Large cardinals beyond HOD

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Introduction

Theorem (Simplified HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds ("*HOD is close to V*").
- Every regular cardinal $\kappa \geq \delta$ is measurable in HOD ("*HOD is far from V*").

Questions

Are there canonical extensions of ZFC that prove that HOD is far from V? Are there such axioms that imply $\mathrm{V} \neq \mathrm{HOD}$?

All standard large cardinal axioms are compatible with the assumption that $\mathrm{V} = \mathrm{HOD}$ and therefore do not provide affirmative answers to these questions.

If we instead ask for extensions of ZF, then large cardinals beyond choice (e.g., Reinhardt cardinals) provide trivial affirmative answers to the second question.

In the following, we will observe that more interesting things can be said about the relationship between V and HOD in this setting.

Definition (Goldberg & Schlutzenberg, ZF)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \beta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ with the property that $\alpha < \mathrm{crit}(j) < \lambda$ and λ is the first non-trivial fixed point of j.

Proposition (GB)

If $j : \mathrm{V} \longrightarrow \mathrm{V}$ is an elementary embedding, then the first non-trivial fixed point of j is a rank-Berkeley cardinal.

Proposition (ZF)

Rank-Berkeley cardinals are cardinals of countable cofinality that are regular in HOD.

Proof.

Assume, towards a contradiction, that a rank-Berkeley cardinal λ is singular in HOD.

Pick $\beta > \lambda$ such that V_β is sufficiently elementary in V.

Then there is an elementary embedding $j : \mathrm{V}_\beta \longrightarrow \mathrm{V}_\beta$ such that $\mathrm{cof}(\lambda)^{\mathrm{HOD}} < \mathrm{crit}(j)$ and λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}} \longrightarrow \lambda$ be the least cofinal function in the canonical well-ordering of HOD.

Then c is definable from the parameter λ and hence $j(c) = c$.

Pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}}$ with $c(\alpha) > \mathrm{crit}(j)$. Then

$$c(\alpha) \ < \ j(c(\alpha)) \ = \ j(c)(j(\alpha)) \ = \ c(\alpha),$$

a contradiction. □

Exacting cardinals

We now want to isolate canonical fragments of rank-Berkeleyness that are compatible with the Axiom of Choice and still allow us to carry out the above argument.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Note that, in order to prove that a cardinal λ is exacting, it suffices to find a single embedding $j : X \longrightarrow \mathrm{V}_\beta$ satisfying $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X \prec \mathrm{V}_\beta \prec_{\Sigma_2} \mathrm{V}$, $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$.

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (Aguilera–Bagaria–L.)

If λ is exacting, then λ is a singular cardinal that is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Corollary

If there is an exacting cardinal above an extendible cardinal, then eventually all regular cardinals are measurable in HOD.

Let λ be an exacting cardinal. Then there is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ and results of Kunen imply $\mathrm{cof}(\lambda) = \omega$.

Assume, towards a contradiction, that λ is singular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$. Then there is $z \in \mathrm{V}_\lambda$ such that λ is singular in $\mathrm{HOD}_{\{z\}}$.

Fix $\beta > \lambda$ such that V_β is sufficiently elementary in V. Pick $X \prec \mathrm{V}_\beta$ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \longrightarrow V_\beta$ with $\mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j(z) = z$.

Results of Kunen imply that λ is the first non-trivial fixed point of j.

Let $c : \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}} \longrightarrow \lambda$ be the least cofinal function with respect to the canonical well-ordering of $\mathrm{HOD}_{\{z\}}$. Then $c \in X$ with $j(c) = c$.

If we pick $\alpha < \mathrm{cof}(\lambda)^{\mathrm{HOD}_{\{z\}}}$ with $c(\alpha) > \mathrm{crit}(j)$, then we have

$$c(\alpha) \ < \ j(c(\alpha)) \ = \ j(c)(j(\alpha)) \ = \ c(\alpha),$$

a contradiction.

We now discuss the naturalness of the notion of exactingness.

First, note that as a fragment of Reinhardtness, this property is phrased in a standard format for large cardinal axioms.

Next, we show that exactingness is equivalent to a natural model-theoretic reflection principle.

For this purpose, remember that a cardinal λ is Jónsson if every structure in a countable first-order language whose domain has cardinality λ has a proper elementary substructure of cardinality λ.

The next result shows that exactingness is equivalent to a strengthening of this property that incorporates external features of the given structure.

Theorem (Aguilera–Bagaria–L.)

The following are equivalent for each cardinal λ with $|V_\lambda| = \lambda$:

- λ is an exacting cardinal.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ contains a proper elementary substructure of cardinality λ isomorphic to a structure in $\mathcal{C}$.
- For every class $\mathcal{C}$ of structures in a countable first-order language that is definable by a formula with parameters in $V_\lambda \cup \{\lambda\}$, every structure of cardinality λ in $\mathcal{C}$ is isomorphic to a proper elementary substructure of a structure of cardinality λ in $\mathcal{C}$.

In another direction, exactingness can also be represented as a natural strengthening of the existence of I3-embeddings:

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- For every non-empty, ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exist $x, y \in A$ and an elementary embedding

$$j : (\mathrm{V}_\lambda, \in, x) \longrightarrow (\mathrm{V}_\lambda, \in, y).$$

The consistency strength of exactingness

Definition

- An I3-embedding is a non-trivial elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ for some limit ordinal λ.
- An I2-embedding is a non-trivial elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{V}_\lambda \subseteq M$, where λ is the first non-trivial fixed point of j.

Theorem (Aguilera–Bagaria–Goldberg–L.)

- If there is an I2-embedding, then there is a transitive ZFC-model with an exacting cardinal.
- If λ is an exacting cardinal, then V_λ is a model of ZFC with a proper class of I3-embeddings.

Theorem

If $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\lambda$ is an I3-embedding with the property that λ has uncountable cofinality in $\mathrm{L}(\mathrm{V}_\lambda)$, then there exists an I3-embedding $i : \mathrm{V}_{\lambda'} \longrightarrow \mathrm{V}_{\lambda'}$ with $\mathrm{crit}(j) < \lambda' < \lambda$.

Theorem (Aguilera–Bagaria–Goldberg–L.)

Let $j : \mathrm{V} \longrightarrow M$ be an I2-embedding with critical point κ and let

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}.$$

If G is generic over V for Prikry forcing with U, then κ is an exacting cardinal in $\mathrm{V}[G]$.

Let $j : \mathrm{V} \longrightarrow M$ be an I2-embedding with critical point κ and least non-trivial fixed point λ. Set $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$.

Results of Martin show that $j \upharpoonright \mathrm{V}_\lambda$ is $(\omega+1)$-iterable. Let $j_\omega : \mathrm{V}_\lambda \longrightarrow M_\omega$ denote the embedding of the first into the ω-th model in this iteration.

Then M_ω is a transitive set with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq M_\omega$, $\mathrm{crit}(j_\omega) = \kappa$ and $j_\omega(\kappa) = \lambda$.

Fix $\rho > \lambda$ such that V_ρ is sufficiently elementary in V and pick an elementary submodel X of V_ρ of cardinality κ with $\mathrm{V}_\kappa \cup \{U\} \subseteq X$.

Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Set $N_* = j_\omega(N)$ and $U_* = j_\omega(\pi(U))$.

Standard arguments then show that $\mathrm{V}_\lambda \subseteq N_*$, $j(N_*) = N_*$ and the critical sequence $\vec{\kappa}$ of j is Prikry generic for U_* over N_*.

We now know that

$$i \;=\; j \restriction N_*[\vec{\kappa}] : N_*[\vec{\kappa}] \longrightarrow N_*[\vec{\kappa}]$$

is an elementary embedding.

Now, in $N_*[\vec{\kappa}]$, fix a non-empty subset A of $\mathrm{V}_{\lambda+1}$ that is definable by a formula with parameter λ. Pick $x \in A$ and set $y = i(x)$. Then $y \in A$ and i induces a non-trivial elementary embedding of $(\mathrm{V}_\lambda, \in, x)$ into $(\mathrm{V}_\lambda, \in, y)$.

Since λ has countable cofinality in $N_*[\vec{\kappa}]$, a well-foundedness argument shows that such an embedding already exists in $N_*[\vec{\kappa}]$.

We then know that λ is exacting in $N_*[\vec{\kappa}]$ and hence elementarity ensures that Prikry forcing with U over V turns κ into an exacting cardinal.

Ultraexacting cardinals

We now consider the possibility of further strengthening the notion of exacting cardinals.

Our motivation for the formulation of stronger notions comes from the observation that certain elements of $\mathrm{H}(\lambda^+)$ have to be missing from the domains of embeddings witnessing the exactingness of a cardinal λ.

The proof of the following result uses ideas from Woodin's proof of the Kunen Inconsistency:

Proposition

If λ is a cardinal, $\zeta > \lambda$ is an ordinal with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$, then $\lambda^+ \nsubseteq X$ and $[\lambda]^\omega \nsubseteq X$.

The above proposition shows that we can strengthen the notion of exacting cardinals by demanding that certain sets are contained in the domains of the elementary embeddings witnessing the given property.

The consistency proof presented earlier shows that initial segments of the given elementary embeddings are canonical examples of sets that are, in general, not contained in their domains.

This motivates the following definition:

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \beta$, there exist

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

As before, this notion is equivalent to a natural strengthening of a rank-into-rank axioms:

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following are equivalent for every cardinal λ:

- λ is an ultraexacting cardinal.
- For every ordinal definable subset A of $\mathrm{V}_{\lambda+1}$, there exists an elementary embedding

$$j : (\mathrm{V}_{\lambda+1}, \in, A) \longrightarrow (\mathrm{V}_{\lambda+1}, \in, A).$$

The consistency strength of ultraexactingness

Definition

An I0-embedding is a non-trivial elementary embedding

$$j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1}),$$

where λ is the first non-trivial fixed point of j.

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following statements are equiconsistent over ZFC:

- There is an ultraexacting cardinal.
- There is an I0-embedding.

Theorem (Aguilera–Bagaria–L.)

If $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$ is an I0-embedding and G is $\mathrm{Add}(\lambda^+, 1)$-generic over V, then $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$ is a model of ZFC and λ is an ultraexacting cardinal in $\mathrm{L}(\mathrm{V}_{\lambda+1}, G)$.

Remember that, given $E \subseteq V_{\lambda+1}$, we let $\Theta^{L(V_{\lambda+1},E)}$ denote the least ordinal γ such that $L(V_{\lambda+1}, E)$ does not contain a surjection from $V_{\lambda+1}$ onto γ.

Definition (Woodin)

The *Internal Axiom I0* holds at a cardinal λ if for all $\lambda < \gamma < \Theta^{L(V_{\lambda+1})}$, there exists a non-trivial elementary embedding

$$j : L_\gamma(V_{\lambda+1}) \longrightarrow L_\gamma(V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$.

Theorem (Woodin)

The following statements are equiconsistent over ZFC:

- There exists an I0-embedding.
- Internal Axiom I0 holds at some cardinal.

We now outline the proof if the result showing that Internal Axiom I0 holds at ultraexacting cardinals.

In the following, fix

- a cardinal λ,
- an ordinal $\zeta > \lambda$ such that V_ζ is sufficiently elementary in V,
- an elementary submodel X of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \lambda \neq \mathrm{id}_\lambda$ and $j \restriction \mathrm{V}_\lambda \in X$.

Remember that, if ξ is an ordinal of countable cofinality and $k : \mathrm{V}_\xi \longrightarrow \mathrm{V}_\xi$ is an elementary embedding, then the map

$$k_+ : \mathrm{V}_{\xi+1} \longrightarrow \mathrm{V}_{\xi+1};\ A \longmapsto \bigcup\{k(A \cap V_\alpha) \mid \alpha < \xi\}$$

is the unique Σ_0-elementary function from $\mathrm{V}_{\xi+1}$ to $\mathrm{V}_{\xi+1}$ extending k.

If $i : \mathrm{V} \longrightarrow M$ is an I2-embedding with least non-trivial fixed point λ, then

$$(i \restriction \mathrm{V}_\lambda)_+ = i \restriction \mathrm{V}_{\lambda+1} : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is Σ_1-elementary. Conversely, every non-trivial Σ_1-elementary function from $\mathrm{V}_{\lambda+1}$ to itself can be extended to an I2-embedding.

Lemma

The map

$$(j \restriction \mathrm{V}_\lambda)_+ : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$$

is an elementary embedding that is contained in X and satisfies

$$(j \restriction \mathrm{V}_\lambda)_+ \restriction (X \cap \mathrm{V}_{\lambda+1}) \ = \ j \restriction (X \cap \mathrm{V}_{\lambda+1}).$$

Proof.

Since $j \restriction \mathrm{V}_\lambda \in X$, elementarity implies that $(j \restriction \mathrm{V}_\lambda)_+ \in X$.

Moreover, the equality

$$(j \restriction \mathrm{V}_\lambda)_+ \restriction (X \cap \mathrm{V}_{\lambda+1}) \ = \ j \restriction (X \cap \mathrm{V}_{\lambda+1}).$$

holds by the definition of $(j \restriction \mathrm{V}_\lambda)_+$.

The elementarity of j and the above equality then imply that $(j \restriction \mathrm{V}_\lambda)_+$ is an elementary embedding of $\mathrm{V}_{\lambda+1}$ into itself in X. Finally, the correctness properties of X ensure that $(j \restriction \mathrm{V}_\lambda)_+$ also has this property in V. $\square$

Lemma

Let γ be an ordinal in X such that there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$. Then there exists a unique function $j_\gamma : \gamma \longrightarrow j(\gamma)$ that is an element of X and satisfies

$$j \restriction (X \cap \gamma) \;=\; j_\gamma \restriction (X \cap \gamma).$$

Proof.

Define $j_\gamma : \gamma \longrightarrow j(\gamma)$ to be the unique function satisfying

$$j_\gamma(s(x)) \;=\; j(s)((j \restriction \mathrm{V}_\lambda)_+(x))$$

for all $x \in \mathrm{V}_{\lambda+1}$. Then j_γ possesses all of the listed properties. □

The basic structure theory of $\mathrm{L}(\mathrm{V}_{\lambda+1}, E)$ now yields the following statement:

Lemma

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)}$ with $j(\gamma) \in X$, then there is a surjection $s : V_{\lambda+1} \longrightarrow \gamma$ in X with $j(s) \in X$.

Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)}$ with $j(\gamma) = \gamma$, then there is a function $j_\gamma : \gamma \longrightarrow \gamma$ in X with

$$j \restriction (X \cap \gamma) \;=\; j_\gamma \restriction (X \cap \gamma).$$

Corollary

If $E \in X \cap V_{\lambda+2}$ with $j(E) = E$ and $\gamma \in X \cap \Theta^{\mathrm{L}(\mathrm{V}_{\lambda+1}, E)}$ is a limit ordinal with $j(\gamma) = \gamma$, then there is a function

$$j^{\gamma} : \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E) \longrightarrow \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)$$

in X with

$$j \restriction (X \cap \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)) \;=\; j^{\gamma} \restriction (X \cap \mathrm{L}_{\gamma}(\mathrm{V}_{\lambda+1}, E)).$$

Corollary

Internal Axiom I0 holds at ultraexacting cardinals.

More applications of ultraexactingness

Definition (Woodin)

An uncountable regular cardinal κ is *ω-strongly measurable in HOD* if there is a cardinal $\delta < \kappa$ such that $(2^\delta)^{\mathrm{HOD}} < \kappa$ and HOD contains no partition of the set E^κ_ω of all elements of κ with cofinality ω into δ-many sets that are all stationary in V.

Lemma (Woodin)

If a cardinal κ is ω-strongly measurable in HOD, then κ is a measurable cardinal in HOD.

Theorem (Aguilera–Bagaria–L.)

If λ is an ultraexacting cardinal, then λ^+ is ω-strongly measurable in HOD.

Theorem (Aguilera–Bagaria–L.)

Let $E \in V_{\lambda+2} \cap X$ be such that $j(E) = E$. If $E^{\#}$ exists, then there is an elementary embedding

$$i : L(V_{\lambda+1}, E) \longrightarrow L(V_{\lambda+1}, E)$$

with $i \upharpoonright V_\lambda = j \upharpoonright V_\lambda$ and $i(E) = E$.

Corollary

If λ is an ultraexacting cardinal with the property that $\mathrm{V}^{\#}_{\lambda+1}$ exists, then there is an I0-embedding $j : \mathrm{L}(\mathrm{V}_{\lambda+1}) \longrightarrow \mathrm{L}(\mathrm{V}_{\lambda+1})$.

A failure of the HOD Conjecture

Theorem (The HOD Dichotomy, Woodin)

If δ is an extendible cardinal, then one of the following statements holds:

- For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\mathrm{HOD}} = \lambda^+$ holds.
- Every regular cardinal greater than or equal to δ is ω-strongly measurable in HOD.

The Weak HOD Conjecture (Woodin)

The theory

$$\mathrm{ZFC} + \text{``}\textit{There is a huge cardinal above an extendible cardinal}\text{''}$$

proves that a proper class of regular cardinals is not ω-strongly measurable in HOD.

Definition

A cardinal λ is *exacting* if for all $\alpha < \lambda < \beta$, there exists

- an elementary submodel X of V_β with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{V}_\beta$ with $\alpha < \mathrm{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem

If λ is exacting, then λ is a singular cardinal that is regular in $\mathrm{HOD}_{\mathrm{V}_\lambda}$.

Corollary

If ZFC is consistent with the existence of an exacting cardinal above an extendible cardinal, then the Weak HOD Conjecture fails.

Definition (GB)

A cardinal κ is *super Reinhardt* if for every ordinal α, there is an elementary embedding $j : \mathrm{V} \longrightarrow \mathrm{V}$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

Theorem (Aguilera–Bagaria–L., BG)

If there is a super Reinhardt cardinal, then there is a model of ZFC with an exacting cardinal above an extendible cardinal.

Thank you for listening!