Strong logics and ordinal definability

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Introduction

The theory of large cardinals and the study of extensions of first-order logic are deeply connected through results showing that the ability to generalize fundamental structural features of first-order logic to stronger logics is equivalent to the existence of large cardinals.

We start by discussing classical results of Magidor that establish such connections for second-order logic $\mathcal{L}^2$.

Definition

- An infinite cardinal κ is a *strong compactness cardinal for* $\mathcal{L}^2$ if every unsatisfiable $\mathcal{L}^2$-theory contains an unsatisfiable subtheory of cardinality less than κ.

- An uncountable cardinal κ is a *Löwenheim–Skolem–Tarski cardinal* for $\mathcal{L}^2$ if for every language τ of size less than κ, every $\mathcal{L}^2(\tau)$-sentence ϕ and every τ-structure N with $N \models \phi$, there exists a substructure M of N of size less than κ with $M \models \phi$.

Theorem (Magidor)

There exists a strong compactness cardinal for $\mathcal{L}^2$ if and only if there exists an extendible cardinal. Moreover, if these cardinals exist, then the least strong compactness cardinal for $\mathcal{L}^2$ is the least extendible cardinal.

Theorem (Magidor)

There exists a Löwenheim–Skolem–Tarski cardinal for $\mathcal{L}^2$ if and only if there exists a supercompact cardinal. Moreover, if these cardinals exist, then the least Löwenheim–Skolem–Tarski cardinal for $\mathcal{L}^2$ is the least supercompact cardinal.

Numerous results of this kind have been proven for various extensions of first-order logic and notions from all regions of the large cardinal hierarchy.

Moreover, it has also been shown that large cardinals can be used to characterize the validity of strong structural properties of *all* extensions of first-order logic.

To formulate these results, we first need to fix a definition of a strong logic.

Definition

An *abstract logic* consists of a class function $\mathcal{L}$ and a binary class relation $\models_{\mathcal{L}}$ satisfying the following statements:

- The domain of $\mathcal{L}$ is the class of all languages.
- If $M \models_{\mathcal{L}} \phi$ holds, then there exists a language τ such that M is a τ-structure and $\phi \in \mathcal{L}(\tau)$.
- $\mathcal{L}$ and $\models_{\mathcal{L}}$ are well-behaved with respect to extensions and renamings of languages as well as isomorphisms of structures.
- There exists an infinite cardinal μ such that for every language τ and every $\phi \in \mathcal{L}(\tau)$, there exists a sublanguage σ of τ of cardinality less than μ with $\phi \in \mathcal{L}(\sigma)$. The least such cardinal $o_{\mathcal{L}}$ is called the *occurrence number* of $\mathcal{L}$.

Definition

Let $\mathcal{L}$ be an abstract logic.

- An infinite cardinal κ is a *strong compactness cardinal for $\mathcal{L}$* if every unsatisfiable $\mathcal{L}$-theory contains an unsatisfiable subtheory of size less than κ.
- An uncountable cardinal κ is a *Löwenheim–Skolem–Tarski cardinal* for $\mathcal{L}$ if for every language τ of size less than κ, every $\phi \in \mathcal{L}(\tau)$ and every τ-structure N with $N \models_{\mathcal{L}} \phi$, there exists a substructure M of N of size less than κ with $M \models_{\mathcal{L}} \phi$.

Theorem (Makowsky, Stavi)

The following schemes are equivalent over ZFC:

- Every abstract logic has a strong compactness cardinal.
- Every abstract logic has a Löwenheim–Skolem–Tarski cardinal.
- *Vopěnka's Principle*, i.e., the scheme of axioms stating that for every proper class of graphs, there are two distinct members of the class with a homomorphism between them.

This characterization of the principle

"Every abstract logic has a strong compactness cardinal"

yields numerous conclusions regarding this principle. For example:

- The principle implies the existence of a proper class of strongly inaccessible cardinals.
- There is no set-theoretic sentence that provably implies this principle.
- The principle implies that there is a proper class of measurable cardinals in HOD.

We now birefly discuss weakenings of the above structural properties of logics that exhibit a different behavior:

Definition

Let $\mathcal{L}$ be an abstract logic.

- An infinite cardinal κ is a *weak compactness cardinal for $\mathcal{L}$* if every unsatisfiable $\mathcal{L}$-theory of size κ contains an unsatisfiable subtheory of size less than κ.
- An uncountable cardinal κ is a *strict Löwenheim–Skolem–Tarski cardinal* for $\mathcal{L}$ if for every language τ of size less than κ, every $\phi \in \mathcal{L}(\tau)$ and every τ-structure N of size κ with $N \models_{\mathcal{L}} \phi$, there exists a substructure M of N of size less than κ with $M \models_{\mathcal{L}} \phi$.

Definition

- "Ord *is essentially faint*" is the scheme of sentences stating that for every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$ and every ordinal ξ, there are ordinals $\xi < \alpha < \beta$ and $A \in E_\beta$ with $A \cap \alpha \in E_\alpha$.

- Given a natural number n, a cardinal κ is $C^{(n)}$-*weakly shrewd* if for every cardinal $\kappa < \theta$ with $\mathrm{V}_\theta \prec_{\Sigma_n} \mathrm{V}$ and every $y \in \mathrm{H}(\theta)$, there exists a cardinal $\bar{\theta}$ with $\mathrm{V}_{\bar{\theta}} \prec_{\Sigma_n} \mathrm{V}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa}+1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $y \in \mathrm{ran}(j)$.

Theorem (L.)

The following schemes are equivalent over ZFC:

- Every abstract logic has a weak compactness cardinal.
- Every abstract logic has a strict Löwenheim–Skolem–Tarski cardinal.
- Ord is essentially faint.
- For every natural number n, there is a proper class of $C^{(n)}$-weakly shrewd cardinals.

As above, this characterization of the principle

"Every abstract logic has a weak compactness cardinal"

allows various conclusions regarding this principle:

- The principle does not imply the existence of a strongy inaccessible cardinal.
- The principle cannot be formulated as a single set-theoretic sentence.
- There are set-theoretic sentences that imply the principle.
- The principle implies the existence of a proper class of strongly inaccessible cardinals in HOD.

Strong forms of Jónssonness

Joint work in progress with Anna Lenz (Hamburg)

Definition

A cardinal κ is *Jónsson* if every structure in a countable first-order language whose domain has cardinality κ has a proper elementary substructure of cardinality κ.

Following the approaches outlined above, we want to consider variations of this properties for stronger logics.

Definition

Let $\mathcal{L}$ be an abstract logic.

- An infinite cardinal κ is an *$\mathcal{L}$-Jónsson cardinal* if for every language τ of size $o_{\mathcal{L}}$, every τ-structure N of cardinality κ and every $\phi \in \mathcal{L}(\tau)$ with $N \models_{\mathcal{L}} \phi$, there exists a proper substructure M of N of cardinality κ with $M \models_{\mathcal{L}} \phi$.

- An infinite cardinal κ is an *$\mathcal{L}$-co-Jónsson cardinal* if for every every language τ of size $o_{\mathcal{L}}$, every τ-structure M of cardinality κ and every $\phi \in \mathcal{L}(\tau)$ with $M \models_{\mathcal{L}} \phi$, there exists a proper superstructure N of M of cardinality κ with $N \models_{\mathcal{L}} \phi$.

Note that, if we denote first-order logic by $\mathcal{L}^1$, then $\mathcal{L}^1$-Jónsson cardinals are just Jónsson cardinals and every infinite cardinal is an $\mathcal{L}^1$-co-Jónsson cardinal.

Proposition

- If there is an I1-embedding $j : \mathrm{V}_{\lambda+1} \longrightarrow \mathrm{V}_{\lambda+1}$, then λ is both a $\mathcal{L}^2$-Jónsson and a $\mathcal{L}^2$-co-Jónsson cardinal.
- If the GCH holds and there is either an $\mathcal{L}^2$-Jónsson or an $\mathcal{L}^2$-co-Jónsson cardinal, then there is an I3-embedding.

We now aim to characterize the existence of $\mathcal{L}$-Jónsson cardinals for all abstract logics $\mathcal{L}$ through the existence of large cardinals.

Definition (Aguilera–Bagaria–L.)

An infinite cardinal λ is *exacting* if for all ordinals $\alpha < \lambda < \zeta$, there is an elementary submodel X of V_ζ with $\mathrm{V}_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \alpha = \mathrm{id}_\alpha$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$.

Definition (Lenz–L.)

An infinite cardinal λ is *weakly exacting* if for all ordinals $\alpha < \lambda < \zeta$, there is an elementary submodel X of V_ζ with $\lambda + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{V}_\zeta$ with $j(\lambda) = \lambda$, $j \restriction \alpha = \mathrm{id}_\alpha$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$.

Theorem (Lenz–L.)

The following schemes are equivalent over ZFC:

- For every abstract logic $\mathcal{L}$, there is a proper class of $\mathcal{L}$-Jónsson cardinals.
- For every abstract logic $\mathcal{L}$, there is a proper class of $\mathcal{L}$-co-Jónsson cardinals.
- There is a proper class of weakly exacting cardinals.

Again, this characterization of the principle

"Every abstract logic $\mathcal{L}$ has an $\mathcal{L}$-Jónsson cardinal"

allows various conclusions regarding this principle:

- The principle does not imply the existence of a strongly inaccessible cardinal.
- The principle can be formulated as a single set-theoretic sentence.
- ZFC together with this principle has consistency strength below I2.
- The principle implies that every abstract logic has a proper class of weak compactness cardinals.
- If the HOD Conjecture is true, then this principle implies that $\mathcal{L}^2$ does not have a strong compactness cardinal.

Lemma

If ζ is a cardinal with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ, ρ is a cardinal with $\rho^+ \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $j(\rho) = \rho$, then $j \restriction \rho^+ = \mathrm{id}_{\rho^+}$.

Corollary

If ζ is a cardinal with $\mathrm{V}_\zeta \prec_{\Sigma_2} \mathrm{V}$, X is an elementary submodel of V_ζ, λ is a cardinal with $\lambda + 1 \subseteq X$ and $j : X \longrightarrow \mathrm{V}_\zeta$ is an elementary embedding with $j(\lambda) = \lambda$ and $j \restriction \lambda \neq \mathrm{id}_\lambda$, then $\lambda = \sup_{n<\omega} \kappa_n$, where $\kappa_0 = \mathrm{crit}(j)$ and $\kappa_{n+1} = j(\kappa_n)$ for all $n < \omega$.

Corollary

Weakly exacting cardinals are regular in HOD.

The following result provides an upper bound for the consistency strength of the existence of $\mathcal{L}$-Jónsson cardinals for every abstract logic $\mathcal{L}$:

Theorem (L.–Poveda)

If there exists an I2-embedding, then there exists a transitive set-sized model of the theory

$$\mathrm{ZFC} \ + \ \text{"}\textit{There is a proper class of exacting cardinals}\text{"}.$$

By definition, exacting cardinals are weakly exacting. The following result provides a characterization of weakly exacting cardinals that are not exacting:

Lemma

The following statements are equivalent for every weakly exacting cardinal λ:

- λ is an exacting cardinal.
- $\mathrm{V}_\lambda \prec_{\Sigma_1} \mathrm{V}$.
- λ is a strong limit cardinal.

Lemma

If λ is an exacting cardinal, $\kappa < \lambda$ is an infinite regular cardinal and G is $\mathrm{Add}(\kappa, \lambda^+)$-generic over V, then λ is weakly exacting in $\mathrm{V}[G]$.

Theorem (Lenz–L.)

If there is a proper class of exacting cardinals, then, in a class forcing extension, there is a proper class of weakly exacting cardinals and no strongly inaccessible cardinals.

Theorem (L.–Poveda)

The following statements are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- λ has countable cofinality and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\lambda}$ *is an exacting cardinal*" holds for every partial order $\mathbb{P} \in H(\lambda)$.
- λ has countable cofinality and there is a partial order $\mathbb{P} \in H(\lambda)$ with the property that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\lambda}$ *is an exacting cardinal*".

Limits of exacting cardinals

Joint work in progress with Alejandro Poveda (Barcelona)

We derived the relative consistency of the existence of the existence of $\mathcal{L}$-Jónsson cardinals for every abstract logic $\mathcal{L}$ with the help of the following result:

Theorem (L.–Poveda)

If there exists an I2-embedding, then there exists a transitive set-sized model of the theory

$$\mathrm{ZFC} + \text{"}\textit{There is a proper class of exacting cardinals}\text{"}.$$

The proof takes a result from joint work with Aguilera, Bagaria and Goldberg that shows that Prikry forcing with the induces normal filter of an I2-embedding makes its critical point singular and combines it with a *Magidor support* iteration.

This construction produces a model with a *discrete* class of exacting cardinals, i.e. no exacting cardinal is a limit of exacting cardinals.

The following observation shows that this is no coincidence:

Proposition

If the HOD Conjecture is true, then ZFC proves that limits of exacting cardinals are not exacting.

Proof.

If λ is an exacting limit of exacting cardinals, then λ is a limit of singular cardinals that are regular in HOD and V_λ is a model of $\mathrm{ZFC}+$ *"There is an extendible cardinal"*. Then, in V_λ, there is a proper class of singular cardinals that are regular in HOD, contradicting the fact that the HOD Hypothesis holds in V_λ. $\square$

Corollary

If the HOD Conjecture is true, then ZFC proves that every cardinal of uncountable cofinality contains a closed unbounded set that does not contain an exacting cardinal.

In contrast, we can use *large cardinals beyond choice* to produce models of ZFC containing exacting limits of exacting cardinals.

Theorem (L.–Poveda, ZF_2)

If there exists a super Reinhardt cardinal, then there exists a set-sized transitive model of ZFC with a proper class of exacting cardinals that are limits of exacting cardinals.

Definition (ZF)

A cardinal κ is called *totally Reinhardt* if for each $A \in \mathrm{V}_{\kappa+1}$, we have

$$\langle \mathrm{V}_\kappa, \mathrm{V}_{\kappa+1} \rangle \models \mathrm{ZF}_2 + \text{"There exists an } A\text{-super Reinhardt cardinal"}.$$

Theorem (L.–Poveda, ZF)

If there is a cardinal that is both supercompact and totally Reinhardt, then, in a forcing extension, there is an inner model of ZFC in which a regular cardinal is a stationary limit of exacting cardinals.

As mentioned earlier, small forcings that preserve cofinalities do not create new exacting cardinals:

Theorem (L.–Poveda)

The following statements are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- λ has countable cofinality and $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\lambda}$ *is an exacting cardinal*" holds for every partial order $\mathbb{P} \in H(\lambda)$.
- λ has countable cofinality and there is a partial order $\mathbb{P} \in H(\lambda)$ with the property that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\lambda}$ *is an exacting cardinal*".

An argument of Laver shows that new I2-embeddings can be forced to exist by collapsing ω_1.

Therefore, we may ask if similar results can be proven for exacting cardinals.

Theorem (L.–Poveda)

Let λ be a cardinal of uncountable cofinality with the property that there exists a partial order $\mathbb{P}$ in $H(\lambda)$ such that

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\check{\lambda} \textit{ is an exacting cardinal}\text{”}.$$

Then every closed unbounded subset of λ contains an exacting cardinal.

Corollary

If the HOD Conjecture is true, then ZFC proves that the following statements are equivalent for every cardinal λ:

- λ is an exacting cardinal.
- There is a partial order $\mathbb{P} \in H(\lambda)$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ ”$\check{\lambda}$ *is an exacting cardinal*”.

Thank you for listening!