cardinal κ with $2^\kappa = \kappa^{++}$ and the property that there is a well-order of $\mathrm{H}(\kappa^+)$ that is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a formula without parameters. It is natural to ask whether this statement is optimal.

Question 7.3. *Is it consistent that there is a measurable cardinal κ such that $2^\kappa > \kappa^{++}$ and there is a well-order of $\mathrm{H}(\kappa^+)$ that is definable in $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a formula* without *parameters?*

The result mentioned above is used in [FH12] to establish the relative consistency of a *definable failure of the Singular Cardinal Hypothesis*, i.e., if the existence of a κ^{++}-strong cardinal is consistent, then it is consistent that $\aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and there is a well-order of $\mathrm{H}(\aleph_{\omega+1})$ that is definable in $\langle \mathrm{H}(\aleph_{\omega+1}), \in\rangle$ by a formula without parameters.

Starting from a supercompact cardinal, it is possible to use the results of this paper to answer the *boldface* version of Question 7.3 positively. Let κ be a supercompact cardinal. Use the *Laver preparation* (see [Lav78]) and κ-Cohen forcing to produce a forcing extension in which κ is still supercompact and $2^\kappa > \kappa^{++}$ holds. Then apply Theorem 1.4 to add a boldface definable well-order of $\mathrm{H}(\kappa^+)$ while preserving the supercompactness of κ and the value of 2^κ.

We may therefore ask whether the existence of a stronger definable failure of the *Singular Cardinal Hypothesis* is consistent.

Question 7.4. *Is it consistent that there is a singular strong limit cardinal ν such that $2^\nu > \nu^{++}$ and there is a well-order of $\mathrm{H}(\nu^+)$ that is definable in $\langle \mathrm{H}(\nu^+), \in\rangle$ by a formula with parameters?*

Finally, we ask whether the existence of a definable well-order of $\mathrm{H}(\aleph_{\omega+1})$ can be forced without applying some variation of Prikry forcing.

Question 7.5. *Is there a partial order $\mathbb{P}$ with cardinality less than the least inaccessible cardinal and the property that, whenever $\mathrm{V}[G]$ is a $\mathbb{P}$-generic extension of the ground model, then there is a well-order of $\mathrm{H}(\aleph_{\omega+1})^{\mathrm{V}[G]}$ that is definable in $\langle \mathrm{H}(\aleph_{\omega+1})^{\mathrm{V}[G]}, \in\rangle$ by a formula with parameters?*

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