On Error-Computation of Symbolic Approximates of Parametrized Dynamical Systems

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Abstract – **This work concerns the model reduction of continuous-time dynamical systems with real parametric uncertainty. An algorithm for the computation of the maximal H_∞ norm difference between the original and the reduced system is given. In contrast to the results in [2] and references not only upper bounds for this number but its exact evaluation is possible.**

The algorithm computes the norm difference via a bisection algorithm using the mixed real and complex structured singular value in every step.

An application of this result is a symbolic-numeric analysis of parametrized linear time-invariant dynamical systems. Especially the symbolic transfer functions of electrical networks are generally very voluminous expressions, and model reduction methods are indispensable.

1 INTRODUCTION

The parametrized dynamical systems in which we are interested are linear, time-invariant, finite-dimensional, evolving in continuous time and depending rationally on finitely many parameters $p_1,...,p_k$. Such a system can be described by a rational matrix valued transfer function $G(s;p_1,...,p_k)$

s is supposed to be the frequency variable.

Further it is assumed that the system is stable for all possible parameter variations in an interval having the nominal value v_i in its center, i.e. $p_i \in I_i = [v_i - m_i, v_i + m_i]$

This means that the transfer function is proper and has no pole in the closed right complex halfplane, Now, the task is to set several parameters p_i to their nominal value v_i, and to determine the maximal error e in the H_∞-norm between the original and the simplified system, i.e.

$$e = \sup_{p_i \in I_i} \left\| G(s;p_1,...,p_k) - G(s;p_1,...,p_r,v_{r+1},...,v_k) \right\|_\infty$$

The H_∞-norm is intensively treated in [9], and is a common distance measure for stable transfer functions.

2 ERROR COMPUTATION

3.1 Parameter Transformation

The foundation of the algorithm is to consider the function

$$\tilde{G}(z;\delta_1,...,\delta_k) = G\left(\frac{1+z}{1-z}; v_1 + m_1\delta_1,..., v_k + m_k\delta_k\right)$$

and write this function in terms of a linear fractional representation, i.e.

$$\tilde{G}(z;\delta_1,...,\delta_k) = F\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \Delta\right) =: D + C\Delta(I - \Delta A)^{-1}B$$

for some real matrices A, B, C, D and the parameters are extracted in

$$\Delta = (zI_n, \delta_1 I_{N_1},..., \delta_1 I_{N_k}) .$$

This is also called realization of the multivariate rational matrix function $\tilde{G}$. It is proven in [3], [5] and [6], that such a representation exists if and only if the multivariate function $\tilde{G}$ is rational and has no pole at the origin. Therefore the given transfer function possesses a realization because it has no poles at the origin due to the fact that stability of G for all allowed parameter variations is assumed. Realization algorithms for this problem are presented e.g. in [3], [5] and [6]. However, for special classes of parametrized dynamical systems like e.g. electrical networks, a realization can be done directly without using these realization algorithms (see [5] and [8] for details).

From the matrices, there can be easily constructed a matrix quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ such that

$$\tilde{G}(z;\delta_1,...,\delta_k) - \tilde{G}(z;\delta_1,...,\delta_r,0,...,0) = F\left(\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}, \Delta_R\right)$$

for

$$\Delta = (zI_n, \delta_1 I_{2N_1},..., \delta_r I_{2N_r}, \delta_{r+1} I_{N_{r+1}},..., \delta_k I_{N_k}) .$$

This construction is without any numerical effort and intensively explained in [9].

The Algorithm is based of the structured singular value, which is defined below.

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Let $\mathbf{\Delta} \subset C^{k \times l}$ and let A be a l×k matrix. Then the structured singular value $\mu_{\mathbf{\Delta}}(A)$ is defined as follows:

$$\mu_{\mathbf{\Delta}}(A) = \frac{1}{\min\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta} : \det\ (I - A\Delta) = 0\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I - A\Delta$ singular, in which case $\mu_{\mathbf{\Delta}}(A) = 0$. More details about the structured singular value can be found in [4].

The following statement is the basis for the formulation of a bisection algorithm which computes

Theorem

Let $\gamma > 0$ and additionally $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{A}) < 1$ for

$$\tilde{\mathbf{\Delta}} = \left\{ \begin{matrix} \text{diag}(zI_n, \delta_1 I_{2N_1}, ..., \delta_r I_{2N_r}, \delta_{r+1} I_{N_{r+1}}, ..., \delta_k I_{N_k}) : \\ \text{where } \delta_i \text{ are real and } z \text{ is complex} \end{matrix} \right\}$$

Then

$$\mu_{\mathbf{\Delta}} \begin{pmatrix} \tilde{A} & \frac{1}{\sqrt{\gamma}} \tilde{B} \\ \frac{1}{\sqrt{\gamma}} \tilde{C} & \frac{1}{\gamma} \tilde{D} \end{pmatrix} < 1 \Leftrightarrow e < \gamma,$$

where e is the error which<h we want to compute and

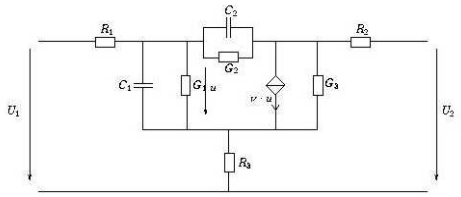
$$\mathbf{\Delta} = \left\{ \begin{matrix} \text{diag}(zI_n, \delta_1 I_{2N_1}, ..., \delta_r I_{2N_r}, \delta_{r+1} I_{N_{r+1}}, ..., \delta_k I_{N_k}, \Delta_1) : \\ \text{where } \delta_i \text{ are real, and } \Delta \text{ is a complex matrix} \end{matrix} \right\}$$

The result of this theorem makes it possible to test if a norm exceeds a given bound via a calculation of the structured singular value. Having an upper and a lower bound for the error, the error e can be determined via a bisection algorithm. A lower bound for e is simply given by 0. An upper bound for e are provided by methods presented in [2], [5] and [7]. There the bound is provided by linear matrix inequality methods.

The condidtion $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{A}) < 1$ is not restrictive since it is fulfilled by nearly all stable systems.

3.1 Example

In the following, we apply the method to the an equivalent circuit diagram of a transistor.



The voltage U_1 is taken as input and the output is U_2. The exact parametrized transfer function is then

$$G(s; \nu, G_1, G_2, G_3, C_1, C_2, R_1, R_2, R_3)$$
$$= -\frac{R_3C_1s + R_1C_1s + R_1C_2s + R_3G_2 + R_3G_1 + R_1G_2 + R_1G_1}{R_3C_2s + R_1C_2s + R_3C_1s + R_1C_1s - R_3C_2\nu s - R_3C_1\nu s + R_3G_1 + R_1G_2 + R_1G_1}$$

The nominal values of all ten parameters are assumed to be 1 and the allowed variations are in the interval [0.9,1.1]. Then

$$\tilde{G}(z; \delta_1, ..., \delta_{10}) = F\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \Delta \right)$$

with $\Delta = diag(z, \delta_1, ..., \delta_{10})$

$$A = \frac{1}{100} \begin{pmatrix} -20 & -27 & 27 & 27 & 0 & -41 & -41 & 27 & 27 & 0 \\ 6 & 8 & 2 & 2 & 0 & 2 & 2 & -8 & -18 & 10 \\ -6 & 2 & -2 & -2 & 0 & -2 & -2 & -2 & -2 & 0 \\ -6 & 2 & -2 & -2 & 0 & -2 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 2 & -2 & -2 & 0 & -2 & -2 & -2 & -2 & 0 \\ -6 & 2 & -2 & -2 & 0 & -2 & -2 & -2 & -2 & 0 \\ 6 & 8 & 2 & 2 & 0 & 2 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 8 & 2 & 2 & 0 & 2 & 2 & -8 & -8 & 0 \end{pmatrix}, B = \frac{1}{10} \begin{pmatrix} -27 \\ 8 \\ 2 \\ 2 \\ 0 \\ 2 \\ 2 \\ 8 \\ 0 \\ 8 \end{pmatrix}$$

$$C = \frac{1}{100}(6 \quad 18 \quad 2 \quad 2 \quad 0 \quad 2 \quad 2 \quad -8 \quad -18 \quad 10), D = 1.8$$

For instance, the approximate transfer function $G(s; 1, G_1, G_2, 1, C_1, C_2, R_1, 1, R_3)$

$$= -\frac{C_1s + C_2s + R_1C_1s + R_1 + G_1 + R_1G_2 + 2}{R_1C_2s + R_1C_1s + G_2 + R_1 + R_1G_1}$$

is considered.

Constructing the matrix quadruple $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ representing the multidimensional realization of the difference system, the Matlab μ-Analysis and Synthesis Toolbox yields

$$\mu_{\mathbf{\Delta}} \begin{pmatrix} \tilde{A} & \frac{1}{\sqrt{\gamma}} \tilde{B} \\ \frac{1}{\sqrt{\gamma}} \tilde{C} & \frac{1}{\gamma} \tilde{D} \end{pmatrix} < 1$$

for γ=0.666 and

$$\mu_{\mathbf{\Delta}} \begin{pmatrix} \tilde{A} & \frac{1}{\sqrt{\gamma}} \tilde{B} \\ \frac{1}{\sqrt{\gamma}} \tilde{C} & \frac{1}{\gamma} \tilde{D} \end{pmatrix} > 1$$

for γ=0.632, thus the error is in the interval [0.632,0.666].

Since the H_∞-norm of the nominal system is approximately 2, the relative error of the simplified transfer function is between 31.6% and 33.3%.

4 CONCLUSIONS

In this paper, a method for the simplification of parametrized transfer function has been presented. The kind of simplification is to set several parameters to their nominal value, i.e. to treat them numerically instead of keeping them as symbolic expressions.

The measure for the error of the simplified transfer function has been the maximal difference of the original and simplified transfer function in the H_∞-norm for the parameters varying in a neighbourhood of their nominal value. This is a reasonable measure since the H_∞-norm expresses the energy gain of a system.

The algorithm to compute this error is a bisection algorithm using a mixed real/complex µ-calculation in every step. The command "mu" of the Matlab µ Analysis and Synthesis Toolbox (see [1]) provides this.

In contrast to other publications like e.g. [2] and [5], not only conservative upper bounds for the error are provided by the algorithm but a calculation of its exact value up to an artbitrary precision.

Acknowledgments

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