Structural Reflection and the HOD Conjecture

2. Lecture: Consistency results

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Introduction

Definition

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Definition

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \upharpoonright \mathrm{V}_\lambda \in X$.

In this lecture, we will discuss results dealing with the relative consistency of these axioms.

For this purpose, we recall the definitions of the rank-into-rank axioms.

Definition

- An I3-embedding is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An I2-embedding is a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{V}_\lambda \subseteq M$, where λ is the first non-trivial fixed point of j.
- An I0-embedding is a non-trivial elementary embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$, where λ is the first non-trivial fixed point of j.

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following statements are equiconsistent over ZFC:

- There is an ultraexacting cardinal.
- There is an I0-embedding.

Theorem (Aguilera–Bagaria–Goldberg–L.)

- If there is an I2-embedding, then there is a transitive ZFC-model with an exacting cardinal.
- If λ is an exacting cardinal, then V_λ is a model of ZFC with a proper class of I3-embeddings.

The consistency of ultraexacting cardinals

The relative consistency of ZFC with an ultraexacting cardinal is established from the consistency of ZFC with an I0-embedding through the following result:

Theorem (Aguilera–Bagaria–L.)

If $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ is an I0-embedding and G is $\mathrm{Add}(\lambda^+, 1)$-generic over V, then $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$ is a model of ZFC and λ is an ultraexacting cardinal in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

We outline the proof of this theorem.

Let $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ be an I0-embedding and let G be $\mathrm{Add}(\lambda^+, 1)$-generic over V.

Then $\mathrm{Add}(\lambda^+, 1) \subseteq \mathrm{L}(\mathrm{V}_{\lambda+1})$ and G is also $\mathrm{Add}(\lambda^+, 1)$-generic over $\mathrm{L}(\mathrm{V}_{\lambda+1})$.

Since $\mathrm{Add}(\lambda^+, 1)$ is $<\lambda^+$-closed in V, it follows that V and $\mathrm{V}[G]$ contain the same λ-sequences of elements of V, and $\mathrm{L}(\mathrm{V}_{\lambda+1})$ and $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$ contain the same λ-sequences of elements of $\mathrm{L}(\mathrm{V}_{\lambda+1})$.

By genericity, the filter G codes a wellordering of $\mathcal{P}(\lambda)$ of order-type λ^+, and it follows that $\mathrm{V}_{\lambda+1}$ can be wellordered in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

This shows that $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$ is a model of ZFC.

Moreover, these observations show that $\mathrm{H}(\lambda^+)^{\mathrm{L}(\mathrm{V}_{\lambda+1}, G)} \subseteq \mathrm{L}(\mathrm{V}_{\lambda+1})$.

In $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$, fix a cardinal $\eta > \lambda$ and an elementary submodel X of V_η of cardinality λ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$.

Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse.

Then $\pi^{-1} : M \longrightarrow \mathrm{V}_\eta$ is an elementary embedding with $\pi^{-1} \restriction \mathrm{V}_\lambda = \mathrm{id}_{\mathrm{V}_\lambda}$ and $M \in \mathrm{H}(\lambda^+)^{\mathrm{L}(\mathrm{V}_{\lambda+1}, G)} \subseteq \mathrm{L}(\mathrm{V}_{\lambda+1})$.

The weak homogeneity of $\mathrm{Add}(\lambda^+, 1)$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ then implies that

$\mathbb{1}_{\mathrm{Add}(\lambda^+,1)} \Vdash$ *There is an elementary embedding* $k : \check{M} \longrightarrow \mathrm{V}_{\check{\eta}}$ *with* $k \restriction \mathrm{V}_{\check{\lambda}} = \mathrm{id}_{\mathrm{V}_{\check{\lambda}}}$.

Since $j(\lambda) = \lambda$ and $j(\mathrm{Add}(\lambda^+, 1)) = \mathrm{Add}(\lambda^+, 1)$, it follows that, in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$, there is an elementary embedding $k : j(M) \longrightarrow \mathrm{V}_{j(\eta)}$ with $k \restriction \mathrm{V}_\lambda = \mathrm{id}_{\mathrm{V}_\lambda}$.

Since $j \restriction M \in \mathrm{L}(\mathrm{V}_{\lambda+1})$, we can conclude that

$$i \;=\; k \circ (j \restriction M) \circ \pi : X \longrightarrow \mathrm{V}_{j(\eta)}^{\mathrm{L}(\mathrm{V}_{\lambda+1}, G)}$$

is an elementary embedding with $i \restriction \mathrm{V}_\lambda = j \restriction \mathrm{V}_\lambda$ in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

Fix a cardinal $\eta > \lambda$ with $j(\eta) = \eta$ and $\mathrm{V}_\eta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, G)} \prec_{\Sigma_2} \mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

In $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$, pick an elementary submodel X of V_η of cardinality λ with $\mathrm{V}_\lambda \cup \{j \upharpoonright \mathrm{V}_\lambda\} \subseteq X$.

The above computations now show that $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$ contains an elementary embedding

$$i : X \longrightarrow \mathrm{V}_\eta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, G)}$$

with $i \upharpoonright \mathrm{V}_\lambda = j \upharpoonright \mathrm{V}_\lambda$.

Since $i(\lambda) = \lambda$ and $j \upharpoonright \mathrm{V}_\lambda \in X$, the existence of this embedding ensures that λ is an ultraexacting cardinal in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

The consistency of exacting cardinals

The following result yields the lower bound for the consistency strength of exacting cardinals:

Theorem

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an I3-embedding with the property that λ has uncountable cofinality in $\mathrm{L}(\mathrm{V}_\lambda)$, then there exists an I3-embedding $i : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ with $\mathrm{crit}(j) < \lambda' < \lambda$.

Let $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ be an I3-embedding with the property that λ has uncountable cofinality in $\mathrm{L}(\mathrm{V}_\lambda)$.

Define T to be the set of all partial elementary embeddings $k : \mathrm{V}_\lambda \xrightarrow{part} \mathrm{V}_\lambda$ with the property that there exists a finite strictly increasing sequence $\langle \kappa_m \mid m \leq n+1 \rangle$ of cardinals below λ with the property that $\mathrm{dom}(k) = \mathrm{V}_{\kappa_n} \cup \{\kappa_n\}$, $\mathrm{ran}(k) \subseteq \mathrm{V}_{\kappa_{n+1}} \cup \{\kappa_{n+1}\}$, $\kappa_0 = \mathrm{crit}(j)$, $k \restriction \kappa_0 = \mathrm{id}_{\kappa_0}$ and $k(\kappa_\ell) = \kappa_{\ell+1}$ for all $\ell \leq n$.

If we order T by inclusion, then we obtain a tree of height at most ω. Moreover, the embedding j induces a cofinal branch through T.

Since T is definable in V_λ, it follows that T is an element of $\mathrm{L}(\mathrm{V}_\lambda)$ and a well-foundedness argument yields a cofinal branch B through T in $\mathrm{L}(\mathrm{V}_\lambda)$.

By our assumption, there is a cardinal $\mathrm{crit}(j) < \lambda' < \lambda$ with the property that $\bigcup B : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ is an I3-embedding. $\square$

The presented upper bound for the consistency strength of exacting cardinals is given by the following result:

Theorem (Aguilera–Bagaria–Goldberg–L.)

Let $j : \mathrm{V} \longrightarrow M$ be an I2-embedding with critical point κ and let

$$U \ = \ \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

If G is generic over V for Prikry forcing with U, then κ is an exacting cardinal in $\mathrm{V}[G]$.

Let $j : \mathrm{V} \longrightarrow M$ be an I2-embedding with critical point κ and least non-trivial fixed point λ. Set $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$.

Results of Martin show that $j \restriction \mathrm{V}_\lambda$ is $(\omega+1)$-iterable. Let $j_\omega : \mathrm{V}_\lambda \longrightarrow M_\omega$ denote the embedding of the first into the ω-th model in this iteration.

Then M_ω is a transitive set with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq M_\omega$, $\mathrm{crit}(j_\omega) = \kappa$ and $j_\omega(\kappa) = \lambda$.

Fix $\rho > \lambda$ such that V_ρ is sufficiently elementary in V and pick an elementary submodel X of V_ρ of cardinality κ with $\mathrm{V}_\kappa \cup \{U\} \subseteq X$.

Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Set $N_* = j_\omega(N)$ and $U_* = j_\omega(\pi(U))$.

Standard arguments then show that $\mathrm{V}_\lambda \subseteq N_*$, $j(N_*) = N_*$ and the critical sequence $\vec{\kappa}$ of j is Prikry generic for U_* over N_*.

We now know that

$$i \ = \ j \restriction N_*[\vec{\kappa}] : N_*[\vec{\kappa}] \longrightarrow N_*[\vec{\kappa}]$$

is an elementary embedding.

Now, in $N_*[\vec{\kappa}]$, fix a non-empty subset A of $\mathrm{V}_{\lambda+1}$ that is definable by a formula with parameter λ. Pick $x \in A$ and set $y = i(x)$. Then $y \in A$ and i induces a non-trivial elementary embedding of $(\mathrm{V}_\lambda, \in, x)$ into $(\mathrm{V}_\lambda, \in, y)$.

Since λ has countable cofinality in $N_*[\vec{\kappa}]$, a well-foundedness argument shows that such an embedding already exists in $N_*[\vec{\kappa}]$.

Results presented yesterday morning now show that λ is exacting in $N_*[\vec{\kappa}]$ and hence elementarity ensures that Prikry forcing with U over V turns κ into an exacting cardinal.

Thank you for listening!